Journal of Combinatorial Theory, Series B **74**, 160–168 (1998)
Article No. TB981838

Short Solution of Kotzig's Problem for Bipartite Graphs

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Received April 2, 1996

In 1975, A. Kotzig posed the following problem: Let G be a t-regular graph which has a proper edge t-coloring, $t \geqslant 4$. Is it possible to obtain, from one proper edge t-coloring of G, any other proper edge t-coloring of G using only transformations of 2-colored and 3-colored subgraphs such that the intermediate colorings are

1. INTRODUCTION

We use Bondy and Murty [4] for terminology and notation not defined here. An edge t-coloring or simply t-coloring of G is a mapping f: $E(G) \to \{1, ..., t\}$. If $e \in E(G)$ and $f(e) = k$ then we say that the edge e is colored k. We shall call a path (k, l)-colored if its edges are alternately colored k and l. The set of edges of color k we denote by $M(f, k)$. A t-coloring of G is called proper if no pair of adjacent edges receive the same color. Clearly f is a proper t-coloring if and only if $M(f, k)$ is a matching for every $k = 1, ..., t$. The minimum number t for which there exists a proper t-coloring of G is called the chromatic index of G and is denoted by $\chi'(G)$. It is clear that $\chi'(G) \geqslant \Delta(G)$ for any graph G. However, the problem of deciding whether $\chi'(G) = \Delta(G)$ is NP-complete even for simple regular graphs [8, 12].

Let f be a proper t-coloring of a graph G. An interchange with respect to colors α and β consists in swapping the two colors on the edges of a connected component of the subgraph induced by the set $M(f, \alpha) \cup M(f, \beta)$, thus obtaining a new proper edge coloring of G using at most t colors. Interchanges play a key role in investigations on edge colorings. Indeed the proofs of many results in this area are based on transformations of one proper edge coloring of a graph G to another using interchanges (see, for instance, [5–7, 13, 14]). Consider, for example, reformulations of three well-known results taking into considerations their proofs.

0095-8956/98 $25.00

THEOREM (König [11]). *For every bipartite graph G,* $\chi'(G)=\Delta(G)$. *Moreover every proper t-coloring of G,* $t>\Delta(G)$, *can be transformed to a proper* $\Delta(G)$*-coloring of G by a sequence of interchanges.*

THEOREM (Folkman and Fulkerson [5]). *Let* $N=(n_1, ..., n_t)$ *and* $Q=(q_1, ..., q_k)$ *be two non-increasing sequences of positive integers such that* $\sum_{i=1}^{t} n_i=\sum_{j=1}^{k} q_j$, $t\leqslant k$ *and* $\sum_{i=1}^{r} n_i\geqslant\sum_{j=1}^{r} q_j$ *for each* $r=1, ..., t$. *If a graph G has a proper t-coloring f such that exactly* n_i *edges are colored i for* $i=1, ..., t$ *then G also has a proper k-coloring g such that exactly* q_j *edges are colored j for* $j=1, ..., k$. *Moreover g can be obtained from f by a sequence of interchanges.*

THEOREM (Vizing [14]). *For every simple graph G,* $\chi'(G)\leqslant\Delta(G)+1$. *Moreover every proper t-coloring of G,* $t>\Delta(G)+1$, *can be transformed to a proper* $(\Delta(G)+1)$*-coloring of G by a sequence of interchanges.*

Actually, for a simple graph G with $\chi'(G)=\Delta(G)+1$ the result of Vising means that any proper edge coloring of G can be transformed to a proper $\chi'(G)$-coloring by using interchanges only. Is there a similar result for a graph G with $\chi'(G)=\Delta(G)$? This problem, posed by Vizing [14], is still open (see [9]).

However, it is known that there exist graphs G such that only interchanges, that is, transformations of 2-colored subgraphs of G, are not enough for obtaining any proper edge coloring of G from any other.

Taking into consideration this fact, Kotzig [10] posed the following problem: Let G be a t-regular graph with $\chi'(G)=t\geqslant 4$. Is it possible to obtain, from one proper t-coloring of G, any other proper t-coloring of G using only transformations of 2-colored and 3-colored subgraphs such that the intermediate colorings are also proper? The author and Mirumian [3] showed that for every $t\geqslant 4$ and $m\geqslant 3$ there exists a t-regular graph on $2m$ vertices where this cannot be done. Consider, for example, the graph K_6 with vertices $x_1, ..., x_6$ and its two proper 5-colorings f and g, where

$$M(f, 1)=\{x_1x_5, x_2x_4, x_3x_6\}, \qquad M(f, 2)=\{x_1x_4, x_2x_6, x_3x_5\},$$
$$M(f, 3)=\{x_1x_6, x_2x_3, x_4x_5\}, \qquad M(f, 4)=\{x_1x_3, x_2x_5, x_4x_6\},$$
$$M(f, 5)=\{x_1x_2, x_3x_4, x_5x_6\}, \qquad M(g, 1)=\{x_1x_3, x_2x_6, x_4x_5\},$$
$$M(g, 2)=\{x_1x_5, x_2x_3, x_4x_6\}, \qquad M(g, 3)=\{x_1x_6, x_2x_4, x_3x_5\},$$
$$M(g, 4)=\{x_1x_4, x_2x_5, x_3x_6\}, \qquad M(g, 5)=\{x_1x_2, x_3x_4, x_5x_6\}.$$

Clearly, g cannot be obtained from f by renaming the colors. On the other hand any proper edge 3-coloring of a subgraph $G(t_1, t_2, t_3)$ induced by the set of edges $M(f, t_1)\cup M(f, t_2)\cup M(f, t_3)$ gives the same partition of edges of $G(t_1, t_2, t_3)$ into perfect matchings, for any $1 \leqslant t_1 < t_2 < t_3 \leqslant 5$. It follows from the fact that $G(t_1, t_2, t_3)$ contains a triangle and edges of this triangle belong to disjoint, uniquely defined perfect matchings. Therefore g cannot be obtained from f by transformations of 2-colored and 3-colored subgraphs.

So, in the general case the answer to Kotzig's question is no. Fortunately, for bipartite graphs the question has an affirmative answer. Two different transformations, named a 2-transformation and a 3-transformation, of proper t-colorings of t-regular bipartite graphs were defined in [1, 2] (see below), where the k-transformation uses a k-colored subgraph for $k = 2, 3$. The following theorem was obtained by using these transformations.

THEOREM 1 (Asratian and Mirumian [1,2]). *Let $t \geqslant 3$ and let G be a t-regular bipartite graph. Then every proper t-coloring of G can be obtained from any other by a sequence of 2- and 3-transformations so that all intermediate colorings are also proper.*

A similar result for arbitrary bipartite graphs follows from Theorem 1.

COROLLARY 2 [1,2]. *Let H be an arbitrary bipartite graph. If f is a proper m-coloring and g is a proper n-coloring of H, then f can be transformed into g such that each intermediate coloring is proper and differs from the previous coloring by a 2- or 3-colored subgraph.*

Note that these results can be useful in practice because many scheduling problems can be reformulated as edge coloring problems in bipartite graphs (see, for example, [15, 16]).

Theorem 1 was first annonced without proof in [1]. The proof given in [2] is complicated.

In this paper we give a simplified proof of Theorem 1. The proof is constructive and contains a polynomial algorithm for transforming one proper edge coloring of G to another. This algorithm can be used for transformations of latin squares because every latin square of order t can be represented as a properly t-colored complete bipartite graph $K_{t,t}$. The paper is concluded with a conjecture.

2. PROOF OF THEOREM 1

We begin by describing the two transformations.

Let G be a t-regular bipartite graph, f a proper t-coloring of G, and let $C = v_0 e_1 v_1 e_2 \cdots e_{2k-1} v_{2k-1} e_{2k} v_0$ be a cycle of G, in which the color of all the even numbered edges is α. If the color of all the odd numbered edges is β, then we could carry out the first of our transformations, a 2-transformation of G (along C), by exchanging the colors of the edges along the cycle. (Actually, this is an interchange along the cycle C.)

Suppose instead that the odd numbered edges are colored with one of two colors, β and γ. Then the cycle C is called a 3-color (α, β, γ)-cycle or sometimes a 3-colored cycle. In this case we can carry out our second type of transformation: divide the set

$$(M(f, \beta) \cup M(f, \gamma) \cup \{e_2, ..., e_{2k}\}) \backslash \{e_1, ..., e_{2k-1}\}$$

into two matchings P_1 and P_2, and define a new proper t-coloring g by

$$g(e) = \begin{cases} \alpha & \text{if } e \in (M(f, \alpha) \backslash \{e_2, ..., e_{2k}\}) \cup \{e_1, ..., e_{2k-1}\}, \\ \beta & \text{if } e \in P_1 \\ \gamma & \text{if } e \in P_2, \\ f(e) & \text{if } f(e) \notin \{\alpha, \beta, \gamma\}. \end{cases}$$

This transformation we shall call a 3-transformation of G (along C). Such a 3-transformation also only changes the original coloring locally: if E_0 denotes the set $M(f, \alpha) \cup M(f, \beta) \cup M(f, \gamma)$, the transformation first changes the matching $M(f, \alpha)$ along the cycle C, and then colors the rest of E_0 with the remaining two colors β and γ.

The effects of this pair of transformations can perhaps be better understood from the 3-colorings f and g of $K_{3,3}$, shown in Fig. 1. It is easy to check that g cannot be obtained from f by a sequence of 2-transformations alone, but g is the result of a 3-transformation of f along the 3-colored cycle C.

Now we need a little more notation and three preliminary lemmas.

Let f and g be two distinct proper t-colorings of G. We shall say that f and g differ by an m-colored subgraph if there is a set of colors S, of size m, so that $M(f, j) \neq M(g, j)$ for each $j \in S$, but $M(f, j) = M(g, j)$ for each $j \notin S$. We denote by $G(f, g, j)$ the colored subgraph induced by the edge subset $M(f, j) \triangle M(g, j) = (M(f, j) \cup M(g, j)) \backslash (M(f, j) \cap M(g, j))$, where each edge $e \in M(f, j) \triangle M(g, j)$ has the color $f(e)$. Since G is regular, all

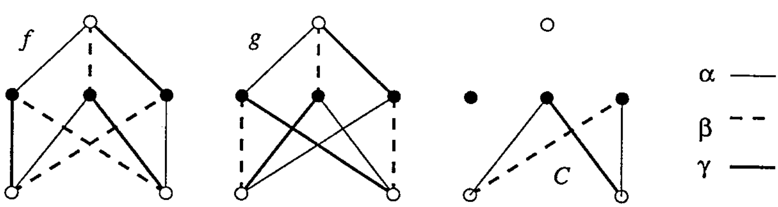

FIGURE 1

the components of $G(f, g, j)$ are cycles of even length, and the following lemma is evident.

LEMMA 3. *If f and g differ by a 2-colored subgraph then g can be obtained from f by a sequence of 2-transformations.*

LEMMA 4. *If f and g differ by a 3-colored subgraph then g can be obtained from f by a sequence of 2- and 3-transformations.*

Proof. Suppose we have colors α, β, and γ so that $M(f, j) \neq M(g, j)$ if and only if $j \in \{\alpha, \beta, \gamma\}$. Now we construct a sequence $\{f_k\}$, $k \geqslant 0$, of proper t-colorings of G in the following way. Put $f_0 = f$. Suppose we have already constructed a sequence $f_0, ..., f_k$ of proper t-colorings of G, where $M(f_k, j) = M(g, j)$ for each $j \notin \{\alpha, \beta, \gamma\}$.

If the coloring f_k differs from g by a 2-colored subgraph then, by the previous lemma, g can be obtained from f_k by a sequence of 2-transformations.

If the coloring f_k differs from g by a 3-colored subgraph then $M(f_k, j) \neq M(g, j)$ for each $j \in \{\alpha, \beta, \gamma\}$ and the subgraph $G(f_k, g, \alpha)$ contains a d-colored cycle C_k for some $d \in \{2, 3\}$. A d-transformation along C_k provides a new proper t-coloring f_{k+1} such that

$$|M(f_{k+1}, \alpha) \cap M(g, \alpha)| > |M(f_k, \alpha,) \cap M(g, \alpha)|.$$

By repeating this process, we can obtain g from f by a sequence of 2- and 3-transformations. ▌

Now we continue with the main lemma.

LEMMA 5. *Let $t \geqslant 4$, and f and g be two distinct proper t-colorings of G. If $M(f, t) \neq M(g, t)$ then for some $k \geqslant 1$ there exists a sequence of proper t-colorings $f_0, f_1, ..., f_k$ such that $f = f_0$ and*

$$|M(f_k, t) \cap M(g, t)| > |M(f, t) \cap M(g, t)|$$

with f_i and f_{i+1} differing by a 2- or 3-colored subgraph, for each $i = 0, 1, ..., k-1$.

Proof. If the subgraph $G(f, g, t)$ contains a 2- or 3-colored cycle C, then we can take $k = 1$, produce f_1 from f by an appropriate 2- or 3-transformation along C, and we are done. If this is not the case we must work a little harder.

Consider a component C of the colored subgraph $G(f, g, t)$. Clearly, C is a cycle. Let $C = v_0 e_1 v_1 e_2 ... v_{2m-1} e_{2m} v_{2m}$, where $v_{2m} = v_0$. Without loss of generality we assume that $f(e_{2j-1}) = t$ for $j = 1, ..., m$, and $f(e_2) \neq f(e_{2m})$. Then there are integers $n_0, n_1, ..., n_{k+1}$ and colors $s_0, s_1, ..., s_k$ such that

$k \geqslant 2$, $s_i \neq s_{i+1}$ for $i = 0, 1, ..., k-1$, $s_0 \neq s_k$, $0 = n_0 < n_1 < \cdots < n_{k+1} = m$, and the path

$$P_i = v_{2n_i} e_{2n_i+1} v_{2n_i+1} e_{2n_i+2} \ldots e_{2n_{i+1}} v_{2n_{i+1}}$$

is a (t, s_i)-colored path, for each $i = 0, 1, ..., k$.

Now we will construct a sequence of proper t-colorings $f_0, f_1, ..., f_{k-1}, f_k$ and a sequence of auxiliary improper colorings $f'_0, f'_1, ..., f'_{k-1}$ in the following way.

Put $f_0 = f$. We denote by f'_0 the improper coloring obtained from f_0 by interchanging the colors s_0 and t along the edges of the path P_0, and by interchanging the colors s_1 and t along the edges of the path P_1. Then f'_0 has the following properties:

(1a) There is no edge incident with v_0 of color t, but there are two edges of color s_0, and one of each color $j \neq s_0$, $1 \leqslant j \leqslant t-1$.

(1b) There is no edge incident with v_{2n_1} of color s_0, but there are two of color s_1, and one of each color $j \neq s_0, s_1$, $1 \leqslant j \leqslant t$.

(1c) There are no edges incident with v_{2n_2} of color s_1, but there are two of color t, and one of each color $j \neq s_1$, $1 \leqslant j \leqslant t-1$.

(1d) At each vertex other than v_0, v_{2n_1}, and v_{2n_2} each color appears on precisely one edge, and $M(f'_0, t) \cap M(g, t) = (M(f, t) \cap M(g, t)) \cup \{e_{2j}: j = 1, 2, ..., n_2\}$.

Suppose that we have already constructed proper t-colorings $f_0, ..., f_i$, and improper t-colorings $f'_0, ..., f'_i$ of G, $0 \leqslant i \leqslant k-2$, such that f'_i satisfies the following conditions:

(2a) There is no edge incident with v_0 of color t, but there are two edges of color s_i, and one of each color $j \neq s_i$, $1 \leqslant j \leqslant t-1$.

(2b) There is no edge incident with $v_{2n_{i+1}}$ of color s_i, but there are two of color s_{i+1}, and one of each color $j \neq s_i, s_{i+1}$, $1 \leqslant j \leqslant t$.

(2c) There are no edges incident with $v_{2n_{i+2}}$ of color s_{i+1}, but there are two of color t, and one of each color $j \neq s_{i+1}$, $1 \leqslant j \leqslant t-1$.

(2d) At each vertex other than v_0, $v_{2n_{i+1}}$, and $v_{2n_{i+2}}$ each color appears on precisely one edge, and

$$M(f'_i, t) \cap M(g, t) = (M(f, t) \cap M(g, t)) \cup \{e_{2j}: j = 1, 2, ..., n_{i+2}\}.$$

Now we construct a proper t-coloring f_{i+1} of G. Consider the subgraph H induced by the sets $M(f'_i, s_i) \cup M(f'_i, s_{i+1})$. It follows from (2a)–(2d) that H contains, with respect to f'_i, an (s_i, s_{i+1})-colored path of even length with end vertices v_0 and $v_{2n_{i+1}}$. Let us interchange the colors s_i and s_{i+1}

along this path to obtain a new t-coloring f'_{i1}. Then f'_{i1} is not proper, and has the following properties:

(3a) There is no edge incident with v_0 of color t, but there are two edges of color s_{i+1}, and one of each color $j \neq s_{i+1}$, $1 \leqslant j \leqslant t-1$.

(3b) There are no edges incident with $v_{2n_{i+2}}$ of color s_{i+1}, but there are two of color t, and one of each color $j \neq s_{i+1}$, $1 \leqslant j \leqslant t-1$.

(3c) At each vertex other than v_0 and $v_{2n_{i+2}}$ each color appears on precisely one edge.

It follows from (3a)–(3c) that there is a (t, s_{i+1})-colored path Q of even length with end vertices v_0 and $v_{2n_{i+2}}$. This path Q finally allows the construction of a proper t-coloring f_{i+1} from f'_{i1} by interchanging the colors t and s_{i+1} along Q. It is clear that $M(f_{i+1}, j) = M(f_i, j)$ for each $j \notin \{s_i, s_{i+1}, t\}$, that is, f_i and f_{i+1} differ by a 2- or 3-colored subgraph. Now we denote by f'_{i+1} an improper t-coloring obtained from f_{i+1} by interchanging the colors t and s_{i+1} along Q (we again obtain f'_{i1}), and also by interchanging the colors t and s_{i+2} along P_{i+2}. It is clear that $M(f'_{i+1}, t) \cap M(g, t) = (M(f, t) \cap M(g, t)) \cup \{e_{2j}\colon j = 1, 2, ..., n_{i+3}\}$.

If $i < k-2$ we construct f_{i+2} from f'_{i+1} by the above procedure.

If $i = k-2$ then $M(f'_{k-1}, t) \cap M(g, t) = (M(f, t) \cap M(g, t)) \cup \{e_{2j}\colon j = 1, ..., m\}$. Moreover, the coloring f'_{k-1} satisfies the following conditions:

(4a) There is no edge incident with v_0 of color s_k, but there are two edges of color s_{k-1}, and one of each color $j \neq s_k, s_{k-1}$, $1 \leqslant j \leqslant t$.

(4b) There is no edge incident with v_{2n_k} of color s_{k-1}, but there are two of color s_k, and one of each color $j \neq s_k, s_{k-1}$, $1 \leqslant j \leqslant t$.

(4c) At each vertex other than v_0 and v_{2n_k} each color appears on precisely one edge.

It follows from (4a)–(4c) that there is an (s_k, s_{k-1})-colored path Q' with end vertices v_0 and v_{2n_k}. By interchanging the colors along Q' we obtain a new proper t-coloring f_k with $M(f_k, t) \cap M(g, t) = (M(f, t) \cap M(g, t)) \cup \{e_{2j}\colon j = 1, ..., m\}$. ▮

Proof of Theorem 1. We shall prove the theorem by induction on t. For $t = 3$ the result follows from Lemmas 3 and 4. Let us turn then to the induction step and suppose that G is a t-regular bipartite graph and that the induction hypothesis holds for $(t-1)$-regular graphs, $t \geqslant 4$.

Let ϕ and ψ be two distinct t-colorings of G. The proof breaks into two cases.

Case 1. $M(\phi, t) = M(\psi, t)$. Then the graph $G' = G - M(\phi, t)$ is $(t-1)$-regular. Let ϕ' and ψ' be the two distinct proper $(t-1)$-colorings of G' induced by ϕ and ψ, respectively. Then, since by the inductive hypothesis

ϕ' can be obtained from ψ' by a sequence of 2- and 3-transformations, the same must be true of ϕ and ψ.

Case 2. $M(\phi, t) \neq M(\psi, t)$. By Lemma 5, we can obtain a sequence of proper t-colorings $\phi = \phi_0, \phi_1, ..., \phi_k$ so that $M(\phi_k, t) = M(\psi, t)$ and ϕ_{i+1} differs from ϕ_i by a 2- or 3-colored subgraph, for each $i = 0, 1, ..., k-1$. Then Lemmas 3 and 4 imply that ϕ_k can be obtained from ϕ by a sequence of 2- and 3-transformations. Finally then, as in Case 1, ψ can be obtained from ϕ_k by the induction hypothesis, and the proof of the theorem is complete. ∎

It is not difficult to see that this proof provides a polynomial algorithm for transforming one proper t-coloring of G into another.

Proof of Corollary 2. Take two disjoint copies H' and H'' of H, with $V(H') = \{x' : x \in V(H)\}$ and $V(H'') = \{x'' : x \in V(H)\}$ and let $t = \max\{m, n\}$. Then we can define a t-regular bipartite graph G obtained from H' and H'' by joining x' and x'' with $t - d_H(x)$ parallel edges, for each vertex $x \in V(H)$. The coloring f of H induces a proper t-coloring ϕ of G in the following way: we color the copies H' and H'' in the same way as H, and then color the set of parallel edges joining x' to x'' with those colors from $\{1, ..., t\}$ which are not used to color an edge incident with x in H. Similarly, the coloring g induces a proper coloring ψ on G. Thus, since the theorem ensures that ϕ can be transformed into ψ by a sequence of 2- and 3-transformations, it is clear that these transformations also define a sequence of proper colorings of H, beginning with f and ending with g. ∎

Finally, I would like to formulate the following conjecture.

Conjecture. Let f and g be two proper t-colorings of a t-regular graph G with $t \geqslant 5$. Then there exists a sequence $f_0, f_1, ..., f_k$ of proper t-colorings of G such that $f_0 = f$, $f_k = g$ and the colorings f_i and f_{i-1} differ by a d_i-colored subgraph, where $d_i \leqslant 4$ for $i = 1, ..., k$.

ACKNOWLEDGMENTS

I thank R. Häggkvist, T. Denley, and A. N. Mirumian for discussions and helpful comments. I also thank the referees for their remarks which led to this improved version.

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