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Eighty Years of Sommerfeld's Radiation Condition

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In 1912 Sommerfeld introduced his radiation condition to ensure the uniqueness of the solution of certain exterior boundary value problems in mathematical physics. In physical applications these problems generally describe wave propagation where an incident time-harmonic wave is scattered by an object, and the resulting diffracted or scattered waves need to be calculated. When formulated mathematically, these problems usually take the form of an exterior Dirichlet or Neumann problem for the Helmholtz partial differential equation. The Sommerfeld condition is applied at infinity and, when added to the statement of the boundary value problem, singles out only the solution which represents "outgoing" (rather than "incoming" or "standing") waves in the physical applications. Since its introduction, the Sommerfeld radiation condition has become indispensable for these types of problems and has stimulated a considerable amount of mathematical research, especially in uniqueness theorems. The present note traces the motivation and reasoning that led Sommerfeld to the original formulation of his radiation condition and surveys the extensions and modifications this condition has undergone since then. © 1992 Academic Press, Inc.

En 1912 Sommerfeld introduisit sa condition de rayonnement pour assurer l'unicité de la solution de certains problèmes extérieurs de valeurs au bord en physique mathématique. Dans les applications physiques ces problèmes décrivent en général des propagations d'ondes, où une onde harmonique incidente est diffractée par un objet et l'on cherche à calculer l'onde diffractée ou dispersée. Quand on les formule mathématiquement, ces problèmes prennent le plus souvent la forme d'un problème de Dirichlet ou de Neumann extérieur pour l'équation aux dérivées partielles de Helmholtz. La condition de Sommerfeld est appliquée à l'infini et, quand on l'ajoute à l'énoncé du problème au bord, elle séléctionne la seule solution qui dans les applications physiques représente les ondes "sortantes" (et non pas les "entrantes" ou "stationnaires"). Depuis son introduction la condition de rayonnement de Sommerfeld est devenue indispensable pour les problèmes de ce type et a stimulé une grande quantité de recherches mathématiques, en particulier des théorèmes d'unicité. La note présente retrace la motivation et le raisonnement qui ont conduit Sommerfeld à la formulation originale de sa condition de rayonnement et passe en revue les extensions et modifications que cette condition a subi depuis. © 1992 Academic Press, Inc.

Im Jahre 1912 führte Sommerfeld die nach ihm benannte Ausstrahlungsbedingung ein, um die Eindeutigkeit der Lösung von gewissen äusseren Randwertaufgaben in der mathematischen Physik zu sichern. In physikalischen Anwendungen beschreiben diese Aufgaben im allgemeinen Wellenausbreitungen, in denen eine eintreffende harmonische Welle durch ein Hindernis gestreut wird und die hierdurch entstehenden gebeugten oder gestreuten Wellen zu berechnen sind. Mathematisch formuliert nehmen diese Aufgaben meistens die Gestalt eines Dirichletschen oder Neumannschen Problems für die Helmholtzsche Schwingungsgleichung in einem Aussengebiet an. Die Sommerfeldsche Bedingung wird im Unendlichen angewandt und wählt, wenn der Formulierung der Randwertaufgabe beigefügt, nur die Lösung aus, die „divergierende" (und nicht etwa „konvergierende" oder „stehende") Wellen in den physikalischen Anwendungen darstellt. Seit ihrer Einführung ist die Sommerfeldsche Ausstrahlungsbedingung für Aufgaben dieser Art unentbehrlich geworden und hat

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auf die mathematische Forschung—besonders die Unitätssätze betreffend—fördernd gewirkt. Die vorliegende Arbeit sucht, die Beweggründe und Überlegungen aufzufinden, die Sommerfeld zur Originalfassung seiner Ausstrahlungsbedingung veranlassten, und gewährt einen Überblick über die Erweiterungen und Änderungen, die diese Bedingung seitdem erfahren hat.

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1. INTRODUCTION

It has been eighty years since Sommerfeld introduced his radiation condition. This condition prescribes the asymptotic behavior of the solutions of exterior boundary value problems for certain classes of partial differential equations in order to ensure the uniqueness of the solution. These boundary value problems generally govern wave propagation where a given acoustic, elastic, or electromagnetic wave encounters an object and it is desired to calculate the reflected, diffracted, or scattered waves which result. Typical examples are the scattering of sound by a small solid sphere, the diffraction of light by a wedge, and the propagation of radio waves along the earth's surface. Under certain simplifying assumptions these problems can all be formulated mathematically as exterior boundary value problems for the Helmholtz equation $\Delta u + k^2u = 0$, where u is the function describing the waves, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator, and k is a positive constant.

One of the difficulties with formulating a wave propagation problem in this way is that the solution may not be unique. Besides the expected outgoing waves which result when the incident wave is scattered by the object, the mathematical solution also provides incoming waves which originate at infinity and move towards the object. These incoming waves are physically meaningless and must be rejected by some criterion built into the mathematical formulation of the problem. Sommerfeld was the first to state a mathematically precise and easily applicable condition which, when added to exterior boundary value problems for the Helmholtz equation, ensures a unique solution. This condition is applied at infinity and for three-dimensional problems requires that the solution u satisfy

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \qquad r = \sqrt{x^2 + y^2 + z^2}, \qquad i = \sqrt{-1}, \tag{1}$$

uniformly with respect to all directions in which the limit is approached.

Since its introduction in 1912 Sommerfeld's radiation condition (1) has become the standard and indispensable criterion used to ensure the uniqueness of the solution for these types of problems in mathematical physics. Moreover, it has also stimulated a considerable amount of research on uniqueness theorems for these problems from a purely mathematical standpoint. As a result, Sommerfeld's condition has been reformulated in a number of equivalent ways, and it has been modified to make it applicable to a wider class of problems. In this note we will attempt to trace the reasoning that led Sommerfeld to the original formulation of the radiation condition and survey some of the extensions and modifications it has undergone since then.

Sommerfeld is known primarily for his contributions to theoretical physics [Laue 1951, Heisenberg 1968] and he is revered as the mentor of a whole generation of physicists [Born 1928]. Since these achievements overshadow his contributions to mathematics, the latter are often overlooked. Thus his discovery of the radiation condition and the impact it had on mathematics are not even mentioned in his biographies [Benz 1975, Eckert *et al.* 1984]. It should be remembered, however, that Sommerfeld was originally trained as a mathematician, and as a student, collaborator, and lifetime admirer of Felix Klein he shared his mentor's view that there should be fruitful cross-fertilization between mathematics and physics and that the two should never become divorced from each other. As Sommerfeld himself asserts in the introduction to his book on partial differential equations in physics:

> We do not really deal with mathematical physics, but with physical mathematics; not with the mathematical formulation of physical facts, but with the physical motivation of mathematical methods. The oft-mentioned "prestabilized harmony" between what is mathematically interesting and is physically important is met at each step and lends an esthetic—I should like to say metaphysical—attraction to our subject." [Sommerfeld 1945 v]

The history of Sommerfeld's radiation condition provides an interesting example of this interplay between physical intuition and rigorous mathematical reasoning.

2. THE EXTERIOR BOUNDARY VALUE PROBLEM FOR THE HELMHOLTZ EQUATION

Although Euler and Lagrange considered the equation $\Delta u + k^2 u = 0$ in connection with sound propagation and vibrating membranes as early as 1759, another century was to elapse before Helmholtz [1860] developed a general solution theory for this equation. Helmholtz studied sound waves in a tube with one open end (organ pipe) and showed that the solution of three-dimensional *interior* boundary value problems for the equation now named after him could be represented in terms of the boundary values u and $\partial u/\partial \nu$ by the formula

$$4\pi u(P) = \int_S \left[\frac{\partial u}{\partial \nu}\frac{e^{ikr}}{r} - u\frac{\partial}{\partial \nu}\left(\frac{e^{ikr}}{r}\right)\right] dS, \tag{2}$$

where P is a point inside the closed surface S, r is the distance from P to S, and $\partial/\partial\nu$ denotes differentiation along the outward normal to S. Weber [1870] extended this result to two dimensions in 1870 and Pockels [1891], on the suggestion of Klein, wrote a monograph on the Helmholtz equation in 1891. The history up to 1900 of boundary value problems for this equation and other partial differential equations is summarized in the *Encyklopädie* article by Sommerfeld [1900]. It is noteworthy that Sommerfeld does not mention *exterior* boundary value problems for the Helmholtz equation in his article since little was known about them at the time.

Exterior boundary value problems for the Helmholtz equation first began to appear in connection with certain physics problems in the final decades of the last century. Important among these are Lord Rayleigh's studies on the scattering of sound by obstacles. These are discussed in his treatise *The Theory of Sound* [Rayleigh 1894], a work which Sommerfeld quotes several times in his encyclopedia article and elsewhere. Rayleigh distinguishes between outgoing and incoming waves and rejects the latter in applications [Rayleigh 1894 II, 109, 238]. For example, in his treatment of what is now called Rayleigh scattering, he considers only "a disturbance due to the presence of the sphere, and radiating outwards from it" [Rayleigh 1894 II, 273]. Since Rayleigh's solution method became a model for treating propagation problems of this type we outline it here.

Rayleigh considers a plane wave incident upon a small solid sphere centered at the origin and assumes that the resulting scattered wave is spherical with amplitude A depending on the angle θ that the direction of the outgoing wave makes with that of the incoming one. Letting a and ω be the propagation speed and frequency, respectively, the composite wave then satisfies the wave equation $\Delta v - (1/a)^2 v_{tt} = 0$ exterior to the sphere and is

$$v = v_{\text{inc}} + v_{\text{scat}} = e^{i(kx-\omega t)} + A(\theta)\frac{e^{i(kr-\omega t)}}{r},$$

where $k = \omega/a$. When the time dependent term $e^{-i\omega t}$ is factored out, the stationary factor

$$u = e^{ikx} + A(\theta)\frac{e^{ikr}}{r}$$

satisfies Helmholtz' equation $\Delta u + k^2 u = 0$. Together with the condition that the sphere is impenetrable ($\partial u/\partial \nu$ given on the sphere) this forms an exterior boundary value problem. By expanding e^{ikx} in terms of spherical harmonics and $A(\theta)e^{ikr}/r$ in terms of spherical harmonics and Bessel functions *and retaining only terms which correspond to outgoing waves*, Rayleigh obtains an approximate solution of this problem in the far field. For spheres which are small compared to the wavelength $\lambda = 2\pi/k$, the scattered amplitude is approximately

$$A(\theta) = -\frac{\pi T}{\lambda^2}\left(1 - \frac{3}{2}\theta\right),$$

where T is the volume of the sphere. From this Rayleigh deduces his scattering law that "the ratio of the scattered and direct waves is in general proportional to the inverse square of the wave-length" [Rayleigh 1894 II, 277]. In a later paper [Rayleigh 1897] he showed that this law also holds for sunlight scattered by the water droplets in the earth's atmosphere. He used this to explain why the sky is blue: The shorter wavelengths (blue) of sunlight are less attenuated and hence pass through the atmosphere more readily.

This example shows how in practical problems a unique solution could be obtained by ruling out incoming waves on physical grounds. Another exclusion argument that was sometimes used [Lamb 1895, 499; Sommerfeld 1912, 331] is based on energy considerations: An outgoing radial wave originates from an energy emitting source at a finite point, whereas an incoming wave would have to originate from such a point at infinity (or a sink at a finite point); but this is physically impossible.

Let us now consider the *general* exterior boundary value problem

$$\begin{cases} \Delta u + k^2 u = 0 & \text{outside a closed surface } \sigma \\ u = f & \text{on } \sigma \left(\text{or } \dfrac{\partial u}{\partial \nu} = g \text{ on } \sigma \right), \end{cases} \tag{3}$$

where f (or g) is a given function. The solution of this problem is not unique. For let σ be the sphere $r = \pi/k$ and let the boundary condition $u = 0$ be given on this sphere. Clearly $u \equiv 0$ and $u = (\sin kr)/r$ are two different solutions of this problem outside of σ. In fact, the second solution represents a standing wave and may be added to *any* solution of this problem. (A somewhat similar counterexample was already given by Pockels [1891, 236], although not in the context of *exterior* boundary value problems.)

The undesirable standing wave solution $u = (\sin kr)/r$ may be decomposed by $u = (\frac{1}{2}i)(u_1 - u_2)$ into two other solutions of the Helmholtz equation, namely $u_1 = e^{ikr}/r$ and $u_2 = e^{-ikr}/r$ (the factors $\pm\frac{1}{2}i$ have been omitted). These correspond to the solutions $v_1 = e^{i(kr-\omega t)}/r$ and $v_2 = e^{-i(kr+\omega t)}/r$ of the wave equation $\Delta v - (1/a^2)v_{tt} = 0$, where $k = \omega/a$, after the time dependence has been factored out by making the substitution $v = ue^{-i\omega t}$. (Since this substitution reduces the wave equation to Helmholtz' equation the latter is also often called the *reduced wave equation.*) The functions v_1 and v_2 represent outgoing and incoming spherical waves, respectively. Hence incoming and standing waves would be ruled out if a condition could be found which when added to the statement of the boundary value problem (3) would reject solutions of the type of u_2. That is, this condition has to rule out not only the simple solution u_2, but also *all* solutions of the Helmholtz equation corresponding to incoming waves. This then was the condition which Sommerfeld set out to find in 1912.

3. SOMMERFELD'S EARLY WORK IN MATHEMATICS AND PHYSICS

In his 1912 paper Sommerfeld [1912] cites two examples of exterior boundary value problems requiring a radiation condition: (1) optical diffraction theory and (2) radio wave propagation. Sommerfeld [1896, 1909] had written important papers on these two topics in 1896 and 1909, respectively, and the uniqueness questions raised therein must have motivated him to search for a mathematically satisfactory way of resolving these questions. Heretofore these uniqueness questions had always been resolved by appealing to physical considerations (inadmissible incoming waves, energy transport from infinity, etc.), but to a person with Sommerfeld's mathematical background these artifices must have seemed contrived and thus encouraged him to look for a single mathematically formulated criterion which when added to the statement of the exterior boundary value problem (3) makes this a "well-posed" problem in the sense of Hadamard. To understand Sommerfeld's motivation and outlook more clearly we first take a look at his mathematical background and experience. Fortunately he left us an autobiographical sketch [Sommerfeld 1951]—written from the perspective of his advanced years—which clearly portrays the formative influences on his work. From this we will select the facts relevant for the discussion of our particular topic.

Arnold Johannes Wilhelm Sommerfeld was born in Königsberg (East Prussia) in 1868 and attended the Gymnasium and studied mathematics at the Albertus Magnus University in his home town. Among his teachers at the university were Hilbert, Hurwitz, and Lindemann, and he completed his doctoral dissertation entitled "Arbitrary Functions in Mathematical Physics" under Lindemann. In 1893 he went to Göttingen, which he calls the "Ort mathematischer Hochkultur" [Sommerfeld 1951, 675] and a year later became Felix Klein's assistant there. This association became crucial for his entire professional development:

> The impression I received of F. Klein's imposing personality through his lectures and in conferences with him was overpowering. Klein resolutely tried to captivate my interest in problems of mathematical physics and to get me to accept his view of these problems which he had expounded in his earlier lectures. I have always regarded Klein as my real teacher, not only in mathematical, but also in mathematical–physical matters, and in the conceptual interpretation of mechanics. The model he provided with his extraordinary expository skill was an incisive influence on my later teaching career. [Sommerfeld 1951, 675]

Under Klein's direction, Sommerfeld [1896] completed his *Habilitationsschrift* in 1896 entitled "Mathematical Theory of Diffraction." In this brilliant paper he gives the first mathematically rigorous solution of diffraction from the straight edge of a screen *as a boundary value problem* (previous theories by Kirchhoff, Fresnel, and others were less precise). Sommerfeld's approach to the problem earned him Poincaré's accolade: "méthode extrêmement ingénieuse" [Sommerfeld 1951, 675].

Assuming the light source to be a large distance from the screen, the incident wave is plane and meets the edge of the screen at an angle. The diffracted electric field is then also parallel to the edge of the screen and the problem becomes two-dimensional; i.e., it can be represented in terms of polar coordinates r and ϕ. The field as modified by the presence of the screen then satisfies (after the time-dependent term has been factored out) a two-dimensional boundary value problem of the type (3) with the boundary condition $u = 0$ given along the edge of the screen. (A weakly singular edge condition $r \operatorname{grad} u \to 0$ must also be imposed to ensure that the edge does not radiate or absorb energy.) Sommerfeld then solves the boundary value problem by the method of images, but with an important and novel modification: one of the image sources is located on the second sheet of a two-sheeted Riemann surface so that it does not interfere with the illuminated part of the field. The field is then represented by the wave packet

$$u = \int A(\beta)e^{-ikr\cos(\phi-\beta)}\,d\beta,$$

where $A(\beta)$ is the amplitude distribution of the field and the integration takes place along a path on the Riemann surface. Sommerfeld chooses this path judiciously so that u satisfies the boundary conditions and *the field has a radiative* (*and not an absorptive*) *character*. Thus he obtains a closed form solution in terms of real definite integrals which can be evaluated and yield approximations agreeing with the results of earlier investigators. This work clearly shows Klein's imprint, and it should be mentioned that it also provides perhaps the earliest example of what von Laue in his eulogy of Sommerfeld calls his "sportive virtuosity in the evaluation of definite integrals in the complex plane" [Laue 1951, 214]. Heisenberg also tells the story that Sommerfeld's students in Munich were advised by the more advanced students to "integrate a few times in the complex plane" to receive a good grade from him [Heisenberg 1968, 530].

After earning his right to teach at the university level, Sommerfeld served as Privatdozent at the Georg August University in Göttingen for five semesters and collaborated with Klein on what was to become their four-volume treatise *Theory of the Spinning Top* (completed in 1910). In 1897 Sommerfeld obtained a professorship in mathematics through Klein's efforts at the comparatively little known Bergakademie Clausthal in the Harz mountains near Göttingen. Here he taught primarily elementary mathematics and spent much time on his duties as one of the editors of the monumental *Encyklopädie der mathematischen Wissenschaften* organized by Klein. The aforementioned encyclopedia article on boundary value problems [Sommerfeld 1900] was written at this time. In 1900 Sommerfeld was appointed to a professorship at the more prominent Technical Institute in Aachen, again through the efforts of Klein. The titles of his papers written during his tenure there suggest that he had to concern himself primarily with engineering problems ("On the Theory of Railroad Brakes" (1902), "On the Hydrodynamic Theory of Lubrication" (1904), etc.).

Finally in 1906 Sommerfeld was appointed to the important chair in theoretical physics at the Ludwig Maximilians University in Munich, a position he was to occupy for the next 32 years. He recalls:

> In Munich I had the opportunity for the first time to lecture on the various fields of theoretical physics and to give special lectures on topics of current interest. From the beginning I have tried and I have spared no effort through my seminars and colloquia to establish a growth center for theoretical physics in Munich. [Sommerfeld 1951, 677]

Indeed, Sommerfeld became the mentor of a whole generation of physicists. Max Born [1928], who took a head count in 1928, found that nearly one third of all the chairs in theoretical physics in the German-speaking countries were occupied by Sommerfeld's students. Among his students and assistants were several who later won Nobel prizes, namely Bethe, Debye, Heisenberg, and Pauli. Research fellows flocked to his institute from all over the world, including the American scientists E. U. Condon, Linus Pauling, and I. I. Rabi [Sommerfeld

1949]. During his years in Munich, Sommerfeld was again able to focus on boundary value problems in mathematical physics. In 1909 he attacked one of the important technical problems of the day as a boundary value problem, namely the influence of the earth's proximity on the propagation of radio waves. This problem was to occupy him and his students for many years, and an entire chapter of more than 50 pages is devoted to it in his text on partial differential equations [Sommerfeld 1945].

In his 1909 paper [Sommerfeld 1909] the mathematical model of radio wave propagation is idealized drastically. The transmitter is represented by an oscillating Hertz dipole placed at a distance h above the earth. The earth is assumed to be flat and homogeneous with constant conductivity, electric permittivity, and magnetic permeability. The atmosphere is also assumed to be homogeneous and reflections from the ionosphere are neglected. If the plane $z = 0$ is taken to represent the air–earth interface in cylindrical coordinates r, ϕ, z, then the boundary value problem becomes (after the time-dependent term has been factored out)

$$\begin{cases} \Delta u + k_1^2 u = 0 & \text{in } z > 0 \\ \Delta u + k_2^2 u = 0 & \text{in } z < 0, \end{cases}$$

where k_1 and k_2 are two different constants which describe the two media, air and earth. In addition, certain boundary conditions hold at the interface $z = 0$. For propagation over long distances the angular coordinate ϕ may be neglected and the field represented by

$$u = \frac{e^{ikR}}{R} = \int_0^\infty J_0(\lambda r) e^{-\mu|z|} \frac{\lambda d\lambda}{\mu}, \qquad R = \sqrt{r^2 + z^2}, \qquad \mu = \sqrt{\lambda^2 - k^2},$$

where J_0 is the Bessel function of order zero. The total disturbance then breaks up into a primary disturbance created by the transmitter and a secondary disturbance due to the currents induced in the ground. Moreover, this secondary disturbance acts differently in $z > 0$ and $z < 0$ because different electric constants describe the two media in these two half-spaces. To single out the waves that propagate in a direction *away from the transmitter*, Sommerfeld splits up the Bessel function $J_0(\lambda r)$ in the integral into the sum of two Hankel functions $H_0^{(1)}(\lambda r)$ and $H_0^{(2)}(\lambda r)$ by $J_0 = \frac{1}{2}(H_0^{(1)} + H_0^{(2)})$ and retains only the part $\frac{1}{2}H_0^{(1)}(\lambda r)$ which corresponds asymptotically to *outgoing* cylindrical waves. (The Hankel functions $H_0^{(1)}(r)$ and $H_0^{(2)}(r)$ behave asymptotically like $e^{ir}/\sqrt{r}$ and $e^{-ir}/\sqrt{r}$, respectively.) The remainder of the solution method need not concern us here; it leads to a convenient approximation formula which qualitatively and even quantitatively yields good results. Sommerfeld, his students, and other investigators later improved the model by taking into account the inhomogeneous nature of the ground (dry ground, wet ground, fresh water, sea water), the curvature of the earth, and—most importantly—the reflections from the ionosphere [Sommerfeld 1945].

4. DERIVATION OF THE RADIATION CONDITION

By 1910 Sommerfeld saw the need for combining the various propagation problems he had encountered in mathematical physics under the single rubric of an exterior boundary value problem and to provide a uniform mathematical treatment therefor. To accomplish this he found it necessary to first derive a condition which would ensure the uniqueness of the solution. He does this in his 1912 paper [Sommerfeld 1912] by first posing a general exterior boundary value problem and then deriving a radiation condition for it and showing that it yields a unique solution (provided a certain Green's function exists). He did not consider the question of existence (or stability) for the boundary value problem, probably because this could be expected to entail difficulties in light of the history of the Dirichlet principle for the potential equation ($k = 0$).

The 1912 paper consists of two parts. The first part constructs Green's functions for *interior* boundary value problems for the Helmholtz equation; it had already been presented at the Annual Meeting of Natural Scientists in Königsberg in 1910 [Sommerfeld 1910]. (An interesting sidelight here is that Sommerfeld uses the picturesque term "Zackenfunktion" (spike function) for $\delta(x) = 0$ for $x \neq 0$, $\delta(x) = \infty$ for $x = 0$, $\int \delta(x)\,dx = 1$. This function was later named after Dirac!) The second part deals with the corresponding exterior boundary value problem and was presented at a meeting of the same group in Münster in 1912.

At the beginning of the second part, Sommerfeld first discusses the lack of uniqueness in exterior boundary value problems for the Helmholtz equation. He then poses the following general problem:

Find u such that

(a) $\Delta u + k^2 u = f$ outside a closed surface σ,
(b) $u = 0$ on σ (or a similar homogeneous boundary condition),
(c) "Finiteness condition" ["Endlichkeitsbedingung"]
$\lim_{r\to\infty} u = 0$ such that ru remains bounded,
(d) "Radiation condition" ["Ausstrahlungsbedingung"]
$\lim_{r\to\infty} r(\partial u/\partial r - iku) = 0$ uniformly with respect to direction.

The function f in (a), Sommerfeld explains, represents the net source strength of any sources which may be located in the finite part of space. For the derivation of the radiation condition these are irrelevant; hence we assume $f \equiv 0$ here. By a "similar homogeneous condition" in (b), Sommerfeld means $\partial u/\partial \nu = 0$ or $u + \alpha \partial u/\partial \nu = 0$, where $0 \leq \alpha < \infty$ and ν is normal to σ. If $f \equiv 0$ in (a) then the boundary condition (b) is usually formulated to be the Dirichlet condition $u = g$ or the Neumann condition $\partial u/\partial \nu = h$ on σ, where g and h are given functions. Actually Sommerfeld states condition (d) in two forms. In the 1912 paper he also writes: "at infinity u must be representable as a sum (or integral) of waves of the divergent traveling type." He then asserts that this is equivalent to (d) as given above. In his 1935 article on electromagnetic oscillations in the *second* (1935) edition of Frank and von Mises' compendium [Sommerfeld 1935] (the first (1927) edition does not contain a discussion of the radiation condition) condition (d) is given only in the more precise form above. In every other respect his discussion in this article parallels that in the 1912 paper. It should also be noted that in either paper Sommerfeld does not explicitly spell out any regularity condition for u (usually u is required to be twice continuously differentiable), nor any restrictions the surface σ must obey.

Sommerfeld now derives conditions (c) and (d) by first obtaining a representation formula for u at a point P outside of σ and inside a sphere Σ with radius r, which is large enough to include σ. This formula is the same as (2), but includes an additional term for Σ, namely

$$\int_\Sigma \left[\frac{\partial u}{\partial r}\frac{e^{ikr}}{r} - u\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right)\right] d\Sigma = \int_\Sigma r\left(\frac{\partial u}{\partial r} - iku\right) e^{ikr}\frac{d\Sigma}{r^2} + \int_\Sigma u e^{ikr}\frac{d\Sigma}{r^2}. \qquad (4)$$

The last integral on the right-hand side of (4) now vanishes by imposing condition (c) and the first integral on this side will vanish if

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \qquad (1)$$

uniformly with respect to all angles through which this limit can be approached. Since e^{ikr}/r was used in (4), this condition singles out only outgoing waves. Because of its simplicity this is the radiation condition most frequently cited and applied.

Sommerfeld points out that if e^{-ikr}/r had been used instead of e^{ikr}/r in (4) then an "absorption condition" ["Einstrahlungsbedingung"]

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} + iku\right) = 0$$

would result which singles out incoming waves. He then proves that the solution of the exterior boundary value problem (a)–(d) is unique "if one postulates the existence of the Green's function for the exterior region" [Sommerfeld 1912, 332]. We do not reproduce this proof here since it is superseded later by proofs which do not require this assumption. The discussion of the radiation condition concludes with the statement of (d) in its two-dimensional form (the factor r in (d) is replaced by $\sqrt{r}$) and in its one-dimensional form (the factor r in (d) is deleted). The remainder of the 1912 paper derives Green's functions for various unbounded regions. Since these Green's functions are often represented by integrals in the complex plane, Sommerfeld shows how to satisfy the radiation condition by an appropriate choice of the path of integration.

5. UNIQUENESS THEOREMS AND THE EXISTENCE QUESTION

Thirty years after Sommerfeld's original paper appeared, Magnus [1942, 1949] gave a uniqueness proof for the exterior boundary value problem (a)–(d) without assuming the existence of Green's function for the exterior region. To do so, he assumes that this problem has two solutions u_1 and u_2 and then shows that $u = u_1 - u_2$ must vanish identically outside of σ. This is accomplished by expanding the first integral on the right-hand side of (2) (the additional integrals (4) vanish by conditions (c) and (d)) in a uniformly convergent series of surface spherical harmonics and powers of $1/r$ and then showing that the boundary condition on σ makes the coefficients in this series vanish. Magnus acknowledges Sommerfeld's "kind interest and valuable advice" in writing this paper [Magnus 1942, 178]. A year later Rellich [1943], who had heard Magnus give a lecture on this subject in Dresden, strengthened this result by proving uniqueness for the problem (a), (b), (d), *without using condition* (c). This shows then that the finiteness condition (c) is superfluous in the formulation of the exterior boundary value problem!

To prove uniqueness, Rellich assumes that the problem (a), (b), (d) has two solutions u_1 and u_2 and forms $u = u_1 - u_2$. Then u also satisfies (a), (b), (d), even if the original boundary condition (b) for u_1 and u_2 is of the Dirichlet type $u_j = g$ ($j = 1, 2$). The conjugate function $\bar{u}$ also satisfies (a) and (b), but (d) is replaced by

$$(\bar{\text{d}}) \quad \lim_{r\to\infty} r\left(\frac{\partial \bar{u}}{\partial r} + ik\bar{u}\right) = 0.$$

Rellich now combines the conditions (d) and ($\bar{\text{d}}$) by forming the integral

$$0 = \lim_{r\to\infty} \int_\Sigma r\left(\frac{\partial u}{\partial r} - iku\right) \cdot r\left(\frac{\partial \bar{u}}{\partial r} + ik\bar{u}\right)\frac{d\Sigma}{r^2} = \lim_{r\to\infty}\int_\Sigma \left|\frac{\partial u}{\partial r} - iku\right|^2 d\Sigma$$

and showing that this implies $\lim_{r\to\infty} \int_\Sigma |u|^2\, d\Sigma = 0$ if $k \neq 0$. From this it follows that $u \equiv 0$ outside of σ, a result that is now known as Rellich's lemma [Rellich 1943, 57; Hellwig 1960, 109].

The proof also shows that Sommerfeld's radiation condition (1) may be replaced by the weaker integral condition

$$\lim_{r\to\infty}\int_\Sigma \left|\frac{\partial u}{\partial r} - iku\right|^2 d\Sigma = 0. \tag{5}$$

In other words, condition (5) is sufficient to ensure uniqueness and may be used to replace (c) and (d). Rellich carried out his uniqueness proof in n dimensions and he also generalized his uniqueness theorem to include the case where σ extends to infinity. For the latter he had to restrict σ to have a paraboloidal shape at large distances.

After his retirement in 1938 Sommerfeld prepared his *Lectures on Theoretical Physics* for publication. They appeared in six volumes [Sommerfeld 1952] during the years 1943–1952 (English edition 1949–1956). The sixth volume is the book on partial differential equations mentioned earlier [Sommerfeld 1945] and it was published two years after Rellich's paper appeared in print. In it Sommerfeld gives a new presentation of the radiation condition. He includes a new uniqueness proof styled after Magnus' and then he acknowledges:

The author's original proof of this uniqueness theorem assumed in addition to the conditions (a), (b), (d) for u, the existence of Green's function for the exterior of the surface and an additional "finality condition" [This is the translator's inaccurate translation of "Endlichkeitsbedingung"]. The fact that the latter is superfluous has been rigorously proven by F. Rellich [1943] even for the case of an arbitrary number of dimensions n where the radiation condition reads

$$\lim_{r\to\infty} r^{(n-1)/2}\left(\frac{\partial u}{\partial r} - iku\right) = 0.$$

[Sommerfeld 1945, 192–193]

These then were the uniqueness proofs influenced directly by Sommerfeld's original work. After the war, a book appeared by Vekua [1967] in which the author presents a uniqueness proof for an exterior boundary value problem and remarks in a footnote that similar proofs had been given earlier by Russian authors. The earliest was given by Kupradze [1934] in a 1934 paper and repeated in Kupradze's book [1956]. However, Vekua cautions, "But the book contains a number of inaccuracies as remarked by the author himself" [Vekua 1967, 318]. In another footnote Vekua points out that he himself had given a proof of a form of Rellich's lemma in 1943 [Vekua 1943]. It seems that in England the wartime papers by Magnus, Rellich, and Vekua had also gone unnoted, for in 1949 Atkinson [1949], who mentions only Sommerfeld's early work, published a paper showing that uniqueness could be proved without assuming the existence of Green's function. In the proof he replaces Sommerfeld's conditions (c), (d) by two equivalent conditions, namely

$$re^{ikr}\left[\left(ik - \frac{1}{r}\right)u - \frac{\partial u}{\partial r}\right] \to 0 \qquad \text{or} \qquad r^3 e^{-ikr}\left[\left(ik - \frac{1}{r}\right)u - \frac{\partial u}{\partial r}\right] \text{bounded},$$

hardly any simpler than Sommerfeld's; however, he allows the constant k in the Helmholtz equation to be a nonzero complex number. A further extension is presented in 1956 by Wilcox [1956], who proves uniqueness, representation, and expansion theorems for complex k, using only Rellich's integral condition (5). Another generalization along these lines is given by Levine [1964] in 1964 who, like Wilcox, allows k to be complex and uses (5), but permits the boundary surface σ to belong to a certain fairly general class of piecewise smooth closed surfaces which may have edges and corners. Furthermore he allows mixed boundary conditions to be given on these surfaces; i.e., u may vanish on some parts of σ and $\partial u/\partial \nu + \alpha u$ vanish over the remainder of σ, with α being in general a (possibly discontinuous) nonnegative function of position on σ. These corners, edges, and boundary conditions occur frequently in applications.

Finally it is interesting to note that uniqueness theorems have also played a role in proving the *existence* of the solution of the exterior boundary value problem for the Helmholtz equation. Although the earlier existence proofs by Weyl [1952], Müller [1957], Kupradze [1956], and others were more complicated, Brakhage and Werner [1965] gave a comparatively simple existence proof in 1965 based on the uniqueness theorem and on the first part of the Fredholm alternative in integral equation theory.

6. EXTENSIONS AND MODIFICATIONS

So far we have discussed only problems for the Helmholtz equation. However, these methods and results have also been extended to other equations and more general boundary value problems. It would take us too far afield to consider all of these here, especially since the subject of scattering and inverse scattering problems has grown very rapidly in recent years. Hence we briefly mention only three extensions of Sommerfeld's original investigations.

First of all, Sommerfeld's condition has been applied to more general equations. Radiation conditions for the iterated Helmholtz equations $(\Delta + k_1^2)(\Delta + k_2^2)u = 0$ and $(\Delta + k^2)^m u = 0$, where m is a positive integer, were given by Subeika [1968] and Vekua [1967]. Kato [1959] found asymptotic growth estimates for the solution of the Helmholtz equation with variable k. Several authors have extended these results to obtain uniqueness theorems for equations where the Laplace operator Δ has been replaced by more general self-adjoint operators. In particular, Jäger [1967] considers exterior boundary value problems for the n-dimensional equation with variable coefficients $a_{lm}(x_1, \ldots, x_n)$:

$$\sum_{l=1}^{n} \sum_{m=1}^{n} \frac{\partial}{\partial x_l} a_{lm} \frac{\partial}{\partial x_m} u + k^2 u = 0.$$

Extensions to certain higher order partial differential equations were also given by Vainberg [1963, 1966], Grushin [1963], and other Russian authors. For the so-called vector Helmholtz equation (where k is a positive constant)

$$\nabla \times (\nabla \times \mathbf{A}) - k^2\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \Delta\mathbf{A} - k^2\mathbf{A} = 0 \tag{6}$$

the radiation condition takes the form

$$\lim_{r\to\infty} r[(\nabla \times \mathbf{A}) \times \mathbf{e}_r - ik\mathbf{A}] = 0, \tag{7}$$

where $\mathbf{e}_r$ is a unit vector in the radial direction. Equation (6) occurs in electromagnetic wave propagation in a homogeneous, isotropic, nonconducting medium. The electromagnetic waves are governed by the time-independent Maxwell equations

$$\nabla \times \mathbf{E} - ik\mathbf{H} = 0 \quad \text{and} \quad \nabla \times \mathbf{H} + ik\mathbf{E} = 0,$$

where $\mathbf{E}$ and $\mathbf{H}$ describe the electric and magnetic fields, respectively, and it follows from these equations that both $\mathbf{E}$ and $\mathbf{H}$ satisfy (6). The radiation condition for the electromagnetic field then follows from (7) and is

$$\lim_{r\to\infty} r(\mathbf{H} \times \mathbf{e}_r - \mathbf{E}) = 0.$$

In electromagnetic theory this is known as the Silver–Müller radiation condition [Silver 1949, Müller 1957].

Second, the surface σ carrying the boundary data has been permitted to extend to infinity, a situation which occurs frequently in diffraction problems. As mentioned earlier, Rellich [1943] had already established uniqueness in the case where the boundary surface σ resembled a paraboloid (more precisely, σ is a surface which has the property that every plane perpendicular to a fixed direction cuts off at most a finite portion of σ and the angle this fixed direction makes with the exterior normal to σ is not less than 90°). Miranker [1957] extended this result to cones with a sufficiently large apex angle, but had to restrict the boundary data given thereon somewhat. The physical reason for these outward flaring shapes is that a surface σ which pinches in toward infinity can trap standing waves and thereby lead to a nonunique solution [Jones 1953]. Other cases of infinite boundaries were considered by Odeh [1963]. Peters and Stoker [1954] deduce a uniqueness theorem tailored for certain two-dimensional optical and water wave diffraction problems where a single half-ray boundary extends to infinity.

The last extension considered here is much more general and far-reaching in that it suggests a wholly new method for deriving and formulating radiation conditions. The three examples of propagation problems considered earlier (Rayleigh scattering, light diffraction, and radio wave propagation) were made amenable to mathematical solution by factoring out the time-dependent term $e^{-i\omega t}$ and thereby reducing a difficult initial-value problem for the wave equation to an easier-to-solve pure boundary value problem for the Helmholtz equation. This simplification was gained at the expense of incurring incoming and/or standing wave solutions which did not arise in the original initial-boundary value problem. These extraneous solutions then had to be eliminated by imposing a radiation condition.

This suggests that one might try to solve the original initial–boundary value problem directly without first reducing it to a pure boundary value problem. One would then assume the incoming wave to have started impulsively at a finite time, say $t = 0$ (after all, no wave can have existed for all time!). If the initial conditions are specified properly, this would produce a time-harmonic wave and a superimposed transient wave, but the transient wave would die out as $t \to \infty$. If properly posed, with only a boundedness condition at infinity, the initial-boundary value problem should have a unique solution, thus obviating the need for an additional radiation condition. In fact, it should be possible to derive a radiation condition, or at least the asymptotic behavior of the solution, by letting $r \to \infty$ (the two limiting processes in time and space may not be interchanged, however).

Although this approach had probably been considered for a long time, Stoker [1956, 1957] may have been the first to propose it formally and actually carry it out. In general these initial–boundary value problems are difficult to solve, but Stoker succeeded in the case of two problems involving water waves. The first entails simple harmonic waves traveling outward from a periodic impulse given to an infinite ocean originally at rest and in the second, unsteady waves are created by a disturbance on the surface of a running stream. A different program of this type was carried out for a purely mathematical problem by Wilcox [1959]. He considers a special initial–boundary value problem where the outgoing wave character of the solution ("radiation function") is built into the formulation of the problem. He then uses this condition to derive four equivalent radiation conditions (the Sommerfeld condition (1) and the Rellich condition (5) are among them).

These extensions illustrate the mathematical development that has radiated outward in time and space from Sommerfeld's 1912 paper. He certainly was correct in believing that this paper "introduces a new and, it would seem, mathematically interesting class of problems" [Sommerfeld 1912, 352].

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