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Spline wavelets in $\ell^2(Z)$ ☆

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Abstract

Compared with the spline wavelet decomposition for the discrete power growth space $\mathcal{F}$ given by Pevnyi and Zheludev, this paper deals with spline wavelet decompositions for the Hilbert space $\ell^2(Z)$. We characterize RTB splines and RTB wavelets, because the space $\ell^2(Z)$ can be represented by them. It turns out that the representation is stable and the convergence is much stronger than the pointwise convergence in $\mathcal{F}$. Finally, a family of symmetric RTB wavelets with finite supports are constructed.

Keywords: RTB spline; RTB wavelet; Discrete Fourier transform; Symmetric wavelet

1. Introduction and preliminary

The spline wavelet theory in $L^2(R^d)$ is important in wavelet analysis, because it has many effective applications. In [1], the authors discuss the so-called discrete splines in $\ell^2(Z)$ through sampling the continuous splines. On the other hand, Pevnyi and Zheludev defined the discrete splines, which is independent of continuous splines in 2000 [8]. Using those discrete splines, they establish the wavelet decomposition for the space of power

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growth $\mathcal{F}$ [9]. Since $\mathcal{F}$ has no norm structure, the convergence there holds only in pointwise sense. However, many practical signals have finite energy and can be considered as elements of $\ell^2(Z)$. Moreover, the norm convergence is much stronger than the pointwise one in $\ell^2(Z)$. Based on this consideration, we shall study the spline wavelet decomposition of $\ell^2(Z)$, by using the knowledge from [8,9]. Wavelet analysis in this Hilbert space can be found in [4,7,10,11], etc.

In [8], the discrete splines on $\mathcal{Z}$ are defined as follows: Let n be a natural number. A discrete B-spline of the first order is the function $B_{1,n}$ on $\mathcal{Z}$, given by

$$B_{1,n}(j) = \begin{cases} 1, & 0 \leqslant j \leqslant n-1; \\ 0, & \text{otherwise.} \end{cases}$$

The higher order B-spline is $B_{p,n} =: B_{p-1,n} * B_{1,n}$ for $p \geqslant 2$. Here and after, $*$ stands for the discrete convolution. The discrete convolution $a * b$ of $a = \{a_n\}$ and $b = \{b_n\}$ is defined by $a * b(n) =: \sum_n a(n-k)b(k)$. It is well known that $a * b \in \ell^2(Z)$, when $a \in \ell(Z)$ and $b \in \ell^2(Z)$. As usual, $\ell(Z) =: \{a(n), \sum_{n\in Z} |a(n)| < +\infty\}$.

Definition 1.1. The upsampling operator $U : \ell^2(Z) \to \ell^2(Z)$ is defined by

$$\big(U^n z\big)(k) =: \begin{cases} z(k/n), & k \text{ is divisible by } n; \\ 0, & \text{otherwise.} \end{cases}$$

This definition can be found in [2,5]. Since $B_{p,n}$ has finite support, $B_{p,n} \in \ell(Z)$ and $U^n c * B_{p,n}(\cdot) = \sum_l c(l) B_{p,n}(\cdot - ln) \in \ell^2(Z)$ for each $c \in \ell^2(Z)$. Then

$$V_{p,n} =: \left\{ \sum_l c(l) B_{p,n}(\cdot - ln), c \in \ell^2(Z) \right\} \tag{1.1}$$

is a subspace of $\ell^2(Z)$. This space is totally different from that in [9], because we assume $c \in \ell^2(Z)$ here and they take c as a power growth sequence therein. It is easy to see that $B_{p,1} = \delta$ and $V_{p,1} = \ell^2(Z)$. We call an element of $V_{p,n}$ a discrete spline of order p.

Note that $B_{p,2n}(j) = \sum_{r=0}^{p} C_p^r B_{p,n}(j - rn),\ j \in Z$ [9, Theorem 5, p. 68]. Then $V_{p,2n}$ is a subspace of $V_{p,n}$. Furthermore, it can be shown that $V_{p,2n}$ is a closed subspace of $V_{p,n}$ by Lemma 2.2. Then there exists the unique closed subspace $W_{p,2n} \subseteq V_{p,n}$ such that

$$V_{p,n} = V_{p,2n} \oplus W_{p,2n}.$$

Recall that $\{x_n\}$ is a Riesz basis of a Hilbert space H, if for each $x \in H$, there exists the unique $c \in \ell^2(Z)$ such that $x = \sum_n c_n x_n$ and $A \sum_n |c_n|^2 \leqslant \|x\|^2 \leqslant B \sum_n |c_n|^2$ with $0 < A \leqslant B$. We introduce the following definition to study the decomposition of $\ell^2(Z)$.

Definition 1.2. A discrete spline $\phi \in V_{p,n}$ is called an RTB spline, if $\{\phi(\cdot - ln)\}_{l\in Z}$ forms a Riesz basis for $V_{p,n}$; a discrete spline $\psi \in W_{p,2n}$ is called an RTB wavelet, if $\{\psi(\cdot - 2ln)\}_{l\in Z}$ forms a Riesz basis for $W_{p,2n}$.

The corresponding discrete spline is said to be TB spline or TB wavelet in [9]. We call it RTB spline or RTB wavelet, because its translations forms a Riesz basis. To characterize

RTB splines and RTB wavelets, we need some other notations, which are introduced in [8,9]: firstly

$$E_{p,n}(x,j) =: \sum_l e^{-ilx} B_{p,n}(j-ln) \tag{1.2}$$

is well defined and

$$E_{p,n}(x,j-kn) = e^{ikx} E_{p,n}(x,j). \tag{1.3}$$

The set $\{n \in \mathcal{Z},\ a(n) \neq 0\}$ is called the support of a, denoted by $\operatorname{supp} a$ in this paper. It is easy to show that $\operatorname{supp} B_{p,n} = \{0,1,2,\ldots,p(n-1)\}$ and $B_{p,n}[p(n-1)-j] = B_{p,n}(j) \geqslant 0,\ j \in \mathcal{Z}$. Let $b_{p,n}(k) =: B_{p,n}(v+kn)$ with $v = \frac{p(n-1)}{2}$. As in [9], we assume that v is an integer in this paper. Then $b_{p,n}$ has finite support, because $B_{p,n}$ does. Moreover,

$$T_{p,n}(x) =: \sum_{-\mu}^{\mu} b_{p,n}(k) e^{ikx} > 0$$

for each real number $x \in R$ [8]. Take $a(x) = e^{ix}(1-e^{-ix})^p T_{2p,n}(x+\pi)$ and define

$$\omega_{p,2n}(x,j) = a\left(\frac{x}{2}\right) E_{p,n}\left(\frac{x}{2}, j\right) + a\left(\frac{x+2\pi}{2}\right) E_{p,n}\left(\frac{x+2\pi}{2}, j\right).$$

Then it is shown that [9]

$$\omega_{p,2n}(x, j-2kn) = e^{ikx} \omega_{p,2n}(x,j). \tag{1.4}$$

In Section 2, we prove that ϕ is an RTB spline if and only if

$$\phi(j) = \frac{1}{2\pi} \int_0^{2\pi} \xi(x) E_{p,n}(x,j)\,dx$$

for $\xi \in L^2[0,2\pi]$ and $a \leqslant |\xi(x)| \leqslant b$ almost everywhere on the whole real line R for some $0 < a \leqslant b < +\infty$.

We use $\xi(x) \sim 1$ to denote $a \leqslant |\xi(x)| \leqslant b$ almost everywhere for some $0 < a \leqslant b < +\infty$. Similar to RTB splines, a characterization for RTB wavelets is given in Section 3: The discrete spline ψ is an RTB wavelet if and only if

$$\psi(j) = \frac{1}{2\pi} \int_0^{2\pi} \tau(x) \omega_{p,2n}(x,j)\,dx$$

for $\tau \in L^2[0,2\pi]$ and $\tau(x) \sim 1$. In the end of the third section, we shall discuss the duals of RTB splines and RTB wavelets.

Finally, we exhibit a family of symmetric RTB wavelets with finite supports in terms of $\tau(x)$, which include Pevnyi and Zheludev's example. It will be proved that the Pevnyi and Zheludev's example has the shortest length among that family. All these will be done in the last section.

For the proofs, we need the discrete Fourier transform $\mathcal{F}:\ell^2(Z)\to L^2[0,2\pi]$, defined by

$$\hat{z}(\theta)=:(\mathcal{F}z)(\theta)=:\sum_k z(k)e^{ik\theta}\in L^2[0,2\pi]$$

for each $z\in\ell^2(Z)$. The inner product in $L^2[0,2\pi]$ is given by

$$\langle u,v\rangle=\frac{1}{2\pi}\int_0^{2\pi}u(x)\overline{v(x)}\,dx.$$

Then the Plancherel formula says $\langle\hat{z},\hat{\omega}\rangle=\sum_k z(k)\overline{\omega(k)}$. On the other hand, it can be checked that $(U^n z)^\wedge(\theta)=\hat{z}(n\theta)$ for each $z\in\ell^2(Z)$ and that $(z*\omega)^\wedge(\theta)=\hat{z}(\theta)\hat{\omega}(\theta)$ for $z\in\ell^2(Z)$ and $\omega\in\ell(Z)$. These can be found in [2,5].

2. Characterization for RTB splines

We shall firstly show that each $\phi\in V_{p,n}$ can be represented by

$$\phi(j)=\frac{1}{2\pi}\int_0^{2\pi}C(x)E_{p,n}(x,j)\,dx$$

with $C(x)\in L^2[0,2\pi]$ in this part and then prove that ϕ is an RTB spline if and only if $C(x)\in L^2[0,2\pi]$ and $C(x)\sim 1$.

Lemma 2.1. *The discrete spline $S\in V_{p,n}$ if and only if*

$$S(j)=\frac{1}{2\pi}\int_0^{2\pi}C(x)E_{p,n}(x,j)\,dx$$

for some $C(x)\in L^2[0,2\pi]$. Moreover, $\hat{S}(x)=C(nx)\hat{B}_{p,n}(x)$.

Proof. Let $S(j)=\frac{1}{2\pi}\int_0^{2\pi}C(x)E_{p,n}(x,j)\,dx$ with $C(x)\in L^2[0,2\pi]$. Then $C(x)=\sum_l c(l)e^{ilx}$ with $\{c(l)\}\in\ell^2(Z)$. Note that (1.2) says $E_{p,n}(x,j)=:\sum_l B_{p,n}(j-ln)e^{-ilx}$. Then, by the Plancherel formula, one has

$$S(j)=\frac{1}{2\pi}\int_0^{2\pi}C(x)E_{p,n}(x,j)\,dx=\sum_l c(l)B_{p,n}(j-ln).\tag{2.1}$$

Hence $S\in V_{p,n}$, according to the definition for $V_{p,n}$ in (1.1).

Conversely, assume $S\in V_{p,n}$. Then there exists $c\in\ell^2(Z)$ such that

$$S(j)=\sum_l c(l)B_{p,n}(j-ln).\tag{2.2}$$

Note that $C(x) =: \sum_l c(l)e^{ilx} \in L^2[0, 2\pi]$ and $E_{p,n}(x, j) =: \sum_l B_{p,n}(j - ln)e^{-ilx}$. Then it follows that $S(j) = \frac{1}{2\pi}\int_0^{2\pi} C(x)E_{p,n}(x, j)\,dx$ from (2.2) and the Plancherel formula.

By (2.1), one knows $S(j) = U^n c * B_{p,n}(j)$ with $\hat{c} = C(x)$. Then $\hat{S}(x) = C(nx)\hat{B}_{p,n}(x)$. □

Lemma 2.2. *The discrete spline $B_{p,n}$ satisfies*

$$\sum_{l=0}^{n-1}\left|\hat{B}_{p,n}\left(x + \frac{2\pi l}{n}\right)\right|^2 \sim 1.$$

Proof. Using the definition of $B_{p,n} = B_{p-1,n} * B_{1,n}$, one has $\hat{B}_{p,n}(x) = [\hat{B}_{1,n}(x)]^p$. Note that $\hat{B}_{1,n}(x) = \sum_{k=0}^{n-1} e^{-ikx}$ and $\hat{B}_{1,1}(x) = 1$. Then Lemma 2.2 is true automatically for $n = 1$. Next, one shows the lemma for $n > 1$. Clearly,

$$\left|\hat{B}_{p,n}(x)\right| = \begin{cases} n^p, & x = 0, 2\pi; \\ \left|\frac{1-e^{-inx}}{1-e^{-ix}}\right|^p, & 0 < x < 2\pi. \end{cases}$$

It is easy to see $\hat{B}_{p,n}(x) \neq 0$ on $[0, \frac{2\pi}{n})$ and hence

$$\sum_{l=0}^{n-1}\left|\hat{B}_{p,n}\left(x + \frac{2\pi l}{n}\right)\right|^2 \geqslant \left|\hat{B}_{p,n}(x)\right|^2 + \hat{B}_{p,n}\left(x + \frac{2\pi(n-1)}{n}\right)\Bigg|^2 > 0$$

on $[0, \frac{2\pi}{n}]$. Since $\sum_{l=0}^{n-1} |\hat{B}_{p,n}(x + \frac{2\pi l}{n})|^2$ is a $\frac{2\pi}{n}$ periodic and continuous function, the desired conclusion follows. □

Now we are in the position to show the main result in this section:

Theorem 2.1. *Let $\phi \in V_{p,n}$. Then ϕ is an RTB spline if and only if*

$$\phi(j) = \frac{1}{2\pi}\int_0^{2\pi} \xi(x)E_{p,n}(x, j)\,dx$$

for $\xi \in L^2[0, 2\pi]$ and $\xi(x) \sim 1$.

Proof. Firstly $\phi \in V_{p,n}$ implies that

$$\phi(j) = \frac{1}{2\pi}\int_0^{2\pi} \xi(x)E_{p,n}(x, j)\,dx \tag{2.3}$$

with $\xi(x) \in L^2[0, 2\pi]$ due to Lemma 2.1.

For the necessary part, assume that ϕ is an RTB spline. Since $\{\phi(\cdot - kn)\}$ is a Riesz basis for $V_{p,n}$, for each $S \in V_{p,n}$,

$$S(\cdot) = \sum_k c(k)\phi(\cdot - kn) \in V_{p,n}$$

and $\|S\| \sim \|c\|$. Furthermore, $S = (U^n c) * \phi$ and $\hat{S}(x) = \hat{c}(nx)\hat{\phi}(x)$. Combining Lemma 2.1 with (2.3), one has $\hat{\phi}(x) = \xi(nx)\hat{B}_{p,n}(x)$. Therefore,

$$\begin{aligned}\|\hat{S}\|^2 &= \left\|\hat{c}(nx)\xi(nx)\hat{B}_{p,n}(x)\right\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|\hat{c}(nx)\right|^2 \left|\xi(nx)\right|^2 \left|\hat{B}_{p,n}(x)\right|^2 dx \\ &= \frac{1}{2\pi n}\int_0^{2\pi n} \left|\hat{c}(y)\right|^2 \left|\xi(y)\right|^2 \left|\hat{B}_{p,n}\left(\frac{y}{n}\right)\right|^2 dy \\ &= \frac{1}{2\pi n}\int_0^{2\pi} \left|\hat{c}(x)\right|^2 \left|\xi(x)\right|^2 \sum_{l=0}^{n-1}\left|\hat{B}_{p,n}\left(\frac{x+2\pi l}{n}\right)\right|^2 dx. \qquad (2.4)\end{aligned}$$

Since $\|S\| \sim \|c\|$, $\|\hat{S}\|^2 \sim \|\hat{c}\|^2$. Note that $\int_0^{2\pi} |\hat{c}(x)|^2 \Gamma(x)\,dx \geqslant 0$ for each $c \in \ell^2(Z)$ implies $\Gamma(x) \geqslant 0$ almost everywhere, because $\Gamma(x) < 0$ on some non-zero measurable set E_0 would lead to a desired contradiction:

$$\int_0^{2\pi} \left|\chi_{E_0}(x)\right|^2 \Gamma(x)\,dx = \int_{E_0} \Gamma(x)\,dx < 0,$$

where $\hat{c}(x) = \chi_{E_0}(x)$. Then

$$\frac{1}{2\pi n}\left|\xi(x)\right|^2 \sum_{l=0}^{n-1}\left|\hat{B}_{p,n}\left(\frac{x+2\pi l}{n}\right)\right|^2 \sim 1.$$

Using Lemma 2.2, one receives $|\xi(x)| \sim 1$.

For the sufficient part, one shows $\{\phi(\cdot - ln)\}_{l\in Z}$ spans the whole $V_{p,n}$, firstly: let $S \in V_{p,n}$. Then Lemma 2.1 says $S(j) = \frac{1}{2\pi}\int_0^{2\pi} C(x)E_{p,n}(x,j)\,dx$ with $C(x) \in L^2[0,2\pi]$. Since $\xi(x) \sim 1$, $\frac{C(x)}{\xi(x)} \in L^2[0,2\pi]$ and

$$S(j) = \frac{1}{2\pi}\int_0^{2\pi} \frac{C(x)}{\xi(x)}\xi(x)E_{p,n}(x,j)\,dx.$$

Let $c(l) = \frac{1}{2\pi}\int_0^{2\pi} \frac{C(x)}{\xi(x)} e^{ilx}\,dx$. Note that (1.3) and (2.3) imply

$$\phi(j-kn) = \frac{1}{2\pi}\int_0^{2\pi} e^{ikx}\xi(x)E_{p,n}(x,j)\,dx.$$

Then it follows that

$$S(\cdot) = \sum_k c(k)\phi(\cdot - kn) \qquad (2.5)$$

from the Plancherel formula. Now it remains to show $\|S\| \sim \|c\|$. Similar to the argument of (2.4), one can prove

$$\|\hat{S}\|^2 = \frac{1}{n}\int_0^{2\pi} \left|\hat{c}(x)\right|^2 \left|\xi(x)\right|^2 \sum_{l=0}^{n-1} \left|\hat{B}_{p,n}\left(\frac{x+2\pi l}{n}\right)\right|^2 dx$$

by using (2.5), (2.3) and Lemma 2.1. Combining Lemma 2.2 with the given condition $\xi(x) \sim 1$, one receives $\|\hat{S}\|^2 \sim \|\hat{c}\|^2$. Furthermore, the desired result $\|S\| \sim \|c\|$ follows. $\square$

3. Characterization for RTB wavelets

The purpose of this section is to characterize all RTB wavelets. The results are similar to that for RTB splines. By those characterizations, we can give spline wavelet decompositions for $\ell^2(Z)$. This will be explained in the end of this section (Remarks 3.1 and 3.2).

As mentioned in the introduction,

$$\omega_{p,2n}(x,j) = a\left(\frac{x}{2}\right)E_{p,n}\left(\frac{x}{2}, j\right) + a\left(\frac{x+2\pi}{2}\right)E_{p,n}\left(\frac{x+2\pi}{2}, j\right)$$

with $a(x) = e^{ix}(1-e^{-ix})^p T_{2p,n}(x+\pi)$. The following lemma comes essentially from Pevnyi and Zheludev's work.

Lemma 3.1. *The discrete spline $R \in W_{p,2n}$ if and only if*

$$R(j) = \frac{1}{2\pi}\int_0^{2\pi} D(x)\omega_{p,2n}(x,j)\,dx$$

for some $D(x) \in L^2[0,2\pi]$. Moreover, $\hat{R}(x) = D(2nx)\hat{\psi}_{p,2n}(x)$, where

$$\psi_{p,2n}(j) = \frac{1}{2\pi}\int_0^{2\pi} \omega_{p,2n}(x,j)\,dx. \tag{3.1}$$

Proof. Note that $D_{2n}(x) \in L^2[0,2\pi]$ in [9, Theorem 7, p. 72], when $C_n(x) \in L^2[0,2\pi]$. Then for each $R \in W_{p,2n}$, there exists some $D(x) \in L^2[0,2\pi]$ such that

$$R(j) = \frac{1}{2\pi}\int_0^{2\pi} D(x)\omega_{p,2n}(x,j)\,dx.$$

Conversely,

$$\omega_{p,2n}(x,j) = a\left(\frac{x}{2}\right)E_{p,n}\left(\frac{x}{2}, j\right) + a\left(\frac{x+2\pi}{2}\right)E_{p,n}\left(\frac{x+2\pi}{2}, j\right)$$

implies

$$\begin{aligned} S(j) &=: \frac{1}{2\pi}\int_0^{2\pi} D(x)\omega_{p,2n}(x,j)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} D(2y)a(y)E_{p,n}(y,j)\,dy + \frac{1}{\pi}\int_\pi^{2\pi} D(2y)a(y)E_{p,n}(y,j)\,dy \\ &= \frac{1}{\pi}\int_0^{2\pi} D(2x)a(x)E_{p,n}(x,j)\,dx \end{aligned}$$

for each $D(x) \in L^2[0,2\pi]$. Note that $a(x) = e^{ix}(1-e^{-ix})^p T_{2p,n}(x+\pi)$ is bounded and $D(x) \in L^2[0,2\pi]$. Then $D(2x)a(x) \in L^2[0,2\pi]$ and, therefore, $S \in V_{p,n}$, due to Lemma 2.1. Moreover, Pevnyi and Zheludev's work [9, Proposition 2, p. 72] shows $S \perp V_{p,2n}$. Hence $S \in W_{p,2n}$.

Now it remains to show $\hat{R}(x) = D(2nx)\hat{\psi}_{p,2n}(x)$: By (1.4), $\omega_{p,2n}(x, j-2kn) = e^{ikx}\omega_{p,2n}(x,j)$. Combining this with (3.1), one has

$$\psi_{p,2n}(j-2kn) = \frac{1}{2\pi}\int_0^{2\pi} \omega_{p,2n}(x,j)e^{ikx}\,dx. \tag{3.2}$$

Let $d_l = \frac{1}{2\pi}\int_0^{2\pi} D(x)e^{ilx}\,dx$. Then

$$R(j) = \frac{1}{2\pi}\int_0^{2\pi} D(x)\omega_{p,2n}(x,j)\,dx = \sum_l d_l\psi_{p,2n}(j-2ln) = U^{2n}d * \psi_{p,2n}(j).$$

Moreover, $\hat{R}(x) = D(2nx)\hat{\psi}_{p,2n}(x)$. This completes the proof. $\square$

Recall that $a(x) = e^{ix}(1-e^{-ix})^p T_{2p,n}(x+\pi)$. Let $G(l)$ denote the Fourier coefficients of $a(x)$, i.e.,

$$G(l) =: \frac{1}{\pi}\int_0^{2\pi} a(x)e^{-ilx}\,dx. \tag{3.3}$$

Then the discrete spline $\psi_{p,2n}(j) = \frac{1}{2\pi}\int_0^{2\pi}\omega_{p,2n}(x,j)\,dx$ can be represented as linear combinations of $B_{p,n}(j-ln)$, in which the coefficients are $G(l)$.

Lemma 3.2. [9] *The discrete spline $\psi_{p,2n}$ has the following properties*:

(i) $\psi_{p,2n}(j) = \sum_l G(l)B_{p,n}(j-ln)$;
(ii) $G(l) = (-1)^p G(-p+2-l)$;

(iii) $\operatorname{supp} G \subseteq [-\mu - p + 1, \mu + 1]$ *and* $G(-\mu - p + 1) \neq 0$, $G(\mu + 1) \neq 0$. *Here,* $\mu = [\frac{p(n-1)}{n}]$ *stands for the integer part of* $\frac{p(n-1)}{n}$.

Now we are ready to give the next lemma, which is important for Theorem 3.1.

Lemma 3.3. *The discrete spline* $\psi_{p,2n}(\cdot)$ *satisfies*

$$\sum_{k=0}^{2n-1} \left|\hat{\psi}_{p,2n}\left(x + \frac{\pi k}{n}\right)\right|^2 \sim 1.$$

Proof. One proves firstly

$$\hat{\psi}_{p,2n}(x) \neq 0 \quad \text{for } x \in \left(0, \frac{\pi}{n}\right]. \tag{3.4}$$

(Here, the authors thank the referee pointing out a mistake in the original proof.) By Lemma 3.2(i), $\psi_{p,2n}(j) = \sum_l G(l) B_{p,n}(j - ln) = U^n G * B_{p,n}$, where U is the upsampling operator defined in Definition 1.1. Moreover, $\hat{\psi}_{p,2n}(x) = \hat{G}(nx)\hat{B}_{p,n}(x)$. Since $G(l)$ is the Fourier coefficient of $a(x)$, one knows $\hat{G}(x) = a(x)$ and $\hat{\psi}_{p,2n}(x) = a(nx)\hat{B}_{p,n}(x)$. Recall that $a(x) =: e^{ix}(1 - e^{-ix})^p T_{2p,n}(x + \pi)$. Then

$$\hat{\psi}_{p,2n}(x) = e^{inx}\left(1 - e^{-inx}\right)^p T_{2p,n}(nx + \pi)\hat{B}_{p,n}(x).$$

To conclude (3.4), one finds that

$$\hat{B}_{p,n}(x) = \left[\hat{B}_{1,n}(x)\right]^p = \left(\sum_{j=0}^{n-1} e^{-ijx}\right)^p = \left(\frac{1 - e^{-inx}}{1 - e^{-ix}}\right)^p \neq 0$$

on $x \in (0, \frac{\pi}{n}]$. On the other hand, $T_{p,n}(x)$ is positive on the real line R (see [8] or [9]). Hence (3.4) is true. Based on (3.4), one can show

$$\sum_{k=0}^{2n-1} \left|\hat{\psi}_{p,2n}\left(x + \frac{\pi k}{n}\right)\right|^2 \sim 1. \tag{3.5}$$

In fact, for $x \in [0, \frac{\pi}{n}]$,

$$\sum_{k=0}^{2n-1} \left|\hat{\psi}_{p,2n}\left(x + \frac{\pi k}{n}\right)\right|^2 \geqslant \left|\hat{\psi}_{p,2n}(x)\right|^2 + \left|\hat{\psi}_{p,2n}\left(x + \frac{\pi}{n}\right)\right|^2 > 0,$$

due to (3.4). Note that $\hat{\psi}_{p,2n}(x)$ is continuous and $\sum_{k=0}^{2n-1} |\hat{\psi}_{p,2n}(x + \frac{\pi k}{n})|^2$ is $\frac{\pi}{n}$ periodic. Then (3.5) holds, which completes the proof. □

Similar to Theorem 2.1, we have a characterization for RTB wavelets.

Theorem 3.1. *Let $\psi \in W_{p,2n}$. Then ψ is an RTB wavelet if and only if*

$$\psi(j) = \frac{1}{2\pi}\int_0^{2\pi} \tau(x)\omega_{p,2n}(x, j)\,dx$$

for $\tau \in L^2[0, 2\pi]$ and $\tau(x) \sim 1$. In particular, $\psi_{p,2n}$ with $\tau(x) = 1$ is an RTB wavelet.

Proof. The proof is similar to that of Theorem 2.1. For the necessary part, assume that $\psi(j) = \frac{1}{2\pi}\int_0^{2\pi} \tau(x)\omega_{p,2n}(x, j)\,dx$ and $\{\psi(\cdot - 2ln)\}_l$ forms a Riesz basis for $W_{p,2n}$. Then for each $S \in W_{p,2n}$, $S(j) = \sum_l c(l)\psi(j - 2ln)$ and $\|\hat{S}\| \sim \|\hat{c}\|$. It is easy to see that $S = U^{2n}c * \psi$ and $\hat{S}(x) = \hat{c}(2nx)\hat{\psi}(x)$. On the other hand, Lemma 3.1 says $\hat{\psi}(x) = \tau(2nx)\hat{\psi}_{p,2n}(x)$. Therefore, $\hat{S}(x) = \hat{c}(2nx)\tau(2nx)\hat{\psi}_{p,2n}(x)$ and

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \left|\hat{S}(x)\right|^2 dx &= \frac{1}{2\pi}\int_0^{2\pi} \left|\hat{c}(2nx)\tau(2nx)\hat{\psi}_{p,2n}(x)\right|^2 dx \\
&= \frac{1}{4\pi n}\int_0^{4\pi n} \left|\hat{c}(x)\tau(x)\hat{\psi}_{p,2n}\left(\frac{x}{2n}\right)\right|^2 dx \\
&= \frac{1}{4\pi n}\int_0^{2\pi} \left|\hat{c}(x)\right|^2 \left|\tau(x)\right|^2 \sum_{l=0}^{2n-1} \left|\hat{\psi}_{p,2n}\left(\frac{x + 2\pi l}{2n}\right)\right|^2 dx.
\end{aligned}$$

Note that $\|\hat{S}\| \sim \|\hat{c}\|$ and $\|\hat{S}\|^2 \sim \|\hat{c}\|^2$ for each $c \in \ell^2(Z)$. Then the desired conclusion $\tau(x) \sim 1$ follows from Lemma 3.3.

Conversely, one knows

$$\psi(j) = \frac{1}{2\pi}\int_0^{2\pi} \tau(x)\omega_{p,2n}(x, j)\,dx \tag{3.6}$$

with $\tau(x) \sim 1$. One shows firstly that $\{\psi(\cdot - 2ln)\}_{l\in Z}$ spans $W_{p,n}$: Let $S \in W_{p,n}$. Then $S(j) = \frac{1}{2\pi}\int_0^{2\pi} D(x)\omega_{p,2n}(x, j)\,dx$ for some $D \in L^2[0, 2\pi]$, due to Lemma 3.1. Since $\tau(x) \sim 1$, $\frac{D(x)}{\tau(x)} \in L^2[0, 2\pi]$ and $\frac{D(x)}{\tau(x)} = \sum_l c(l)e^{ilx}$ with $c \in \ell^2(Z)$. Combining (3.6) with (1.4), one has $\psi(j - 2ln) = \frac{1}{2\pi}\int_0^{2\pi} \tau(x)\omega_{p,2n}(x, j)e^{ilx}\,dx$. Furthermore,

$$\begin{aligned}
S(\cdot) &= \frac{1}{2\pi}\int_0^{2\pi} D(x)\omega_{p,2n}(x, j)\,dx = \frac{1}{2\pi}\int_0^{2\pi} \frac{D(x)}{\tau(x)}\tau(x)\omega_{p,2n}(x, j)\,dx \\
&= \sum_k c(k)\psi(\cdot - 2kn).
\end{aligned}$$

Next one shows the existence of Riesz bounds: note that Lemma 3.1 says that $\hat{\psi}(x) = \tau(2nx)\hat{\psi}_{p,2n}(x)$. Then $\hat{\psi}(x) \sim \hat{\psi}_{p,2n}(x)$, due to the given condition $\tau(x) \sim 1$. On the other hand, Lemma 3.3 tells $\sum_{l=0}^{2n-1} |\hat{\psi}_{p,n}(x + \frac{\pi l}{n})|^2 \sim 1$. Hence

$$\sum_{l=0}^{2n-1} \left|\hat{\psi}\left(x + \frac{\pi l}{n}\right)\right|^2 \sim \sum_{l=0}^{2n-1} \left|\hat{\psi}_{p,n}\left(x + \frac{\pi l}{n}\right)\right|^2 \sim 1. \tag{3.7}$$

Take $S(\cdot) = \sum_k c(k)\psi(\cdot - 2kn)$. Then $\hat{S}(x) = \hat{c}(2nx) \cdot \hat{\psi}(x)$. Furthermore, it is easy to see that

$$\|\hat{S}\|^2 = \left\|\hat{c}(2nx)\hat{\psi}(x)\right\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left|\hat{c}(2nx)\hat{\psi}(x)\right|^2 dx$$

$$= \frac{1}{4\pi n} \int_0^{2\pi} \left|\hat{c}(x)\right|^2 \sum_{l=0}^{2n-1} \left|\hat{\psi}\left(\frac{x + 2\pi l}{2n}\right)\right|^2 dx \sim \|\hat{c}\|^2$$

by (3.7). Hence $\|S\| \sim \|c\|$. Now the proof of Theorem 3.1 is finished. $\square$

We give two more concluding remarks, before closing this section:

Remark 3.1. Let $\psi_{p,2}$ be an RTB wavelet in $W_{p,2}$ and $\phi_{p,2}$ be an RTB spline in $V_{p,2}$. Recall that

$$V_{p,n} = V_{p,2n} \oplus W_{p,2n} \quad \text{and} \quad V_{p,1} = \ell^2(Z).$$

Then $\{\phi_{p,2}(\cdot - 2k), \psi_{p,2}(\cdot - 2k)\}_{k\in Z}$ must be a Riesz basis for $\ell^2(Z)$: In fact, the desired result follows from $\ell^2(Z) = V_{p,2} \oplus W_{p,2}$ and

$$\left\|\sum_k c(k)\phi(\cdot - 2k) + \sum_k d(k)\psi(\cdot - 2k)\right\|^2$$

$$= \left\|\sum_k c(k)\phi(\cdot - 2k)\right\|^2 + \left\|\sum_k d(k)\psi(\cdot - 2k)\right\|^2 \sim \|c\|^2 + \|d\|^2.$$

In general, we have

$$\ell^2(Z) = V_{p,2^n} \oplus W_{p,2^n} \oplus W_{p,2^{n-1}} \oplus \cdots \oplus W_{p,2}$$

and $\{\phi_{p,2^n}(\cdot - 2^n k), \psi_{p,2^n}(\cdot - 2^n k), \psi_{p,2^{n-1}}(\cdot - 2^{n-1}k), \ldots, \psi_{p,2}(\cdot - 2k)\}_{k\in Z}$ forms a Riesz basis for $\ell^2(Z)$.

There are many references for this kind of basis in $\ell^2(Z)$, see [3–6], etc. In particular, it is called p stage wavelet basis in [5]. In this case, each element of $\ell^2(Z)$ can be represented by this kind of stable basis. The convergence holds in $\ell^2(Z)$ sense, which can be compared with the pointwise convergence in [9]. On the other hand, since a Riesz basis is not necessarily orthogonal, we need to find the dual to give the expansion.

Remark 3.2. A detailed observation on [9, Theorems 4 and 9] shows that the similar dual results hold for RTB splines and RTB wavelets in $\ell^2(Z)$. We mention the conclusions without repeating the proofs:

Let

$$\phi(x)=\frac{1}{2\pi}\int_0^{2\pi} C(x)E_{p,n}(x,j)\,dx \quad \text{and} \quad \tilde{\phi}(x)=\frac{1}{2\pi}\int_0^{2\pi} D(x)E_{p,n}(x,j)\,dx$$

be RTB splines. Then ϕ and $\tilde{\phi}$ are dual each other, i.e., $\sum_j \phi(j-kn)\overline{\tilde{\phi}(j-qn)}=\delta(k-q)$ if and only if

$$C(x)\overline{D(x)}T_{2p,n}(x)=1.$$

Similarly, assume

$$\psi(x)=\frac{1}{2\pi}\int_0^{2\pi} C(x)\omega_{p,2n}(x,j)\,dx \quad \text{and} \quad \tilde{\psi}(x)=\frac{1}{2\pi}\int_0^{2\pi} D(x)\omega_{p,2n}(x,j)\,dx$$

be RTB wavelets. Then ψ and $\tilde{\psi}$ are dual each other, i.e., $\sum_j \psi(j-2kn)\overline{\tilde{\psi}(j-2ln)}=\delta(k-l)$, if and only if

$$C(x)\overline{D(x)}\Omega(x)=1,$$

where $\Omega(x)=4T_{2p,n}(\frac{x}{2})T_{2p,n}(\frac{x}{2}+\pi)T_{2p,2n}(x)$.

4. A class of symmetric RTB wavelets

It is easy to see that $\psi_{p,2n}$ is symmetric (or antisymmetric) and has finite support from Lemma 3.2. In this part, we shall give a family of symmetric RTB wavelets with finite supports, which include $\psi_{p,2n}$ as an example. It turns out that $\psi_{p,2n}$ has the shortest support among that class.

Theorem 4.1. *Let the function $\tau(x)=\sum_{m=1}^{N} c_m e^{-ik_m x}$ with $c_1\neq 0$ and $c_N\neq 0$ satisfies that*

(i) $\tau(x)\neq 0$ *on the whole real line* R;
(ii) $c_l=c_{N-l+1}$ *for each* $l\in\{1,2,\dots,N\}$ *and*
(iii) k_l+k_{N-l+1} *is a fixed constant for each* $l\in\{1,2,\dots,N\}$.

Then

$$\psi(j)=\frac{1}{2\pi}\int_0^{2\pi} \tau(x)\omega_{p,2n}(x,j)\,dx$$

is a symmetric (*or antisymmetric*) *RTB wavelet with finite support.*

If $N = c_1 = 1$ and $k_1 = 0$, then $\tau(x) = 1$ and the corresponding $\psi = \psi_{p,2n}$. We have much more choices, according to this theorem.

Proof. Since $\tau(x)$ is a trigonometric polynomial, the condition (i) implies that $\tau(x) \sim 1$ and $\tau(x) \in L^2[0, 2\pi]$. Moreover, it follows that the corresponding ψ is an RTB wavelet from Theorem 3.1.

Recall that

$$\omega_{p,2n}(x, j) = a\left(\frac{x}{2}\right)E_{p,n}\left(\frac{x}{2}, j\right) + a\left(\frac{x+2\pi}{2}\right)E_{p,n}\left(\frac{x+2\pi}{2}, j\right).$$

Then

$$\psi(j) = \frac{1}{2\pi}\int_0^{2\pi} \tau(x)\left[a\left(\frac{x}{2}\right)E_{p,n}\left(\frac{x}{2}, j\right) + a\left(\frac{x+2\pi}{2}\right)E_{p,n}\left(\frac{x+2\pi}{2}, j\right)\right] dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} + \int_{\pi}^{2\pi} \tau(2y)a(y)E_{p,n}(y, j)\, dy\right] = \frac{1}{\pi}\int_0^{2\pi} \tau(2x)a(x)E_{p,n}(x, j)\, dx.$$

Substituting $E_{p,n}(x, j) = \sum_l B_{p,n}(j - ln)e^{-ilx}$ and $\tau(x) = \sum_{m=1}^{N} c_m e^{-ik_m x}$ into this above identity, it reduces to

$$\psi(j) = \frac{1}{\pi}\sum_{m=1}^{N} c_m \int_0^{2\pi} a(x)\sum_l B_{p,n}(j - ln)e^{-i(l+2k_m)x}\, dx.$$

By the definition of $G(l)$ given in (3.3), one has

$$\psi(j) = \sum_l B_{p,n}(j - ln)\sum_{m=1}^{N} c_m G(l + 2k_m). \tag{4.1}$$

Furthermore, because both $B_{p,n}(\cdot)$ and $G(\cdot)$ have finite supports, $\psi(\cdot)$ does as well.

Now, one shows

$$\psi\left[2n - p - 2(k_1 + k_N)n - j\right] = (-1)^p\psi(j)$$

to conclude the symmetric (or antisymmetric) property for ψ: Note that

$$2n - p - 2(k_1 + k_N)n - j - ln = p(n - 1) + (2 - p - l)n - 2(k_1 + k_N)n - j$$

and

$$B_{p,n}\left[p(n - 1) - j\right] = B_{p,n}(j).$$

Then

$$B_{p,n}\left[2n - p - 2(k_1 + k_N)n - j - ln\right] = B_{p,n}\left[j + 2(k_1 + k_N)n - (2 - p - l)n\right].$$

This together with (4.1) leads to

$$\psi\big[2n-p-2(k_1+k_N)n-j\big] = \sum_l B_{p,n}\big[j+2(k_1+k_N)n-(2-p-l)n\big]\sum_{m=1}^N c_m G(l+2k_m).$$

By taking a variable change $s=2-p-l-2(k_1+k_N)$, one has

$$\psi\big[2n-p-2(k_1+k_N)n-j\big] = \sum_s B_{p,n}(j-sn)\sum_{m=1}^N c_m G\big[-s+2-p-2(k_1+k_N)+2k_m\big].$$

Note that condition (iii) says $-2(k_1+k_N)+2k_m=-2k_{N-m+1}$. Then

$$G\big[-s+2-p-2(k_1+k_N)+2k_m\big] = G(-s+2-p-2k_{N-m+1}) = (-1)^p G(s+2k_{N-m+1}),$$

by Lemma 3.2(ii). Hence

$$\psi\big[2n-p-2(k_1+k_N)n-j\big] = \sum_s B_{p,n}(j-sn)\sum_{m=1}^N c_m(-1)^p G(s+2k_{N-m+1}) = (-1)^p\sum_s B_{p,n}(j-sn)\sum_{l=1}^N c_{N-l+1}G(s+2k_l).$$

Using condition (ii), $c_{N-l+1}=c_l$ and (4.1), one reaches the desired conclusion

$$\psi\big[2n-p-2(k_1+k_N)n-j\big]=(-1)^p\psi(j).$$

The proof is done. □

Given a sequence $a=(a(n))$ with finite support. The length of the support is defined as $\max\{n, a(n)\neq 0\}-\min\{n, a(n)\neq 0\}$, denoted by $|\operatorname{supp} a|$.

Theorem 4.2. *Let $\tau(x)=\sum_{m=1}^N c_m e^{-ik_m x}$ with $c_1\neq 0$, $c_N\neq 0$ and $k_1\leqslant k_2\leqslant\cdots\leqslant k_N$. Then the corresponding RTB wavelet*

$$\psi_N(j)=\frac{1}{2\pi}\int_0^{2\pi}\tau(x)\omega_{p,2n}(x,j)\,dx$$

satisfies $|\operatorname{supp}\psi_N|\geqslant|\operatorname{supp}\psi_{p,2n}|$.

Note that the assumption $k_1\leqslant k_2\leqslant\cdots\leqslant k_N$ is not essential, since we may make a rearrangement for non-zero terms of the given $\tau(x)$, if it is not in such a case.

Proof. By using (4.1), $\psi_N(j)=\sum_l B_{p,n}(j-ln)\sum_{m=1}^N c_m G(l+2k_m)$, one shows firstly

$$\operatorname{supp}\psi_N\subseteq\big[(-\mu-p+1-2k_N)n,\ p(n-1)+(\mu+1-2k_1)n\big]. \tag{4.2}$$

For each $j \in \operatorname{supp}\psi_N$, since $\operatorname{supp} B_{p,n} \subseteq [0, p(n-1)]$, there exists some l such that $ln \leqslant j \leqslant p(n-1)+ln$. On the other hand, $\operatorname{supp} G \subseteq [-\mu-p+1, \mu+1]$ implies $-\mu-p+1 \leqslant l+2k_m \leqslant \mu+1$ for some $m \in \{1,2,\dots,N\}$. Hence $-\mu-p+1-2k_N \leqslant l \leqslant \mu+1-2k_1$. Altogether one receives (4.2). Note that $\psi_N = \psi_{p,2n}$, when $N=1$ and $k_1 = k_N = 0$. Then

$$\operatorname{supp}\psi_{p,2n} \subseteq \big[(-\mu-p+1)n,\ p(n-1)+(\mu+1)n\big]. \tag{4.3}$$

Next one proves

$$\psi_N\big[(-\mu-p+1-2k_N)n\big] \neq 0 \quad \text{and} \quad \psi_N\big[p(n-1)+(\mu+1-2k_1)n\big] \neq 0. \tag{4.4}$$

Using the expression for $\psi_N(j)$ in the beginning of the proof, one has

$$\begin{aligned}&\psi_N\big[(-\mu-p+1-2k_N)n\big]\\&\quad=\sum_l B_{p,n}\big[(-\mu-p+1-2k_N)n-ln\big]\sum_{m=1}^{N} c_m G(l+2k_m).\end{aligned}$$

Since $\operatorname{supp} B_{p,n} \subseteq [0, p(n-1)]$, only $l \leqslant -\mu-p+1-2k_N$ contributes to this above summation. Similarly $\operatorname{supp} G \subseteq [-\mu-p+1, \mu+1]$ requires $l+2k_N \geqslant -\mu-p+1$ or $l \geqslant -\mu-p+1-2k_N$. Therefore, $l = -\mu-p+1-2k_N$ and furthermore $l+2k_m = -\mu-p+1-2k_N+2k_m < -\mu-p+1$ for $1 \leqslant m \leqslant N-1$. Then

$$\psi_N\big[(-\mu-p+1-2k_N)n\big] = B_{p,n}(0)C_N G(-\mu-p+1) \neq 0.$$

Similarly $\psi_N[p(n-1)+(\mu+1-2k_1)n] \neq 0$ and hence (4.4) is proved.

According to (4.2), (4.4) and (4.3), one has

$$\begin{aligned}|\operatorname{supp}\psi_N| &= \big|p(n-1)+(\mu+1-2k_1)n-(-\mu-p+1-2k_N)n\big|\\&= \big|2(\mu+p)n-p+2(k_N-k_1)n\big| \geqslant 2(\mu+p)n-p \geqslant |\operatorname{supp}\psi_{p,2n}|.\end{aligned}$$

This completes the proof. □

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