NORTH-HOLLAND

On Perturbation Bounds for the QR Factorization

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ABSTRACT

Certain new perturbation bounds of the orthogonal factor in the QR factorization of a real matrix are derived. The bounds of this note improve the known bounds in the literature.

1. INTRODUCTION

Let A be a real $m \times n$ matrix ($A \in \mathcal{R}^{m\times n}$) with rank $A = n$. The QR factorization of A is a decomposition of the form $A = QR$, in which $R \in \mathcal{R}^{n\times n}$ is an upper triangular matrix with positive diagonal elements, and the matrix $Q \in \mathcal{R}^{m\times n}$ satisfies $Q^{\mathrm{T}}Q = I$, the identity matrix. Here Q^{T} is the transpose of Q. The matrix Q is referred to as the orthogonal factor, and R the triangular factor in the QR factorization of A. It is well known that the QR factorization of a full-column-rank matrix is unique, and the QR factorization is one of the most important decompositions of a matrix (see [3]).

Let $E \in \mathcal{R}^{m\times n}$ with rank$(A + E) = n$, and let $A + E = (Q + W)(R + F)$ be the QR factorization of $A + E$. A number of upper bounds on $\|F\|/\|R\|$ and $\|W\|$ in terms of $\|E\|$ for a certain norm $\|\ \|$ have been derived by Stewart [6] and Sun [8]. Recently, Stewart [7] gave asymptotic perturbation bounds on $\|F\|/\|R\|$ and $\|W\|$, and Bhatia and Mukherjea [2] presented a new bound on $\|W\|$. Besides, Sun [9] gives componentwise perturbation bounds of $|F|$ and $|W|$, where the matrix $|F|$ is defined by $|F| = (|f_{ij}|)$ for $F = (f_{ij})$. It is worthwhile to point out that the bounds of $\|W\|$ (or $|W|$) given by [6–9] are derived from certain known bounds of $\|F\|$

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(or $|F|$) and the relation

$$W = [E - (Q + W)F]R^{-1}.$$

In this note we derive several new bounds of $\|W\|$ directly. Roughly speaking, the new bounds of [2] and this note improve the results of [7] and [8] by a factor $1 + 1/\sqrt{2}$:

$$\begin{aligned} \|W\|_{\rm F} &\lesssim (1+\sqrt{2})\|A^{\dagger}\|_2\|E\|_{\rm F} \qquad \text{(by the results of [7, 8])}, \\ \|W\|_{\rm F} &\lesssim \sqrt{2}\left\|A^{\dagger}\right\|_2\|E\|_{\rm F} \qquad \text{(by the new bounds)}, \end{aligned}$$

where $A^{\dagger}$ denotes the Moore-Penrose inverse of A, and $\|\ \|_2$ and $\|\ \|_F$ stand for the spectral norm and the Frobenius norm, respectively.

Note that the methods and results of this note are different from those of [2]. Bhatia and Mukherjea [2] apply calculus on manifold and matrix Lie groups to consider complex square matrices. The following inequality was proved in [2]:

$$\|\widetilde{Q} - Q\|_{\rm F} \le \max_{0\le t\le 1} \left\|A(t)^{-1}\right\|_2 \|E\|_{\rm F},$$

where $A(t) = A + tE$, $0 \le t \le 1$. Remark that the quantity $\max_{0\le t\le 1} \|A(t)^{-1}\|_2$ is not convenient to calculate. In this note we apply fixed-point theory and elementary calculus to consider real rectangular matrices. The upper bounds of $\|\widetilde{Q} - Q\|_{\rm F}$ obtained in this note are computable.

We shall use $I^{(n)}$ to denote the identity matrix of order n, and 0 the null matrix. $P_A = AA^{\dagger}$ denotes the orthogonal projection onto the column space of A. The symbol $\lambda(\ )$ denotes the set of the eigenvalues of a matrix or an operator. $\mathcal{U}^{n\times n}$ is the set of real $n\times n$ upper triangular matrices, and $\mathcal{U}_{\rm s}^{m\times n}$, $\mathcal{L}_{\rm s}^{m\times n}$, $\mathcal{D}^{m\times n}$ are sets defined by

$$\begin{aligned} \mathcal{U}_{\rm s}^{m\times n} &= \left\{A = (\alpha_{ij}) \in \mathcal{R}^{m\times n} : \alpha_{ij} = 0\ \forall i \ge j\right\}, \\ \mathcal{L}_{\rm s}^{m\times n} &= \left\{A = (\alpha_{ij}) \in \mathcal{R}^{m\times n} : \alpha_{ij} = 0\ \forall i \le j\right\}, \\ \mathcal{D}^{m\times n} &= \left\{A = (\alpha_{ij}) \in \mathcal{R}^{m\times n} : \alpha_{ij} = 0\ \forall i \ne j\right\}. \end{aligned}$$

Obviously, an $X \in \mathcal{R}^{m\times n}$ can be split uniquely as

$$X = X_L + X_D + X_U, \qquad X_L \in \mathcal{L}_{\rm s}^{m\times n}, \quad X_D \in \mathcal{D}^{m\times n}, \quad X_U \in \mathcal{U}_{\rm s}^{m\times n}. \tag{1.1}$$

The matrices X_L, X_D, X_U of (1.1) will be denoted by

$$X_L = \mathrm{low}(X), \qquad X_D = \mathrm{diag}(X), \qquad X_U = \mathrm{up}(X). \tag{1.2}$$

The relation (1.2) gives the definitions of the operators low( ), diag( ), and up( ) defined on $\mathcal{R}^{m\times n}$ (Reference [4]).

In Section 2 we derive perturbation equations. In Section 3 we discuss some basic properties of the operator **L** [defined below by (2.8)] and the function $l(R)$ [defined below by (3.8)], which are important for studying perturbation bounds for the orthogonal factor in the QR factorization. In Sections 4 and 5 we apply fixed-point theory and elementary calculus to derive perturbation bounds of the orthogonal factor, respectively, and in Section 6 we give a numerical example.

2. PERTURBATION EQUATIONS

Let $A \in \mathcal{R}^{m\times n}$ with rank $A = n$, and let $\widetilde{A} = A + E$, where $E \in \mathcal{R}^{m\times n}$ satisfies

$$\epsilon_2 \equiv \left\|A^\dagger\right\|_2 \|E\|_2 < 1. \tag{2.1}$$

Then obviously rank $\widetilde{A} = n$. Let $A = QR, \widetilde{A} = \widetilde{Q}\widetilde{R}$ be the QR factorizations of $A, \widetilde{A}$, respectively, and let

$$E = \widetilde{A} - A, \qquad W = \widetilde{Q} - Q, \qquad F = \widetilde{R} - R. \tag{2.2}$$

Then it is easy to verify that the perturbation matrices W, F satisfy the equation

$$E = WR + \widetilde{Q}F. \tag{2.3}$$

Take a matrix $\widetilde{P} \in \mathcal{R}^{m\times(m-n)}$ such that $\widetilde{U} = (\widetilde{Q}, \widetilde{P})$ is an orthogonal matrix, and let

$$\widetilde{E} = \widetilde{U}^{\mathrm{T}}E, \qquad X = \widetilde{U}^{\mathrm{T}}W = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}, \quad X^{(1)} \in \mathcal{R}^{n\times n}. \tag{2.4}$$

Then from (2.3)

$$XR + \begin{pmatrix} F \\ 0 \end{pmatrix} = \widetilde{E}. \tag{2.5}$$

Moreover, from the relations $Q^{\mathrm{T}}Q = I$ and

$$\begin{pmatrix} I \\ 0 \end{pmatrix} - X = \widetilde{U}^{\mathrm{T}}Q$$

we get

$$X^{(1)} + X^{(1)^{\mathrm{T}}} = X^{\mathrm{T}}X. \tag{2.6}$$

Let X be a solution to the equations (2.5)–(2.6). Define the matrices X_L, X_D, X_U by (1.1)–(1.2). Then from (2.5) and $\mathrm{low}(XR) = \mathrm{low}(X_L R)$ it follows that

$$\mathrm{low}(X_L R) = \mathrm{low}(\widetilde{E}). \tag{2.7}$$

Define the operator $\mathbf{L} : \mathcal{L}_s^{m\times n} \to \mathcal{L}_s^{m\times n}$ by

$$\mathbf{L}X_L = \mathrm{low}(X_L R), \qquad X_L \in \mathcal{L}_s^{m\times n}. \tag{2.8}$$

Then the equation (2.7) can be rewritten as

$$\mathbf{L}X_L = \mathrm{low}(\widetilde{E}). \tag{2.9}$$

Moreover, from (2.6)

$$X_D = \begin{pmatrix} \frac{1}{2}\,\mathrm{diag}(X^{\mathrm{T}}X) \\ 0 \end{pmatrix}, \qquad X_U = \begin{pmatrix} \mathrm{up}(X^{\mathrm{T}}X) - X_L^{(1)^{\mathrm{T}}} \\ 0 \end{pmatrix}, \tag{2.10}$$

where $X_L^{(1)} = \mathrm{low}(X^{(1)})$.

In Section 4 we shall prove that there is a unique solution X to the equations (2.9)–(2.10) in the case of small $\|E\|_{\mathrm{F}}$, and derive upper bounds of $\|X\|_{\mathrm{F}}$ for the solution X.

3. THE OPERATOR L AND FUNCTION $l(R)$

Before we go on to derive perturbation bounds of $\|X\|_{\mathrm{F}}$ from the equations (2.9)–(2.10), it will be necessary to discuss the basic properties of the operator $\mathbf{L}$ defined by (2.8).

We first consider the relation

$$Y = XR, \tag{3.1}$$

where $X, Y \in \mathcal{R}^{m\times n}$, $R = (r_{ij}) \in \mathcal{U}^{n\times n}$. Let

$$X = (x_1, \ldots, x_n) = (\xi_{ij}), \quad Y = (y_1, \ldots, y_n) = (\eta_{ij}),$$
$$x = \left(x_1^{\mathrm{T}}, \ldots, x_n^{\mathrm{T}}\right)^{\mathrm{T}}, \qquad y = \left(y_1^{\mathrm{T}}, \ldots y_n^{\mathrm{T}}\right)^{\mathrm{T}}.$$

The relation (3.1) can be rewritten as

$$y = \left(R^{\mathrm{T}} \otimes I^{(m)}\right)x,$$

where $A \otimes B \equiv (\alpha_{ij}B)$ is a Kronecker product. Write

$$L \equiv R^{\mathrm{T}} \otimes I^{(m)} = (L_{ij}), \qquad L_{ij} \in \mathcal{R}^{m\times m}, \quad i, j = 1, \ldots, n; \tag{3.2}$$

we have

$$L_{ij} = \begin{cases} 0, & i < j, \\ r_{ii}I^{(m)}, & i = j, \\ r_{ji}I^{(m)}, & i > j. \end{cases} \tag{3.3}$$

Let

$$Y_L = \mathbf{L}X_L \equiv \mathrm{low}(X_L R). \tag{3.4}$$

Define $x_j^{(L)}$, $y_j^{(L)}$, $x^{(L)}$, $y^{(L)}$ by

$$\begin{aligned} x_j^{(L)} &= \begin{pmatrix} \xi_{j+1,j} \\ \vdots \\ \xi_{mj} \end{pmatrix}, \quad y_j^{(L)} = \begin{pmatrix} \eta_{j+1,j} \\ \vdots \\ \eta_{mj} \end{pmatrix}, \qquad j = 1, 2, \ldots, n_1, \\ x^{(L)} &= \left(x_1^{(L)^{\mathrm{T}}}, \ldots, x_{n_1}^{(L)^{\mathrm{T}}}\right)^{\mathrm{T}}, \; y^{(L)} = \left(y_1^{(L)^{\mathrm{T}}}, \ldots, y_{n_1}^{(L)^{\mathrm{T}}}\right)^{\mathrm{T}}, \end{aligned}$$

where

$$n_1 = \begin{cases} n & \text{if} \quad m > n, \\ n - 1 & \text{if} \quad m = n. \end{cases} \tag{3.5}$$

The relation (3.4) can be rewritten as

$$y^{(L)} = L^{(L)}x^{(L)}, \tag{3.6}$$

where

$$\begin{aligned} L^{(L)} = \left(L_{ij}^{(L)}\right), \qquad L_{ij}^{(L)} = L_{ij}\begin{pmatrix} i+1, i+2, \ldots, m \\ j+1, j+2, \ldots, m \end{pmatrix}, \\ i, j = 1, 2, \ldots, n_1, \end{aligned} \tag{3.7}$$

in which n_1 is defined by (3.5), and

$$L_{ij}\begin{pmatrix} i+1, i+2, \ldots, m \\ j+1, j+2, \ldots, m \end{pmatrix}$$

denotes the submatrix of L_{ij} consisting of rows $i+1, i+2, \ldots, m$ and columns $j+1, j+2, \ldots, m$. Combining (3.7) with (3.3), we see that

$$\lambda\left(L^{(L)}\right) = \bigcup_{j=1}^{n_1} \lambda\left(L_{jj}^{(L)}\right) = \bigcup_{j=1}^{n_1} \underbrace{\{r_{jj}, \ldots, r_{jj}\}}_{m-j},$$

where n_1 is defined by (3.5). Since (3.6) is equivalent to (3.4), we have the following

THEOREM 3.1. *Let* $R = (r_{ij}) \in \mathcal{U}^{n\times n}$, *and let* $\mathbf{L}$ *be the operator defined by* (2.8). *Then the eigenvalues of* $\mathbf{L}$ *are*

$$\underbrace{r_{11},\ldots,r_{11}}_{m-1},\underbrace{r_{22},\ldots,r_{22}}_{m-2},\ldots,\underbrace{r_{nn},\ldots,r_{nn}}_{m-n}.$$

Let $\mathbf{L}$ be the operator defined by (2.8). Now we define the function $l(R)$ by

$$l(R) = \inf_{\substack{X_L\in\mathcal{L}_s^{m\times n}\\ \|X_L\|_F=1}} \|\mathbf{L}X_L\|_F. \tag{3.8}$$

It is easy to verify that

$$l(R) = \begin{cases} \|\mathbf{L}^{-1}\|^{-1}, & 0 \notin \lambda(\mathbf{L}),\\ 0, & 0 \in \lambda(\mathbf{L}),\end{cases} \tag{3.9}$$

where $\|\mathbf{L}^{-1}\|$ is the subordinate operator norm defined by

$$\|\mathbf{L}^{-1}\| = \max_{\substack{Y_L\in\mathcal{L}_s^{m\times n}\\ \|Y_L\|_F=1}} \|\mathbf{L}^{-1}Y_L\|_F. \tag{3.10}$$

Therefore, from Theorem 3.1 we get the following

COROLLARY 3.2. *Let* $l(R)$ *be the function defined by* (3.8). *If* R *is nonsingular, then*

$$0 < l(R) \le \min_{1\le i\le n_1} |r_{ii}|,$$

where n_1 *is defined by* (3.5).

The following result gives a relation between $l(R)$ and $1/\|A^\dagger\|_2$.

THEOREM 3.3. *Let* $A \in \mathcal{R}^{m\times n}$ *with* rank $A = n$, $A = QR$ *be the QR factorization of* A, *and let* $\mathbf{L}$, $l(R)$ *be defined by* (2.8), (3.8), *respectively. Then*

$$l(R) \ge \frac{1}{\|A^\dagger\|_2}. \tag{3.11}$$

PROOF. By (3.9) and $\|A^\dagger\|_2 = \|R^{-1}\|_2$, the inequality (3.11) is equivalent to

$$\|\mathbf{L}^{-1}\| \le \|R^{-1}\|_2. \tag{3.12}$$

Now we are going to prove (3.12).

Let $X = (x_1, x_2, \ldots, x_n) \in \mathcal{R}^{m\times n}$, where

$$x_j = \begin{pmatrix} x_j^{(U)} \\ x_j^{(L)} \end{pmatrix}, \quad x_j^{(U)} = \begin{pmatrix} \xi_{1j} \\ \vdots \\ \xi_{jj} \end{pmatrix}, \quad x_j^{(L)} = \begin{pmatrix} \xi_{j+1,j} \\ \vdots \\ \xi_{mj} \end{pmatrix}, \qquad j = 1, \ldots, n_1,$$

where n_1 is defined by (3.5), and $x_n^{(U)} = x_n$ if $m = n$. By (1.1)–(1.2) $X_L = (x_1^{(0)}, \ldots, x_n^{(0)}) \in \mathcal{R}^{m\times n}$, where

$$x_j^{(0)} = \begin{pmatrix} 0 \\ x_j^{(L)} \end{pmatrix}, \quad j = 1, \ldots, n_1, \qquad x_n^{(0)} = 0 \quad \text{if } m = n.$$

Let $x = \left(x_1^{\mathrm{T}}, \ldots, x_n^{\mathrm{T}}\right)^{\mathrm{T}}$, and let

$$x^{(L)} = \left(x_1^{(L)^{\mathrm{T}}}, \ldots, x_n^{(L)^{\mathrm{T}}}\right)^{\mathrm{T}}, \qquad x^{(U)} = \left(x_1^{(U)^{\mathrm{T}}}, \ldots, x_n^{(U)^{\mathrm{T}}}\right)^{\mathrm{T}}.$$

Moreover, we use the vectors $y \in \mathcal{R}^{mn}$, $y^{(L)} \in \mathcal{R}^{mn_1 - n_1(n_1+1)/2}$, $y^{(U)} \in \mathcal{R}^{n(n+1)/2}$ to correspond to $Y \in \mathcal{R}^{m\times n}$. Then the relations

$$Y = XR, \qquad Y_L = \mathbf{L}X_L \equiv \mathrm{low}\,(X_L R)$$

are equivalent to

$$y = Lx, \qquad y^{(L)} = L^{(L)} x^{(L)}, \tag{3.13}$$

respectively, when L and $L^{(L)}$ are expressed by (3.2)–(3.3) and (3.7). From the first relation of (3.13)

$$\frac{\partial y_k}{\partial x_j} = \begin{cases} 0, & k < j, \\ \begin{pmatrix} r_{jj} I^{(j)} & 0 \\ 0 & r_{jj} I^{(m-j)} \end{pmatrix}, & k = j, \\ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, & k > j. \end{cases}$$

Thus, we have $\partial y_k^{(L)} / \partial x_j^{(U)} = 0 \; \forall k, j$, and

$$\begin{pmatrix} y^{(L)} \\ y^{(U)} \end{pmatrix} = \begin{pmatrix} L^{(L)} & 0 \\ * & * \end{pmatrix} \begin{pmatrix} x^{(L)} \\ x^{(U)} \end{pmatrix}.$$

This means that there is a permutation matrix P such that

$$P(R^{\mathrm{T}} \otimes I^{(m)})P^{\mathrm{T}} = \begin{pmatrix} L^{(L)} & 0 \\ * & * \end{pmatrix},$$

i.e.,

$$P\left(R^{\mathrm{T}} \otimes I^{(m)}\right)^{-1} P^{\mathrm{T}} = \begin{pmatrix} L^{(L)^{-1}} & 0 \\ * & * \end{pmatrix}. \tag{3.14}$$

Observe that from the second relation of (3.13) $x^{(L)} = L^{(L)^{-1}} y^{(L)}$; hence by the definition (3.10) and (3.14) we have

$$\|\mathbf{L}^{-1}\| = \left\|L^{(L)^{-1}}\right\|_2 \leq \left\|(R^{\mathrm{T}} \otimes I^{(m)})^{-1}\right\|_2 = \|R^{-1}\|_2.$$

The inequality (3.12) is proved. ■

The following result shows that the function $l(R)$ is insensitive to perturbations of R.

THEOREM 3.4. *Let R, $M \in \mathcal{U}^{n\times n}$. Then*

$$l(R) - \|M\|_2 \leq l(R+M) \leq l(R) + \|M\|_2. \tag{3.15}$$

PROOF. By the definition (3.8), we have

$$\begin{aligned} l(R) &= \min\left\{\|\mathrm{low}\,(X_L R)\,\|_{\mathrm{F}} : X_L \in \mathcal{L}_{\mathrm{s}}^{m\times n},\ \|X_L\|_{\mathrm{F}} = 1\right\} \\ &= \left\|\mathrm{low}\left(X_L^* R\right)\right\|_{\mathrm{F}}, \qquad X_L^* \in \mathcal{L}_{\mathrm{s}}^{m\times n}, \quad \left\|X_L^*\right\|_{\mathrm{F}} = 1, \end{aligned}$$

and

$$\begin{aligned} l(R+M) &= \min\left\{\left\|\mathrm{low}\,(X_L(R+M))\right\|_{\mathrm{F}} : X_L \in \mathcal{L}_{\mathrm{s}}^{m\times n},\ \|X_L\|_{\mathrm{F}} = 1\right\} \\ &\leq \left\|\mathrm{low}\left(X_L^*(R+M)\right)\right\|_{\mathrm{F}} \\ &\leq \left\|\mathrm{low}\left(X_L^* R\right)\right\|_{\mathrm{F}} + \left\|\mathrm{low}\left(X_L^* M\right)\right\|_{\mathrm{F}} \\ &\leq l(R) + \|M\|_2. \end{aligned}$$

Similarly, we can prove the first inequality of (3.15). ■

Let

$$X = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix} \in \mathcal{R}^{m\times n} \qquad \text{with} \quad \mathrm{rank}\, X = n,$$

where $X^{(1)} \in \mathcal{R}^{n\times n}$. Define the operator $\mathbf{L}_1 : \mathcal{L}_s^{n\times n} \to \mathcal{L}_s^{n\times n}$ by

$$\mathbf{L}_1 X_L^{(1)} = \mathrm{low}\left(X_L^{(1)} R\right), \qquad X_L^{(1)} \in \mathcal{L}_s^{n\times n}, \tag{3.16}$$

and define the function $l_1(R)$ by

$$l_1(R) = \inf_{\substack{X_L^{(1)} \in \mathcal{L}_s^{n\times n} \\ \|X_L^{(1)}\|_F = 1}} \|\mathbf{L}_1 X_L^{(1)}\|_F. \tag{3.17}$$

Observe that $\mathbf{L}_1$ is just $\mathbf{L}$ for the case $m = n$; hence the operator $\mathbf{L}_1$ and the function $l_1(R)$ have the same properties as $\mathbf{L}$ and $l(R)$ stated by Theorem 3.1, Corollary 3.2, and Theorems 3.3–3.4.

4. PERTURBATION THEOREMS (I)

Now we are going to discuss the solution X to the equations (2.9)–(2.10) under the assumption that all the diagonal elements of R are positive.

By the assumption and Theorem 3.1, the operators $\mathbf{L}$ and $\mathbf{L}_1$ defined by (2.8) and (3.16) are nonsingular. Moreover, from (2.4)–(2.5)

$$X_L^{(1)} = \mathbf{L}_1^{-1}\left[\mathrm{low}(\widetilde{Q}^T E)\right].$$

Thus, the equations (2.9)–(2.10) can be rewritten as a continuous mapping $\Phi : \mathcal{R}^{m\times n} \to \mathcal{R}^{m\times n}$ expressed by

$$\begin{aligned} X_L &= \mathbf{L}^{-1}\left[\mathrm{low}(\widetilde{E})\right], \\ X_D &= \begin{pmatrix} \frac{1}{2}\,\mathrm{diag}\left(X^{\mathrm{T}}X\right) \\ 0 \end{pmatrix}, \\ X_U &= \begin{pmatrix} \mathrm{up}(X^{\mathrm{T}}X) - \mathbf{L}_1^{-1}[\mathrm{low}(\widetilde{Q}^{\mathrm{T}}E)] \\ 0 \end{pmatrix}, \end{aligned}$$

or simply

$$X = \phi(X) + G, \tag{4.1}$$

where

$$\begin{aligned} \phi(X) &= \begin{pmatrix} \frac{1}{2}\,\mathrm{diag}(X^{\mathrm{T}}X) + \mathrm{up}(X^{\mathrm{T}}X) \\ 0 \end{pmatrix}, \\ G &= \mathbf{L}^{-1}[\mathrm{low}(\widetilde{E})] - \begin{pmatrix} (\mathbf{L}_1^{-1}[\mathrm{low}(\widetilde{Q}^{\mathrm{T}}E)])^{\mathrm{T}} \\ 0 \end{pmatrix}. \end{aligned} \tag{4.2}$$

Let

$$\eta = \frac{\sqrt{5}}{2}, \qquad \gamma = \sqrt{\left(\frac{\|E\|_F}{l(R)}\right)^2 + \left(\frac{\|P_{\widetilde{A}}E\|_F}{l_1(R)}\right)^2}, \tag{4.3}$$

where $l(R)$ and $l_1(R)$ are the functions defined by (3.8) and (3.17), respectively. One can verify the following inequalities:

(1) $\|\phi(X)\|_F \leq \eta\|X\|_F^2$,
(2) $\|\phi(X) - \phi(Y)\|_F \leq 2\eta \max\{\|X\|_F, \|Y\|_F\}\|X - Y\|_F$,
(3) $\|G\|_F \leq \gamma$.

Hence, by Stewart [5, Theorem 3.1], if $4\gamma\eta < 1$, then the mapping Φ expressed by (4.1)–(4.2) has a unique fixed point X in the neighborhood

$$\mathcal{S}(0; 2\gamma) \equiv \{X \in \mathcal{R}^{m\times n} : \|X\|_F < 2\gamma\}$$

of the origin $\mathcal{R}^{m\times n}$, and

$$\|X\|_F \leq \frac{2\gamma}{1 + \sqrt{1 - 4\gamma\eta}}.$$

Let

$$\epsilon_{l,l_1} \equiv \frac{\gamma}{\sqrt{2}} = \sqrt{\frac{1}{2}\left[\left(\frac{\|E\|_F}{l(R)}\right)^2 + \left(\frac{\|P_{\widetilde{A}}E\|_F}{l_1(R)}\right)^2\right]}. \tag{4.4}$$

Then the condition $4\gamma\eta < 1$ can be rewritten as

$$2\sqrt{10}\epsilon_{l,l_1} < 1. \tag{4.5}$$

Observe that from (2.4) and (2.2)

$$\|X\|_F = \|W\|_F = \|\widetilde{Q} - Q\|_F.$$

Hence, we get the following perturbation theorem.

THEOREM 4.1. *Let $A \in \mathcal{R}^{m\times n}$ with* rank $A = n$, *$A = QR$ be the QR factorization of A, and $A = A + E$, and let ϵ_2 and ϵ_{l,l_1} be defined by* (2.1) *and* (4.4), *respectively. If $\epsilon_2 < 1$ and ϵ_{l,l_1} satisfies the condition* (4.5), *then* rank $\widetilde{A} = n$ *and the QR factorization $\widetilde{A} = \widetilde{Q}\widetilde{R}$ satisfies*

$$\begin{aligned}\|\widetilde{Q} - Q\|_F &\leq \frac{2\sqrt{2}\epsilon_{l,l_1}}{1 + \sqrt{1 - 2\sqrt{10}\epsilon_{l,l_1}}} \equiv b(\epsilon_{l,l_1}) \\ &= \sqrt{2}\epsilon_{l,l_1} + \sqrt{5}\epsilon_{l,l_1}^3 + 5\sqrt{2}\epsilon_{l,l_1}^3 + O(\epsilon_{l,l_1}^4).\end{aligned} \tag{4.6}$$

By Theorem 3.3

$$\epsilon_{l,l_1} \le \|A^\dagger\|_2 \sqrt{\frac{\|E\|_F^2 + \|P_{\widetilde{A}} E\|_F^2}{2}} \equiv \widetilde{\epsilon}. \tag{4.7}$$

Moreover, we have

$$2\sqrt{10}\widetilde{\epsilon} \ge 2\sqrt{5}\|A^\dagger\|_2\|E\|_F \ge \|A^\dagger\|_2\|E\|_2 = \epsilon_2.$$

Hence, Theorem 4.1 gives the following

COROLLARY 4.2. *Let A, Q, $\widetilde{A}$, E be as in Theorem* 4.1, *and let $\widetilde{\epsilon}$ be defined by* (4.7). *If*

$$2\sqrt{10}\widetilde{\epsilon} < 1,$$

then $\operatorname{rank}\widetilde{A} = n$, *and the QR factorization $\widetilde{A} = \widetilde{Q}\widetilde{R}$ satisfies*

$$\begin{aligned} \|\widetilde{Q} - Q\|_F &\le \frac{2\sqrt{2}\widetilde{\epsilon}}{1 + \sqrt{1 - 2\sqrt{10}\widetilde{\epsilon}}} \equiv b(\widetilde{\epsilon}) \\ &= \sqrt{2}\widetilde{\epsilon} + \sqrt{5}\widetilde{\epsilon}^2 + 5\sqrt{2}\widetilde{\epsilon}^3 + O\left(\widetilde{\epsilon}^4\right). \end{aligned} \tag{4.8}$$

5. PERTURBATION THEOREMS (II)

Let $A \in \mathcal{R}^{m\times n}$ with $\operatorname{rank}(A) = n$, $A = QR$ be the QR factorization of A, $A(t) = A + tE$, $0 \le t \le 1$, $\widetilde{A} = A(1)$, and let ϵ_2 be defined by (2.1). It is known that if $\epsilon_2 < 1$, then $\operatorname{rank} A(t) = n$, and $A(t)$ has the unique QR factorization

$$A(t) = Q(t)R(t) \tag{5.1}$$

for each $t \in [0, 1]$, where $R(t) \in \mathcal{R}^{n\times n}$ is a upper triangular matrix with positive diagonal elements, and $Q(t) \in \mathcal{R}^{m\times n}$ satisfy

$$Q(t)^T Q(t) = I. \tag{5.2}$$

In this section we shall apply elementary calculus [1, 8] to derive perturbation bounds of the orthogonal factor in the QR factorization of A. Differentiating (5.1) and (5.2), we get

$$E\,\mathrm{d}t = \mathrm{d}Q(t)\,R(t) + Q(t)\,\mathrm{d}R(t) \tag{5.3}$$

and

$$\mathrm{d}Q(t)^T\,Q(t) + Q(t)^T\,\mathrm{d}Q(t) = 0. \tag{5.4}$$

Take $P(t) \in \mathcal{R}^{m\times(m-n)}$ so that $U(t) = (Q(t), P(t))$ is orthogonal, and let

$$\delta A(t) = U(t)^{\mathrm{T}} E\,\mathrm{d}t, \qquad \delta X(t) = U(t)^{\mathrm{T}}\,\mathrm{d}Q(t) = \begin{pmatrix}\delta X^{(1)}(t)\\ \delta X^{(2)}(t)\end{pmatrix}, \quad \delta X^{(1)}(t) \in \mathcal{R}^{n\times n}. \tag{5.5}$$

Then (5.3) and (5.4) can be rewritten as

$$\delta X(t) = \delta A(t)\,R(t)^{-1} - \begin{pmatrix}\mathrm{d}R(t)\,R(t)^{-1}\\ 0\end{pmatrix} \tag{5.6}$$

and

$$\delta X^{(1)}(t)^{\mathrm{T}} + \delta X^{(1)}(t) = 0. \tag{5.7}$$

By (1.1)–(1.2), the matrix $\delta X(t)$ can be split uniquely as

$$\delta X(t) = \delta X(t)_L + \delta X(t)_D + \delta X(t)_U.$$

Thus, the relation (5.7) implies $\delta X(t)_D = 0$. Moreover, from (5.5)–(5.7) we get

$$\delta X(t)_L = \mathrm{low}\big(\delta A(t)\,R(t)^{-1}\big)$$

and

$$\begin{aligned}\delta X(t)_U &= \begin{pmatrix}\delta X^{(1)}(t)_U\\ 0\end{pmatrix} = \begin{pmatrix}-[\delta X^{(1)}(t)_L]^{\mathrm{T}}\\ 0\end{pmatrix}\\ &= \begin{pmatrix}-[\mathrm{low}(Q(t)^{\mathrm{T}}ER(t)^{-1})]^{\mathrm{T}}\,\mathrm{d}t\\ 0\end{pmatrix}.\end{aligned} \tag{5.8}$$

Hence,

$$\begin{aligned}\|\mathrm{d}Q(t)\|_{\mathrm{F}} &= \|\delta X(t)\|_{\mathrm{F}} = \sqrt{\|\delta X(t)_L\|_{\mathrm{F}}^2 + \|\delta X(t)_U\|_{\mathrm{F}}^2}\\ &\le \sqrt{2}\|E\|_{\mathrm{F}}\|R(t)^{-1}\|_2\,\mathrm{d}t = \sqrt{2}\|E\|_{\mathrm{F}}\|A(t)^{\dagger}\|_2\,\mathrm{d}t.\end{aligned} \tag{5.9}$$

Consequently,

$$\begin{aligned}\|\widetilde{Q} - Q\|_{\mathrm{F}} &= \left\|\int_0^1 \mathrm{d}Q(t)\right\|_{\mathrm{F}} \le \int_0^1 \|\mathrm{d}Q(t)\|_{\mathrm{F}}\\ &\le \sqrt{2}\|E\|_{\mathrm{F}}\int_0^1 \|A(t)^{\dagger}\|_2\,\mathrm{d}t = \sqrt{2}\|E\|_{\mathrm{F}}\int_0^1 \frac{\mathrm{d}t}{\sigma_n(t)},\end{aligned} \tag{5.10}$$

where $\sigma_n(t)$ is the smallest singular value of the matrix $A(t)$. Let σ_n be the smallest singular value of A. Combining (5.10) with the well-known relation $\sigma_n(t) \geq \sigma_n - \|E\|_2 t$, we get

$$\begin{aligned}\|\widetilde{Q} - Q\|_F &\leq \sqrt{2}\|E\|_F \int_0^1 \frac{dt}{\sigma_n - \|E\|_2 t} \\ &= \frac{\sqrt{2}\|E\|_F}{\|E\|_2} \ln \frac{1}{1 - \|A^\dagger\|_2\|E\|_2} \\ &= \sqrt{2}\omega(\epsilon_2)\epsilon_f,\end{aligned}$$

where ϵ_2 is defined by (2.1), and

$$\epsilon_f = \|A^\dagger\|_2\|E\|_F, \qquad \omega(\epsilon) = \frac{1}{\epsilon}\ln\frac{1}{1-\epsilon}, \quad 0 < \epsilon < 1. \tag{5.11}$$

Hence, we have proved the following result.

THEOREM 5.1. *Let $A, Q, \widetilde{A}, E$ be as in Theorem* 4.1, *and let ϵ_2, ϵ_f, and $\omega(\epsilon)$ be defined by* (2.1) *and* (5.11), *respectively. If $\epsilon_2 < 1$, then* $\operatorname{rank} A = n$, *and the QR factorization $\widetilde{A} = \widetilde{Q}\widetilde{R}$ satisfies*

$$\begin{aligned}\|\widetilde{Q} - Q\|_F &\leq \sqrt{2}\omega(\epsilon_2)\epsilon_f \equiv b_1(\epsilon_2, \epsilon_f) \\ &= \sqrt{2}\epsilon_f + \frac{\sqrt{2}}{2}\epsilon_2\epsilon_f + \frac{\sqrt{2}}{3}\epsilon_2^2\epsilon_f + O\left(\epsilon_f^4\right). \\ &\leq \frac{\sqrt{2}\epsilon_f}{\sqrt{1-\epsilon_2}} \equiv b_2(\epsilon, \epsilon_f) \\ &= \sqrt{2}\epsilon_f + \sqrt{2}\epsilon_2\epsilon_f + \sqrt{2}\epsilon_2^2\epsilon_f + O\left(\epsilon_f^4\right).\end{aligned} \tag{5.12}$$

We note that the perturbation bound (5.12) can be improved by observing that from (5.8)

$$\begin{aligned}\|\delta X(t)_U\|_F &\leq \left\|Q(t)^T E R(t)^{-1}\, dt\right\|_F \leq \left\|R(t)^{-1}\right\|_2 \left\|Q(t)^T E\right\|_F dt \\ &= \left\|A(t)^\dagger\right\|_2 \left\|\left(\int_0^t dQ(\tau) + Q\right)^T E\right\|_F dt \\ &\leq \left\|A(t)^\dagger\right\|_2 \left(\|E\|_2 \int_0^t \|dQ(\tau)\|_F + \|P_A E\|_F\right) dt.\end{aligned}$$

Combining it with

$$\int_0^t \|\mathrm{d}Q(\tau)\|_{\mathrm{F}} \le \int_0^1 \|\mathrm{d}Q(\tau)\|_{\mathrm{F}} \le \sqrt{2}\omega(\epsilon_2)\epsilon_f \qquad \text{[see (5.10)–(5.12)]},$$

we get

$$\|\delta X(t)_U\|_{\mathrm{F}} \le \left\|A(t)^\dagger\right\|_2 \left[\sqrt{2}\|E\|_2\omega(\epsilon_2)\epsilon_f + \|P_A E\|_{\mathrm{F}}\right]\mathrm{d}t.$$

Substituting it into (5.9), we obtain

$$\begin{aligned}
\|\mathrm{d}Q(t)\|_{\mathrm{F}} &\le \left\|A(t)^\dagger\right\|_2 \sqrt{\|E\|_{\mathrm{F}}^2 + (\sqrt{2}\|E\|_2\omega(\epsilon_2)\epsilon_f + \|P_A E\|_{\mathrm{F}})^2}\,\mathrm{d}t \\
&\le \left\|A(t)^\dagger\right\|_2 \left(\sqrt{\|E\|_{\mathrm{F}}^2 + \|P_A E\|_{\mathrm{F}}^2} + \sqrt{2}\|E\|_2\omega(\epsilon_2)\epsilon_f\right)\mathrm{d}t \\
&= \sqrt{2}\left\|A(t)^\dagger\right\|_2 \left[\|A^\dagger\|_2^{-1}\epsilon + \|E\|_2\omega(\epsilon_2)\epsilon_f\right]\mathrm{d}t,
\end{aligned}$$

where

$$\epsilon \equiv \left\|A^\dagger\right\|_2 \frac{\sqrt{\|E\|_{\mathrm{F}}^2 + \|P_A E\|_{\mathrm{F}}^2}}{2} \tag{5.13}$$

Hence,

$$\begin{aligned}
\|\widetilde{Q} - Q\|_{\mathrm{F}} &\le \int_0^1 \|\mathrm{d}Q(t)\|_{\mathrm{F}} \\
&\le \sqrt{2}\left[\|A^\dagger\|_2^{-1}\epsilon + \|E\|_2\omega(\epsilon_2)\epsilon_f\right] \int_0^1 \left\|A(t)^\dagger\right\|_2 \mathrm{d}t \\
&\le \sqrt{2}\omega(\epsilon_2)\left[\epsilon + \omega(\epsilon_2)\epsilon_2\epsilon_f\right].
\end{aligned}$$

Thus, we have proved the following

THEOREM 5.2. *Let $A, Q, \widetilde{A}, E$ be as in Theorem* 4.1, *and let $\epsilon_2, \epsilon_f, \omega(\epsilon), \epsilon$ be defined by* (2.1), (5.11), (5.13). *If $\epsilon_2 < 1$, then* $\operatorname{rank}\widetilde{A} = n$, *and the QR factorization $\widetilde{A} = \widetilde{Q}\widetilde{R}$ satisfies*

$$\begin{aligned}
\|\widetilde{Q} - Q\|_{\mathrm{F}} &\le \sqrt{2}\omega(\epsilon_2)\left[\epsilon + \omega(\epsilon_2)\epsilon_2\epsilon_f\right] \equiv b_0(\epsilon, \epsilon_2, \epsilon_f) \\
&= \sqrt{2}\epsilon + \frac{\sqrt{2}}{2}\epsilon_2(\epsilon + 2\epsilon_f) + \frac{\sqrt{2}}{3}\epsilon_2^2(\epsilon + 3\epsilon_f) + O\left(\epsilon_f^4\right).
\end{aligned} \tag{5.14}$$

6. AN EXAMPLE

The following result has been proved in [8]: *Let $A, Q, R, \widetilde{A}, E$ be as in Theorem* 4.1, *and let ϵ_2, ϵ_f be defined by* (2.1), (5.11). *If $\epsilon_2 < 1$, then there is a unique QR* *factorization* $\widetilde{A} = \widetilde{Q}\widetilde{R}$, *and*

$$\begin{aligned}\|\widetilde{Q} - Q\|_F &\le (1+\sqrt{2})\omega(\epsilon_2)\epsilon_f \equiv \beta_1(\epsilon_2, \epsilon_f) \\ &\le \frac{(1+\sqrt{2})\epsilon_f}{1-\epsilon_2} \equiv \beta_2(\epsilon_2, \epsilon_f). \qquad (6.1)\end{aligned}$$

It is easy to verify that there are the following relations between the upper bounds $b(\epsilon_{l,l_1})$, $b(\widetilde{\epsilon})$, $b_1(\epsilon_2, \epsilon_f)$, and $\beta_1(\epsilon_2, \epsilon_f)$:

$$b\left(\epsilon_{l,l_1}\right) \le b(\widetilde{\epsilon}), \qquad b_1\left(\epsilon_2, \epsilon_f\right) \le \beta_1\left(\epsilon_2, \epsilon_f\right).$$

Moreover, the bound $b_0(\epsilon, \epsilon_2, \epsilon_f)$ expressed by (5.14) is better than $b_1(\epsilon_2, \epsilon_f)$ for very small perturbations of A, and by the author's numerical tests the bound $b(\epsilon_{l,l_1})$ is better than all the others.

Now we give a numerical example.

EXAMPLE 6.1. Let

$$A = \begin{pmatrix} R \\ 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & -2 & 1 & 2 & 3 \\ 0 & 2 & 4 & 1 & -5 \\ 0 & 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

and let $\widetilde{A} = A + E$, $E = \tau E_0$ with $\tau > 0$, and

$$E_0 = \begin{pmatrix} 0.2 & -0.5 & 0.3 & 0.1 & 0.4 \\ -0.1 & 0.4 & 0.1 & -0.3 & 0.2 \\ 0.5 & 0.7 & -0.2 & 0.1 & 0.6 \\ 0.3 & -0.6 & 0.1 & -0.1 & 0.2 \\ 0.2 & 0.1 & 0.7 & 0.3 & -0.4 \\ 0.4 & 0.8 & -0.2 & 0.1 & 0.3 \\ 0.6 & -0.1 & -0.5 & 0.1 & -0.2 \\ 0.1 & -0.3 & 0.2 & 0.6 & 0.7 \end{pmatrix}.$$

TABLE 1.

τ	1.000000e-01	1.000000e-02	1.000000e-05	1.000000e-08
$\|E\|_F$	2.431049e-01	2.431049e-02	2.431049e-05	2.431049e-08
$\|\widetilde{Q}-Q\|_F$	3.355805e-01	3.162668e-02	3.138194e-05	3.138170e-08
$b(\epsilon_{l,l_1})$		9.834899e-02	8.765590e-05	8.764743e-08
$b(\widetilde{\epsilon})$		1.037806e-01	9.189167e-05	9.188238e-08
$b_0(\epsilon, \epsilon_2, \epsilon_f)$	2.104630e+00	9.913509e-02	9.188933e-05	9.188238e-08
$b_1(\epsilon_2, \epsilon_f)$	1.397628e+00	1.065221e-01	1.040365e-04	1.040340e-07
$b_2(\epsilon_2, \epsilon_f)$	1.939075e+00	1.090902e-01	1.040389e-04	1.040341e-07
$\beta_1(\epsilon_2, \epsilon_f)$	2.385901e+00	1.818447e-01	1.776013e-04	1.775972e-07
$\beta_2(\epsilon_2, \epsilon_f)$	3.310208e+00	1.862287e-01	1.776055e-04	1.775972e-07

Obviously, A has the QR factorization $A = QR$ with

$$Q = \begin{pmatrix} I \\ 0 \end{pmatrix} \in \mathcal{R}^{8\times 5}.$$

By MATLAB we get the QR factorization $\widetilde{A} = \widetilde{Q}\widetilde{R}$ of $\widetilde{A}$ for small τ. Some numerical results by using MATLAB are listed in Table 1, where the scalars $\epsilon_2, \epsilon_{l,l_1}, \widetilde{\epsilon}, \epsilon_f, \epsilon$ are defined by (2.1), (4.4), (4.7), (5.11), (5.13), and the bounds $b(\epsilon_{l,l_1}), b(\widetilde{\epsilon}), b_i(\epsilon_2, \epsilon_f)\,(i = 1, 2), b_0(\epsilon, \epsilon_2, \epsilon_f), \beta_i(\epsilon_2, \epsilon_f)(i = 1, 2)$ are defined by (4.6), (4.8), (5.12), (5.14), (6.1), respectively.

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