Properties of Schur complements in partitioned idempotent matrices

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Abstract

Related to a complex partitioned matrix \( P \), having \( A, B, C, \) and \( D \) as its consecutive \( m \times m, m \times n, n \times m, \) and \( n \times n \) submatrices, are generalized Schur complements \( S = A - BD^{-1}C \) and \( T = D - CA^{-1}B \), where the minus superscript denotes a generalized inverse of a given matrix. In the first part of the present paper, we aim at specifying conditions under which certain properties of \( P \) hold also for \( S \) and \( T \) when \( P \) is an idempotent matrix (i.e., represents a projector) or a Hermitian idempotent matrix (i.e., represents an orthogonal projector). Among the properties considered are: the idempotency itself, existence of an eigenvalue equal to zero, and relationships between eigenvectors of \( P \) and those of \( S \) and \( T \), corresponding to this eigenvalue. The second part of the paper deals with two partitioned idempotent matrices \( P_1 \) and \( P_2 \). We indicate conditions under which the idempotency of the sum \( P_1 + P_2 \) and the difference \( P_1 - P_2 \) is inherited by the sums and differences of the related Schur complements \( S_1, S_2 \) and \( T_1, T_2 \). The inheritance property of such a type is also discussed in the context of matrix partial orderings, with the emphasis laid on the minus (rank subtractivity) ordering.

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1. Introduction

Let $C_{m,n}$ denote the set of $m \times n$ complex matrices and let $C_{m,m}^\geq$ be the subset of $C_{m,m}$ consisting of Hermitian nonnegative definite matrices. The symbols $K^*$, $\mathcal{R}(K)$, $\mathcal{N}(K)$, and $r(K)$ will stand for the conjugate transpose, range (column space), null space, and rank, respectively, of $K \in C_{m,n}$, while $\text{tr}(K)$ and $I_m$ will denote the trace of $K \in C_{m,m}$ and the identity matrix of order $m$. Further, let $K[1]$ and $K[1,2]$ be the sets of all generalized inverses and all reflexive generalized inverses of $K \in C_{m,n}$, i.e.,

$$K[1] = \{ K^- \in C_{n,m} : KK^- K = K \},$$

$$K[1,2] = \{ K^\ominus \in C_{n,m} : KK^\ominus K = K, K^\ominus KK^\ominus = K^\ominus \}.$$

and let $K^+$ denote the Moore–Penrose inverse of $K$, i.e., the unique solution to the equations

$$KK^+ K = K, \quad K^+ KK^+ = K^+, \quad KK^+ = (KK^+)^*, \quad K^+ K = (K^+ K)^*.$$  

It is well known (cf. Theorem 5.1.1 in [17]) that a matrix $K \in C_{m,m}$ is a projector in $C_{m,1}$ if and only if it is idempotent, i.e., $K = K^2$. Consequently, the terms “a projector” and “an idempotent matrix” will be used interchangeably.

The present paper is concerned with idempotent matrices $P \in C_{m+n,m+n}$ partitioned as

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

(1.1)

where $A \in C_{m,m}$ and $D \in C_{n,n}$. Related to this matrix are generalized Schur complements: of $D$ in $P$ and $A$ in $P$, defined by the formulae

$$S = A - BD^+ C \quad \text{and} \quad T = D - CA^+ B,$$

(1.2)

respectively, where $D^-$ and $A^-$ are any given generalized inverses of $D$ and $A$. Let us recall that the term “Schur complement” was introduced by Haynsworth [12, p. 74] with reference to the matrix $T$ with $A^-$ replaced by $A^{-1}$, as in the original paper by Schur [18]. Carlson et al. [4] generalized this concept by relaxing the assumption of the nonsingularity and referring to the notion of the Moore–Penrose inverse, which leads to

$$S = A - BD^+ C \quad \text{and} \quad T = D - CA^+ B.$$  

(1.3)

A generalization of (1.3) to (1.2), with $D^- \in D[1]$ and $A^- \in A[1]$ used instead of $D^+$ and $A^+$, is due to Marsaglia and Styan [14,15]. An exhaustive survey of results concerning Schur complements has been given by Ouellette [16]; see also [19].

Throughout this paper, we will often refer to two particular classes of matrices of the form (1.1): with submatrices $B, C, D$ satisfying the inclusions

$$\mathcal{R}(B^*) \subseteq \mathcal{R}(D^*) \quad \text{and} \quad \mathcal{R}(C) \subseteq \mathcal{R}(D),$$

(1.4)
equivalent to
\[ BD^- D = B \quad \text{and} \quad DD^- C = C, \]  
(1.5)
and with submatrices \( A, B, C \) satisfying the inclusions
\[ \mathcal{R}(C^*) \subseteq \mathcal{R}(A^*) \quad \text{and} \quad \mathcal{R}(B) \subseteq \mathcal{R}(A), \]  
(1.6)
equivalent to
\[ CA^- A = C \quad \text{and} \quad AA^- B = B, \]  
(1.7)
where the choices of \( D^- \in D[1] \) and \( A^- \in A[1] \) in (1.5) and (1.7) are arbitrary. It should be emphasized that the pairs of inclusions (1.4) and (1.6) admit a very natural interpretation from the view-point of generalized Schur complements specified in (1.2). Namely, they are necessary and sufficient for \( S \) and \( T \) to be independent of the choice of \( D^- \in D[1] \) and \( A^- \in A[1] \), respectively; cf., e.g., [17, pp. 21 and 43]. In particular, from Theorem 1 of Albert [1] it follows that all four inclusions in (1.4) and (1.6) are satisfied by \( A = A^*, D = D^*, \) and \( C = B^* \) when a partitioned matrix

\[ \tilde{P} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \]  
(1.8)
is Hermitian nonnegative definite. Then the requirement of idempotency means that \( \tilde{P} \) is an orthogonal projector (with the orthogonality understood according to the standard inner product).

It seems that the purpose of our considerations may well be reflected by the word “inheritance”. We aim at specifying conditions under which certain properties of idempotent matrices \( P \) partitioned as in (1.1) or Hermitian idempotent matrices \( \tilde{P} \) partitioned as in (1.8) hold also for related generalized Schur complements of \( D \) and \( A \) in \( P \), defined in (1.2), or \( D \) and \( A \) in \( \tilde{P} \) specified as

\[ \tilde{S} = A - BD^- B^* \quad \text{and} \quad \tilde{T} = D - B^* A^- B, \]  
(1.9)
respectively, the expressions in (1.9) being actually independent of the choice of \( D^- \in D[1] \) and \( A^- \in A[1] \) due to the assumption that \( \tilde{P} \in C_{m+n}^{m+n} \). In Section 2, we derive criteria for \( S \) and \( T \) to be idempotent and observe that they are fulfilled in each case when considering \( \tilde{S} \) and \( \tilde{T} \). This observation is useful in establishing relationships between the ranks and traces of principal submatrices of \( \tilde{P} \). In Section 3, we provide conditions under which \( S \) and \( T \) contain at least one eigenvalue equal to zero and investigate possible relationships between eigenvectors corresponding to zero eigenvalue of \( P \) on the one hand and those corresponding to zero eigenvalues of \( S \) and \( T \) on the other. The next two sections deal with two partitioned idempotent matrices \( P_1 \) and \( P_2 \). In the first of them, we aim at characterizing situations where the properties which ensure the idempotency of \( P_1 + P_2 \) and \( P_1 - P_2 \) are inherited by \( S_1, S_2 \) and \( T_1, T_2 \). These results are accompanied by examples showing that inheritance of such a type does not hold for other linear combinations of \( P_1 \) and \( P_2 \) and for their product \( P_1 P_2 \). In the last section, we indicate conditions under which matrix partial orderings, especially the minus (rank subtractivity) ordering, between \( P_1 \) and \( P_2 \) remain valid for corresponding generalized Schur complements.
2. General properties

We begin with the problem of characterizing the idempotency of generalized Schur complements related to an arbitrary partitioned idempotent matrix.

**Theorem 2.1.** Let \( P \in \mathbb{C}^{m+n,m+n} \) be a partitioned idempotent matrix of the form (1.1) and let \( S \in \mathbb{C}^{m,m} \) and \( T \in \mathbb{C}^{n,n} \) be generalized Schur complements defined in (1.2). Then \( S \) is idempotent if and only if

\[
B(I_n - D^*D)(I_n - DD^*)C = 0, \tag{2.1}
\]

provided that

\[
BD^*DD^*C = BD^*C, \tag{2.2}
\]

and \( T \) is idempotent if and only if

\[
C(I_m - A^*A)(I_m - AA^*)B = 0, \tag{2.3}
\]

provided that

\[
CA^*AA^*B = CA^*B. \tag{2.4}
\]

**Proof.** It can easily be verified that \( P \) is idempotent if and only if

\[
A = A^2 + BC, \quad B = AB + BD, \quad C = CA + DC, \quad D = CB + D^2. \tag{2.5}
\]

Hence, in view of (1.2),

\[
S^2 = A^2 - ABD^*C - BD^*CA + BD^*CBD^*C
\]

\[
= A - BC - (B - BD)D^*C - BD^*(C - DC) + BD^*(D - D^2)D^*C
\]

\[
= A - BC - BD^*C + BDD^*C - BD^*C + BD^*DC
\]

\[
+ BD^*DD^*C - BD^*D^2D^*C,
\]

which under condition (2.2) simplifies to the form

\[
S^2 = A - BD^*C - B(I_n - DD^*) - D^*D + D^*D^2D^*C.
\]

This shows that (2.1) is necessary and sufficient for \( S = S^2 \) in all cases where a generalized inverse \( D^* \) satisfies (2.2), and by analogous arguments it follows that (2.3) is necessary and sufficient for \( T = T^2 \) whenever \( A^* \) satisfies (2.4). \( \square \)

It should be noticed that, in view of the equivalences of (1.4) to (1.5) and (1.6) to (1.7), conditions (2.1) and (2.2) are fulfilled when either of the two inclusions in (1.4) holds and, similarly, conditions (2.3) and (2.4) are fulfilled when either of the
two inclusions in (1.6) holds. In particular, since all these inclusions hold when \( P \) is a Hermitian nonnegative definite matrix, Theorem 2.1 leads immediately to the following.

**Corollary 2.1.** For every partitioned Hermitian idempotent matrix \( \tilde{P} \in \mathbb{C}_{m+n}^2 \) of the form (1.8), each of the two generalized Schur complements \( \tilde{S} \in \mathbb{C}_m^2 \) and \( \tilde{T} \in \mathbb{C}_n^2 \) defined in (1.9) is also a Hermitian idempotent matrix.

Another question concerning Theorem 2.1 is whether conditions (2.1) and (2.3) alone are sufficient for the idempotency of \( S \) and \( T \), respectively. It appears that the answer is negative. For instance, if \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), then choosing \( D^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) leads to the generalized Schur complement

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix},
\]

which is not an idempotent matrix despite the fact that condition (2.1) is satisfied.

Moreover, it is noteworthy that conditions (2.2) and (2.4) in Theorem 2.1 are fulfilled by all reflexive generalized inverses of \( D \) and \( A \), respectively. In particular, this observation results in the following.

**Corollary 2.2.** Let \( P \in \mathbb{C}_{m+n,m+n} \) be a partitioned idempotent matrix of the form (1.1) and let \( \tilde{S} \in \mathbb{C}_{m,m} \) and \( \tilde{T} \in \mathbb{C}_{n,n} \) be the generalized Schur complements defined in (1.3). Then \( \tilde{S} \) is idempotent if and only if

\[
B(I_n - D^+D)(I_n - DD^+)C = 0,
\]

and \( \tilde{T} \) is idempotent if and only if

\[
C(I_m - A^+A)(I_m - AA^+)B = 0.
\]

An immediate consequence of Corollary 2.2 is a solution to Problem 90 of Graybill [6, pp. 365–366].

**Corollary 2.3.** Let \( P \in \mathbb{C}_{m+n,m+n} \) be a partitioned upper triangular matrix of the form (1.1) with all diagonal elements of \( A \in \mathbb{C}_{m,m} \) equal to unity and all diagonal elements of \( D \in \mathbb{C}_{n,n} \) equal to zero. Then \( P \) is idempotent if and only if \( A = I_m \) and \( D = 0 \).

**Proof.** The assumption that \( P \) is an upper triangular matrix entails \( C = 0 \), which in view of Corollary 2.2 implies that both \( \tilde{S} = A \) and \( \tilde{T} = D \) are idempotent matrices.
Since $A$ is nonsingular, it can be idempotent only when $A = I_m$. On the other hand, since $\text{tr}(D) = 0$ and the trace of any idempotent matrix is equal to its rank, it follows that $D = 0$, thus completing the proof of necessity. Sufficiency of the conditions $A = I_m$ and $D = 0$ is clear. □

It is known that the orthogonal projector onto the range of a given matrix $X$ admits a representation

$$P_X = X(X^*X)^{-1}X^*;$$

(cf., e.g., [17, p. 111]). The next result is obtained by applying Corollary 2.1 to the projector related to a row-partitioned matrix $X = (X^*_1 : X^*_2)^\dagger$.

**Corollary 2.4.** For given $X_1 \in \mathbb{C}^{m,p}$ and $X_2 \in \mathbb{C}^{n,p}$, let $Y \in \mathbb{C}^{p,q}$ be any matrix satisfying

$$YY^* = (X^*_1X_1 + X^*_2X_2)^+.$$  

Then the matrices

$$P_{12} = X_1Y(I_q - P_{Y^*X_1^*})Y^*X_1^* \quad \text{and} \quad P_{21} = X_2Y(I_q - P_{Y^*X_2^*})Y^*X_2^*$$

represent the orthogonal projectors onto $\mathbb{R}[X^*_1Y(Y^*X_1^*)^\perp]$ and $\mathbb{R}[X^*_2Y(Y^*X_2^*)^\perp]$, respectively, where $(Y^*X_1^*)^\perp$ is any matrix such that $\mathbb{R}[(Y^*X_i^*)^\perp] = \mathbb{N}(X_i,Y), \ i = 1,2$. 

**Proof.** Since $P_X$ in (2.6) is independent of the choice of $(X^*X)^{-1} \in (X^*X)^{1}$, it may be expressed as $P_X = X(X^*X)^{1}X^*$. Substituting to this formula $X = (X^*_1 : X^*_2)^\dagger$ and $(X^*X)^{1}$ represented as in (2.7) yields

$$P_X = \begin{pmatrix} X_1YY^*X_1^* & X_1YY^*X_2^* \\ X_2YY^*X_1^* & X_2YY^*X_2^* \end{pmatrix}. $$

Then $P_X$ corresponds to $\tilde{P}$ of the form (1.8), while $P_{12}$ and $P_{21}$ given in (2.8) correspond to $\tilde{S}$ and $\tilde{T}$ of the form (1.9), respectively. Consequently, on account of Corollary 2.1, these matrices are the orthogonal projectors onto their ranges. Since $I_q - P_{Y^*X_i^*}$ represents the orthogonal projector onto $\mathbb{N}(X_i,Y)$ and since

$$\mathbb{R}(P_{ij}) = \mathbb{R}[X_iY(I_q - P_{Y^*X_j^*})], \quad i, j = 1, 2; \ i \neq j,$$

it follows that the subspaces (2.9) admit characteristics given in this corollary. □

From the first and fourth equalities in (2.5), with $C$ replaced by $B^*$, it follows that if a matrix $\tilde{P}$ of the form (1.8) is an orthogonal projector, then neither of its principal submatrices can itself be a projector except only for the trivial case when $B = 0$, i.e., when $\tilde{P}$ is a block-diagonal matrix. This observation is closely related to Problem 81 in [6, p. 365], stating that when $A$ is an orthogonal projector, then $\tilde{P}$ is an orthogonal projector if and only if also $D$ is an orthogonal projector and $B = 0$. 


Actually, it is possible to establish a stronger result, which concerns the equalities \( r(A) = \text{tr}(A) \) and \( r(D) = \text{tr}(D) \) being well known necessary conditions for the idempotency of \( A \) and \( D \).

**Theorem 2.2.** If \( \tilde{P} \in \mathbb{C}^{m+n} \) is a partitioned Hermitian idempotent matrix of the form (1.8), then

\[
\begin{align*}
r(A) &= \text{tr}(A) + \text{tr}(B^*A^+B) \quad \text{and} \quad r(D) = \text{tr}(D) + \text{tr}(BD^*B^*).
\end{align*}
\]

Consequently,

\[
\begin{align*}
r(A) &\geq \text{tr}(A) \quad \text{and} \quad r(D) \geq \text{tr}(D),
\end{align*}
\]

with the equality holding in each case if and only if \( B = 0 \).

**Proof.** From Corollary 19.1 of Marsaglia and Styan [14] it is known that, for any Hermitian nonnegative definite matrix \( \tilde{P} \) of the form (1.8) and the corresponding generalized Schur complements specified in (1.9),

\[
r(\tilde{P}) = r(A) + r(\tilde{T}) = r(D) + r(\tilde{S});
\]

see also Theorem 9.6.1 in [11]. On the other hand, however, if \( \tilde{P} \) is idempotent, then

\[
r(\tilde{P}) = \text{tr}(\tilde{P}) = \text{tr}(A) + \text{tr}(D).
\]

From Corollary 2.1 it follows that the idempotency of \( \tilde{P} \) entails the idempotency of \( \tilde{S} \) and \( \tilde{T} \), and hence

\[
r(\tilde{S}) = \text{tr}(A) - \text{tr}(BD^*B^*) \quad \text{and} \quad r(\tilde{T}) = \text{tr}(D) - \text{tr}(B^*A^+B),
\]

Combining (2.13) with (2.12) modified with the use of (2.14) yields equalities (2.10). Inequalities (2.11) are immediate consequences of the fact that \( \text{tr}(B^*A^+B) \) and \( \text{tr}(BD^*B^*) \) [equal to \( \text{tr}(B^*A^+B) \) and \( \text{tr}(BD^+B^*) \), respectively] are squares of the Frobenius norm of the matrices \((A^+)^{1/2}B\) and \((D^+)^{1/2}B^*\). Furthermore, this observation shows that the equalities in (2.11) hold if and only if \((A^+)^{1/2}B = 0\) and \((D^+)^{1/2}B^* = 0\), or, equivalently, \(AA^+B = 0\) and \(BD^+D = 0\). Since, on the other hand, the nonnegative definiteness of \( \tilde{P} \) entails \( \Re(B) \subseteq \Re(A) \) and \( \Re(B^*) \subseteq \Re(D^*) \), i.e., \( AA^+B = B \) and \( BD^+D = B \), it follows that each of the equalities \( r(A) = \text{tr}(A) \) and \( r(D) = \text{tr}(D) \) holds exclusively when \( B = 0 \). □

3. Eigenvalues and eigenvectors

It is known that the eigenvalues of an idempotent matrix are equal to zero or unity. Theorem 2.1 ensures that if a partitioned matrix \( P \) of the form (1.1) is a projector satisfying either of the conditions in (1.4), then also \( S \) specified in (1.2) is a projector, and an analogous statement is valid with regard to \( T \) and the conditions in (1.6).
Theorem 3.1 shows that if a projector $P$ is not the identity matrix, thus having at least one eigenvalue equal to zero, then in most cases also the Schur complements $S$ and $T$ have at least one such eigenvalue.

**Theorem 3.1.** Let $P \in \mathbb{C}_{m+n,m+n}$ be a partitioned idempotent matrix of the form (1.1) and let $S \in \mathbb{C}_{m,m}$ and $T \in \mathbb{C}_{n,n}$ be generalized Schur complements defined in (1.2). Then the condition $\mathfrak{R}(C) \subseteq \mathfrak{R}(D)$, under which $S = S^2$, ensures also that $S$ has at least one eigenvalue equal to zero if and only if

$$A \neq I_m \text{ or } BD^{-1}C \neq 0.$$  \hfill (3.1)

Similarly, the condition $\mathfrak{R}(B) \subseteq \mathfrak{R}(A)$, under which $T = T^2$, ensures also that $T$ has at least one eigenvalue equal to zero if and only if

$$D \neq I_n \text{ or } CA^{-1}B \neq 0.$$  \hfill (3.2)

**Proof.** In view of the equivalence of the latter parts of (1.4) and (1.6) to those in (1.5) and (1.7), a consequence of Theorem 2.1 is that if $\mathfrak{R}(C) \subseteq \mathfrak{R}(D)$ and $\mathfrak{R}(B) \subseteq \mathfrak{R}(A)$, then the idempotency of $P$ entails the idempotency of $S$ and $T$, respectively. Since an idempotent matrix is nonsingular if and only if it is the identity matrix, it follows that $S$ and $T$ have no zero eigenvalue exclusively when

$$A = I_m \text{ and } D = I_n, \quad \text{or } \quad BD^{-1}C = I_m \quad \text{and} \quad CA^{-1}B = I_n.$$  \hfill (3.3)

In view of conditions (2.5) characterizing the idempotency of $P$, premultiplying the equalities in (3.3) by $A$ and $D$, respectively, leads to

$$A^2 - (B - BD)D^{-1}C = A^2 + BC \quad \text{and} \quad D^2 - (C - CA)A^{-1}B = CB + D^2.$$  

Hence $BD^{-1}C = 0$ in the former case and $CA^{-1}B = 0$ in the latter, thus showing that the two equalities in (3.3) correspond to the pairs

$$A = I_m, \quad BD^{-1}C = 0 \quad \text{and} \quad D = I_n, \quad CA^{-1}B = 0.$$  \hfill (3.4)

Consequently, conditions (3.1) and (3.2) are obtained just by negating the conjunctions in (3.4). \hfill $\Box$

It has already been pointed out that both inclusions appearing as assumptions in the two parts of Theorem 3.1 hold when considering an orthogonal projector $P$ of the form (1.8). Then the conditions $BD^{-1}C \neq 0$ and $CA^{-1}B \neq 0$ take the forms $BD^{-1}B^* \neq 0$ and $B^*A^{-1}B \neq 0$. Since each of them is equivalent to $B \neq 0$, Theorem 3.1 leads in particular to the following.

**Corollary 3.1.** Let $P \in \mathbb{C}_{m+n,m+n}$ be a partitioned Hermitian idempotent matrix of the form (1.8) and let $\tilde{S} \in \mathbb{C}_{m}^{m}$ and $\tilde{T} \in \mathbb{C}_{n}^{n}$ be the generalized Schur complements defined in (1.9). Then $\tilde{S}$ has at least one eigenvalue equal to zero if and only if $A \neq I_m$ or $B \neq 0$ and, similarly, $\tilde{T}$ has this property if and only if $D \neq I_n$ or $B \neq 0$.\hfill \Box
The next result is concerned with certain relationships between eigenvectors of $\mathbf{P}$ and those of $\mathbf{S}$ and $\mathbf{T}$, which correspond to zero eigenvalue.

**Theorem 3.2.** Let $\mathbf{P} \in \mathbb{C}_{m+n,m+n}$ be a partitioned idempotent matrix of the form (1.1) and let $\mathbf{S} \in \mathbb{C}_{m,m}$ and $\mathbf{T} \in \mathbb{C}_{n,n}$ be generalized Schur complements defined in (1.2). If $\mathbf{x} = (\mathbf{u}^* : \mathbf{v}^*)^* \in \mathbb{C}_{m+n,1}$ is an eigenvector of $\mathbf{P}$ corresponding to zero eigenvalue (i.e., $\mathbf{P}\mathbf{x} = \mathbf{0}$) such that $\mathbf{u} \neq \mathbf{0}$ and if the submatrices of $\mathbf{P}$ fulfill conditions (1.4) and (3.1), then $\mathbf{u}$ is an eigenvector of $\mathbf{S}$ also corresponding to zero eigenvalue (i.e., $\mathbf{S}\mathbf{u} = \mathbf{0}$), and, on the other hand, if a nonzero vector $\mathbf{u} \in \mathbb{C}_{m,1}$ satisfies $\mathbf{S}\mathbf{u} = \mathbf{0}$, then it can be supplemented by $\mathbf{v} = -\mathbf{D}^* \mathbf{C} \mathbf{u}$ to form the vector $\mathbf{x} = (\mathbf{u}^* : \mathbf{v}^*)^*$, which satisfies $\mathbf{P}\mathbf{x} = \mathbf{0}$. An analogous result holds with respect to $\mathbf{v}$ and $\mathbf{T}$ when $\mathbf{v} \neq \mathbf{0}$ and the submatrices of $\mathbf{P}$ fulfill conditions (1.6) and (3.2), a supplement of $\mathbf{v}$ to $\mathbf{x}$ then being $\mathbf{u} = -\mathbf{A}^* \mathbf{B} \mathbf{v}$.

**Proof.** On account of Theorem 3.1, the assumptions (1.4) with (3.1) and (1.6) with (3.2) ensure that at least one eigenvalue of $\mathbf{S}$ and $\mathbf{T}$, respectively, is equal to zero. For $\mathbf{P}$ of the form (1.1), $\mathbf{P}\mathbf{x} = \mathbf{0}$ if and only if

$$
\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{v} = \mathbf{0} \quad \text{and} \quad \mathbf{C}\mathbf{u} + \mathbf{D}\mathbf{v} = \mathbf{0}.
$$

(3.5)

Since $\mathcal{R}(\mathbf{B}^*) \subseteq \mathcal{R}(\mathbf{D}^*) \Leftrightarrow \mathbf{BD}^{-1} = \mathbf{B}$, it follows that combining the former equality in (3.5) with the latter premultiplied by $\mathbf{BD}^{-1}$ leads to $\mathbf{Su} = \mathbf{0}$. Similarly, since $\mathcal{R}(\mathbf{C}^*) \subseteq \mathcal{R}(\mathbf{A}^*) \Leftrightarrow \mathbf{CA}^{-1} = \mathbf{C}$, it follows that combining the latter equality in (3.5) with the former premultiplied by $\mathbf{CA}^{-1}$ leads to $\mathbf{Tv} = \mathbf{0}$. On the other hand, since $\mathcal{R}(\mathbf{C}) \subseteq \mathcal{R}(\mathbf{D}) \Leftrightarrow \mathbf{DD}^{-1} = \mathbf{C}$, it is seen that for any $\mathbf{u}$ satisfying $\mathbf{S}\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = -\mathbf{D}^* \mathbf{C} \mathbf{u}$ we have

$$
\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{v} = \mathbf{Su} = \mathbf{0} \quad \text{and} \quad \mathbf{C}\mathbf{u} + \mathbf{D}\mathbf{v} = (\mathbf{I}_n - \mathbf{DD}^{-1})\mathbf{Cu} = \mathbf{0},
$$

which is (3.5). Similarly, since $\mathcal{R}(\mathbf{B}) \subseteq \mathcal{R}(\mathbf{A}) \Leftrightarrow \mathbf{AA}^{-1} = \mathbf{B}$, it is seen that Eqs. (3.5) are also fulfilled by any $\mathbf{v}$ satisfying $\mathbf{Tv} = \mathbf{0}$ and $\mathbf{u} = -\mathbf{A}^* \mathbf{B} \mathbf{v}$. □

In a comment to Theorem 3.2 it can be noticed that when considerations are restricted to a partitioned orthogonal projector $\tilde{\mathbf{P}}$ of the form (1.8) and the generalized Schur complements $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{T}}$ defined in (1.9), then assumptions (1.4), (3.1) and (1.6), (3.2) reduce to $\mathbf{A} \neq \mathbf{I}_m$ or $\mathbf{B} \neq \mathbf{0}$ and $\mathbf{D} \neq \mathbf{I}_n$ or $\mathbf{B} \neq \mathbf{0}$, respectively, as in Corollary 3.1.

4. The sum, difference, and product

In this section, we consider two partitioned idempotent matrices

$$
\mathbf{P}_1 = \begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{pmatrix} \quad \text{and} \quad \mathbf{P}_2 = \begin{pmatrix} \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{C}_2 & \mathbf{D}_2 \end{pmatrix},
$$

(4.1)
Let

\begin{align*}
\text{Theorem 4.1.} & \quad \text{An answer given below reveals an essential role played in a solution by inclusions of assumptions (4.5), it follows that} \\
& \quad \text{where } A_i \in C_{m,m} \text{ and } D_i \in C_{n,n}, \ i = 1, 2. \text{ It is known that} \\
& \quad \begin{align*}
& \quad P_1 + P_2 = (P_1 + P_2)^2 \iff P_1 P_2 = 0 = P_2 P_1 \quad (4.2) \\
& \quad \text{and} \\
& \quad P_1 - P_2 = (P_1 - P_2)^2 \iff P_1 P_2 = P_2 = P_2 P_1; \quad (4.3)
\end{align*}
\end{align*}

cf. Theorem in [7, §42] or Theorems 5.1.2 and 5.1.3 in [17]. Our purpose is to find conditions under which the properties of $P_1$ and $P_2$ characterized in (4.2) and (4.3) are inherited by generalized Schur complements

\begin{align*}
& \quad S_i = A_i - B_i D_i^+ C_i \quad \text{and} \quad T_i = D_i - C_i A_i^- B_i, \quad i = 1, 2. \quad (4.4)
\end{align*}

An answer given below reveals an essential role played in a solution by inclusions of the type (1.4) and (1.6).

\textbf{Theorem 4.1.} \quad \text{Let } P_1 \text{ and } P_2 \text{ be partitioned idempotent matrices of the forms given in (4.1), and let } S_i \text{ and } T_i, \ i = 1, 2, \text{ be generalized Schur complements defined in (4.4). Then the conditions}

\begin{align*}
& \quad \mathcal{R}(B_i^+) \subseteq \mathcal{R}(D_i^+) \quad \text{and} \quad \mathcal{R}(C_i) \subseteq \mathcal{R}(D_i), \quad i = 1, 2, \quad (4.5) \\
& \quad \text{under which } S_1 = S_1^2 \text{ and } S_2 = S_2^2, \text{ ensure also that}
\end{align*}

\begin{align*}
& \quad P_1 + P_2 = (P_1 + P_2)^2 \Rightarrow S_1 + S_2 = (S_1 + S_2)^2, \quad (4.6) \\
& \quad P_1 - P_2 = (P_1 - P_2)^2 \Rightarrow S_1 - S_2 = (S_1 - S_2)^2, \quad (4.7)
\end{align*}

\text{and the conditions}

\begin{align*}
& \quad \mathcal{R}(C_i^+) \subseteq \mathcal{R}(A_i^+) \quad \text{and} \quad \mathcal{R}(B_i) \subseteq \mathcal{R}(A_i), \quad i = 1, 2, \quad (4.8) \\
& \quad \text{under which } T_1 = T_1^2 \text{ and } T_2 = T_2^2, \text{ ensure also that}
\end{align*}

\begin{align*}
& \quad P_1 + P_2 = (P_1 + P_2)^2 \Rightarrow T_1 + T_2 = (T_1 + T_2)^2, \quad (4.9) \\
& \quad P_1 - P_2 = (P_1 - P_2)^2 \Rightarrow T_1 - T_2 = (T_1 - T_2)^2, \quad (4.10)
\end{align*}

\textit{irrespective of the choices of } $D_i^+ \in D_i \{1\}$ \text{ and } $A_i^- \in A_i \{1\}$ \text{ in (4.4).}

\textbf{Proof.} \quad \text{In view of (4.2), the sum } P_1 + P_2 \text{ of matrices specified in (4.1) is idempotent if and only if}

\begin{align*}
& \quad A_1 A_2 + B_1 C_2 = 0 = A_2 A_1 + B_2 C_1, \quad (4.11) \\
& \quad A_1 B_2 + B_1 D_2 = 0 = A_2 B_1 + B_2 D_1, \quad (4.12) \\
& \quad C_1 A_2 + D_1 C_2 = 0 = C_2 A_1 + D_2 C_1, \quad (4.13) \\
& \quad C_1 B_2 + D_1 D_2 = 0 = C_2 B_1 + D_2 D_1. \quad (4.14)
\end{align*}

Consequently, on account of the equalities $B_i D_i^+ D_i = B_i$ \text{ and } $D_i D_i^+ C_i = C_i, \ i = 1, 2, \text{ which according to the equivalence (1.4) } \Leftrightarrow (1.5) \text{ are consequences of the assumptions (4.5), it follows that}
\[ S_1S_2 = A_1A_2 - A_1B_2D_2^T C_2 - B_1D_1^T C_1B_2D_2^T C_2 + B_1D_1^T C_1B_2D_2^T C_2 = 0 \]

and, by analogous arguments (just by interchanging the subscripts “1” and “2”), \( S_2S_1 = 0 \). Similarly, on account of the equalities \( C_iA_i^T A_i = C_i \) and \( A_iA_i^T B_i = B_i \), \( i = 1, 2 \), which according to the equivalence (1.6) \( \Leftrightarrow \) (1.7) are consequences of the assumptions (4.8), it follows that

\[ T_1T_2 = D_1D_2 - D_1C_2A_2^T B_2 - C_1A_1^T B_1D_2 + C_1A_1^T B_1C_2A_2^T B_2 = 0 \]

and, in the same way, \( T_2T_1 = 0 \). This establishes (4.6) and (4.9).

In the second part of the proof first observe that, on account of (4.3), the difference \( P_1 - P_2 \) is idempotent if and only if conditions (4.11)–(4.14) hold in the modified versions, with zero matrices in the middle replaced consecutively by \( A_2, B_2, C_2, \) and \( D_2 \). Then, under assumptions (4.5),

\[ S_1S_2 = A_2 - B_1C_2 - (B_2 - B_1D_2)D_2^T C_2 - B_1D_1^T (C_2 - D_1C_2) + B_1D_1^T (D_2 - D_1D_2)D_2^T C_2 = A_2 - B_1C_2 - B_2D_2^T C_2 + B_1C_2 + B_1D_1^T C_2 - B_1C_2 + B_1D_1^T C_2 - B_1C_2 = S_2 \]

and, by analogous arguments, \( S_2S_1 = S_2 \), which establishes (4.7). Similarly it follows that assumptions (4.8) entail \( T_1T_2 = T_2 = T_2T_1 \), thus proving (4.10). □

Baksalary and Baksalary [3] considered the problem of idempotency of linear combinations of two nonzero idempotent matrices \( P_1 \) and \( P_2 \) (\( P_1 \neq P_2 \)), determined by nonzero \( c_1, c_2 \in \mathbb{C} \). Their Corollary 2 asserts that if \( P_1 \) and \( P_2 \) are Hermitian, then there is no idempotent matrix of the form \( c_1P_1 + c_2P_2 \) other than \( P_1 + P_2 \) (in cases characterized in (4.2)), \( P_1 - P_2 \) (in cases characterized in (4.3)), and \( P_2 - P_1 \) (in cases characterized by \( P_1P_2 = P_1 = P_2P_1 \)). For orthogonal projectors

\[ \tilde{P}_1 = \begin{pmatrix} A_1 & B_1 \\ B_1^* & D_1 \end{pmatrix} \quad \text{and} \quad \tilde{P}_2 = \begin{pmatrix} A_2 & B_2 \\ B_2^* & D_2 \end{pmatrix}, \quad \text{(4.15)} \]

the conditions in (4.5) and (4.8) reduce to \( \mathcal{H}(B_1^*) \subseteq \mathcal{H}(D_1) \) and \( \mathcal{H}(B_2) \subseteq \mathcal{H}(A_i) \), respectively, \( i = 1, 2 \), and are automatically satisfied on account of the nonnegative definiteness of \( \tilde{P}_1 \) and \( \tilde{P}_2 \). Consequently, Theorem 4.1 leads immediately to the following.
Corollary 4.1. Let \( \tilde{P}_1 \) and \( \tilde{P}_2 \) be partitioned Hermitian idempotent matrices of the forms given in (4.15), and let
\[
\tilde{S}_i = A_i - B_i D_i^* B_i^* \quad \text{and} \quad \tilde{T}_i = D_i - B_i^* A_i B_i, \quad i = 1, 2
\]
(independent of the choice of \( D_i \in D_i \{1\} \) and \( A_i \in A_i \{1\} \)) be the generalized Schur complements of \( D_i \) and \( A_i \) in \( \tilde{P}_i \), respectively. Then for each of the following three pairs of scalars:
\[
c_1 = 1, c_2 = 1; \quad c_1 = 1, c_2 = -1; \quad c_1 = -1, c_2 = 1;
\]
the equality
\[
c_1 \tilde{P}_1 + c_2 \tilde{P}_2 = (c_1 \tilde{P}_1 + c_2 \tilde{P}_2)^2
\]
implies
\[
c_1 \tilde{S}_1 + c_2 \tilde{S}_2 = (c_1 \tilde{S}_1 + c_2 \tilde{S}_2)^2 \quad \text{and} \quad c_1 \tilde{T}_1 + c_2 \tilde{T}_2 = (c_1 \tilde{T}_1 + c_2 \tilde{T}_2)^2.
\]

Theorem in [3] indicates that in the general case there is an additional possibility of \( c_1 \tilde{P}_1 + c_2 \tilde{P}_2 \) being idempotent, characterized by the conditions
\[
P_1 P_2 = P_2 P_1, \quad (P_1 - P_2)^2 = 0, \quad \text{and} \quad c_1 + c_2 = 1.
\]
It is natural to ask, therefore, whether the inheritance property holds also in this situation. The answer appears to be negative.

Remark 4.1. For partitioned idempotent matrices \( P_1 \) and \( P_2 \) of the forms given in (4.1) such that \( P_1 P_2 \neq P_2 P_1 \), \( (P_1 - P_2)^2 = 0 \), and \( c_1 + c_2 = 1 \). It is natural to ask, therefore, whether the inheritance property holds also in this situation. The answer appears to be negative.

For partitioned idempotent matrices \( P_1 \) and \( P_2 \) of the forms given in (4.1) such that \( P_1 P_2 \neq P_2 P_1 \), conditions (4.5) are in general insufficient for the idempotency of a linear combination \( c_1 P_1 + c_2 P_2 \) to imply the idempotency of the corresponding linear combination \( c_1 S_1 + c_2 S_2 \) of generalized Schur complements \( S_1 \) and \( S_2 \) defined in (4.4); and an analogous statement is valid with regard to conditions (4.8) and the complements \( T_1 \) and \( T_2 \).

A justification of this remark is provided by idempotent matrices \( P_1 \) and \( P_2 \) composed according to (4.1) with
\[
A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}
\]
(4.17)
and
\[
A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
(4.18)
Since these matrices satisfy \( P_1 P_2 \neq P_2 P_1 \) and \( (P_1 - P_2)^2 = 0 \), it follows that \( c_1 P_1 + (1 - c_1) P_2 \) is idempotent for every choice of \( c_1 \in \mathbb{C} \). However, although the matrices in (4.17) and (4.18) fulfil conditions (4.5), their generalized Schur complements
\[
S_1 = \begin{pmatrix} 0 & -1/2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 0 & 0 \\ 1/2 & 1 \end{pmatrix}
\]
lead to
\[ c_1 S_1 + (1 - c_1) S_2 = \begin{pmatrix} 0 & -\frac{1}{2} c_1 \\ \frac{1}{2} - \frac{1}{2} c_1 & 1 \end{pmatrix}, \]
which is not an idempotent matrix except only for the trivial cases where \( c_1 = 0 \) or \( c_1 = 1 \).

The last part of this section is concerned with the product of two idempotent matrices. It is known that if \( P_1 = P_1^2 \) and \( P_2 = P_2^2 \), then
\[ P_1 P_2 = P_2 P_1 \Rightarrow P_1 P_2 = (P_1 P_2)^2; \]
cf. Theorem in [7, §42] and Theorem 5.1.4 in [17]. The commutativity property \( P_1 P_2 = P_2 P_1 \) is in general sufficient only. However, it becomes necessary and sufficient when both projectors involved are orthogonal; cf., e.g., Theorem 1 in [2]. We again ask about the corresponding inheritance property and find that also in this case the answer is negative.

Remark 4.2. For partitioned Hermitian idempotent matrices \( T_1 \) and \( T_2 \) of the form (4.15), the commutativity property \( T_1 T_2 = T_2 T_1 \), which is necessary and sufficient for \( T_1 T_2 \) to be idempotent, does not in general imply that \( \tilde{S}_1 \tilde{S}_2 \) and \( \tilde{T}_1 \tilde{T}_2 \) are idempotent.

As a justification of this remark consider \( \tilde{T}_1 \) and \( \tilde{T}_2 \) composed according to (4.15) with
\[ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \quad D_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \]
and
\[ A_2 = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \quad D_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}. \]
They satisfy \( \tilde{T}_1 \tilde{T}_2 = \tilde{T}_2 \tilde{T}_1 \), which means that \( \tilde{T}_1 \tilde{T}_2 \) is a projector, but the Schur complements
\[ \tilde{S}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{S}_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \]
do not satisfy \( \tilde{S}_1 \tilde{S}_2 = \tilde{S}_2 \tilde{S}_1 \), and therefore \( \tilde{S}_1 \tilde{S}_2 \) is not a projector.

5. Partial orderings

Also considerations of the present section are concerned with two partitioned idempotent matrices \( P_1 \) and \( P_2 \) of the form (4.1). Hartwig and Styan [10] provided an exhaustive study of three matrix partial orderings under the assumption that the
matrices involved are idempotent. An impression following from their study is that the most natural ordering in this particular situation is the one introduced by Hartwig [8], which is now known as minus ordering or rank-subtractivity ordering, and for \( K, L \in C_{m,n} \) can be defined by

\[
K \preceq L \quad \text{whenever} \quad r(L - K) = r(L) - r(K);
\]

see also [8,9] for alternative versions of the definition.

For idempotent matrices \( P_1, P_2 \in C_{m+n,m+n} \) the minus ordering admits a characterization

\[
P_1 \preceq P_2 \iff (I_{m+n} - P_2)P_1 = 0 = P_1(I_{m+n} - P_2);
\]

(5.1)
cf. Theorem 5.1(e) of Hartwig and Styan [10]. We will show that under the same conditions as in Theorem 4.1 the property that \( P_1 \) and \( P_2 \) partitioned as in (4.1) are minus-ordered is inherited by generalized Schur complements \( S_1, S_2 \) and \( T_1, T_2 \).

**Theorem 5.1.** Let \( P_1 \) and \( P_2 \) be partitioned idempotent matrices of the forms given in (4.1), and let \( S_i \) and \( T_i \), \( i = 1, 2 \), be generalized Schur complements defined in (4.4). Then conditions (4.5), under which \( S_1 = S_1^2 \) and \( S_2 = S_2^2 \), ensure also that

\[
P_1 \preceq P_2 \Rightarrow S_1 \preceq S_2,
\]

(5.2)
and conditions (4.8), under which \( T_1 = T_1^2 \) and \( T_2 = T_2^2 \), ensure also that

\[
P_1 \preceq P_2 \Rightarrow T_1 \preceq T_2;
\]

(5.3)
irrespective of the choices of \( D_i^{-} \in D_i \{1\} \) and \( A_i^{-} \in A_i \{1\}, i = 1, 2 \).

**Proof.** In view of the assumptions \( B_i D_i^{-} D_i = B_i \) and \( D_i D_i^{-} C_i = C_i, i = 1, 2 \), the idempotency of \( S_1 \) and \( S_2 \) follows immediately from Theorem 2.1. Substituting \( P_1 \) and \( P_2 \) of the form (4.1) into (5.1) shows that \( P_1 \preceq P_2 \) if and only if the following eight equalities hold:

\[
(I_n - A_2)A_1 = B_2C_1, \quad (I_n - A_2)B_1 = B_2D_1, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (5.4)
\]

\[
(I_n - D_2)C_1 = C_2B_1, \quad (I_n - D_2)B_1 = C_2B_1, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (5.5)
\]

\[
A_1(I_n - A_2) = B_2C_2, \quad C_1(I_n - A_2) = D_1C_2, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (5.6)
\]

\[
B_1(I_n - D_2) = A_1B_2, \quad D_1(I_n - D_2) = C_1B_2. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (5.7)
\]

Then from (5.4) and (5.5) it follows that

\[
(I_n - S_2)S_1 = B_2C_1 - B_2D_1 D_1^{-} C_1 + B_2 D_2^{-} C_2 A_1 - B_2 D_2^{-} C_2 B_1 D_1^{-} C_1
\]

\[
= B_2C_1 - B_2C_1 + B_2 D_2^{-} (I_n - D_2) C_1 - B_2 D_2^{-} (I_n - D_2) D_1 D_1^{-} C_1
\]

\[
= B_2 D_2^{-} C_1 - B_2 D_2^{-} C_1 + B_2 C_1
\]

\[
= 0 \quad \quad \quad \quad (5.8)
\]
and, from (5.6) and (5.7),
\[ S_1(I_m - S_2) = B_1C_2 - B_1D_1^*D_1C_2 + A_1B_2D_2^*C_2 - B_1D_1^*C_1B_2D_2^*C_2 \]
\[ = B_1C_2 - B_1C_2 + B_1(I_n - D_2)D_2^*C_2 - B_1D_1^*D_1(I_n - D_2)D_2^*C_2 \]
\[ = B_1D_1^*C_2 - B_1C_2 + B_1D_1^*C_2 + B_1C_2 \]
\[ = 0. \]  
(5.9)

In view of (5.1), equalities (5.8) and (5.9) establish (5.2). Implication (5.3) can be proved in a similar way. \(\square\)

It is known that
\[ \tilde{P}_1 \preceq L \tilde{P}_2 \iff \tilde{P}_1^* \preceq \tilde{P}_2^* \iff \tilde{P}_1 - \tilde{P}_2 \]  
(5.10)
for any orthogonal projectors \(\tilde{P}_1\) and \(\tilde{P}_2\) (cf. Theorem 5.8 in [10]), while in general
\[ P_1 \preceq P_2 \Rightarrow P_1^* \preceq P_2^* \Rightarrow P_1 \preceq L P_2 \]  
(5.11)
for any Hermitian matrix \(P_1\) and any Hermitian nonnegative definite matrix \(P_2\) (cf. Theorem 2.1 in [10]), and, on the other hand,
\[ P_1 \preceq L P_2 \Rightarrow P_1^* \preceq P_2^* \Rightarrow P_1 \preceq P_2 \]  
(5.12)
for any idempotent matrices \(P_1\) and \(P_2\) (cf. Theorem 2.2 in [10]). The symbols \(\preceq_L\) and \(\preceq^*\) in (5.10)–(5.12) stand for the Löwner partial ordering (cf. [13, p. 177]) and Drazin’s “star ordering” (cf. [5]), which are defined as follows: for \(K, L \in \mathbb{C}_{m,m}\),
\[ K \preceq_L L \quad \text{whenever} \quad L - K \in \mathbb{C}^{\geq}_{m}, \]
and, for \(K, L \in \mathbb{C}_{m,n}\),
\[ K \preceq^* L \quad \text{whenever} \quad K^*K = K^*L \quad \text{and} \quad KK^* = LK^*. \]

In view of (5.10), an immediate consequence of Theorem 5.1 is the following.

**Corollary 5.1.** Let \(\tilde{P}_1\) and \(\tilde{P}_2\) be partitioned Hermitian idempotent matrices of the form (4.15), and let \(\tilde{S}_i\) and \(\tilde{T}_i\), \(i = 1, 2\), be the generalized Schur complements defined in (4.16). Then \(\tilde{S}_i\) and \(\tilde{T}_i\) are orthogonal projectors and any of the orderings in (5.10) implies each of the orderings
\[ \tilde{S}_1 \preceq_L \tilde{S}_2 \iff \tilde{S}_1 \preceq^* \tilde{S}_2 \iff \tilde{S}_1 \preceq \tilde{S}_2 \]
and
\[ \tilde{T}_1 \preceq_L \tilde{T}_2 \iff \tilde{T}_1 \preceq^* \tilde{T}_2 \iff \tilde{T}_1 \preceq \tilde{T}_2. \]
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