Embeddings into normal first countable spaces

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Abstract

In this paper we construct, in response to a question of Arhangel’skiı̆, a zero-dimensional first countable space which cannot be embedded into a normal first countable space. © 2001 Elsevier Science B.V. All rights reserved.

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In [1] Arhangel’skiı̆ posed the following problem: is it possible to embed every Tychonoff first countable space into a normal first countable space?

In this paper we give a negative answer to the above question.

We recall that a space $X$ is wD if for every infinite closed discrete subspace $D$ of $X$ there are an infinite subset $D'$ of $D$ and a discrete family $\{U_d: d \in D'\}$ of open sets in $X$ such that $U_d \cap D' = \{d\}$ for every $d \in D'$ (see, e.g., [3]).

The reader is referred to [2] for notations and terminology not explicitly given.

Our first result gives a necessary condition on a space $X$ to be embeddable into a normal first countable space.

Lemma. Every space which is embeddable into a normal first countable space is wD.

Proof. Let $X$ be a normal first countable space. Let us show that every subspace $Y$ of $X$ is wD. Let $D$ be an infinite closed discrete subspace of $Y$. If $D$ is closed in $X$, we are done (every normal space is wD, [3]). Otherwise let us take a point $p \in (\text{cl}_X D) \setminus D$. It is possible to define inductively two sequences $\{U_n\}_{n \in \omega}$, $\{V_n\}_{n \in \omega}$ of open subsets of $X$ and an infinite subset $D' = \{d_n\}_{n \in \omega}$ of $D$ in such a way that:

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(i) \( \{V_n\}_{n \in \omega} \) is a base for \( X \) at the point \( p \) such that \( \overline{V}_{n+1} \subset V_n \);

(ii) \( d_n \in U_n, U_n \cap \overline{V}_n = \emptyset \) and \( U_{n+1} \subset V_n \) for every \( n \in \omega \).

Then \( \{U_n \cap Y\}_{n \in \omega} \) is a discrete family of open subsets of \( Y \) such that \( U_n \cap D' = \{d_n\} \) for every \( n \). \( \square \)

**Remark.** A space \( X \) has countable pseudocharacter if every point of \( X \) is a \( G_\delta \)-set. The proof of the above lemma shows that the following more general statement is true: every space which is embeddable into a normal space with countable pseudocharacter is wD.

**Example.** A zero-dimensional first countable space which is not embeddable into a normal first countable space.

Let \( X = \omega^\omega \cup \{p\} \) be the one-point compactification of the discrete space \( \omega^\omega \), let \( Y \) be the ordinal space \( \omega \) and set \( (Z, \tau) = X \times Y \).

Let \( \sigma \) be the topology on \( Z \) generated by the base \( \tau \cup \{V(n, k): n, k \in \omega\} \), where \( V(n, k) = \{\langle \phi, n \rangle: \phi(n) \geq k\} \cup \{(p, n)\} \).

Clearly each point of \( \omega^\omega \times \omega \) is isolated. Moreover \( \{\phi\} \times (\omega + 1) \) is a clopen subset of \( (Z, \sigma) \) homeomorphic to \( \omega + 1 \), and \( \{V(n, k): k \in \omega\} \) is a base for \( (Z, \sigma) \) at the point \( \langle p, n \rangle \) for each \( n \in \omega \). Let \( S \) be the subspace \( Z \setminus \{(p, \omega)\} \) of \( Z = (Z, \sigma) \). We claim that \( S \) has the required properties.

Clearly \( S \) is first countable, moreover it is zero-dimensional. In fact, \( (Z, \tau) \) is zero-dimensional and every \( V(n, k) \) is closed in \( Z \), so \( Z \) is zero-dimensional too.

It remains to show that \( S \) is not embeddable into a normal first countable space. By the lemma, it is enough to check that \( S \) is not wD.

Let

\[ D = \{\langle p, n \rangle: n \in \omega\} \]

be an infinite closed discrete subspace of \( S \) and let

\[ D' = \{\langle p, n_i \rangle: i \in \omega\} \]

be an infinite subset of \( D \). We claim that every family \( \mathcal{U} = \{U_i: i \in \omega\} \) of open sets in \( S \) such that \( U_i \cap D' = \{\langle p, n_i \rangle\} \) for every \( i \), is not discrete.

We may assume that \( U_i = V(n_i, k_i) \) for every \( i \in \omega \). Let us show that there is a point \( \langle \phi, \omega \rangle \) such that every neighbourhood of it meets infinitely many members of \( \{U_i: i \in \omega\} \) (so \( \mathcal{U} \) is not even locally finite).

Choose a function \( \phi: \omega \to \omega \) such that \( \phi(n_i) = k_i \), for every \( i \in \omega \). Let \( V = \{\langle \phi, m\rangle: m \geq \overline{m}\} \cup \{\langle \phi, \omega\rangle\} \) be a neighbourhood of \( \langle \phi, \omega \rangle \) in \( S \), then \( \langle \phi, n_i \rangle \in U_i \cap V \) for every \( i \) with \( n_i \geq \overline{m} \).

**Remark.** It is worth noting that the space \( Z \) constructed in the example is a normal Fréchet space.

(1) \( Z \) is Fréchet. It is enough to show that if \( \langle p, \omega \rangle \in (cl_Z A) \setminus A \) for some \( A \subset Z \), then there is a sequence \( \{a_n\}_{n \in \omega} \subset A \) converging to \( \langle p, \omega \rangle \).

If \( A \cap (\{p\} \times \omega) \) or \( A \cap (X \times \{\omega\}) \) is infinite we are done (in fact \( \{p\} \times \omega \) and \( X \times \{\omega\} \) are homeomorphic to \( \omega \) and \( X \), respectively).
Otherwise choose a sequence \( \{(\phi_i, n_i)\}_{i \in \omega} \subset A \) such that \( \phi_i \neq \phi_j \) and \( n_i \neq n_j \) whenever \( i \neq j \).

(2) \( Z \) is normal. Let \( C_1, C_2 \) be two disjoint closed subsets of \( Z \).

(i) \( (p, \omega) \notin C_1 \cup C_2 \). Then there are a \( k \in \omega \) and a finite subset \( F \) of \( \omega^\omega \) such that \( C_1 \cup C_2 \subset Z \setminus V(F, k) \), where \( V(F, k) = (X \setminus F) \times (Y \setminus k) \). Now observe that

\[
Z \setminus V(F, k) = \bigcup \{ X \times \{ n \} : n < k \} \cup \{ \{ \phi \} \times Y : \phi \in F \}.
\]

Every \( X \times \{ n \} \) is clopen and normal (every point different from \( (p, n) \) is isolated), and every \( \{ \phi \} \times Y \) is homeomorphic to \( Y \), therefore it is compact and clopen. Hence \( Z \setminus V(F, k) \) is a finite union of normal and clopen subsets, therefore it is normal and clopen. So \( C_1 \) and \( C_2 \) can be separated in \( Z \) by disjoint open subsets.

(ii) \( (p, \omega) \in C_1 \). Choose a \( k \in \omega \) and a finite subset \( F \) of \( Y \) so that \( C_2 \) does not meet \( V(F, k) \). Now we can split \( Z \) in the clopen sets \( Z \setminus V(F, k) \) and \( V(F, k) \). Since \( Z \setminus V(F, k) \) is normal and \( V(F, k) \) does not meet \( C_2 \), we are done.

The above remark leads us to pose the following problem:

**Problem.** Is it possible to embed every Tychonoff first countable space into a normal Fréchet space?

**References**

