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# NOTE

# Multidimensional Ehrhart Reciprocity[1]

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In an earlier paper (1999, *Electron. J. Combin.* **6**, R37), the author generalized



just a single dilation factor, we allow different dilation factors for each of these facets. We proved that, if our polytope is a simplex, the lattice point counts in the interior and closure of such a *vector-dilated* simplex are quasipolynomials satisfying an Ehrhart-type reciprocity law. This generalizes the classical reciprocity law for rational polytopes. In the present paper we complete the picture by extending this result to general rational polytopes. As a corollary, we also generalize a reciprocity theorem of Stanley. 

## 1. INTRODUCTION

Based on the elementary identity

$$\left[\frac{t-1}{a}\right] = -\left[\frac{-t}{a}\right] - 1, \tag{1}$$

where $a \in \mathbb{N}$, $t \in \mathbb{Z}$, and $[x]$ is the greatest integer function, we proved in [1] a generalization of the Ehrhart–Macdonald reciprocity law for rational polytopes. (A **rational polytope** is a polytope whose vertices are rational.) More precisely, let $\mathscr{P}$ be an $n$-dimensional rational polytope in $\mathbb{R}^n$. For a positive integer $t$, let

$$L(\mathscr{P}^\circ, t) = \#(t\mathscr{P}^\circ \cap \mathbb{Z}^n) \qquad \text{and} \qquad L(\bar{\mathscr{P}}, t) = \#(t\bar{\mathscr{P}} \cap \mathbb{Z}^n)$$

[1] This work is part of the author's Ph.D. thesis.



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denote the number of integer points ("lattice points") in the interior of the dilated polytope $t\mathscr{P} = \{tx\colon x \in \mathscr{P}\}$ and its closure, respectively. Ehrhart, who initiated the study of the lattice point count in dilated polytopes [2], proved that $L(\mathscr{P}^\circ, t)$ and $L(\bar{\mathscr{P}}, t)$ are quasipolynomials in $t$. (A **quasipolynomial** is an expression of the form

$$c_n(t)\, t^n + \cdots + c_1(t)\, t + c_0(t),$$

where $c_0, ..., c_n$ are periodic functions in $t$.) He conjectured the following reciprocity law, which was first proved by Macdonald [3]:

THEOREM 1 Ehrhart–Macdonald Reciprocity Law. *Suppose the rational polytope $\mathscr{P}$ is homeomorphic to an n-manifold. Then*

$$L(\mathscr{P}^\circ, -t) = (-1)^n\, L(\bar{\mathscr{P}}, t).$$

In [1], we generalized the notion of dilated polytopes: we use the description of a convex polytope as the intersection of halfspaces, which determine the facets of the polytope. Instead of dilating the polytope by a single factor, we allow different dilation factors for each facet, such that the combinatorial type of the polytope does not change. Recall that two polytopes are **combinatorially equivalent** if there exists a bijection between their faces that preserves the inclusion relation.

It is a crucial fact that rational polytopes can be described by inequalities with integer coefficients. The following definition appeared in [1] only for simplices:

DEFINITION 1. Let the convex rational polytope $\mathscr{P}$ be given by

$$\mathscr{P} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\},$$

with $\mathbf{A} \in M_{m\times n}(\mathbb{Z})$, $\mathbf{b} \in \mathbb{Z}^m$. Here the inequality is understood componentwise. For $\mathbf{t} \in \mathbb{Z}^m$, define the **vector-dilated polytope** $\mathscr{P}^{(\mathbf{t})}$ as

$$\mathscr{P}^{(\mathbf{t})} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leqslant \mathbf{t}\}.$$

For those $\mathbf{t}$ for which $P^{(\mathbf{t})}$ is combinatorially equivalent to $\mathscr{P} = \mathscr{P}^{(\mathbf{b})}$, we define the number of lattice points in the interior and closure of $\mathscr{P}^{(\mathbf{t})}$ as

$$i_{\mathscr{P}}(\mathbf{t}) = \#(\mathscr{P}^{(\mathbf{t})\circ} \cap \mathbb{Z}^n) \quad \text{and} \quad j_{\mathscr{P}}(\mathbf{t}) = \#(\mathscr{P}^{(\mathbf{t})} \cap \mathbb{Z}^n),$$

respectively.

Geometrically, we fix for a given polytope the normal vectors to its facets and consider all possible positions of the facets that do not change the face structure of the polytope. Note that the dimension of $\mathbf{t}$ is the number of facets of the polytope. The previously defined quantities $L(\mathscr{P}^\circ, t)$ and $L(\bar{\mathscr{P}}, t)$ can be recovered from this new definition by choosing $\mathbf{t} = t\mathbf{b}$. In [1], we obtained a reciprocity law for vector-dilated *simplices*:

THEOREM 2. *Let $\mathscr{S}$ be an n-dimensional rational simplex. Then $i_{\mathscr{S}}(\mathbf{t})$ and $j_{\mathscr{S}}(\mathbf{t})$ are quasipolynomials in $\mathbf{t} \in \mathbb{Z}^{n+1}$, satisfying*

$$i_{\mathscr{S}}(-\mathbf{t}) = (-1)^n \, j_{\mathscr{S}}(\mathbf{t}).$$

A **quasipolynomial** in the $d$-dimensional variable $\mathbf{t} = (t_1, ..., t_d)$ is the natural generalization of a quasipolynomial in a 1-dimensional variable: namely, an expression of the form

$$\sum_{0 \leqslant k_1, ..., k_d \leqslant n} c_{(k_1, ..., k_d)} \, t_1^{k_1} \cdots t_d^{k_d},$$

where $c_{(k_1, ..., k_d)} = c_{(k_1, ..., k_d)}(t_1, ..., t_d)$ is periodic in $t_1, ..., t_d$. In [1], we gave an actual example of such a quasipolynomial arising from a lattice point count in a polytope.

In the present paper, we finish the picture by extending Theorem 2 to general rational polytopes. We should extend Definition 1 to non-convex polytopes. This can be done naturally in an additive way: write the polytope as the union of convex polytopes, and apply the above Definition 1 to these components. More thoroughly, we make the following

DEFINITION 2. Let $\mathscr{P}$ be a rational polytope. Write $\mathscr{P} = \bigcup_{k=1}^r \mathscr{P}_k$, where $\mathscr{P}_k$ are convex rational polytopes, say,

$$\mathscr{P}_k = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}_k \mathbf{x} \leqslant \mathbf{b}_k\},$$

with $\mathbf{b}_k \in \mathbb{Z}^{m_k}$. Given $\mathbf{t} \in \mathbb{Z}^m$, where $m = m_1 + \cdots + m_r$, combine the first $m_1$ components of $\mathbf{t}$ in a vector $\mathbf{t}_1$, the next $m_2$ components in $\mathbf{t}_2$, etc. Define the **vector-dilated polytope** $\mathscr{P}^{(\mathbf{t})}$ as

$$\mathscr{P}^{(\mathbf{t})} = \bigcup_{k=1}^r \mathscr{P}_k^{(\mathbf{t}_k)}.$$

For those $\mathbf{t}$ for which $P^{(\mathbf{t})}$ is combinatorially equivalent to $\mathscr{P}$, we define as above

$$i_{\mathscr{P}}(\mathbf{t}) = \#(\mathscr{P}^{(\mathbf{t})\circ} \cap \mathbb{Z}^n) \quad \text{and} \quad j_{\mathscr{P}}(\mathbf{t}) = \#(\mathscr{P}^{(\mathbf{t})} \cap \mathbb{Z}^n).$$

Finally, we derive a generalization of the following theorem of Stanley ([4]) in terms of vector-dilated polytopes. The Ehrhart–Macdonald reciprocity law compares the lattice point count of the polytope with that of the interior, that is, the polytope with all its facets removed. Stanley's theorem tells us what to expect if we only remove *some* of the facets.

THEOREM 3 (Stanley). *Suppose the rational polytope $\mathcal{P}$ is homeomorphic to an n-manifold. Denote the set of all (closed) facets of $\mathcal{P}$ by F, and let T be a subset of F, such that $\bigcup_{\mathcal{F} \in T} \mathcal{F}$ is homeomorphic to an $(n-1)$-manifold. Let*

$$j_{\mathcal{P},T}(t) = \#\left( t\left(\mathcal{P} - \bigcup_{\mathcal{F} \in T} \mathcal{F}\right) \cap \mathbb{Z}^n \right)$$

*and*

$$i_{\mathcal{P},T}(t) = \#\left( t\left(\mathcal{P} - \bigcup_{\mathcal{F} \in F-T} \mathcal{F}\right) \cap \mathbb{Z}^n \right).$$

*Then*

$$i_{\mathcal{P},T}(-t) = (-1)^n\, j_{\mathcal{P},T}(t).$$

Note that Theorem 1 is the special case $T = \varnothing$ of Theorem 3. For an example that this result does not hold in general, see [4].

## 2. EXTENDING EHRHART RECIPROCITY

In [1], we remarked that Theorem 1 follows directly from Theorem 2. Since we will use Theorem 1 to show the main result of this paper, we start by actually proving this remark.

*Proof of Theorem* 1. We use double induction on the dimension of the polytope $\mathcal{P}$ and on the number of $n$-dimensional simplices which triangulate $\mathcal{P}$. It is easy to see ([1]) that Theorem 1 follows for 1-dimensional polytopes (that is, intervals) from (1). Also, Theorem 1 holds for simplices, as a special case of Theorem 2. For a general $\mathcal{P}$ satisfying the hypotheses of the statement, write

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2,$$

where $\mathcal{P}_1$ is an $n$-dimensional simplex such that $\mathcal{P}_2 := \overline{\mathcal{P} - \mathcal{P}_1}$ is again a polytope homeomorphic to an $n$-manifold. Note that the conditions on $\mathcal{P}$

imply that $\mathscr{P}_1$ and $\mathscr{P}_2$ share an $(n-1)$-dimensional polytopal boundary, which we denote by $\mathscr{P}_3$. Hence

$$L(\bar{\mathscr{P}}, t) = L(\bar{\mathscr{P}}_1, t) + L(\bar{\mathscr{P}}_2, t) - L(\bar{\mathscr{P}}_3, t)$$

and

$$L(\mathscr{P}^\circ, t) = L(\mathscr{P}_1^\circ, t) + L(\mathscr{P}_2^\circ, t) + L(\mathscr{P}_3^\circ, t).$$

By induction, we can apply Theorem 1 to $\mathscr{P}_1$, $\mathscr{P}_2$, and $\mathscr{P}_3$:

$$\begin{aligned} L(\bar{\mathscr{P}}, -t) &= L(\bar{\mathscr{P}}_1, -t) + L(\bar{\mathscr{P}}_2, -t) - L(\bar{\mathscr{P}}_3, -t) \\ &= (-1)^n L(\mathscr{P}_1^\circ, t) + (-1)^n L(\mathscr{P}_2^\circ, t) - (-1)^{n-1} L(\mathscr{P}_3^\circ, t) \\ &= (-1)^n L(\mathscr{P}^\circ, t). \quad \blacksquare \end{aligned}$$

From the Ehrhart–Macdonald reciprocity law we can now conclude a generalized version of Theorem 2:

THEOREM 4. *Suppose the rational polytope $\mathscr{P}$ is homeomorphic to an n-manifold. Then $i_{\mathscr{P}}(\mathbf{t})$ and $j_{\mathscr{P}}(\mathbf{t})$ are quasipolynomials in $\mathbf{t} \in \mathbb{Z}^m$, satisfying*

$$i_{\mathscr{P}}(-\mathbf{t}) = (-1)^n j_{\mathscr{P}}(\mathbf{t}).$$

*Proof.* It suffices to prove that $i_{\mathscr{P}}(\mathbf{t})$ and $j_{\mathscr{P}}(\mathbf{t})$ are quasipolynomials. In fact, *once we know this*, the statement follows from Theorem 1:

$$i_{\mathscr{P}}(-\mathbf{t}) = L(\mathscr{P}^{(\mathbf{t})\circ}, -1) = (-1)^n L(\overline{\mathscr{P}^{(\mathbf{t})}}, 1) = (-1)^n j_{\mathscr{P}}(\mathbf{t}).$$

To show that our lattice point count operators are quasipolynomials, it clearly suffices to prove that $i_{\mathscr{P}}(\mathbf{t})$ and $j_{\mathscr{P}}(\mathbf{t})$ are quasipolynomials in one of the components of $\mathbf{t}$, say $t_1$. Because we leave only this one component variable, we may also assume that $\mathscr{P}$ is convex. We make a similar unimodular transformation (which leaves the lattice invariant) as in [1]: we may assume that the defining inequalities for $\mathscr{P}^{(\mathbf{t})}$ are

$$\begin{aligned} a_{11}x_1 \qquad\qquad\qquad &\leqslant t_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &\leqslant t_2 \\ &\vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n &\leqslant t_m. \end{aligned}$$

(Actually, we could obtain a lower triangular form.) Viewing these inequalities as

$$\begin{aligned} x_1 \qquad\qquad &\leqslant \frac{t_1}{a_{11}} \\ a_{22}x_2 + \cdots + a_{2n}x_n &\leqslant t_2 - a_{21}x_1 \\ &\vdots \\ a_{m,2}x_2 + \cdots + a_{m,n}x_n &\leqslant t_m - a_{m,1}x_1, \end{aligned}$$

we can compute the number of lattice points in the interior and closure of $\mathscr{P}^{(\mathbf{t})}$ as

$$i_{\mathscr{P}}(\mathbf{t}) = \sum_{k=s_1}^{\left[\frac{t_1-1}{a_{11}}\right]} i_{\mathscr{Q}}(t_2 - a_{21}k, ..., t_m - a_{m,1}k) \tag{2}$$

and

$$j_{\mathscr{P}}(\mathbf{t}) = \sum_{k=s_2}^{\left[\frac{t_1}{a_{11}}\right]} j_{\mathscr{Q}}(t_2 - a_{21}k, ..., t_m - a_{m,1}k), \tag{3}$$

respectively. Here $s_1$ and $s_2$ are rational numbers not depending on $t_1$, and the $(n-1)$-dimensional polytope $\mathscr{Q}^{(\mathbf{b})}$ is given by

$$\mathscr{Q}^{(\mathbf{b})} = \{\mathbf{x} \in \mathbb{R}^{n-1} : \mathbf{Bx} \leqslant \mathbf{b}\},$$

where

$$\mathbf{B} = \begin{pmatrix} a_{22} & \cdots & a_{2n} \\ & \vdots & \\ a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \in M_{(m-1)\times(n-1)}(\mathbb{Z}).$$

The functions $i_{\mathscr{Q}}$ and $j_{\mathscr{Q}}$, over which the summations in (2) and (3) range, are constant in $t_1$. Thus we only need a weak form of Lemma 4 in [1] to deduce that $i_{\mathscr{P}}(\mathbf{t})$ and $j_{\mathscr{P}}(\mathbf{t})$ are quasipolynomials in $t_1$. ∎

At this point, we find it appropriate to remark why we did not simply start the notion of vector-dilated polyotopes with this proof, assuming classical Ehrhart–Macdonald reciprocity. The point of [1] (or at least half of it) was really to give an *elementary* proof of Theorem 1. It is for this reason that we chose to build our proof of Theorem 4 upon the work in [1]. The course of the proof looks like the following diagram:

$$(1) \overset{[1]}{\Rightarrow} \text{Theorem 2} \Rightarrow \text{Theorem 1} \Rightarrow \text{Theorem 4}.$$

## 3. EXTENDING STANLEY'S THEOREM

We conclude by proving the appropriate generalization of Theorem 3, essentially in the same way Stanley deduced Theorem 3 from Theorem 1.

Corollary 5. *Suppose the rational polytope $\mathscr{P}$ is homeomorphic to an n-manifold. Denote the set of all (closed) facets of $\mathscr{P}$ by $F$, and let $T$ be a subset of $F$, such that $\bigcup_{\mathscr{F} \in T} \mathscr{F}$ is homeomorphic to an $(n-1)$-manifold. Let*

$$j_{\mathscr{P},T}(\mathbf{t}) = \#\left(\left(\mathscr{P}^{(\mathbf{t})} - \bigcup_{\mathscr{F} \in T} \mathscr{F}^{(\mathbf{t})}\right) \cap \mathbb{Z}^n\right)$$

*and*

$$i_{\mathscr{P},T}(\mathbf{t}) = \#\left(\left(\mathscr{P}^{(\mathbf{t})} - \bigcup_{\mathscr{F} \in F-T} \mathscr{F}^{(\mathbf{t})}\right) \cap \mathbb{Z}^n\right).$$

*Then*

$$i_{\mathscr{P},T}(-\mathbf{t}) = (-1)^n\, j_{\mathscr{P},T}(\mathbf{t}).$$

Again, note that Theorem 4 is the special case $T = \varnothing$ of this corollary.

*Proof.* By definition,

$$j_{\mathscr{P},T}(\mathbf{t}) = j_{\mathscr{P}}(\mathbf{t}) - \sum_{\mathscr{F} \in T} j_{\mathscr{F}}(\mathbf{t})$$

and

$$i_{\mathscr{P},T}(\mathbf{t}) = j_{\mathscr{P}}(\mathbf{t}) - \sum_{\mathscr{F} \in F-T} j_{\mathscr{F}}(\mathbf{t}) = i_{\mathscr{P}}(\mathbf{t}) + \sum_{\mathscr{F} \in T} i_{\mathscr{F}}(\mathbf{t}).$$

Hence by Theorem 4,

$$i_{\mathscr{P},T}(-\mathbf{t}) = (-1)^n\, j_{\mathscr{P}}(\mathbf{t}) + \sum_{\mathscr{F} \in T} (-1)^{n-1}\, j_{\mathscr{F}}(\mathbf{t}) = (-1)^n\, j_{\mathscr{P},T}(\mathbf{t}). \quad \blacksquare$$

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## REFERENCES

1. M. Beck, A closer look at lattice points in rational simplices, *Electronic J. Combin.* **6** (1999), R 37.
2. E. Ehrhart, Sur un problème de géométrie diophantienne linéaire, II, *J. Reine Angew. Math.* **227** (1967), 25–49.
3. I. G. Macdonald, Polynomials associated with finite cell complexes, *J. London Math. Soc.* **4** (1971), 181–192.
4. R. P. Stanley, Combinatorial reciprocity theorems, *Adv. Math.* **14** (1974), 194–253.