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Integrability and Linearizability of the Lotka–Volterra System with a Saddle Point with Rational Hyperbolicity Ratio[1]

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In this paper, we consider normalizability, integrability and linearizability properties of the Lotka–Volterra system in the neighborhood of a singular point with eigenvalues 1 and $-\lambda$. The results are obtained by generalizing and expanding two methods already known: the power expansion of the first integral or of the linearizing transformation and the transformation of the saddle into a node. With these methods we find conditions that are valid for $\lambda \in \mathbb{R}^+$ or $\lambda \in \mathbb{Q}$. These conditions will allow us to find all the integrable and linearizable systems for $\lambda = \frac{p}{2}$ and $\frac{2}{p}$, with $p \in \mathbb{N}^+$. © 2002 Elsevier Science (USA)

Key Words: Lotka–Volterra system; linearizability; integrability; normalizability.

1. INTRODUCTION

This paper sits within the framework of a program aimed at a better understanding of normalizable, integrable and linearizable strata of families of polynomial systems:

$$\begin{aligned} \dot{x} &= x + f(x, y) = x + o(|(x, y)|), \\ \dot{y} &= -\lambda y + g(x, y) = -\lambda y + o(|(x, y)|) \end{aligned} \tag{1.1}$$

in the neigborhood of the origin. We work mainly on the Lotka–Volterra system:

$$\begin{aligned} \dot{x} &= x(1 + ax + by), \\ \dot{y} &= y(-\lambda + cx + dy). \end{aligned} \tag{1.2}$$

This family is sufficiently general to give important information on the organization of strata in families of polynomial systems and, we hope, reveal

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universal phenomena. One can find parameters such that system (1.2) is not normalizable, normalizable but not integrable, integrable but not linearizable, or linearizable. We expect, in particular, that for λ rational this family is not integrable and not normalizable almost everywhere. The choice of the Lotka–Volterra family is also supported by its simplicity, which allows to bring general calculations to an end.

DEFINITION 1.1. System (1.1) is normalizable at the origin if and only if there exists an analytic change of variables

$$(X, Y) = (x + \phi(x, y), y + \psi(x, y)) = (x + o(|(x, y)|), y + o(|x, y|)) \quad (1.3)$$

bringing the system to its normal form, that is, for $\lambda = \frac{p}{q} > 0$,

$$\dot{X} = X \sum_{i=0}^{\infty} \varepsilon_{Xi} U^i,$$
$$\dot{Y} = -\lambda Y \sum_{i=0}^{\infty} \varepsilon_{Yi} U^i \quad (1.4)$$

with

$$U = X^p Y^q \qquad \text{and} \qquad \varepsilon_{X0} = \varepsilon_{Y0} = 0.$$

DEFINITION 1.2. The coefficients $\varepsilon_{Xi}, \varepsilon_{Yi}$ are called "coefficients of the normal form of order i".

DEFINITION 1.3. System (1.1) is integrable at the origin if and only if the change of coordinates (1.3) transforms system (1.1) into

$$\dot{X} = Xh(X, Y) = X(1 + O(|(X, Y)|),$$
$$\dot{Y} = -\lambda Yh(X, Y) = -\lambda Y(1 + O(|(X, Y)|). \quad (1.5)$$

This definition is equivalent to ask that $\varepsilon_{Xi} = \varepsilon_{Yi}\ \forall i$.

DEFINITION 1.4. We call "saddle quantity of order i" the quantity $\varepsilon_{Xi} - \varepsilon_{Yi}$, whose vanishing $\forall i$ guarantees integrability.

DEFINITION 1.5. A first integral of the vector field (1.1) is a function $H(x, y)$ such that

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0. \quad (1.6)$$

The trajectories of the system are curves with the equation $H(x, y) = C$. For rational λ, system (1.1) is integrable if and only if it has an analytic first integral.

DEFINITION 1.6. System (1.1) is linearizable at the origin if and only if there exists a change of coordinates (1.3) which linearizes the system, that is, all the coefficients of the normal form vanish.

A linearizable system is therefore always integrable.

Necessary and sufficient conditions for integrability and linearizability are already known in the case $\lambda \in \mathbb{N}$.

Indeed, Fronville *et al.* [FSZ] have shown that System (1.2) with $\lambda \in \mathbb{N}$ is integrable if and only if one of the following conditions is satisfied:

(I) $ma + c = 0,\ m = 0, \ldots, \lambda - 2$;
(II) $\lambda ab - (\lambda - 1)ad - cd = 0$.

Furthermore, Christopher *et al.* [CMR] proved that system (1.2) with $\lambda \in \mathbb{N}$ is linearizable if and only if one of the following conditions is satisfied:

(i) $ma + c = 0,\ m = 0, \ldots, \lambda - 2$;
(ii) $b = d = 0$;
(iii) $a - c = b - d = 0$;
(iv) $b = (\lambda - 1)a + c = 0$.

Note that the conditions just stated yield conditions for the case $\lambda = \frac{1}{n}$, $n \in \mathbb{N}$. Indeed,

Remark 1. Family (1.2) is closed under

$$(x, y, \lambda, t, a, b, c, d) \to \left(y, x, \frac{1}{\lambda}, -\lambda t, d, c, b, a\right). \tag{1.7}$$

Thus, any condition valid for λ yields a new condition for $\frac{1}{\lambda}$.

Remark 2. Family (1.2) is closed under a dilation:

$$(x, y) \to (AX, BY). \tag{1.8}$$

Indeed, system (1.2) becomes

$$\begin{aligned} \dot{X} &= X(1 + aAX + bBY), \\ \dot{Y} &= Y(-\lambda + cAX + dBY). \end{aligned} \tag{1.9}$$

System (1.2) is therefore basically a two-parameter family. Thus, if $ab \neq 0$, we shall sometimes use the system obtained through the change of coordinates as

$$(x, y) = (X/a, Y/b). \tag{1.10}$$

Assuming that $\alpha = c/a$ and $\beta = d/b$, system (1.2) becomes (renaming (X, Y) by (x, y))

$$\begin{aligned} \dot{x} &= x(1 + x + y), \\ \dot{y} &= y(-\lambda + \alpha x + \beta y). \end{aligned} \tag{1.11}$$

In this paper, we are interested in the general case with $\lambda \in \mathbb{Q}^+$, and, occasionally, with $\lambda \in \mathbb{R}^+$. The methods introduced by Fronville *et al.* [FSZ] and Christopher *et al.* [CMR] for $\lambda \in \mathbb{N}$ allow to produce certain sufficient conditions for general λ. However, new methods are needed in order to find all the necessary and sufficient conditions. In this paper, we give methods which allow to find the complete set of necessary and sufficient conditions for integrability and linearizability of system (1.2) with $\lambda = \frac{p}{2}$ and $\frac{2}{p}$.

2. RESULTS

We prove new sufficient conditions with methods introduced by Fronville *et al.* [FSZ] and Christopher *et al.* [CMR]. The first method is the search of the first integral in the form of a series. When applied with $\lambda \in \mathbb{Q}^+$, it leads to Theorem D. The transformation into a node applied with $\lambda \in \mathbb{R}^+$ leads to Theorems E and F. With the help of a cardinality based proof, we shall show that the conditions stated in Theorems C–F are the necessary conditions for integrability and linearizability in the case $\lambda = \frac{p}{2}$.

THEOREM A. *System* (1.2) *with* $\lambda = \frac{p}{2}$ *is integrable if and only if one of the following conditions is satisfied*:

A1. $b = 0$;

A2. $pa(d - b) + 2d(c - a) = 0$;

A3. $2c + (n - 1)a = 0, \ n = 0, 1, \ldots, p - 2$.

THEOREM B. *System* (1.2) *with* $\lambda = \frac{p}{2}$ *is linearizable if and only if one of the following conditions is satisfied*:

B1. $b = 0$;

B2. $d = b, \ c = a$;

B3. $(p-1)a+2c=0,\ \ pb+d=0$;

B4. $2c+(n-1)a=0,\ \ n=0,1,\ldots,p-2$.

The next theorems are valid with $\lambda \in \mathbb{Q}$ or $\lambda \in \mathbb{R}$.

THEOREM C. *System* (1.2) *is integrable if* $\lambda a(d-b)=d(a-c)$.

Proof. This result is already known [CMR, FSZ]. Using the Darboux method explained in [CMR], one can find that, under the condition $\lambda a(d-b)=d(a-c)$, the system has a first integral of the form

$$H(x,y)=\frac{x^{\lambda}y}{(1+ax-\frac{d}{\lambda}y)^{\sigma}}, \tag{2.1}$$

where $\sigma=\lambda+\frac{c}{a}$ if $a\neq 0$ and $\sigma=\frac{b}{d}\lambda+1$ if $d\neq 0$. When $a=d=0$, the first integral is given by

$$H(x,y)=\frac{x^{\lambda}y}{\exp(cx-by)}. \qquad \blacksquare \tag{2.2}$$

THEOREM D. *System* (1.2) *is linearizable if* $\lambda \in \mathbb{Q}$ *and* $\frac{c}{a}+\lambda=k\in\mathbb{N}$, $2\leqslant k<\lambda+1$.

There exists a symmetric condition for $0<\lambda<1$ *by means of change* (1.7).

THEOREM E. *System* (1.2) *is linearizable if* $\lambda>1$, $-\frac{c}{a}=n\in\mathbb{N}^*$ *and one of the following conditions is satisfied*:

E1. $\lambda\in\mathbb{R}\backslash\mathbb{Q}$ *and* $1\leqslant n<\lambda$;

E2. $\lambda\in\mathbb{Q}$ *and* $1\leqslant n<\lambda-1$;

E3. $\lambda\in\mathbb{Q},\ \lambda-1<n<\lambda$ *and* $\lambda\neq n+\frac{1}{q}$;

E4. $\lambda=\frac{p}{q}=n+\frac{1}{q}$ *and* $\frac{d}{b}=\frac{-p}{q-1}$.

Change (1.7) *gives the corresponding conditions for* $0<\lambda<1$.

THEOREM F. *System* (1.2) *is linearizable if one of the following conditions is satisfied*:

F1. $c=0$ *and* $0<\lambda\neq\frac{1}{n}\quad\forall n\in\mathbb{N}$;

F2. $a=c,\ b=d$.

Other cases are obtained through change (1.7).

Theorem D is proved in Section 4, and Theorems E and F are proved in Section 5. These theorems, along with Theorem C, prove the sufficiency of all conditions appearing in Theorems A and B. The proof that these conditions are necessary is completed in Section 6.

3. NORMALIZATION ALGORITHM

We want to linearize system (1.2) up to degree n, so that it becomes

$$\begin{aligned} \dot{x} &= x + O(|(x, y)|^{n+1}), \\ \dot{y} &= -\lambda y + O(|(x, y)|^{n+1}). \end{aligned} \tag{3.1}$$

To do so, let us use a change of variable tangent to the identity

$$(X, Y) = \left(x + \sum_{k=2}^{n} P_k(x, y), y + \sum_{k=2}^{n} Q_k(x, y) \right) \tag{3.2}$$

with

$$\begin{aligned} P_k(x, y) &= \sum_{i+j=k} \phi_{ij} x^i y^j, \\ Q_k(x, y) &= \sum_{i+j=k} \psi_{ij} x^i y^j. \end{aligned} \tag{3.3}$$

One can also find an equivalent change of variable as a composition of $(n-1)$ polynomial changes of the form of the identity plus a homogeneous part. This way of doing things makes it easier to understand the idea of the algorithm. Indeed, one can change the system

$$\begin{aligned} \dot{x} &= x + \sum_{i+j=n} \alpha_{ij} x^i y^j + O(|(x, y)|^{n+1}), \\ \dot{y} &= -\lambda y + \sum_{i+j=n} \beta_{ij} x^i y^j + O(|(x, y)|^{n+1}) \end{aligned}$$

into

$$\begin{aligned} \dot{X} &= X + O(|(X, Y)|^{n+1}), \\ \dot{Y} &= -\lambda Y + O(|(X, Y)|^{n+1}), \end{aligned}$$

using the change of variables:

$$\begin{aligned} X &= x + P_n(x, y), \\ Y &= y + Q_n(x, y). \end{aligned}$$

The coefficients of P_n and Q_n then satisfy

$$\phi_{ij}(i - \lambda j - 1) = \alpha_{ij},$$
$$\psi_{ij}(i - \lambda j + \lambda) = \beta_{ij}. \tag{3.4}$$

When λ is irrational, all these coefficients are well determined, but the convergence of the composition of these transformations is not guaranteed. However, if $\lambda = \frac{p}{q}$, one can see that, in general, there will not exist a formal linearizing change of coordinates, as there exist i and j such that $i - \lambda j - 1 = 0$ and $i - \lambda j + \lambda = 0$. Thus, we meet an obstruction for each of the so-called resonant term of the form $x^{kp+1}y^{kq}$ in $\dot{x}$ and of the form $x^{kp}y^{kq+1}$ in $\dot{y}$, for $k \in \mathbb{N}^*$. Iteration of the process brings the system to the form (1.4). The coefficients $\alpha_{kp+1,kq} = \varepsilon_{Xk}$ and $\beta_{kp,kq+1} = -\lambda\varepsilon_{Yk}$, which are the normal form coefficients, must vanish for the system to be linearizable. For integrability, one must find, $\forall k$, $\varepsilon_{Xk} - \varepsilon_{Yk} = 0$, which are the saddle quantities of order k.

4. POWER EXPANSION OF THE FIRST INTEGRAL

4.1. Proof of Integrability Under the Conditions of Theorem D

We first show that system (1.2) is integrable when the conditions of Theorem D are satisfied. The method used has been first introduced by Fronville *et al.* [FSZ].

LEMMA 4.1. *System* (1.2) *is integrable if* $\lambda \in \mathbb{Q}$ *and* $\frac{c}{a} + \lambda = k \in \mathbb{N},\ 2 \leqslant k < \lambda + 1$.

Proof. We can assume that the first integral has the form

$$H(x, y) = x^{\lambda} y \sum_{i=0}^{\infty} H_i(x) y^i \tag{4.1}$$

with

$$H_0(0) = 1.$$

We ask

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0,$$

and we find the recursive differential equation

$$\begin{aligned} x(1+ax)H_j' &+ [-j\lambda + (\lambda a + (j+1))cx]H_j \\ &+ bxH_{j-1}' + (\lambda b + jd)H_{j-1} = 0. \end{aligned} \tag{4.2}$$

The solution is given in the integral form

$$\begin{aligned} H_j = &- x^{j\lambda}(1+ax)^{-(j+1)(\lambda+c/a)} \int x^{-j\lambda-1}(1+ax)^{(j+1)(\lambda+c/a)-1} \\ &\times [bxH_{j-1}' + (\lambda b + jd)H_{j-1}]\, dx. \end{aligned} \tag{4.3}$$

We are interested in the case $a \neq 0$. As we want to avoid any $\log x$ in H_j (which would imply the non-analyticity of H at the origin), we look for conditions ensuring the vanishing of the coefficient of the x^{-1} term in the integrand.

For $j = 0$ and 1, we find

$$\begin{aligned} H_0 &= (1+ax)^{-(\lambda+c/a)} \\ H_1 &= x^{\lambda}(1+ax)^{-2(\lambda+c/a)} \int x^{-\lambda-1}(1+ax)^{(\lambda+c/a)-2}((ad-bc)x + (\lambda b + d))\, dx. \end{aligned}$$

Let $\lambda + \frac{c}{a} = k \in \mathbb{N}$ with $k \geqslant 2$. Then $(1+ax)^{k-2}$ is a polynomial. We prove by induction that

$$H_j = (1+ax)^{-(j+1)k} \sum_{i=0}^{j(k-1)} a_{ij}x^i. \tag{4.4}$$

Assuming (4.4) for a given j implies that we find integrals of the form $\int x^{-\gamma}P_m(x)\, dx$ in the expression for H_{j+1}, where $P_m(x)$ is a polynomial of degree $(j+1)(k-1)$ and $\gamma = (j+1)\lambda + 1$.

Setting

$$-(j+1)\lambda - 1 + (j+1)(k-1) < -1 \Leftrightarrow k < \lambda + 1,$$

guarantees that there is no possible obstruction and H_{j+1} has the form (4.4). Hence, the system is integrable when

$$2 \leqslant k < \lambda + 1. \qquad \blacksquare \tag{4.5}$$

As we know that the system is integrable, we know that there exists a change of variables bringing the system to (1.5). We therefore only need to linearize one of the coordinates to show linearizability.

4.2. Proof of Theorem D

We use a method similar to the one just presented in Lemma 4.1. Let

$$X(x, y) = x + \sum_{i=0}^{\infty} U_i(x) y^i, \tag{4.6}$$

where $U_0 = O(x^2)$ and $U_1 = O(x)$, so that x is the only linear term.

We suppose $\dot{X} = X$, yielding the following differential equations:

$$\begin{aligned} x(1+ax)U_0' + ax^2 - U_0 &= 0, \\ x(1+ax)U_1' + (cx - \lambda - 1)U_1 + bx(U_0' + 1) &= 0, \\ x(1+ax)U_i' + (icx - i\lambda - 1)U_i + bxU_{i-1}' + d(i-1)U_{i-1} &= 0. \end{aligned}$$

We compute easily the solution of the first equation. If $a \neq 0$, we find

$$U_0(x) = -ax^2(1+ax)^{-1}.$$

We also express U_1 and U_i in the integral form

$$U_1(x) = -bx^{\lambda+1}(1+ax)^{-(k+1)} \int x^{-(\lambda+1)}(1+ax)^{(k-2)}\, dx,$$

$$\begin{aligned} U_i(x) = {} & -x^{(i\lambda+1)}(1+ax)^{-(ik+1)} \int x^{-(i\lambda+2)}(1+ax)^{ik} \\ & \times [bxU_{i-1}' + d(i-1)U_{i-1}]\, dx, \end{aligned} \tag{4.7}$$

where, again, $k = \lambda + \frac{c}{a}$. Now, as before, we prove by induction that

$$U_i(x) = x(1+ax)^{-(ik+1)} \sum_{j=0}^{i(k-1)-1} a_{ij} x^j. \tag{4.8}$$

The same arguments as in Lemma 4.1 can be used here: we let $k \geqslant 2$ so that the expression $(1+ax)^{(k-2)}$, in U_1, is a finite polynomial. Also, the condition $k < \lambda + 1$ prevents the existence of a x^{-1} term in the integrand, for all i. The system is therefore linearizable under condition (4.5). ∎

4.3. The Case $a = d = 0$

We show here a lemma which will be necessary in Section 6.

LEMMA 4.2 *System* (1.2) *with* $a = d = 0$, $bc \neq 0$ *and* $\lambda = \frac{p}{2}$ *is integrable but not linearizable.*

Proof. The system is integrable as it satisfies the hypothesis of Theorem C.

We show that it is not possible to linearize one of the variables. To do so, we use a change of variables of the form (4.6) and show that the series cannot be analytic.

We can suppose, by transformation (1.8), that $b = c = 1$. The system is therefore

$$\begin{aligned} \dot{x} &= x(1 + y), \\ \dot{y} &= y(-\lambda + x). \end{aligned} \tag{4.9}$$

We let $\dot{X} = X$ and obtain

$$\begin{aligned} U_0'x - U_0 &= 0, \\ U_1'x + U_1(x - \lambda - 1) + U_0'x + x &= 0, \\ U_i'x + U_i(ix - i\lambda - 1) + U_{i-1}'x &= 0. \end{aligned}$$

The solution of the first equation is $U_0(x) = Ax$, but we set $A = 0$ because $U_0(x) = O(x^2)$. We find also

$$U_1(x) = -e^{-x}x^{\lambda+1} \int e^x x^{-\lambda-1}\, dx. \tag{4.10}$$

If $\lambda \in \mathbb{N}$ (i.e. p even), there is clearly a x^{-1} term in the series, leading to an obstruction. Otherwise, when p is odd, the integrand can be expressed as a series. We find

$$U_1(x) = -xe^{-x} \sum_{i=0}^{\infty} \frac{x^i}{i!(i - \lambda)} \tag{4.11}$$

and

$$U_1'(x) = e^{-x} \sum_{i=0}^{\infty} \frac{(\lambda + 1)x^i}{i!(i - 1 - \lambda)(i - \lambda)}.$$

U_2 is computed in the same way:

$$\begin{aligned} U_2(x) = &- e^{-2x}x^{2\lambda+1} \int x^{-2\lambda} e^{2x} U_1'\, dx \\ = &- e^{-2x}x^{2\lambda+1} \int x^{-2\lambda} \left(\sum_{j=0}^{\infty} \frac{x^j}{j!} \right) \left(\sum_{i=0}^{\infty} \frac{(\lambda + 1)x^i}{i!(i - 1 - \lambda)(i - \lambda)} \right) dx \\ = &- e^{-2x}x^{2\lambda+1} \int x^{-2\lambda} \left(\sum_{k=0}^{\infty} a_k x^k \right) dx, \end{aligned}$$

where

$$a_k = \sum_{i=0}^{k} \frac{\lambda + 1}{i!(k - i)!(i - 1 - \lambda)(i - \lambda)}. \tag{4.12}$$

As $2\lambda = p \in \mathbb{N}$, there may be a x^{-1} term in the integrand in the expression of U_2. We check that the coefficient of this term, a_{p-1}, does not vanish:

$$a_{p-1} = \sum_{i=0}^{p-1} \frac{2(p+2)}{i!(p-1-i)!(2i-p-2)(2i-p)}.$$

This expression is always finite, as p is odd. We shift the index of the sum with the change $j = i - (p+1)/2$:

$$a_{p-1} = \frac{2(p+2)}{(p-1)!} \sum_{j=-(p+1)/2}^{(p-3)/2} \binom{p-1}{j+(p+1)/2} \frac{1}{4j^2-1}. \tag{4.13}$$

We show that this sum is strictly negative. Let $\binom{m}{n}$ be zero when $n > m$ or $n < 0$. We then express a_{p-1} as an infinite series, in which the only negative term is $-\binom{p-1}{(p+1)/2}$, corresponding to $j = 0$:

$$\begin{aligned}\sum_{j=-(p+1)/2}^{(p-3)/2} \binom{p-1}{j+(p+1)/2} \frac{1}{4j^2-1} &= \sum_{j=-\infty}^{\infty} \binom{p-1}{j+(p+1)/2} \frac{1}{4j^2-1} \\ &= -\binom{p-1}{(p+1)/2} + \sum_{j=1}^{\infty} \left[\binom{p-1}{j+(p+1)/2} \right. \\ &\quad \left. + \binom{p-1}{-j+(p+1)/2} \right] \frac{1}{4j^2-1}.\end{aligned}$$

We compare this series with the following telescoping series:

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \frac{1}{4j^2-1} &= -1 + 2\sum_{j=1}^{\infty} \frac{1}{4j^2-1} \\ &= -1 + \sum_{j=1}^{\infty} \left(\frac{1}{2j-1} - \frac{1}{2j+1} \right) = 0.\end{aligned}$$

Thus

$$-\binom{p-1}{(p+1)/2} + 2\sum_{j=1}^{\infty} \binom{p-1}{(p+1)/2} \frac{1}{4j^2-1} = 0.$$

It therefore suffices to show that

$$2\binom{p-1}{(p+1)/2} > \binom{p-1}{j+(p+1)/2} + \binom{p-1}{-j+(p+1)/2} \qquad \forall j \in \mathbb{N}.$$

This relation is obviously satisfied except for $j = 1$, as the only term greater than $\binom{p-1}{(p+1)/2}$ is $\binom{p-1}{(p-1)/2}$. We then only have to show that

$$2\binom{p-1}{(p-1)/2+1} > \binom{p-1}{(p-1)/2+2} + \binom{p-1}{(p-1)/2}. \qquad (4.14)$$

Using the properties of binomial coefficients, we find that (4.14) is equivalent to

$$2\left[\binom{p-2}{(p-1)/2} + \binom{p-2}{(p+1)/2}\right] > \left[\binom{p-2}{(p+1)/2} + \binom{p-2}{(p+3)/2}\right] + 2\binom{p-2}{(p-1)/2}$$

i.e.

$$\binom{p-2}{(p+1)/2} > \binom{p-2}{(p+3)/2},$$

which is obviously satisfied. a_{p-1} is therefore strictly negative $\forall p = 1, 3, 5, \ldots$ ∎

5. TRANSFORMATION OF THE SADDLE INTO A NODE

The idea of this method (presented in [CMR]) consists in transforming the saddle point of system (1.2) in the (x, y) plane into a node in the (u, v) plane. Through the inverse tranformation, we can find conditions such that system (1.2) is linearizable. If there exists a change of variable linearizing the node, we find the system

$$\begin{aligned} \dot{U} &= U, \\ \dot{V} &= \Lambda V \end{aligned} \qquad (5.1)$$

with $\Lambda > 0$. Then, we can bring the system to the form

$$\begin{aligned} \dot{X} &= X, \\ \dot{Y} &= -\lambda Y. \end{aligned} \qquad (5.2)$$

Using the composition of all changes of coordinates, i.e.,

$$(x, y) \to (u, v) \to (U, V) \to (X, Y),$$

one can often find an analytic transformation $(x, y) \to (X, Y)$ which linearizes system (1.2). Indeed, if we find conditions such that the node is linearizable, then the same conditions guarantee the linearizability of the initial system.

Let

$$(u, v) = (x^{\xi} y^{\eta}, y^{\eta}), \qquad \xi, \eta \in \mathbb{Q}. \tag{5.3}$$

Through this transformation, system (1.2) becomes

$$\begin{aligned} \dot{u} &= u\left[(\xi - \eta\lambda) + \left(\frac{u}{v}\right)^{1/\xi}(\xi a + \eta c) + v^{1/\eta}(\xi b + \eta d)\right], \\ \dot{v} &= v\left[-\eta\lambda + \eta c\left(\frac{u}{v}\right)^{1/\xi} + \eta\, d v^{1/\eta}\right]. \end{aligned} \tag{5.4}$$

We search conditions on ξ, η, a, b, c, d such that system (5.4) has an analytic node at the origin. Christopher *et al.* [CMR] has introduced this method with $\xi = 1$, $\eta \in \mathbb{Q}^{+}$. Here, we allow $\xi, \eta \in \mathbb{Q}$ and find new conditions.

We first show that a family of polynomial systems (defined in Lemma 5.1) is linearizable.

5.1. Case F1 of Theorem F

This condition implies that system (1.2) with $bc = 0$ is linearizable provided λ is not an integer or the inverse of an integer.

LEMMA 5.1. *Any system of the form*

$$\begin{aligned} \dot{x} &= x\left(1 + \sum_{i=1}^{m} a_i y^i + x \sum_{i=0}^{m-1} b_i y^i\right), \\ \dot{y} &= y\left(-\lambda + \sum_{i=1}^{m} c_i y^i\right) \end{aligned} \tag{5.5}$$

is linearizable if $\lambda \neq \frac{1}{n}$, $n \in \mathbb{N}$.

Proof. The new system in the plane (u, v) resulting of transformation (5.3) (with $\xi = -1$ and $\eta = 1$) is always analytic at the origin. Indeed,

$$\begin{aligned} \dot{u} &= -(1 + \lambda)u + \sum_{i=1}^{m}(c_i - a_i)uv^i - \sum_{i=1}^{m} b_{i-1} v^i, \\ \dot{v} &= v\left(-\lambda + \sum_{i=1}^{m} c_i v^i\right). \end{aligned} \tag{5.6}$$

After a proper time scaling, the eigenvalues become 1 and $\frac{\lambda}{1+\lambda}$. A node with eigenvalues 1 and Λ is linearizable when $\Lambda \neq m$ or $\frac{1}{m}$, $m \in \mathbb{N}$. We supposed that $\lambda \neq \frac{1}{n}$. System (5.6) is therefore always linearizable. We can assume that the change of variables which linearizes the system in the (u, v) plane is of the form

$$U = u + \sum_{i=1}^{\infty} f_i v^i + \sum_{i=1}^{\infty} g_i u v^i,$$
$$V = v + \sum_{i=2}^{\infty} e_i v^i.$$

Thus, the resulting change of variables is analytic. Indeed, we find

$$\begin{aligned} X &= \frac{V}{U} = \frac{v + \sum_{i=2}^{\infty} e_i v^i}{u + \sum_{i=1}^{\infty} f_i v^i + \sum_{i=1}^{\infty} g_i u v^i} \\ &= \frac{y + \sum_{i=2}^{\infty} e_i y^i}{\frac{y}{x} + \sum_{i=1}^{\infty} f_i y^i + \sum_{i=1}^{\infty} g_i \frac{y}{x} y^i} \\ &= \frac{x + x\sum_{i=2}^{\infty} e_i y^{i-1}}{1 + x\sum_{i=1}^{\infty} f_i y^{i-1} + \sum_{i=1}^{\infty} g_i y^i} \end{aligned} \tag{5.7}$$

and

$$Y = V = v + \sum_{i=2}^{\infty} e_i v^i = y + \sum_{i=2}^{\infty} e_i y^i. \quad \blacksquare \tag{5.8}$$

As system (1.2) with $c = 0$ is of the form (5.5), case F1 of Theorem F is proved. ∎

5.2. *Case F2 of Theorem F: a blow up*

Choosing $\xi = 1$, $\eta = -1$, we find that the Lotka–Volterra system with $c = a$ and $d = b$ is the only polynomial system which is successfully transformed into an analytic node. This node is located in $(0, \frac{d}{\lambda})$

$$\begin{aligned} \dot{u} &= u(1 + \lambda), \\ \dot{v} &= -d + \lambda v - cu. \end{aligned} \tag{5.9}$$

Let us bring the node to the origin with the change of variables $(U, V) = (u, v - \frac{d}{\lambda})$:

$$\begin{aligned} \dot{U} &= (1 + \lambda)U, \\ \dot{V} &= \lambda V - cU. \end{aligned} \tag{5.10}$$

We then diagonalize the linear part with the change $(U_1, V_1) = (U, V + \frac{c}{\lambda}U)$:

$$\dot{U}_1 = (1+\lambda)U_1,$$
$$\dot{V}_1 = \lambda V_1. \tag{5.11}$$

We finally apply the inverse transformation $(U_1, V_1) = \left(\frac{X}{Y}, \frac{1}{Y}\right)$ to get the linear system

$$\dot{X} = X,$$
$$\dot{Y} = -\lambda Y. \tag{5.12}$$

We show that the composition of these changes of coordinates is analytic:

$$X = \frac{U_1}{V_1} = \frac{U}{V + \frac{c}{\lambda}U} = \frac{u}{-\frac{d}{\lambda} + v + \frac{c}{\lambda}u} = \frac{x}{1 - \frac{d}{\lambda}y + \frac{c}{\lambda}x}$$
$$Y = \frac{1}{V_1} = \frac{1}{V + \frac{c}{\lambda}U} = \frac{1}{-\frac{d}{\lambda} + v + \frac{c}{\lambda}u} = \frac{y}{1 - \frac{d}{\lambda}y + \frac{c}{\lambda}x}. \qquad \blacksquare \tag{5.13}$$

5.3. *Theorem E: a blow-down*

We first present a result proved in [CMR]. Let $(u, v) = (xy^\eta, y^\eta)$ be a change of variable with $\eta = \frac{m}{n}$. Then a polynomial system

$$\dot{x} = x + \sum_{k=2}^{m} f_k(x, y) = x + \sum_{i+j=2}^{m} a_{i,j}x^i y^j,$$
$$\dot{y} = -\lambda y + \sum_{k=2}^{m} g_k(x, y) = -\lambda y + \sum_{i+j=2}^{m} b_{i,j}x^i y^j \tag{5.14}$$

is transformed through this tranformation into an analytic system with a node at the origin if the following conditions are satisfied:

(i) $b_{ij} = 0$ if

$$j < \eta(i-1) + 1 \quad \text{or} \quad j \not\equiv 1 \bmod m;$$

(ii) for $i = 0$,

$$a_{0,j} = 0 \quad \text{when} \quad j \not\equiv 0 \bmod m.$$

for $i > 0$,

$$a_{ij} + \eta b_{i-1,j+1} = 0 \quad \text{when} \quad j < \eta(i-1) \quad \text{and when} \quad j \not\equiv 0 \bmod m;$$

(iii)

$$\lambda\eta - 1 > 0.$$

We assume $m = 1$ and $n > 0$. Conditions (ii) and (iii) yield $\frac{c}{a} = -n$ and $n < \lambda$. System (1.2) then becomes, after a linear transformation and a time scaling $(u, v, t) \to \left(u, au + v, \left(\frac{\lambda}{n} - 1\right)t\right)$:

$$\begin{aligned}\dot{u} &= u + \frac{(nb + d)}{(n - \lambda)} u\left(v + \frac{c}{n}u\right)^n,\\ \dot{v} &= \frac{\lambda}{\lambda - n} v + \frac{(-bcu + dv)}{\lambda - n}\left(v + \frac{c}{n}u\right)^n.\end{aligned} \tag{5.15}$$

An obstruction in the linearization process is possible only if $\frac{\lambda}{\lambda - n} = \Lambda \in \mathbb{N}$. System (5.15) is therefore always linearizable when $\lambda \in \mathbb{R}/\mathbb{Q}$. Part E1 of Theorem E is then proved.

Let $\lambda \in \mathbb{Q}$. Now, for some λ, the eigenvalue $\frac{\lambda}{\lambda - n} = \Lambda$ can be an integer. In any case, it is always possible to transform a node to its normal form by an analytic change of coordinates. Thus, one can bring system (5.15) to the form

$$\begin{aligned}\dot{U} &= U,\\ \dot{V} &= \Lambda V + \gamma U^{\Lambda}.\end{aligned} \tag{5.16}$$

From the fact that the lowest degree of the nonlinear terms in (5.15) is $n + 1$, we choose the normalizing change of coordinates as

$$(U, V) = \left(u + \sum_{i+j=n+1}^{\infty} a_{ij} u^i v^j, v + \sum_{i+j=n+1}^{\infty} b_{ij} u^i v^j\right). \tag{5.17}$$

Thus, we find that if $\Lambda < n + 1$, then $\gamma = 0$, i.e. the system is linearized.

$$\frac{\lambda}{\lambda - n} = \Lambda < n + 1,$$

$$\Updownarrow$$

$$n < \lambda - 1.$$

This completes the proof of case E2. ∎

Now, we show that the system is linearizable in some cases when $\Lambda \geqslant n + 1$. Indeed, the degree of each term in system (5.15) is congruent to 1 mod n. If we apply to this system a transformation of the form

$$u_1 = u + P_{kn+1}(u, v),$$
$$v_1 = v + Q_{kn+1}(u, v), \tag{5.18}$$

we obtain the new system

$$\dot{u}_1 = (1 + \partial_u P_{kn+1})\dot{u} + \partial_v P_{kn+1}\dot{v},$$
$$\dot{v}_1 = (1 + \partial_v Q_{kn+1})\dot{v} + \partial_u Q_{kn+1}\dot{u}. \tag{5.19}$$

When the right-hand side of this system is expressed as a function of u and v, we find that the degree of each term is still congruent to 1 mod n. Thus, it is sufficient to keep only terms of degree $kn + 1$, $(k \in \mathbb{N})$ in the normalizing transformation:

$$(U, V) = \left(u + \sum_{\substack{k=1 \\ i+j=kn+1}}^{\infty} a_{ij}u^i v^j, v + \sum_{\substack{k=1 \\ i+j=kn+1}}^{\infty} b_{ij}u^i v^j \right). \tag{5.20}$$

Hence, an obstruction can exist only if

$$\frac{\lambda}{\lambda - n} = \Lambda = kn + 1,$$
$$\Updownarrow$$
$$\lambda = n + \frac{1}{k} = n + \frac{1}{q}.$$

The system is therefore linearizable if $-\frac{c}{a} = n$, $\lambda - 1 < n < \lambda$ and $\lambda \neq n + \frac{1}{q}$. This proves case E3. ∎.

Corollary 5.1. *System* (1.2) *is normalizable if* $-\frac{c}{a} = n, 1 \leqslant n < \lambda$.

The proof of Corollary 5.1 is in [CMR]. It comes from the fact that the node of (5.15) is always normalizable.

5.4. Transformation of an Integrable Saddle into a Node

Let us suppose that $\lambda = n + \frac{1}{q}$, and that transformation (5.3) with $\xi = 1$ and $\eta = \frac{1}{n}$ of system (1.2) yields a normalizable node of the form (5.16). Then, one can still find conditions such that $\gamma = 0$.

The idea of this section is based on the fact that a node cannot be integrable unless it is linearizable. Thus, we may ask what happens to an integrable saddle transformed into a node.

PROPOSITION 5.1. *Let system* (5.14) *be a polynomial system satisfying conditions* (i)–(iii) *of Section* 5.3, *and such that* $\lambda = n + \frac{1}{q}, n, q \in \mathbb{N}$.

Then the system is linearizable if and only if it is integrable.

Proof. Under these conditions, the inverse change of coordinates $(X, Y) = \left(\frac{u}{v}, v^n\right)$ transforms system (5.16) into

$$\dot{X} = X\left(1 + \frac{\gamma}{qn} X^{qn+1} Y^q\right),$$
$$\dot{Y} = -\lambda Y\left(1 + \frac{\gamma}{qn+1} X^{qn+1} Y^q\right), \tag{5.21}$$

which is the normal form of (5.14). It is shown in [CMR] that the transformation $(x, y) \to (X, Y)$ is analytic. As we supposed that system (5.14) is integrable, we must have

$$\frac{\gamma}{qn} = \frac{\gamma}{qn+1} \quad \Leftrightarrow \quad \gamma = 0.$$

The system is therefore linearizable. ∎

Proof of part E4 of Theorem E. System (1.2) with $\lambda = n + \frac{1}{q}$ is integrable if $\lambda a(d - b) = d(a - c)$ (by Theorem C). When $\frac{c}{a} = -n$, it also satisfies condition (ii) of Section 5.3, thus satisfying the hypothesis of Proposition 5.1. Intersecting the conditions $\frac{c}{a} = -n$ with $\lambda a(d - b) = d(a - c)$, we find

$$c + an = 0, \qquad (q - 1)\, d + pb = 0. \quad ∎ \tag{5.22}$$

Remark 3. If we set $q = 1$, we find condition (iv) of linearizability of [CMR] with $\lambda = n + 1 \in \mathbb{N}$.

When $\lambda \neq n + \frac{1}{q}$, system (1.2) with $\frac{c}{a} = -n$ $(\lambda - 1 \leqslant n < \lambda)$ is linearizable for all $\frac{d}{b}$. On the bifurcation diagram (as we define it in Section 7), this condition is the whole line $\frac{c}{a} = -n$. In the particular cases $\lambda = n + \frac{1}{q}$, the condition of linearizability reduces to a point $\left(\frac{c}{a} = -n, \frac{d}{b} = \frac{nq+1}{1-q}\right)$ on this line. The condition on d and b can be expressed as

$$\frac{b}{d} = 1 - \frac{n+1}{n + \frac{1}{q}} = 1 - \frac{n+1}{\lambda}.$$

The first form shows that the limit of this expression is $\frac{1}{n}$ as $q \to \infty$. The second form shows the curve $\frac{b}{d}(\lambda)$ on which system (1.2) with $\frac{c}{a} = -n$ is linearizable, $\forall \lambda$ $(n < \lambda \leqslant n + 1)$.

Remark 4. We can use the idea of tranformation of an integrable saddle point into a node to prove the case $b = d = 0$, $\lambda = p \in \mathbb{N}$. Through transformation (1.7), this case is equivalent to $c = a = 0$ and $\lambda = \frac{1}{p}$. System (1.2) is integrable under this condition, because it satisfies the hypothesis of theorem C. Once again, we use the change of coordinates $(u, v) = \left(\frac{y}{x}, y\right)$. We obtain a node with eigenvalues 1 and $\frac{1}{p+1}$. We can bring the system to its normal form

$$\begin{aligned} \dot{U} &= (1+p)U + \gamma V^{p+1}, \\ \dot{V} &= V. \end{aligned} \tag{5.23}$$

Then, the inverse change of coordinates $(X, Y) = \left(\frac{U}{V}, V\right)$ yields the saddle in its normal form:

$$\begin{aligned} \dot{X} &= X\left(1 - \frac{\gamma}{p} XY^p\right), \\ \dot{Y} &= \frac{-1}{p} Y. \end{aligned} \tag{5.24}$$

As the system is integrable, we must have $\gamma = 0$. Thus, system (1.2) is linearizable if $d = b = 0$ and $\lambda = p \in \mathbb{N}$. ∎

5.5. Node Linearizability using Invariant Analytic Curves

We can find new conditions of linearizability of the node of (5.15) by searching for an invariant analytic curve through the node. This curve, if it exists, should be of the form

$$v = \sum_{i=1}^{\infty} b_i u^{in+1}.$$

The existence of two invariant analytic curves through the node (the first one being $u = 0$) guarantees linearizability, as system (5.16) with $\gamma \neq 0$ has only one analytic invariant curve ($U = 0$) through the node.

We substitute (5.15) into the equation

$$\dot{v} = \left(\sum_{i=1}^{\infty} (in+1) b_i u^{in}\right) \dot{u}$$

and find the coefficients b_i as functions of $(b_1, b_2, \ldots, b_{i-1})$. We find expressions of the form

$$b_k(k\lambda - kn - 1) = f(b_1, b_2, \ldots, b_{k-1}).$$

Notice that $k\frac{p}{q} = kn + 1 \Leftrightarrow k = q$, as $kn + 1$ has no common factor with k. Thus, we must set $n = \frac{p-1}{q}$, otherwise the node is linearizable.

One can compute $f(b_1, b_2, \ldots, b_{q-1})$ for $q = 2, 3, 4, \ldots$ to find linearizability conditions when $\lambda = n + \frac{1}{q}$. If $k = 2$, we only get condition E4. But as q increases, we find more and more conditions for which system (1.2) is linearizable. These conditions can be expressed as points of the form $\left(\frac{c}{a} = -n, \frac{d}{b}\right)$ (see Section 7). When $q = 3$, we find two solutions for $\frac{d}{b}$ which are rational expressions in n. When $q = 4$, we find three points which are the roots of a polynomial of degree three. Two of those roots are irrational. When $q = 5$, we get four roots, of which two are complex. It is possible to compute these conditions for any λ, but the calculations become very complex and tedious, and we have not found a general pattern.

6. PROOF OF THEOREMS A AND B

Theorems C–F yield the sufficiency of the conditions of Theorems A and B. We only need to show the necessity of these conditions.

6.1. Proof of Theorem A

We know that the coefficients of the normal form of order k, $(\varepsilon_{Xk}, \varepsilon_{Yk})$, have degree $k(p+q)$ in a, b, c, d. An integrable system is a system in which all saddle quantities vanish, i.e. each pair of coefficients of order k satisfies the relation $\varepsilon_{Xk} - \varepsilon_{Yk} = 0$. The saddle quantity of order k has therefore degree $k(p+q)$ in a, b, c, d. We have shown that system (1.2) with $\lambda = \frac{p}{2}$ is integrable if one of the conditions of Theorem A is satisfied. The system is thus integrable if

$$b(pa(d-b) + 2d(c-a))\left(\prod_{n=0}^{p-2} 2c + (n-1)a\right) = 0,$$

and we must have

$$\varepsilon_{Xk} - \varepsilon_{Yk} = b(pa(d-b) + 2d(c-a))\left(\prod_{n=0}^{p-2} 2c + (n-1)a\right) R_k(a, b, c, d) \qquad (6.1)$$

with $R_k(a, b, c, d)$ a homogeneous polynomial the degree of which depends on k.

When $k = 1$, the saddle quantity must have degree $p+2$. Expression (6.1) without R_k already has degree $p+2$, we conclude that R_1 has degree 0. It has been shown in [CMR] that there exists at least one system (1.2) which is not integrable, $\forall\lambda$. Thus, we have $R_1 \not\equiv 0$ and system (1.2) with $\lambda = \frac{p}{2}$ cannot be integrable unless it satisfies one of the conditions of Theorem A. ∎

6.2. Proof of Theorem B

We use the same kind of argument as for the proof of Theorem A to show that the system cannot be linearizable unless the conditions of Theorem B are fulfilled. Let I and L be the manifolds defined, respectively, by conditions A and B. As the system cannot be linearizable if it is not integrable, there only remains to prove that, for $\lambda = \frac{p}{2}$, there are no linearizable points in $I \backslash L$, which is equivalent to showing that there are only two linearizable points (B2 and B3) on Condition A2 ($pa(d-b) + 2d(c-a) = 0$) that are not included in conditions B1 and B4.

Normal form coefficients of $\dot{x}$ in system (1.2) must be of the form

$$\varepsilon_{Xk} = b\left(\prod_{n=0}^{p-2} 2c + (n-1)a\right) S_k(a,b,c,d),$$
$$\varepsilon_{Yk} = b\left(\prod_{n=0}^{p-2} 2c + (n-1)a\right) T_k(a,b,c,d) \tag{6.2}$$

and we must have

$$\lambda S_1(a,b,c,d) + T_1(a,b,c,d) = pa(d-b) + 2d(c-a) \tag{6.3}$$

so $\varepsilon_{X1} - \varepsilon_{Y1}$ vanishes when condition A2 is respected.

The degrees of $S_1(a,b,c,d)$ and $T_1(a,b,c,d)$ is less or equal to 2 as ε_{X1} and ε_{Y1} are of degree $p+q$, whilst $b\prod_{n=0}^{p-2} 2c + (n-1)a$ is of degree $p+q-2$. However they must vanish if (B2) or (B3) are fulfilled. As we would like to use the degree of these polynomials to show these are the only points where they can vanish, we must prove that they do not vanish identically. This is shown in Lemma 6.1. Then we will know that their degree is 2.

We can show that there does not exist a linearizable point in $I \backslash L$ if $a = 0$. Indeed, condition A2 is then satisfied. If $bc \neq 0$ and $a = 0$ then $d = 0$ and Lemma 4.2 proves that the system with $\lambda = \frac{p}{2}$ is not linearizable.

There only remains to prove that there does not exist a linearizable point with $a \neq 0$ on $I \backslash L$. Scaling $a = b = 1$ and $c = \alpha, d = \beta$, we get system (1.11). We then lose the homogeneity of the normal form coefficients, but the breaking of symmetry between the degree of the coefficients in $\dot{x}$ and $\dot{y}$ allows us to prove some useful properties.

Indeed, one of the consequences of corollary 6.1 below, is that $S_1(1,1,\alpha,\beta)$ is of degree less than or equal to 1.

As Eq. (6.3) remains of degree 2 when $a = b = 1$, $T_1(1,1,\alpha,\beta)$ must be of degree 2. $S_1(1,1,\alpha,\beta)$ cannot be of degree zero, and is therefore of degree 1, as it must vanish for $\beta = \alpha = 1$ but cannot vanish identically according to Lemma 6.1. For the system to be linearizable on $ab \neq 0$, $S_1(1,1,\alpha,\beta)$ and $T_1(1,1,\alpha,\beta)$ must vanish together. After substitution of the condition given by $S_1(1, 1, \alpha, \beta) = 0$, equation $T_1(1, 1, \alpha, \beta) = 0$ becomes a quadratic equation in only one variable which does not vanish identically unless $S_1(1, 1, \alpha, \beta)$ is a factor of $T_1(1, 1, \alpha, \beta)$. Let us show that this does not happen. Otherwise

$$T_1(1, 1, \alpha, \beta) = S_1(1, 1, \alpha, \beta)\, K_1(\alpha, \beta).$$

Since

$$\lambda S_1(1, 1, \alpha, \beta) + T_1(1, 1, \alpha, \beta) = p(\beta - 1) + 2\beta(\alpha - 1)$$

then

$$(\lambda + K_1(\alpha, \beta))S_1(1, 1, \alpha, \beta) = 2\beta\alpha + p\beta - (p + 2)$$

We see that the right-hand term is irreducible when $p > 0$, yielding a contradiction.

Hence $T_1(1, 1, \alpha, \beta)|_{S_1(1,1,\alpha,\beta)}$ is a polynomial of degree 2 in one variable which does not vanish identically. It therefore admits at most two roots. As the two conditions B2 and B3 cannot be contained in another factor of the coefficient of the normal form, we already know those two roots. Therefore the first coefficients of the normal form cannot vanish elsewhere than on L. ∎

LEMMA 6.1. *There is at least one integrable system with the form* (1.2) *whose coefficients of the normal form of order* 1 (*of degree* $p + q + 1$) *are nonvanishing.*

Proof. Let us consider the integrable system

$$\begin{aligned} \dot{x} &= x(1 + x + y), \\ \dot{y} &= -\frac{p}{q}y(1 + x + y). \end{aligned} \tag{6.4}$$

It was shown in [CMR] that the coefficients of the normal form of order 1 are nonvanishing.

The proof uses the fact that there is an obstruction to the existence of a linearizing change of coordinates of the form

$$(X, Y) = (xe^{f(x,y)}, ye^{-\lambda f(x,y)}). \quad ∎ \tag{6.5}$$

In order to prove Corollary 6.1, we first prove a more general property of a set of systems that contains system (1.11).

DEFINITION 6.1. Let E be the set of systems

$$\dot{x} = x + x\sum_i \varepsilon_{Xi}(\alpha,\beta)U^i + \sum_{i+j=k}^{\infty} A_{ij}(\alpha,\beta)x^i y^j,$$
$$\dot{y} = -\frac{p}{q}y + y\sum_i \varepsilon_{Yi}(\alpha,\beta)U^i + \sum_{i+j=k}^{\infty} B_{ij}(\alpha,\beta)x^i y^j, \qquad (6.6)$$

where $U = x^p y^q$ and where the coefficients $\varepsilon(\alpha,\beta)$, $A(\alpha,\beta)$ and $B(\alpha,\beta)$ of $x^k y^{t-k}$ are of degree

- 0 or $t-2$ in $\dot{x}$
- 0 or $t-1$ in $\dot{y}$

$\forall(t,k)$.

LEMMA 6.2. *The set E is closed with respect to application of the normalization algorithm.*

Proof. We discussed in Section 3 the possibility of splitting the normalization algorithm as a compositions of steps that normalize one degree at a time. We can actually use smaller steps to normalize one term at a time. The normalization algorithm then becomes a composition of changes of coordinates of the form

$$X = x + \phi_{j,k-j}x^j y^{k-j},$$
$$Y = y \qquad (6.7)$$

or

$$X = x,$$
$$Y = y + \psi_{j,k-j}x^j y^{k-j}. \qquad (6.8)$$

As the degrees of $\phi_{j,k-j}$ of $\psi_{j,k-j}$ are given by (3.4), straightforward calculations show that the normalized system is in E. ∎

COROLLARY 6.1. *Consider system* (1.11). *Degrees in α,β of the coefficients of the normal form of order k* (*of degree $k(p+q)+1$ in x, y*) *are*:

- 0 or $k(p+q)$ in $\dot{y}$;
- 0 or $k(p+q)-1$ in $\dot{x}$.

Proof. As the initial system is in E, so is the normalized system.

7. QUALITATIVE RESULTS

We are interested to see where the strata of linearizable and integrable systems sit in the bifurcation diagram of family (1.2). Let us first show general properties of the Lotka–Volterra system.

7.1. Invariant Lines

The projection of system (1.2) onto the Poincaré sphere yields four finite singular points and three on the "equator" (see Fig. 1).

The system always has two invariant lines, $x = 0$ and $y = 0$. A third one may appear under special conditions. It passes through singular points **B** : $\left(-\frac{1}{a}, 0\right)$ and **C** : $(0, \frac{\lambda}{d})$, as it could not cross the axes elsewhere, therefore, the equation of this line is $dy - \lambda ax - \lambda = 0$. Invariance of this line implies the integrability of system (1.2) (see Theorem C) and may also yield a bifurcation. The line is invariant under the condition

$$\lambda a(d - b) = d(a - c). \tag{7.1}$$

7.2. Bifurcation Diagram

Bifurcations of the Lotka–Volterra system are shown in Fig. 2. As we discussed above, it is possible, for fixed λ, to present the parameter-space as a two-dimensional space. The invariance property of system (1.2) under dilation (transformation (1.9)) allows us to choose the product of two circles as parameter-space. We scale so that $(a, c) \in \mathbb{S}^1$ and $(b, d) \in \mathbb{S}^1$. However, we

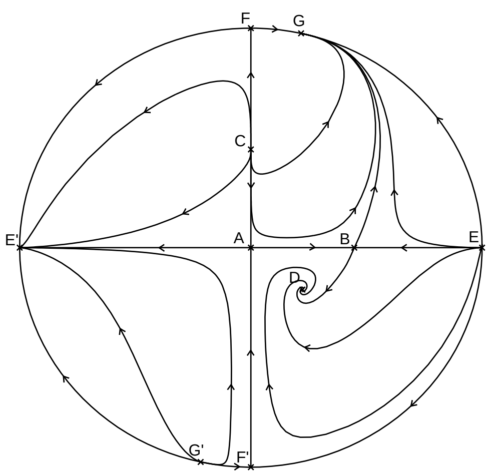

FIG. 1. Example of the phase portrait of a system (1.2).

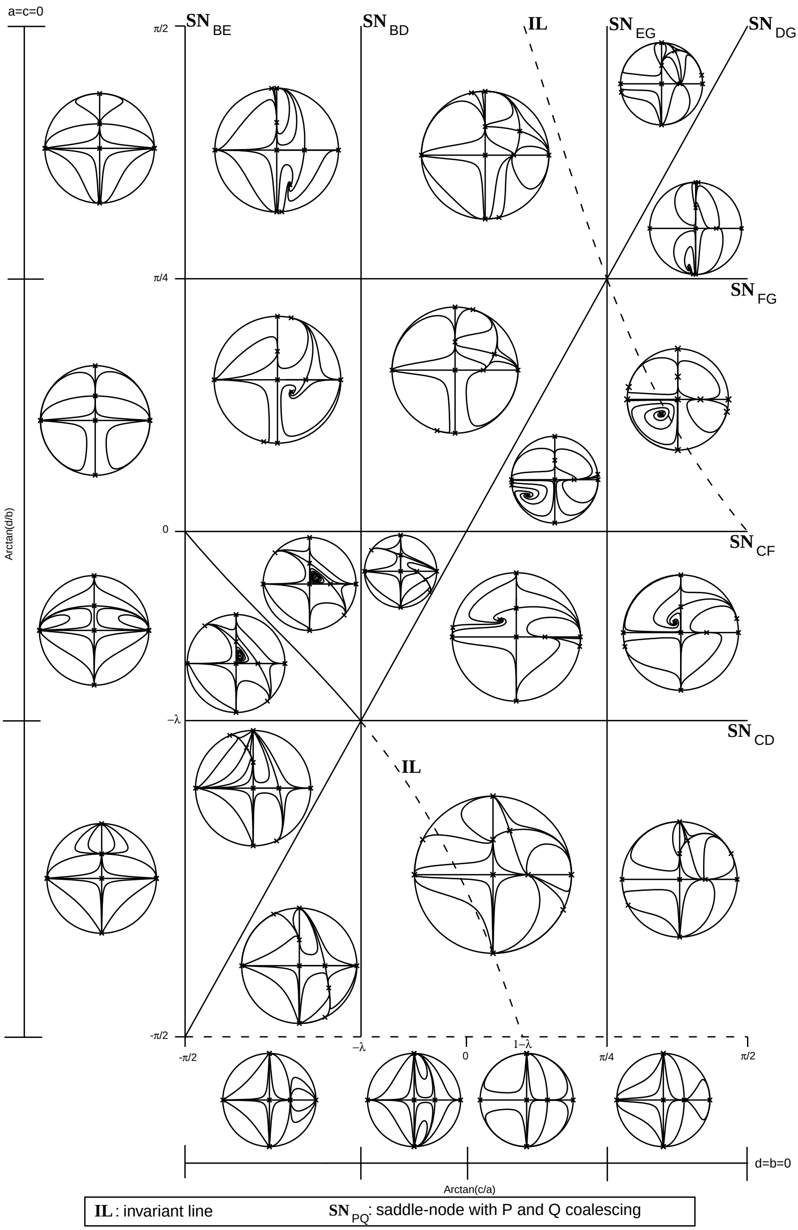

FIG. 2. Bifurcation diagram of system (1.2).

must add the cases $a = c = 0$ and $b = d = 0$ ($a = b = c = d = 0$ is trivial and therefore avoided). Thus, the "natural" parameter-space for family (1.2) is $(\mathbb{S}^1_{a,c} \times \mathbb{S}^1_{b,d}) \cup (\mathbb{S}^1_{a,c} \times \{0, 0\}) \cup (\{0, 0\} \times \mathbb{S}^1_{b,d})$, where $\mathbb{S}^1_{a,b}$ is the unit circle in the space a, b.

7.3. Organization of the Strata

Using the theorems shown above, we are now able to draw some conclusions about the global organization of the strata of system (1.2).

In spite of Theorems C and E, which yield general results, the behavior of system (1.2) is not well understood when λ is irrational. However, when λ is rational, it seems clear that the number of normalizability, integrability and linearizability conditions increase as p and q increase. In fact, the degree of the first coefficients of the normal form is $p + q$. There seems also to be an increase in "complexity" in the conditions when q increases (with $p > q$). Indeed, when $\lambda = p$ or $\lambda = \frac{p}{2}$, all conditions of linearizability and integrability are polynomials in a, b, c, d with rational coefficients. However, other calculations yielded some conditions in the form of an isolated point on the bifurcation diagram whose coordinates are irrational or complex (see Section 5.5).

Integrability and linearizability conditions obtained for all λ are essentially continuous. The condition of Theorem C is continuous for all λ and conditions in Theorems D and E have discontinuities only when $\lambda \in \mathbb{N}$, due to the appearance of a new line of linearizable points. We may suppose that another discontinuity always arises when $\lambda = n + \frac{1}{q}$, as the system transformed into a node is not linearizable in general (condition E4). For other λ, the linearizable stratum is a whole line in the bifurcation diagram ($\frac{c}{a} = -n$) while, for $\lambda = n + \frac{1}{q}$, the stratum seems to reduce to a finite number of points on this line. When $q = 2$, we proved that there is only one such point $\left(\frac{c}{a} = -n, \frac{d}{b} = \frac{-p}{q-1}\right)$, but we found that other points appear when q increases.

Comparing conditions of Theorems A and B, one could think that the only linearizable isolated points on the bifurcation diagram arise on curves of integrable systems. However, calculation of a few coefficients of the normal form with $\lambda = \frac{p}{q}$ $(q > 2)$ yielded isolated points for both linearizability and integrability.

In the case $\lambda = \frac{p}{2}$, we have observed that the system is nonintegrable almost everywhere. As expected, our results yielded systems which are normalizable but not integrable, integrable but not linearizable and linearizable. Many integrable systems are also linearizable. We also noticed that most conditions obtained were valid only for $\lambda > 1$ or $\lambda < 1$. Thus, the particular symmetry of the Lotka–Volterra system (transformation (1.7)) does not imply the coexistence of two "mirror" conditions for a given lambda, except for the $bc = 0$ condition.

8. CONCLUSION AND RESEARCH AVENUES

The work done here is far from a complete investigation of the Lotka–Volterra family. Here are some questions which we consider to be worth investigating.

8.1. Developments

While we needed only two methods to prove all the integrability and linearizability conditions in the case $\lambda \in \mathbb{N}$, that is the transformation into a node and the Darboux method, we had to add the power series method in order to complete the case $\lambda = \frac{p}{2}$. For the case $\lambda = \frac{p}{3}$, these methods already are not sufficient. New methods or generalization of the existing methods are necessary. For example, the transformation of integrable systems into nodes could probably be used to find other conditions.

Even if we tried to use (5.3) with negative ξ and η without finding new cases, other forms of transformations, for example a study of the system at point **D** remain unexplored.

In the case of a general quadratic system, the first conjecture below let us dream of the existence of a more general method than those that we used.

8.2. Conjectures

1. Theorem D can also be applied to the system:

$$\begin{aligned} \dot{x} &= x(1 + ax + by), \\ \dot{y} &= y(-\lambda + cx + dy) + fx^2. \end{aligned} \tag{8.1}$$

We noticed that the condition given by Theorem D ensures the vanishing of the first few coefficients of the normal form for all the λ we tried. However, we could not find a way to modify our proof so as to cover this case.

2. System (1.2) with $a = d = 0$ is integrable, but not linearizable, for all rational λ. The system is always integrable in that case. We were however only able to find a proof that it is not linearizable in the case $\lambda = \frac{p}{2}$. The simplicity of the system (there is no free parameter) should allow various ways of tackling the problem.

3. There exists a point in parameter-space such that system (1.2) with $\lambda = n + \frac{1}{q}$ is normalizable, but not integrable. There is a transformation into a resonant node which ensures the normalizability of the system. It would be

surprising if the coefficients of the normal form of the node vanished identically.

In addition, here are some questions to which we could not find an answer, and that could be used as a starting point for further research.

8.3. Questions

1. Can Theorem D be generalized to the case $\lambda \in \mathbb{R}$? Linearizability for all rational λ make us believe in the convergence of series (4.1) for irrational λ.

2. Is there any point a, b, c, d that is not integrable $\forall \lambda \in \mathbb{Q}^+$?

3. What happens when $p + q$ goes to infinity? Will the varieties of integrable and linearizable systems remain in a particular area, or will they be distributed randomly or uniformly over the parameter-space?

4. When using the normalizing algorithm for a given λ, it is sometimes necessary to annihilate more than one saddle quantity in order to find the integrability conditions. How many saddle quantities are required to determine the sufficient conditions for integrability, for a given $\lambda \in \mathbb{Q}$?

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