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Continuity of the temperature in boundary heat control problems

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Abstract

Motivated by the boundary heat control problems formulated in the book of Duvaut and Lions, we study a boundary Stefan problem and a boundary porous media problem. We prove continuity of the solution with the appropriate modulus. We also extend the results to the fractional order case and to the anomalous diffusion problems.

1. Introduction

In this paper we study initial–boundary problems with nonlinear Neumann data on part of its boundary, i.e.,

$$\begin{cases} H_\alpha u(x,t) := \Delta u(x,t) - \alpha u_t(x,t) = 0, & (x,t) \in Q := \Omega \times (0,T], \\ -u_\nu(x,t) \in \beta_t\big(u(x,t)\big), & (x,t) \in \Gamma \times (0,T], \\ u(x,t) = 0, & (x,t) \in (\partial\Omega - \Gamma) \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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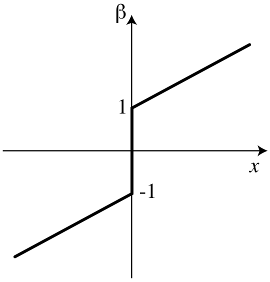
where Ω is a bounded smooth domain in $\mathbb{R}^{N+1}$ ($N \geqslant 1$), Γ is a smooth submanifold of $\partial\Omega$, $T > 0$, u_0 is a given smooth function, and β is either (i) a multivalued mapping

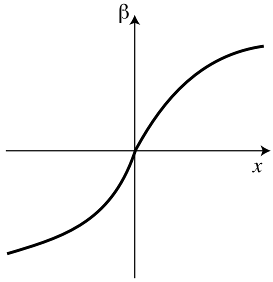
$$\beta(x) := \begin{cases} ax - 1, & x < 0 \ (a > 0), \\ [-1, 1], & x = 0, \\ bx + 1, & x > 0 \ (b > 0) \end{cases} \tag{1.2}$$

or (ii) a continuous increasing real-valued satisfying

$$\begin{cases} \text{(a)} \quad \beta'(x) \text{ exists for all } x \neq 0, \\ \text{(b)} \quad \beta(0) = 0, \\ \text{(c)} \quad \beta'(x) \geqslant c_1 > 0 \text{ for some constant } c_1 \text{ and } x \neq 0, \\ \text{(d)} \quad \beta'(x) \leqslant C(\varepsilon) \text{ for } x \in (-\frac{1}{\varepsilon}, -\varepsilon) \cup (\varepsilon, \frac{1}{\varepsilon}) \text{ and } \varepsilon > 0, \end{cases} \tag{1.3}$$

that is,

(i) 

(ii) 

Problems (1.1), (1.2) can be thought of as "Boundary Stefan Problem" while (1.1), (1.3) is a boundary version of a singular equation which includes the porous media equation.

Problems like these occur in boundary heat control and are formulated in Duvaut and Lions book (see [7]). They prove existence and uniqueness in some particular cases in the proper spaces. A general existence theory was developed in [8].

For simplicity we shall assume that Γ lies on the hyperplane $\mathbb{R}^N$. Our results can be extended to hold for more general Γ such as Lipschitz manifolds.

The main result in this paper asserts that u is a continuous function of x and t up to the boundary. Its modulus of continuity will depend, of course, on β. If, in addition, we assume that β of case (ii) has near zero a homogeneous behavior such as that of the porous media, i.e., $\beta(u) \sim u^{1/m}$, $m > 1$, then we obtain a Hölder modulus of continuity. We also show how our methods can be modified to prove boundedness and continuity to more general anomalous diffusion problems, i.e.,

$$\begin{cases} -(-\Delta)^{\delta} u(x,t) \in \beta_t\big(u(x,t)\big), & (x,t) \in \mathbb{R}^N \times (0,\infty), 0 < \delta < 1, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1.4}$$

with the same β's as before and to Initial–Boundary Value Problems of the type

$$\begin{cases} \dfrac{1}{y^\gamma}\nabla\cdot\big(y^\gamma\nabla u(x,y,t)\big)-\alpha u_t(x,y,t)=0, & (x,y,t)\in\mathbb{R}^N\times\mathbb{R}^+\times(0,\infty),\\ +\lim\limits_{y\to0^+} y^\gamma u_y(x,y,t)\in\beta_t\big(u(x,0,t)\big), & (x,t)\in\mathbb{R}^N\times(0,\infty),\\ u(x,0,t)\underset{|x|\to\infty}{\longrightarrow}0, & (x,t)\in\mathbb{R}^N\times(0,\infty),\\ u(x,0)=u_0(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.5}$$

where $\gamma=1-2\delta$.

2. Normalized lemmas

Our approach is that of DeGiorgi's method in his celebrated paper [6], and it is based on a combination of the methods in [1] for the usual Stefan problem and [4] for the treatment of non-local evolution problems. As it was done in the paper of Caffarelli and Evans (see [1]) we approximate the β by smooth functions β_ε but keeping its basic structure and derive our estimates independently of ε. We note that the bounds on the L^∞ norm of u^ε and the L^2 norm of Du^ε can be obtained independently of $\varepsilon>0$.

Let $0<\varepsilon<1$ and define for the case (i)

$$\beta_\varepsilon(s)=\begin{cases} as-1 & \text{for } s<\frac{-\varepsilon}{1-\varepsilon a}\ (a>0),\\ \frac{1}{\varepsilon}s & \text{for } -\frac{\varepsilon}{1-\varepsilon a}\leqslant s\leqslant\frac{\varepsilon}{1-\varepsilon b},\\ bs+1 & \text{for } s>\frac{\varepsilon}{1-\varepsilon b}\ (b>0) \end{cases}$$

and similar for the case (ii).

Consider, now, the approximate problem

$$\begin{cases} \Delta u^\varepsilon(x,t)-\alpha u^\varepsilon_t(x,t)=0, & (x,t)\in\Omega\times(0,T],\\ -u^\varepsilon_\nu(x,t)=\beta'_\varepsilon\big(u^\varepsilon(x,t)\big)u^\varepsilon_t(x,t), & (x,t)\in\Gamma\times(0,T],\\ u^\varepsilon(x,t)=0, & (x,t)\in(\partial\Omega-\Gamma)\times(0,T],\\ u^\varepsilon(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

where β_ε is either of the above.

Proposition 2.1. *Suppose $u_0\in C^1(\bar\Omega)$ and $|u_0|$ and $|Du_0|$ are bounded. Then there exists a unique function u^ε such that $u^\varepsilon\in C((\Omega\cup\Gamma)\times(0,T])$, $\Delta u^\varepsilon\in L^2((\Omega\cup\Gamma)\times(0,T])$ solving*

$$\int_0^T\int_\Omega\big(\alpha u^\varepsilon\zeta_t-\nabla u^\varepsilon\nabla\zeta\big)\,dx\,dt+\int_0^T\int_\Gamma\beta_\varepsilon\big(u^\varepsilon\big)\zeta_t\,dS\,dt +\alpha\int_\Omega u_0(x)\zeta(x,0)\,dx+\int_\Gamma\beta_\varepsilon\big(u_0(x)\big)\zeta(x,0)\,dx=0$$

for all $\zeta\in C^1(\bar\Omega\times[0,T])$ with $\zeta=0$ on $(\partial\Omega-\Gamma)\times\{t=T\}$. Furthermore

$$\|u^\varepsilon\|_{L^\infty(\Omega\times(0,T])},\ \|\nabla u^\varepsilon\|_{L^2(\Omega\times(0,T))}<C,$$

where C is independent of ε.

Proof. The assertions follow from [7] and [8]. □

In order to simplify our approach we start with a normalized situation in a cylindrical domain Q_k^1 (defined below) assuming that the oscillation, $\operatorname{osc}(u^\varepsilon) \leqslant 1$ and we obtain in the interior of the cylinder a decay in the oscillation $\operatorname{osc}(u^\varepsilon) \leqslant 1-\gamma$ for some $0<\gamma<1$ independent of ε. Then we rescale and repeat. But we are confronted here with two conflicting rescalings. Inside the cylinder, the heat equation implies parabolic rescaling. On the boundary, in principle, hyperbolic scaling is required. It turns out that hyperbolic rescaling is the right one. Although the time derivative of the heat equation disappears in this rescaling, we still obtain the continuity of solutions. This is not surprising in view of the recent paper [4] where there is no time derivative in Ω. Therefore we shall normalize our approximate solution u^ε, i.e., we take $0<u^\varepsilon<1$ in appropriate rectangular cylinders whose one side lies on $\Gamma \subset \mathbb{R}^N$ and the side normal to Γ is small compared to the others. More precisely, we set $B_R' := (-R,R)^N \subset \mathbb{R}^N$, $Q_R' := B_R' \times (-R,0]$, $B_R := B_R' \times (0,1)$, and $Q_R := Q_R' \times (0,1)$ where $R \geqslant 2(N+7)\log 2$. Notice that because of this normalization $\beta(0)$ is not necessarily zero any more.

Before we state our first lemma we define two "comparison" functions and the parabolic Poisson kernel $H(y)(x,t)$, which we use in our proofs. The first one is precisely the one used in [4], i.e.,

Elliptic barrier, b:

$$b(x,y) := 2\cos y e^{-x}, \quad x,y \in \mathbb{R}.$$

This function is positive harmonic in $\{x>0, 0<y<1\}$ and bounded therein by $2e^{-x}$. Also, b is larger than one on $\{x=0, 0\leqslant y \leqslant 1\}$ and positive on $\{x\in\mathbb{R}\colon\ y=0, y=1\}$.

The second one is a parabolic variation of the one in [4]:

Parabolic barrier, a:

$$a(x,y,t) := 2^{N+1}\prod_{i=1}^{N}\cos x_i \cos y e^{-t}, \quad x=(x_1,x_2,\ldots,x_N),\ y,t\in\mathbb{R}.$$

This function is a positive supercaloric in $(-1,1)^N \times (0,1) \times (0,\infty)$ and bounded therein by $2^{N+1}e^{-t}$. On the bottom of this domain, i.e., on $(-1,1)^N \times (0,1) \times \{0\}$, $a(x,y,0)$ is larger than one and is positive on the rest of the parabolic boundary.

Finally by parabolic Poisson kernel $H(y)(x,t)$ we mean that extends by convolution data prescribed in the hyperplane $y=0$, $t>0$, as a caloric function in $y>0$, $t>0$:

$$H(y)(x,t) := \frac{2}{\pi^{\frac{N+1}{2}}}\frac{y}{(4t)^{\frac{N+3}{2}}}e^{-\frac{|x|^2+y^2}{4t}}, \quad x\in\mathbb{R}^N,\ y,t\geqslant 0.$$

Observe that

$$\big\|H(y)(x,t)\big\|_{L^\infty(\{y\geqslant 1\})} = \frac{2}{\pi^{\frac{N+1}{2}}}\left(\frac{N+3}{2e}\right)^{\frac{N+3}{2}}, \quad x\in\mathbb{R}^N,\ t\geqslant 0$$

and

$$\big\|H(y)(x,t)\big\|_{L^1(Q_R)} \leqslant \sqrt{\frac{R}{\pi}}.$$

2.1. Part 1: Smallness in average implies uniform decay

With these definitions at hand we pass to the first lemma. It says that given u, $0 \leqslant u \leqslant 1$ in a hypercube in (x, y, t), setting in $(R^{n+1})^+$ against the hyperplane $y = 0$ and going backwards in time from 0 to $-R$, if u is very tiny "most of the time", then in a smaller cube, into the future from $-R$, u goes down from 1 to 7/8.

Lemma 2.2. *Let $Q_R \subset Q := \Omega \times (-T, T]$ where $Q_R := B_R \times (-R, 0]$, $B_R := B'_R \times (0, 1)$, $B'_R := \{(x_1, \dots, x_N)\colon |x_i| < R,\ i = 1, \dots, N\}$ and $Q'_R : B'_R \times (-R, 0]$. Suppose that*

$$0 < u^\varepsilon < 1$$

in Q_R then there exists a constant $\sigma > 0$ independent of ε such that

$$\fint_{Q'_R} u^\varepsilon \, dx\, dt + \fint_{Q_R} \left(u^\varepsilon\right)^2 dx\, dy\, dt < \sigma$$

implies that

$$u^\varepsilon \leqslant \frac{7}{8}$$

in $Q_{R/8} := B'_{R/8} \times (0, \frac{1}{8}) \times (-\frac{R}{8}, 0]$.

Proof. Step 1 – Energy inequality.
We start by developing the necessary energy inequalities associated to the structure of these equations. We assume that β_ε are smooth approximations to β satisfying $\beta'_\varepsilon \geqslant c_1 > 0$, and β_ε locally bounded on $\mathbb{R}$. For simplicity, we drop the ε subscript then, in the interior and on $\mathbb{R}^N$, u satisfies

$$\begin{aligned} \Delta u - \alpha u_t &= 0 \quad \text{in } \Omega \times (-T, T], \\ -u_\nu &= \beta'(u) u_t \quad \text{on } \Gamma \times (-T, T]. \end{aligned}$$

Choose a smooth cutoff function ζ vanishing near the parabolic boundary of Q_R, i.e., the lateral sides and the bottom except that of Q'_R and $k \geqslant 0$. We multiply the above equations by $\zeta^2 (u - k)^+$ and integrate by parts to get

$$\begin{aligned} &\alpha \int_{Q_R} \zeta^2 (u - k)^+ u_t \, dx\, dy\, dt + \int_{Q'_R} \zeta^2 (u - k)^+ \beta'(u) u_t \, dx\, dt + \int_{Q_R} \zeta^2 \left|\nabla (u - k)^+\right|^2 dx\, dy\, dt \\ &\quad = -2 \int_{Q_R} \left(\zeta \nabla \zeta (u - k)^+ \nabla u\right) dx\, dy\, dt. \end{aligned}$$

Define

$$B\big((u-k)^+\big)=\int_k^u \beta'(s)(s-k)\,ds=\int_0^{(u-k)^+}\beta'(k+\tau)\tau\,d\tau$$

and inserting it in the above we have

$$\begin{aligned}
&\frac{\alpha}{2}\int_{Q_R}\big(\zeta^2\big[(u-k)^+\big]^2\big)_t\,dx\,dy\,dt+\int_{Q'_R}\big(\zeta^2B\big((u-k)^+\big)\big)_t\,dx\,dt\\
&\qquad+\int_{Q_R}\big|\nabla\big(\zeta(u-k)^+\big)\big|^2\,dx\,dy\,dt\\
&\quad=+\alpha\int_{Q_R}\big[(u-k)^+\big]^2\zeta\zeta_t\,dx\,dy\,dt+2\int_{Q'_R}B\big((u-k)^+\big)\zeta\zeta_t\,dx\,dt\\
&\qquad+\int_{Q}\big[(u-k)^+\big]^2|\nabla\zeta|^2\,dx\,dy\,dt.
\end{aligned}$$

Since, by the properties of β,

$$B\big((u-k)^+\big)\geqslant c_1\int_0^{(u-k)^+}\tau\,d\tau=\frac{c_1}{2}\big[(u-k)^+\big]^2,$$

$$B\big((u-k)^+\big)\leqslant (u-k)^+\int_0^{(u-k)^+}\beta'(k+\tau)\,d\tau\leqslant\big(\beta(1)-\beta(0)\big)(u-k)^+$$

and by replacing $t=0$ with any $-R<t\leqslant 0$ as the upper limit of integration, we obtain by standard estimates

$$\begin{aligned}
&\frac{c_1}{2}\max_{-R\leqslant t\leqslant 0}\int_{B'_R}\big[(u-k)^+\zeta\big]^2\,dx\\
&\qquad+\frac{\alpha}{2}\max_{-R\leqslant t\leqslant 0}\int_{B_R}\big|(u-k)^+\zeta\big|^2\,dx\,dy+\int_{Q_R}\big|\nabla\big((u-k)^+\zeta\big)\big|^2\,dx\,dy\,dt\\
&\quad\leqslant 2\big(\beta(1)-\beta(0)\big)\int_{Q'_R}(u-k)^+|\zeta_t|\,dx\,dt+\int_{Q_R}\big[(u-k)^+\big]^2\big(\alpha|\zeta_t|+|\nabla\zeta|^2\big)\,dx\,dy\,dt.
\end{aligned}$$

Since $0\leqslant\alpha\leqslant 1$ and the second term above is nonnegative we have

$$\max_{-R<t\leqslant 0}\int_{B_R}\big|(u-k)^+\zeta\big|^2\,dx+\int_{Q_R}\big|\nabla\big(\big(u-k^+\big)\zeta\big)\big|^2\,dx\,dy\,dt$$
$$\leqslant C\bigg(\int_{Q'_R}(u-k)^+|\zeta_t|\,dx\,dt+\int\big[(u-k)^+\big]^2\big(|\zeta_t|+|\nabla\zeta|^2\big)\,dx\,dy\,dt\bigg) \tag{2.2}$$

where $C=\frac{2}{c_1}\max\{2(\beta(1)-\beta(0)),1\}$.

Step 2 – DeGiorgi type iteration.

Now that we have our energy inequality (2.2), we propose to obtain an iterative sequence of inequalities. We distinguish two cases: $\alpha=0$ and $0<\alpha\leqslant 1$. In both cases and in particular $\alpha=0$ it follows the general lines of Lemma 6 in [4]. We work in detail case $0<\alpha\leqslant 1$. For simplicity in this case we can take $\alpha=1$ in α-heat equation without effecting our estimates; as a matter of fact they are improving as α is getting smaller. Note that α does not appear in the energy equation (2.2).

We recall that the method consists in taking a sequence of decreasing cut offs in space and time ζ_m that converge to the indicator function of $Q_{R/4}$, and simultaneously a series of cut offs of the graph of u, u_m that converge to $(u-7/8)^+$ and prove by iteration that in the limit $\lim_{Q_{R/2}}(u-7/8)^+\equiv 0_-$. In this proof we follow closely the corresponding argument in [4].

To this end we define for $m=0,1,2,3,\dots,$

$$k_m:=\frac{9}{16}+\frac{1}{16}\big(1-2^{-m}\big),\qquad R_m:=\frac{R}{4}\bigg(1+\frac{1}{2^m}\bigg),$$
$$Q'_m:=\big\{(x_1,\dots,x_N,t)\colon -R_m\leqslant x_i\leqslant R_m,\ -R_m\leqslant t\leqslant 0\big\}$$

and we choose the cutoff functions ζ_m to depend only on x and t such that

$$\chi_{Q'_{m+1}}\leqslant\zeta_m\leqslant\chi_{Q'_m},$$
$$|\nabla\zeta_m|\leqslant C2^m,\qquad \big|(\zeta_m)_t\big|\leqslant C2^m.$$

We set $u_m:=(u-k_m)^+$ and we denote

$$I_m:=\iint(\zeta_m u_m)^2\,dx\,dt+\iiint_0^{\delta^m/2}\big|\nabla(\zeta_m u_m)\big|^2\,dx\,dy\,dt$$

where $0<\delta<1$ is chosen such that

$$2^{N+1}\cdot 2^{-\frac{(N+7)2^{-m-1}}{\delta^m}}\leqslant 2^{-m-7} \tag{2.3}$$

holds. We also choose M to satisfy

$$2^{N+2}M^{-\frac{m}{2}}\big(\delta^{N+1}\big)^{-m-1}\leqslant 2^{-m-6}, \tag{2.4}$$
$$M^{-m}\geqslant C\,4^{m(1+\frac{1}{N})}M^{-(m-3)(1+\frac{1}{N})},\quad m\geqslant 14N. \tag{2.5}$$

Such choices of δ and M are permissible as it is shown in Lemma 7 of [4].

Now, we want to prove simultaneously that for every $m \geqslant 0$

$$I_m \leqslant M^{-m}, \tag{2.6}$$

$$u_m = 0 \quad \text{on } Q'_m \times \left\{\frac{\delta^m}{2}\right\}. \tag{2.7}$$

We prove them, inductively.

Step 2a. We prove in this substep that (2.6) is verified for $0 \leqslant m \leqslant 14N$ and that (2.7) is verified for $m = 0$. Substituting k_m for k, ζ_m for ζ we see that for $0 \leqslant m \leqslant 14N$ if we take σ such that

$$2^{28N}\sigma \leqslant M^{-14N}$$

(2.6) is verified, where we used that $|\nabla\zeta_m|^2 \leqslant C2^{28N}$ for $0 \leqslant m \leqslant 14N$. Now, by maximum principle, we have in $Q_R := B_R \times (0,1) \times (-R, 0]$

$$u \leqslant (u\chi_{Q'_R}) * H(y) + y + a\left(\frac{x}{R}, y, (t+R)\right) + w(x, y)$$

where

$$w(x, y) = \sum_{i=1}^{N}\left\{b(x_i + R, y) + b(-x_i + R, y)\right\}.$$

Now, for $t \geqslant -\frac{R}{2}$ we have

$$a\left(\frac{x}{R}, y, (t+R)\right) \leqslant 2^{N+1}e^{-\frac{R}{2}} \leqslant 2^{N+1}e^{-(N+7)\log 2} = \frac{1}{2^6}$$

for $-\frac{R}{2} \leqslant x_i \leqslant \frac{R}{2}$ for all $i = 1, \ldots, N$,

$$w(x, y) \leqslant 4Ne^{-\frac{R}{2}} \leqslant 4Ne^{-(N+7)\log 2} < \frac{1}{2^5}$$

and

$$\begin{aligned}\left\|u\chi_{Q'_R} * H(y)\right\|_{L^\infty(\{y \geqslant \frac{1}{2}\})} &\leqslant \left\|H(y)\right\|_{L^\infty(\{y \geqslant \frac{1}{2}\})} \int\limits_{Q'_R} u(x,t)\,dx\,dt \\ &\leqslant \frac{2^{N+3}}{\pi^{\frac{N+1}{2}}}\left(\frac{N+3}{2e}\right)^{\frac{N+3}{2}} \left|Q'_R\right| \cdot \sigma < \frac{1}{64}\end{aligned}$$

if we choose σ small enough. Therefore

$$u \leqslant \frac{9}{16} \quad \text{for } y = \frac{1}{2},\ x \in B'_{R/2},\ t \geqslant -\frac{R}{2}.$$

Hence

$$u_0 := \left(u - \frac{9}{16}\right)^+ \leqslant 0 \quad \text{for } y = \frac{1}{2},\ x \in B'_{R/2},\ t \geqslant -\frac{R}{2}.$$

That is

$$\zeta_0 u_0 = 0 \quad \text{on } \partial_p Q_0$$

where $Q_0 := Q'_0 \times [0, \frac{\delta^0}{2}]$.

Step 2b. We assume in this substep that (2.6) and (2.7) hold true for m and we want to show that (2.7) is true for $m+1$. Now, again by maximum principle in Q_m, we have

$$u_m \leqslant \zeta_m u_m * H(y) + a\left(\frac{x}{R_m}, \frac{2y}{\delta^m}, \frac{2(t+R_m)}{\delta^m}\right) + \sum_{i=1}^{N}\left[b\left(\frac{2(x_i+R_m)}{\delta^m}, \frac{2y}{\delta^m}\right) + b\left(\frac{2(-x_i+R_m)}{\delta^m}, \frac{2y}{\delta^m}\right)\right].$$

So in Q_{m+1} we have

$$a \leqslant 2^{N+1} e^{-\frac{2(-R_{m+1}+R_m)}{\delta^m}} = 2^{N+1} e^{-\frac{R\,2^{-m-2}}{\delta^m}} \leqslant 2^{-m-7}$$

thanks to $(*)$ and the third term is bounded by

$$4N e^{-\frac{R 2^{-m-1}}{\delta^m}} \leqslant 2^{-m-7}.$$

By $(**)$ we have for $y = \frac{\delta^{m+1}}{2}$

$$\begin{aligned}\left\|\zeta_m u_m * H(y)\right\| &\leqslant I_m^{1/2}\left\|H(y)\right\|_{L^2(\{y \geqslant \frac{\delta^{m+1}}{2}\})} \\ &\leqslant \frac{2^{N+2} M^{-m/2}}{(\delta^{N+1})^{m+1}}\left\|H(1/2)\right\|_{L^2} \\ &\leqslant 2^{-m-6}.\end{aligned}$$

So in Q_{m+1}

$$u_{m+1} \leqslant \left(u_m - 2^{-m-5}\right)^+$$

or

$$u_{m+1} \leqslant \left(\zeta_m u_m * H(y) - 2^{-m-6}\right)^+,$$

i.e.,

$$\zeta_{m+1} u_{m+1} \leqslant \left(\zeta_m u_m * H(y) - 2^{-m-6}\right)^+.$$

In particular

$$\zeta_{m+1}u_{m+1} \leqslant \big(\zeta_m u_m * H(y)\big)\zeta_{m+1}. \tag{2.8}$$

Therefore

$$\zeta_{m+1}u_{m+1} = 0 \quad \text{on } \partial_p Q_{m+1}$$

where $Q_m := Q'_m \times [0, \frac{\delta^m}{2}]$.

Step 2c. By the previous steps we have that (2.7) is true up to $m = 14N + 1$, (2.6) up to $m = 14N$ and (2.8) up to $m = 14N$. In this step we show that if (2.7) is true for $m - 3$ and (2.6) for $m - 3$, $m - 2$, $m - 1$ then (2.6) is true for m. Since by Step 2 (2.7) is also true for $m - 2$, $m - 1$, m we only have to show that

$$I_m \leqslant C4^{m(1+\frac{1}{N})} I_{m-3}^{1+\frac{1}{N}}, \quad m \geqslant 14N + 1.$$

For, by (2.2)

$$I_m \leqslant C2^m \int \zeta_{m-1}u_m \,dx\,dt + \big(C2^m\big)^2 \int (\zeta_{m-1}u_m)^2 \,dx\,dy\,dt.$$

Since $u_m < u_{m-1}$ and $\{u_m \neq 0\} = \{u_{m-1} > 2^{-m-4}\}$ the integral of the first term on the right is bounded by

$$\begin{aligned}
&\frac{1}{2}\int (\zeta_{m-1}u_m)^2 \,dx\,dt + \frac{1}{2}\big|\{u_m \neq 0\} \cap Q'_{m-1}\big| \\
&\quad \leqslant \frac{1}{2}\int (\zeta_{m-1}u_{m-1})^2 \,dx\,dt + 2^{m+3}\int (\zeta_{m-1}u_{m-1})^2 \,dx\,dt \\
&\quad \leqslant \frac{1}{2}\big(1 + 2^{m+4}\big)\int (\zeta_{m-1}u_{m-1})^2 \,dx\,dt.
\end{aligned}$$

By (2.8) the integral of the second term above is bounded by

$$\int \big|\zeta_{m-2}u_{m-2} * H(y)\big|^2 \,dx\,dy\,dt \leqslant \|H\|^2_{L^1(Q_R)} \int (\zeta_{m-2}u_{m-2})^2 \,dx\,dt.$$

Therefore

$$\begin{aligned}
I_m &\leqslant C4^m \int (\zeta_{m-2}u_{m-2})^2 \,dx\,dt \\
&\leqslant C4^m \bigg(\int (\zeta_{m-2}u_{m-2})^{2\cdot\frac{N+1}{N}} \,dx\,dt\bigg)^{\frac{N}{N+1}} \cdot \big|\{u_{m-2} \neq 0\} \cap Q'_{m-2}\big|^{\frac{1}{N+1}} \\
&\leqslant C4^{m(1+\frac{1}{N})} \int (\zeta_{m-3}u_{m-3})^{2\cdot\frac{N+1}{N}} \,dx\,dt.
\end{aligned}$$

By Sobolev's inequality

$$I_m \leqslant C4^{m(1+\frac{1}{N})}\left(\int (\zeta_{m-3}u_{m-3})^2\,dx\,dt + \int \left|\Lambda^{1/2}(\zeta_{m-3}u_{m-3})\right|^2 dx\,dt\right)^{\frac{N+1}{N}}$$

where $\Lambda(\zeta_{m-3}u_{m-3}) = -\frac{\partial}{\partial y}(\zeta_{m-3}u_{m-3})$. Since

$$\int \left|\Lambda^{1/2}(\zeta_{m-3}u_{m-3})\right|^2 dx\,dt \leqslant \int \left|\nabla(\zeta_{m-3}u_{m-3})\right|^2 dx\,dy\,dt$$

we have

$$I_m \leqslant C4^{m(1+\frac{1}{N})} I_{m-3}^{1+\frac{1}{N}}, \quad m \geqslant 14N+1,$$

i.e., $I_m \to 0$ as $m \to \infty$ provided

$$I_0 \leqslant C^{-N}4^{-N(N+1)} = \frac{1}{4^{N(N+1)}}\left(\frac{c_1}{2\max\{2(\beta(1)-\beta(0)),1\}}\right)^N =: \sigma.$$

To complete the proof of our lemma consider the function v defined by

$$\begin{aligned} &\Delta v - v_t = 0 && \text{in } Q_{R/4},\\ &v = 1 && \text{on } \partial_p \bar{Q}_{R/4} \setminus \{y=0\},\\ &v = \tfrac{5}{8} && \text{on } Q'_{R/4}. \end{aligned}$$

Then $v < \frac{7}{8}$ in $Q_{R/8}$ and by maximum principle $u \leqslant v$. $\square$

Our next result, Lemma 2.4, relies on a "parabolic" version of DeGiorgi's isoperimetric lemma. This lemma is proved in [4] and with minor adjustments applies to our situation. We state it as our next lemma, Lemma 2.3.

Lemma 2.3. *Given $\sigma_1 > 0$ there exists a $\delta_1 > 0$ such that for every subsolution u^ε to* (2.1) *with $\beta'_\varepsilon \leqslant C$ satisfying*

$$0 < u^\varepsilon < 1 \quad \text{in } Q_R,$$
$$\left|\left\{(x,y,t) \in Q_R\colon u^\varepsilon = 0\right\}\right| \geqslant \sigma_1 |Q_R|$$

if

$$\left|\left\{(x,y,t) \in Q_R\colon 0 < u^\varepsilon < \frac{1}{2}\right\}\right| < \delta_1 |Q_R|$$

then

$$\fint_{Q'_{R/4}} \left(u^\varepsilon - \frac{1}{2}\right)^+ dx\,dt + \fint_{Q_{R/4}} \left[\left(u^\varepsilon - \frac{1}{2}\right)^+\right]^2 dx\,dy\,dt \leqslant C\sqrt{\sigma_1}$$

where C depends on the bound of β'_ε but not on the "ε".

In order to complete the proof of oscillation decay we have to consider two alternatives. One is when u is, on average, very close to the singular value of β and the second when it is far from it. In the next lemma we handle the second more delicate alternative situation to Lemma 2.2.

Lemma 2.4. *Let Q_R and σ be as in Lemma* 2.2 *and*

$$0 < u^\varepsilon < 1 \quad \textit{in } Q_R$$

a solution to (2.1) *with $\beta'_\varepsilon(x) < C$, C independent of ε for $x < 1/4$. Then, if*

$$⨍_{Q'_R} u^\varepsilon \, dx\, dt + ⨍_{Q_R} (u^\varepsilon)^+ \, dx\, dy\, dt \geqslant \sigma, \tag{2.9}$$

$u^\varepsilon \geqslant C\sigma$ for every $(x, y, t) \in Q_{R/32}$.

Proof. For simplicity again we drop the "ε". Now, if (2.9) holds then it follows that

$$\left|\left\{u > \frac{\sigma}{4}\right\} \cap Q_R\right| \geqslant c_0\sigma|Q_R|$$

for some $c_0 < 1$. Therefore we define

$$w := \frac{4}{\sigma}\left(u - \frac{\sigma}{4}\right)^-$$

and we observe that w is a subsolution to problem (2.1). Following DeGiorgi's method we will consider a dyadic sequence of normalized truncations, i.e.,

$$w_k := 2^k\bigl(w - \bigl(1 - 2^{-k}\bigr)\bigr)^+$$

still subsolutions to (2.1). We will show that in a finite number of steps $k_0 = k_0(\delta_1)$ (where δ_1 is defined in Lemma 2.3 with $C\sqrt{\sigma_1} \leqslant \sigma$) that

$$\bigl|\{w_{k_0} > 0\}\bigr| = 0.$$

Note that for every k, $0 \leqslant w_k \leqslant 1$ and $|\{w_k = 0\} \cap Q_R| \geqslant \sigma_1|Q_R|$. Assume, now, that for every k $|\{0 < w_k < \frac{1}{2}\} \cap Q_R| \geqslant \delta_1|Q_R|$. Then for every k

$$\bigl|\{w_k = 0\}\bigr| = \bigl|\{w_{k-1} = 0\}\bigr| + \left|\left\{0 < w_{k-1} < \frac{1}{2}\right\}\right| \geqslant \bigl|\{w_{k-1} = 0\}\bigr| + \delta_1|Q_R|.$$

Hence after a finite number of steps, say $k_0 \geqslant 1/\delta$,

$$\bigl|\{w_{k_0} = 0\}\bigr| \geqslant |Q_R|.$$

Therefore

$$w_{k_0} < 0$$

or

$$2^{k_0}\big(w - \big(1 - 2^{-k_0}\big)^+\big) = 0,$$

i.e.,

$$w < 1 - 2^{-k_0}.$$

Suppose, now, that there exists k', $0 \leqslant k' \leqslant k_0$ such that

$$\left|\left\{0 < w_{k'} < \frac{1}{2}\right\}\right| < \delta_1.$$

By Lemma 2.3 applied to $w_{k'}$ and consequently by Lemma 2.2 applied to $w_{k'+1}$ we have

$$w_{k'+1} \leqslant \frac{7}{8}$$

in $Q_{R/32}$, i.e.,

$$w < 1 - \frac{1}{8} \cdot 2^{-(k'+1)}.$$

A fortiori, in both cases we have

$$w < 1 - 2^{-(k_0+4)} \quad \text{in } Q_{R/32}$$

that is

$$u \geqslant 2^{-k_0-5}\sigma$$

in $Q_{R/32}$. □

We conclude this section by proving our normalized oscillations decay. Lemma 2.5 below encompasses both alternatives.

Lemma 2.5. *Let u^ε be a solution to* (2.1) *with*

$$0 < u^\varepsilon < 1 \quad \textit{in } Q_R$$

and suppose that $\beta_\varepsilon'(x) \leqslant C$ (C independent of ε) for $x \leqslant \frac{1}{4}$ or $x \geqslant \frac{3}{4}$. Then

$$\operatorname*{osc}_{Q_{R/32}} u \leqslant 1 - C\sigma.$$

Remark. Depending if the singularity of β falls above or below $1/2$ one of the alternatives holds.

Proof. If u is close in measure of order σ to zero or to one then by Lemma 2.2 applied to u or to $1-u$ we obtain

$$\operatorname*{osc}_{Q_{R/8}} u \leqslant \frac{7}{8}.$$

If not then by Lemma 2.4 applied to u or $1-u$ we obtain

$$\operatorname*{osc}_{Q_{R/32}} u \leqslant 1 - C\sigma$$

provided that $\frac{1}{8} \leqslant C\sigma < 1$. □

2.2. Part 2: Oscillation decay: Iteration

The estimates we obtained in the previous section, apart that they are independent of "ε", are, also, independent of the "α", the coefficient to u_t in the equation. This allows us to scale hyperbolically without effecting the estimates and consequently we obtain a modulus of continuity.

We would like now to iterate the lemmas above to force the oscillation of u to decrease to zero along this in a dyadic sequence of decreasing hypercubes to obtain continuity of u. Since the estimates at hand will deteriorate as β'_ε goes to infinity, our modulus will not be Hölder, except in case (ii) where we have an extra rescaling invariance.

Proposition 2.6. *Let u^ε be a solution to problem* (2.1) *in Q_R. Suppose that*

$$\beta_\varepsilon\Big(\sup_{Q_R} u^\varepsilon\Big) - \beta_\varepsilon\Big(\inf_{Q_R} u^\varepsilon\Big) \leqslant K \quad \text{and} \quad \inf_{Q_R} \beta'_\varepsilon \geqslant \delta > 0$$

where K and δ are independent of ε. Then

$$\big|u^\varepsilon(x,y,t) - u^\varepsilon(0,0,0)\big| \leqslant \omega\big(|x|,|y|,|t|\big)$$

where ω is a modulus of continuity (i.e., ω monotone and $\omega(0)=0$) depending only on K and δ.

Proof. We drop again "ε" from our notation. Set

$$Q_k := Q_{R/(32)^k} = \left(-\frac{R}{(32)^k}, \frac{R}{(32)^k}\right) \times \left(0, \frac{1}{(32)^k}\right) \times \left(-\frac{R}{(32)^k}, 0\right]$$

and $m_k := \inf_{Q_k} u$, $M_k := \sup_{Q_k} u$.

Define

$$v := \frac{u_k - m_k}{M_k - m_k}$$

where $u_k(x,y,t) := u(\frac{x}{(32)^k}, \frac{y}{(32)^k}, \frac{t}{(32)^k})$. Then v verifies

$$\Delta v - \frac{\alpha}{(32)^k} v_t = \quad \text{in } Q_R,$$
$$-v_\nu = \bar{\beta}'(v) v_t \quad \text{on } Q'_R$$

where $\bar{\beta}(v) = \frac{1}{(M_k - m_k)} \beta((M_k - m_k)v + m_k)$. Now we apply Lemma 2.5 to v to obtain

$$\operatorname*{osc}_{Q_{R/32}} v \leqslant (1 - C\sigma)$$

where $\sigma := \frac{\inf_{Q_R} \bar{\beta}(v)}{\bar{\beta}(1) - \bar{\beta}(0)}$. Hence, in our original setting we have

$$\operatorname*{osc}_{Q_{k+1}} u \leqslant \mu_k \operatorname*{osc}_{Q_k} u$$

where $\mu_k := (1 - \frac{C\delta}{K} \operatorname{osc}_{Q_k} u)$. We see, therefore, that $\mu_k \underset{k\to\infty}{\longrightarrow} 1$ only when $\operatorname{osc}_{Q_k} u \underset{k\to\infty}{\longrightarrow} 0$ which yields our modulus of continuity. □

As mentioned in the Introduction with the additional assumption on β of case (ii), which, of course, includes the porous media case, i.e., $\beta(u) \sim u^{1/m}$, $m > 1$, we can achieve a Hölder modulus of continuity. This was achieved by a different approach for the porous media equation in [2].

Proposition 2.7. *Let u^ε be a solution to problem* (2.1) *in Q_R with β_ε being as the one in case* (ii). *Suppose that for any $m < M$*

$$\frac{(\inf_{[m,M]} \beta'_\varepsilon) \cdot (M - m)}{\beta(M) - \beta(m)} \geqslant \ell$$

where ℓ is a positive constant independent of ε then

$$\left| u^\varepsilon(x, y, t) - u^\varepsilon(0, 0, 0) \right| \leqslant C \big(|x| + |y| + |t| \big)^\gamma$$

where $\gamma = \gamma(\ell)$.

Proof. As in the proof of the preceding proposition we arrive at

$$\operatorname*{osc}_{Q_{k+1}} u \leqslant (1 - C\ell) \operatorname*{osc}_{Q_k} u$$

or

$$\operatorname*{osc}_{Q_k} u \leqslant (1 - C\ell)^k \operatorname*{osc}_{Q_R} u. \qquad \square$$

Theorem 2.8. *Let u be solution to* (1.1) *with β satisfying* (1.2) *or* (1.3) *then u is continuous with a modulus depending on the nature of the singularity of β.*

Proof. By Propositions 2.1 and 2.6, or Proposition 2.7, we can extract a subsequence u^{ε_m} which, by standard methods, converges uniformly to our solution u. □

3. Fractional diffusion case

The purpose of this section is to show how our methods of Section 2 can be generalized to yield continuity of the solutions to problems (1.4) and (1.5).

According to an extension theorem of [3] problem (1.4) is equivalent to problem (1.5) when we set $\alpha = 0$. We point out that the adaptation of the methods of [4] to general fractional diffusion ($\gamma > 0$) was carried out by Constantin and Wu [5]. Therefore it is enough to treat only problem (1.5). We approximate again the β by smooth β_ε and we note that the bounds on the L^∞ norm of u^ε and the L^2 norm of Du^ε can be obtained by standard methods independently of $\varepsilon > 0$. More precisely, we consider the problem

$$\begin{cases} \dfrac{1}{y^\gamma}\nabla\big(y^\gamma \nabla u^\varepsilon(x,y,t)\big) - \alpha u_t^\varepsilon(x,y,t) = 0, & (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^+ \times (0,\infty), \\ + \lim\limits_{y \to o^+} y^\gamma u_y^\varepsilon(x,y,t) = \beta_\varepsilon'\big(u^\varepsilon(x,0,t)\big)u_t^\varepsilon(x,0,t), & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u^\varepsilon(x,y,t) \underset{|x|\to\infty}{\longrightarrow} 0, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u^\varepsilon(x,y,0) = u_0^\varepsilon(x,y), & (x,y) \in \mathbb{R}^N \times \mathbb{R}^+, \end{cases} \tag{3.1}$$

where $\gamma \in (-1,1)$ ($\gamma = 1 - 2\delta$) and β_ε is the one of Section 2.

We shall use, in the proofs which will follow, again two comparison functions and the fractional parabolic Poisson kernel. That is, we consider a function $b^{(\gamma)}(x,y)$ defined by

$$\begin{aligned} & b_{xx}^{(\gamma)} + b_{yy}^{(\gamma)} + \frac{\gamma}{y} b_y^{(\gamma)} = 0, && \text{in } (0,\infty) \times (0,1), \\ & b^{(\gamma)}(0,y) = 1, && 0 \leqslant y \leqslant 1, \\ & b^{(\gamma)}(x,0) = b^{(\gamma)}(x,1) = 0, && 0 < x < +\infty. \end{aligned}$$

Then there exists a universal constant $C_\gamma < 1$ such that

$$\big|b^{(\gamma)}(x,y)\big| \leqslant C_\gamma e^{-\frac{\sqrt{(1-\gamma)(5-\gamma)}}{2}x}.$$

As a matter of fact, by the method of separation of variables

$$b^{(\gamma)}(x,y) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n x} \left(\frac{y}{1-\gamma}\right)^{\frac{1-\gamma}{2}} J_{\frac{1-\gamma}{2}}(\lambda_n y)$$

where $J_{\frac{1-\gamma}{2}}$ is the Bessel function of the first kind of order $\frac{1-\gamma}{2}$ and c_n the corresponding Fourier coefficients. It is well known (see [9, p. 485]) that $\lambda_1 > \frac{\sqrt{(1-\gamma)(5-\gamma)}}{2}$.

The second function is

$$a^{(\gamma)}(x,y,t) := 2^N \prod_{i=1}^{N} \cos x_i e^{-t}, \quad x = (x_1,\dots,x_N),\ t \geqslant 0$$

which is positive supersolution

$$\frac{1}{y^\gamma}\nabla\big(y^\gamma \nabla a^{(\gamma)}\big) - a_t^{(\gamma)} \leqslant 0$$

in $(-1,1)^N \times (0,1) \times (0,\infty)$. On the bottom of this domain, i.e., on $(-1,1)^N \times (0,1) \times \{0\}$ $a^{(\gamma)}(x,y,0)$ is larger than one and positive on the rest of its parabolic boundary.

Finally by fractional parabolic Poisson kernel we mean

$$H^{(\gamma)}(y)(x,t) := \frac{2}{\pi^{\frac{N+1-\gamma}{2}}} \frac{y^{1-\gamma}}{(4t)^{\frac{N+3-\gamma}{2}}} e^{-\frac{|x|^2+y^2}{4t}}, \quad x \in \mathbb{R}^N,\ y,t \geqslant 0.$$

Observe that

$$\big\|H^{(\gamma)}(y)(x,t)\big\|_{L^\infty(\{y\geqslant 1\})} = \frac{2}{\pi^{\frac{N+1-\gamma}{2}}} \cdot \left(\frac{N+3-\gamma}{2e}\right)^{\frac{N+3-\gamma}{2}}$$

and

$$\big\|H^{(\gamma)}(y)(x,t)\big\|_{L^1(Q_R)} \leqslant \frac{2}{2-\gamma}\sqrt{\frac{R}{\pi^{1-\gamma}}}.$$

Lemma 3.1. *Let u^ε be a solution to* (3.1) *with*

$$0 < u^\varepsilon < 1$$

in Q_R with $R \geqslant \frac{2(N+7)\log 2}{\sqrt{1-\gamma}}$. Then there exist a $\sigma > 0$ and a $0 < \lambda < 1$ independent of ε such that

$$\fint_{Q'_R} u^\varepsilon\,dx\,dt + \fint_{Q_R} y^\gamma \big(u^\varepsilon\big)^2\,dx\,dy\,dt < \sigma$$

implies that

$$u^\varepsilon \leqslant 1-\lambda$$

in $Q_{R/8} := B'_{R/8} \times (0,\frac{1}{8}) \times (-\frac{R}{8},0]$.

Proof. We follow the steps of Lemma 2.2 and we arrive at the estimate

$$\begin{aligned}
&\max_{-R\leqslant t\leqslant 0} \int_{B'_R} \big|\zeta(u-k)^+\big|^2\,dx + \int_{Q_R} y^\gamma \big|\nabla\big(\zeta(u-k)^+\big)\big|^2\,dx\,dy\,dt \\
&\quad\leqslant C\bigg(\int_{Q'_R} |\zeta_t|(u-k)^+\,dx\,dt + \int_{Q_R} \big(|\zeta_t| + |\nabla\zeta|^2\big) y^\gamma \big|(u-k)^+\big|^2\,dx\,dy\,dt\bigg) \qquad (3.2)
\end{aligned}$$

where

$$C := \frac{\max(2(\beta(1) - \beta(0)), 1)}{\min(\frac{c_1}{2}, 1)}.$$

Again, we distinguish two cases: $\alpha = 0$ and $0 < \alpha \leqslant 1$. Both cases can be treated as in Lemma 6 of [4]. In particular $\alpha = 0$ has been worked out in Proposition 3.3 of [5]. We treat here the case $0 < \alpha \leqslant 1$. For simplicity, as before, we take $\alpha = 1$ and we obtain an iterative sequence of inequalities with

$$k_m := 1 - \frac{\lambda^*}{2}\big(1 + 2^{-m}\big), \qquad R_m := \frac{R}{4}\left(1 + \frac{1}{2^m}\right)$$

for $m = 0, 1, 2, \ldots$, where $0 < \lambda^* < 1$ a constant to be defined below.

The cutoff function ζ_m depends on x and t only with

$$\chi_{Q'_{m+1}} \leqslant \zeta_m \leqslant \chi_{Q'_m}, \qquad |\nabla \zeta_m| \leqslant C2^m, \qquad \big|(\zeta_m)_t\big| \leqslant C2^m$$

where $Q'_m := \{(x_1, \ldots, x_N, 0, t)\colon\ -R_m \leqslant x_i \leqslant R_m,\ -R_m \leqslant t \leqslant 0,\ i = 1, \ldots, N\}$ and $u_m := (u - k_m)^+$. Since the second integral differs, our new I_m is defined as:

$$I_m := \iint (\zeta_m u_m)^2\, dx\, dt + \iiint_0^{\delta^m/2} y^\gamma \big|\nabla(\zeta_m u_m)\big|^2\, dx\, dy\, dt$$

where $0 < \delta < 1$ is chosen such that

$$2^N 2^{-\frac{(N+7)2^{-m-1}}{\delta^m\sqrt{1-\gamma}}} \leqslant \lambda^* 2^{-m-4} \tag{3.3}$$

holds, and we choose M to satisfy

$$M^{-m/2}\big\|H^{(\gamma)}(1/2)\big\|_{L^2}\big(\delta^{N+5}\big)^{-m-1} \leqslant \lambda^* 2^{-m-3}, \tag{3.4}$$

$$M^{-m} \geqslant C\big(\lambda^*\big)^{-\frac{2(1-\gamma)}{N}} 4^{m(1+\frac{1-\gamma}{N})} M^{-(m-3)(1+\frac{1-\gamma}{N})}, \quad m \geqslant 14N. \tag{3.5}$$

We want, again, to prove simultaneously that for every m

$$I_m \leqslant M^{-m} \tag{3.6}$$

and

$$\zeta_m u_m = 0 \quad \text{for } y = \frac{\delta^m}{2}. \tag{3.7}$$

We prove them inductively.

Step 1. In this step we prove that (3.6) is verified for $0 \leqslant m \leqslant 14N$ and that (3.7) is verified for $m = 0$. Substituting in (3.2) k_m for k, ζ_m for ζ we see that for $0 \leqslant m \leqslant 14N$, if we take σ such that

$$2^{28N}\sigma \leqslant M^{-14N}$$

(3.6) is verified. Now by the maximum principle we have in $Q_R := B_R \times (0,1) \times (-R, 0]$

$$u \leqslant (u\chi_{Q'}) * H^{(\gamma)}(y) + y^{1-\gamma} + a^{(\gamma)}\left(\frac{x}{R}, y, t+R\right) + w^{(\gamma)}(x,y)$$

where

$$w^{(\gamma)}(x,y) := \sum_{i=1}^{N}\bigl\{b^{(\gamma)}(x_i + R, y) + b^{(\gamma)}(-x_i + R, y)\bigr\}.$$

Now, for $t \geqslant -\frac{R}{2}$ we have

$$a^{(\gamma)}\left(\frac{x}{R}, y, t+R\right) \leqslant 2^N e^{-\frac{(N+7)\log 2}{\sqrt{1-\gamma}}} \leqslant \frac{1}{2^{\frac{7}{\sqrt{1-\gamma}}}}$$

for $-\frac{R}{2} \leqslant x_i \leqslant \frac{R}{2}$, $i = 1, \dots, N$,

$$w^{(\gamma)}(x,y) \leqslant 2NC_\gamma e^{-\frac{(N+7)\log 2}{\sqrt{1-\gamma}}} \leqslant 2^{N+1} 2^{-\frac{N+7}{\sqrt{1-\gamma}}} \leqslant \frac{1}{2^{\frac{6}{\sqrt{1-\gamma}}}}$$

and

$$\begin{aligned}
\bigl\|(u\chi_{Q'_R}) * H^{(\gamma)}(y)\bigr\|_{L^\infty(\{y \geqslant \frac{1}{2}\})} &\leqslant \bigl\|H^{(\gamma)}(y)\bigr\|_{L^\infty(\{y \geqslant \frac{1}{2}\})} \int_{Q'_R} u(x,y)\,dx\,dy \\
&\leqslant \frac{2^{N+3}}{\pi^{\frac{N+1-\gamma}{2}}}\left(\frac{N+3-\gamma}{2e}\right)^{\frac{N+3-\gamma}{2}} \bigl|Q'_R\bigr|\sigma < \frac{1}{2^{\frac{7}{\sqrt{1-\gamma}}}}
\end{aligned}$$

if we choose σ small enough. Now, we define λ^* by

$$\lambda^* := 1 - \left(\frac{1}{2^{\frac{5}{\sqrt{1-\gamma}}}} + \frac{1}{2^{1-\gamma}}\right)$$

and $0 < \lambda^* < 1$ when $\gamma < 1$. Therefore

$$u \leqslant 1 - \lambda^* \quad \text{for } y = \frac{1}{2},\ x \in B'_{R/2},\ t \geqslant -\frac{r}{2}.$$

Hence

$$u_0 := \bigl(u - \bigl(1 - \lambda^*\bigr)\bigr)^+ = 0 \quad \text{for } y = \frac{1}{2},\ x \in B'_{R/2},\ t \geqslant -\frac{R}{2},$$

i.e.,

$$\zeta_0 u_0 = 0 \quad \text{on } \partial_p Q_0$$

where $Q_0 := Q_0' \times [0, \frac{\delta^0}{2}]$.

Step 2. We assume that (3.3) and (3.4) hold true for m and we want to show that (3.4) is true for $m+1$. Again by maximum principle in Q_m we have

$$u_m \leqslant (\zeta_m u_m) * H^{(\gamma)}(y) + a^{(\gamma)}\left(\frac{x}{R_m}, \frac{2y}{\delta^m}, \frac{2(t+R_m)}{\delta^m}\right) + \sum_{i=1}^N \left[b^{(\gamma)}\left(\frac{2(x_i+R_m)}{\delta^m}, \frac{2y}{\delta^m}\right) + b^{(\gamma)}\left(\frac{2(-x_i+R_m)}{\delta^m}, \frac{2y}{\delta^m}\right)\right].$$

So in Q_{m+1} we have

$$a^{(\gamma)} \leqslant 2^N e^{-\frac{2(-R_{m+1}+R_m)}{\delta^m}} = 2^N 2^{-\frac{R2^{-m-1}}{\delta^n}} \leqslant 2^N e^{-\frac{(N+7)2^{-m-1}}{\delta^m\sqrt{1-\gamma}}} \leqslant \lambda^* 2^{-m-4}$$

thanks to (3.3) and the third term is bounded by

$$2NC_\gamma e^{-\frac{\sqrt{(1-\gamma)(5-\gamma)}R}{\zeta^m 2^{m-3}}} \leqslant \lambda^* 2^{-m-4}.$$

By (3.4) we have for $y = \delta^{m+1}/2$

$$\begin{aligned}\left\|(\zeta_m u_m) * H^{(\gamma)}(y)\right\| &\leqslant I_m^{1/2}\left\|H^{(\gamma)}(y)\right\|_{L^2(\{y \geqslant \frac{\delta^{m+1}}{2}\})} \\ &\leqslant \frac{M^{-m/2}}{(\delta^{N+5})^{m+1}}\left\|H^{(\gamma)}(1/2)\right\|_{L^2} \\ &\leqslant \lambda^* 2^{-m-3}\end{aligned}$$

so in Q_{m+1}

$$u_{m+1} \leqslant \left(u_m - \lambda^* 2^{-m-2}\right)^+$$

or

$$u_{m+1} \leqslant \left(\zeta_m u_m * H^{(\gamma)}(y) - \lambda^* 2^{-m-3}\right)^+,$$

i.e.,

$$\zeta_{m+1} u_{m+1} \leqslant \left(\zeta_m u_m * H^{(\gamma)}(y) - \lambda^* 2^{-m-3}\right)^+.$$

In particular

$$\zeta_{m+1}u_{m+1} \leqslant \big(\zeta_m u_m * H^{(\gamma)}(y)\big)\zeta_{m+1}. \tag{3.8}$$

Therefore

$$\zeta_{m+1}u_{m+1} = 0 \quad \text{on } \partial_p Q_{m+1}$$

where

$$Q_m := Q'_m \times \left[0, \frac{\delta^m}{2}\right].$$

Step 3. So, by the previous steps we have (3.7) true up to $m = 14N + 1$, (3.6) up to $m = 14N$, and (3.8) up to $m = 14N$. We will show here that if (3.7) is true for $m - 3$ and (3.6) for $m - 3$, $m - 2$, $m - 1$ then (3.6) is true for m. Since by Step 2 (3.7) is also true for $m - 2$, $m - 1$, m we only have to show that

$$I_m \leqslant C.$$

By (3.2)

$$I_m \leqslant C2^m \int (\zeta_{m-1}u_m)\,dx\,dt + \big(C2^m\big)^2 \int y^{(\gamma)}(\zeta_{m-1}u_m)^2\,dx\,dy\,dt$$

since $u_m < u_{m-1}$ and $\{u_m \neq 0\} = \{u_{m-1} > \lambda^* 2^{-m-1}\}$ the integral of the first term is bounded by

$$\begin{aligned}
&\frac{1}{2}\int (\zeta_{m-1}u_{m-1})^2\,dx\,dt + \frac{1}{2}\big|\{u_m \neq 0\} \cap Q'_{m-1}\big| \\
&\quad\leqslant \frac{1}{2}\int (\zeta_{m-1}u_{m-1})^2\,dx\,dt + \frac{2^m}{\lambda^*}\int (\zeta_{m-1}u_{m-1})^2\,dx\,dt \\
&\quad= \frac{1}{2}\left(1 + \frac{2^{m-1}}{\lambda^*}\right)\int (\zeta_{m-1}u_{m-1})^2\,dx\,dt.
\end{aligned}$$

By (3.8) the integral of the second term is bounded by

$$\int y^{\gamma}\big((\zeta_{m-2}u_{m-2}) * H^{(\gamma)}(y)\big)^2\,dx\,dy\,dt. \leqslant \big\|H^{(\gamma)}\big\|_{L^1}^2 \int (\zeta_{m-2}u_{m-2})^2\,dx\,dt.$$

Therefore

$$\begin{aligned}
I_m &\leqslant C4^m \int (\zeta_{m-2}u_{m-2})^2\,dx\,dt \\
&\leqslant C4^m \left(\int (\zeta_{m-2}u_{m-2})^{2\cdot\frac{N+1-\gamma}{N}}\,dx\,dt\right)^{\frac{N}{N+1-\gamma}} \cdot \big|\{u_{m-2} \neq 0\} \cap Q'_{m-2}\big|^{\frac{1-\gamma}{N+1-\gamma}} \\
&\leqslant \frac{C4^{m(1+\frac{1-\gamma}{N})}}{(\lambda^*)^{\frac{2(1-\gamma)}{N}}} \cdot \int (\zeta_{m-3}u_{m-3})^{2\cdot\frac{N+1-\gamma}{N}}\,dx\,dt.
\end{aligned}$$

By Sobolev's inequality

$$I_m \leqslant \frac{C4^{m(1+\frac{1-\gamma}{N})}}{(\lambda^*)^{\frac{2(1-\gamma)}{N}}}\left(\int (\zeta_{m-3}u_{m-3})^2\,dx\,dt + \int \left|\Lambda^{\frac{1-\gamma}{2}}(\zeta_{m-3}u_{m-3})\right|^2 dx\,dt\right)^{\frac{N+1-\gamma}{N}}$$

where $\Lambda^{1-\gamma}(\zeta_{m-3}u_{m-3}) := -\lim_{y\to 0^+} y^\gamma \frac{\partial}{\partial y}(\zeta_{m-3}u_{m-3})$. Since

$$\int \left|\Lambda^{\frac{1-\gamma}{2}}(\zeta_{m-3}u_{m-3})\right|^2 dx\,dt \leqslant \int y^\gamma \left|\nabla(\zeta_{m-3}u_{m-3})\right|^2 dx\,dy\,dt$$

we have

$$I_m \leqslant \frac{C4^{m(1+\frac{1-\gamma}{N})}}{(\lambda^*)^{\frac{2(1-\gamma)}{N}}} I_{m-3}^{1+\frac{1-\gamma}{N}} \quad \text{for } m \geqslant 14N+1,$$

i.e., $I_m \to 0$ as $m \to \infty$ provided

$$I_0 \leqslant \frac{C^{-\frac{N}{1-\gamma}}}{(\lambda^*)^{-2}} 4^{-\frac{N}{1-\gamma}(1+\frac{N}{1-\gamma})} = \frac{(\lambda^*)^2}{4^{\frac{N}{1-\gamma}(\frac{N}{1-\gamma}+1)}}\left(\frac{c_1}{2\max(2(\beta_{(1)}-\beta_{(0)}),1)}\right)^{\frac{N}{1-\gamma}} = 2\sigma.$$

Finally, consider the function defined by

$$\begin{aligned}
&\frac{1}{y^\gamma}\nabla\big(y^\gamma\nabla v\big) - v_t = 0 && \text{in } Q_{R/4},\\
&v = 1 && \text{on } \partial_p Q_{R/4}\setminus\{y=0\},\\
&v = 1-\frac{\lambda^*}{2} && \text{on } Q'_{R/4},
\end{aligned}$$

then $v < 1-\frac{\lambda^*}{4}$ in $Q_{R/8}$. Since by maximum principle $u \leqslant v$ by setting $\lambda := \lambda^*/4$ the proof of the lemma is complete. □

The next lemma is a weighted version of Lemma 2.3. Its proof is essentially that of Lemma 8 of [4] (see, also, Proposition 3.4 in [5]).

Lemma 3.2. *Given $\sigma_1 > 0$ there exists a $\delta_1 > 0$ such that for every subsolution u^ε to* (3.1) *with $\beta'_\varepsilon \leqslant C$ satisfying*

$$0 < u^\varepsilon < 1 \quad \text{in } Q_R,$$
$$\left|\left\{(x,y,t)\in Q_R\colon\ 0 < u^\varepsilon < \frac{1}{2}\right\}\right| < \delta_1 |Q_R|$$

then

$$\fint_{Q'_{R/4}} \left(u^\varepsilon - \frac{1}{2}\right)^+ dx\,dt + \fint_{Q_{R/4}} y^\gamma \left[\left(u^\varepsilon - \frac{1}{2}\right)^+\right]^2 dx\,dy\,dt \leqslant C\sqrt{\sigma_1}.$$

To proceed we observe that the analog to Lemmas 2.4 and 2.5 as well as to Propositions 2.6 and 2.7 are straightforward. Therefore we have completed the continuity of our solutions, i.e.,

Theorem 3.3. *Let u be a solution to problem* (1.4) *or to problem* (1.5)*. Then u is continuous with a modulus depending on the nature of the singularity of β.*

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