A note on the integrability of the classical portfolio selection model

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We revisit the classical Merton portfolio selection model from the perspective of integrability analysis. By an application of a nonlocal transformation the nonlinear partial differential equation for the two-asset model is mapped into a linear option valuation equation with a consumption dependent source term. This result is identical to the one obtained by Cox–Huang [J.C. Cox, C.-f. Huang, Optimal consumption and portfolio policies when asset prices follow a diffusion process, J. Econom. Theory 49 (1989) 33–88], using measure theory and stochastic integrals. The nonlinear two-asset equation is then analyzed using the theory of Lie symmetry groups. We show that the linearization is directly related to the structure of the generalized symmetries.

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1. Introduction

A classical financial problem is the modeling of optimal investment–consumption decisions under uncertainty [1–5]. This was originally solved as an application of dynamic programming by Merton [4,5] and later by others using approaches based on martingales and stochastic integrals [2,3]. Of these Merton [5] and Cox–Huang [2] develop an identical problem: The optimal behaviour of an investor who wishes to maximize lifetime utility of consumption, \( \int_0^T U(C, t) \, dt \), plus a final bequest, \( V(W, T) \), subject to wealth, \( W \), being invested in a portfolio of assets modeled as geometric Brownian motions. A dichotomy which arises is that in the Merton result [5] a nonlinear differential equation is derived on the optimal controls and in the Cox–Huang result [2] a linear differential equation is derived on the optimal controls. The simplest instances of the models ([5] and [2]) are for a portfolio comprising a stock following a geometric Brownian motion with return, \( \alpha \), and volatility, \( \sigma \), and a bond with interest rate, \( r \). In [5] this is the nonlinear equation,

\[
J_t + (rW - C)J_W - \frac{(\alpha - r)^2}{2\sigma^2}J_{WW} + U(C, t) = 0,
\]

while [2] gives the linear equation,

\[
F_t + \left( \frac{(\alpha - r)^2}{\sigma^2} - r \right)ZF_Z + \frac{(\alpha - r)^2}{2\sigma^2}Z^2F_{ZZ} = rF - C(Z, t).
\]

The purpose of the present paper is to study the integrability properties of Eq. (1) and to demonstrate that (1) and (2) are in fact equivalent and can be mapped into each other by a generalized transformation on the optimal controls. The transformation is implemented by differentiating (1) with respect to \( W \), setting \( Z = J_W \) and inverting the roles of \( W \) and \( Z \).

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as dependent and independent variables. This provides a way of linking the results directly and goes beyond the explanation of the relationship between the models given in [2]. The transformation is then explored within the context of Lie symmetry groups [6–8]. We note that the point symmetry structure of (1) was considered in Perets and Yashiv [9] who calculated the point symmetries of (1) and proved that the solution obtained by Merton [5] for the hyperbolic risk aversion class of utility functions is based on symmetry. Here, we show how the generalized infinitesimal Lie symmetries of (1) contain (2) inside the symmetry structure and that this structure allows one to construct the linearization. This provides a theoretical basis for our result.

We review the Merton model for the derivation of (1) in Section 2. Section 3 contains the Lie symmetry analysis and introduces the relevant concepts. Section 4 contains the conclusion.

2. The classical Merton model

We briefly review the two-asset version of classical portfolio selection model [5]. It is assumed that \(U(C, t)\) and \(V(W, T)\) are given functions. Recall that \(C\) is consumption and \(W\) total wealth. Furthermore wealth is completely invested in a portfolio composed of a bond and a stock and there are no exogenous sources of income. The controls which are to be optimized are consumption, \(C\), and the fraction of total wealth invested in each asset: \(\pi\) for the stock and \(1 - \pi\) for the bond. The underlying market is modeled by assuming that the share price, \(S\), and the bond price, \(B\), follow

\[
dS_t = \alpha S_t \, dt + \sigma S_t \, dZ_t
\]

(3)

and

\[
dB_t = rB_t \, dt,
\]

(4)

respectively, and that \(Z_t\) is a standard Brownian motion. The parameters \(\alpha, \sigma\) and \(r\) are constants. This leads to the budget constraint on wealth [5]

\[
dW_t = [(r + (\alpha - r)\pi)W_t - C] \, dt + \pi \sigma W_t \, dZ_t.
\]

(5)

The portfolio optimization problem may then be formulated as

\[
J(W, S, t) = \max_{\pi, \, C} \left[ \int_0^t U(C, t) \, dt + V(W, T) \mid W_t = W(t), S_t = S(t) \right]
\]

(6)

subject to the constraints (3) and (5). Since the parameters are constant we may set \(J(W, S, t) = J(W, t)\) with an implicit dependence on the share price and the constraint (3) through \(W\).

Applying dynamic programming on (5) and (6) we have, [5],

\[
J_t + \max_{\pi, \, C} \left[ (r + (\alpha + (\alpha - r)\pi)W - C)J_W + \frac{1}{2} \pi^2 W^2 \sigma^2 J_{WW} \right] + U(C, t) = 0.
\]

(7)

The first-order maximization conditions are, [5],

\[
U_C = J_W
\]

(8)

so that

\[
C = U_C^{-1}[J_W]
\]

(9)

and

\[
\pi = -\frac{1}{\sigma^2} \cdot \frac{J_W}{W J_{WW}}
\]

(10)

are the optimal controls. Substitution of this into (7) gives the Hamilton–Jacobi–Bellman equation (1) subject to the terminal condition

\[
J(W, T) = V(W, T).
\]

(11)

Note that a nonzero initial wealth, \(W(0) = W_0 > 0\), is also required.
3. Reduction of the nonlinear Merton equation

We now give the mapping from (1) to (2). It may be seen if one differentiates Eq. (1) totally with respect to \( W \) to give

\[
J_{WW} + (r - \mu^2) J_W + \left( rW - C + \frac{\partial U}{\partial C} \frac{\partial C}{\partial J_W} - \frac{\partial C}{\partial J_W} \right) J_{WW} + \frac{\mu^2}{2} \frac{J_W^2}{J_{WW}} J_{WWW} = 0, \tag{12}
\]

where we have used \( C = C(J_W, t) \) and set \( \mu^2 = (\alpha - r)^2 / \sigma^2 \). Although this may seem to complicate the situation it is this step that leads to the linearization. Eq. (12) may be written as the potential system

\[
Z = J_W, \tag{13}
\]

\[
Z_t + (r - \mu^2) Z + (rW - C) Z_W + \frac{\mu^2}{2} \frac{Z^2}{Z_W} Z_{WWW} + \frac{\partial C}{\partial Z} Z_W \left[ \frac{\partial U}{\partial C} - Z \right] = 0. \tag{14}
\]

The optimality condition (8), \( U_\mathcal{C} = J_W = Z \), gives

\[
Z_t + (r - \mu^2) Z + (rW - C) Z_W + \frac{\mu^2}{2} \frac{Z^2}{Z_W} Z_{WWW} = 0. \tag{15}
\]

The task is now to integrate (15) which may be regarded as an independent equation in \( Z \) and \( W \). The change of variables

\[
W = F(Z, t) \tag{16}
\]

inverts the roles of \( W \) and \( Z \). The derivatives transform as

\[
F_t = -\frac{Z_t}{Z_W}, \quad F_Z = \frac{1}{Z_W}, \quad F_{ZZ} = -\frac{Z_{WWW}}{Z_W^2}. \tag{17}
\]

We note that these are exactly the changes of variables used in [2]. Substitution of (16) and (17) into (15) gives the linear equation

\[
F_t + \left( \mu^2 - r \right) ZF_Z + \frac{\mu^2}{2} Z^2 F_{ZZ} = rF - C(Z, t). \tag{18}
\]

In addition to being identical to (2) this is clearly, apart from the term involving \( C(Z, t) \), a form of the Black–Scholes equation [10] in which the left hand side has wealth as an option on marginal utility (since \( F = U_\mathcal{C} \)) with a risk premium related to Sharpe’s ratio via \( \mu^2 \) [11] and the right hand side is equivalent to a return on a risk-free asset when the consumption strategy is given. It is subject to the initial condition

\[
F(Z, 0) = W_0 \tag{19}
\]

and the terminal condition

\[
F(Z, T) = V_\mathcal{W}^{-1}(W, T). \tag{20}
\]

The latter follows from \( F = W \) and the terminal condition on (1), \( J(W, T) = V(W, T) \), and we have also assumed that \( V \) is invertible and differentiable. Finally we remark that this transformation also holds for the time-independent, \( t \to \infty \), case of the Merton problem as it appears in for example [12].

4. Connection to Lie symmetry groups

The transformation we have used above can be explained in terms of Lie symmetry groups for differential equations [6,8]. These are well known tools from mathematical physics which have appeared on occasion in various financial contexts [13,3,14–18]. One of the earliest indications of the rôle of symmetries in finance are the comments in the paper by Pliska and Selby [15] which highlight them as a revolutionary way forward in terms of research. The usual results in the literature are to use point symmetries to derive the transformational properties of given differential equations and to study the integrability properties of models. Largely these have been concerned with local point symmetry properties of the underlying space [13,14,17,18]. This is the calculation in [9] for Eq. (1). Since the transformation we have used depends upon differentiation, point symmetries do not give sufficient information to lead to (2). A broader class of symmetries comprises generalized symmetries [8,6] which expand the space of point symmetries (independent and dependent variables) to include derivatives of the dependent variables. It is these that contain the information relevant for the linearization described in Section 3. We give a brief calculation below.

First for aesthetic reasons we rewrite (1) as

\[
u_t + (\nu - C) \nu_x - K \frac{\nu_x^2}{\nu_{xx}} + u(C, t) = 0. \tag{21}
\]
We are interested in transformations which are functions of the variables \((x, t, u, u_x)\). The corresponding generalized symmetry is a vector field of the form
\[
\mathbf{v} = \phi(x, t, u, u_x) \partial_u + \tau(x, t, u, u_x) \partial_t + \xi(x, t, u, u_x) \partial_x.
\] (22)
The symmetry condition is
\[
\mathbf{v}^{[2]} E = 0,
\] (23)
where \(E\) is (21) and \(\mathbf{v}^{[2]}\) is the second prolongation of the vector field, i.e.
\[
\mathbf{v}^{[2]} = \phi \partial_u + \tau \partial_t + \xi \partial_x + \phi^t \partial_{u_t} + \phi^x \partial_{u_x} + \phi^{xx} \partial_{u_{xx}} + \phi^{tt} \partial_{u_{tt}} + \phi^{tx} \partial_{u_{xt}}.
\] (24)
Since the equation is independent of \(u_{xx}\) and \(u_{tt}\), these terms fall away immediately. The remaining terms \(\phi^t, \phi^x\) and \(\phi^{xx}\) are defined (in the notation of Olver [8]) as
\[
\phi^t = D_t (\phi - \xi u_t - \tau u_t) + \xi u_{xt} + \tau u_{tt},
\] (25)
\[
\phi^x = D_x (\phi - \xi u_x - \tau u_x) + \xi u_{xx} + \tau u_{tx},
\] (26)
\[
\phi^{xx} = D_x^2 (\phi - \xi u_x - \tau u_x) + \xi u_{xxx} + \tau u_{txx}.
\] (27)
\(D_x\) and \(D_t\) are total differentiation operators. Condition (23) and the optimality condition (8), now \(U_C = u_x\) so that \(C = C(u_x, t)\), give
\[
\phi^t + (rx - C) \phi^x - 2K \frac{u_x}{u_{xx}} \phi^x + K \frac{u_{xx}^2}{u_{xx}} \phi^{xx} = 0.
\] (28)
The full set of generalized symmetries corresponding to (21) is obtained from (28) using (25)--(27), expanding the derivatives and solving a concomitant system of partial differential equations. The full set of symmetries is not required here. We merely wish to demonstrate how the transformation we obtained is related to the theory. Setting \(\phi = \tau = 0\) we have
\[
\phi^t = -\xi_t u_x,
\] (29)
\[
\phi^x = -\xi_x u_x,
\] (30)
\[
\phi^{xx} = -\xi_{xx} u_x - 2\xi_x u_{xx}.
\] (31)
The symmetry condition is
\[
-\xi_t u_x - (rx - C) \xi_x u_x + 2K \frac{u_x}{u_{xx}} \xi_x u_x + r\xi u_x + K \frac{u_{xx}^2}{u_{xx}} (\xi_{xx} u_x - 2\xi_x u_{xx}) = 0.
\] (32)
Collecting terms and dividing by \(u_x\) we obtain
\[
-\xi_t - (rx - C) \xi_x - K \frac{u_{xx}^2}{u_{xx}} \xi_{xx} + r\xi = 0.
\] (33)
The added constraint \(\xi = \xi(t, x, u_x)\) gives
\[
-\frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial u_x} u_{xt} - (rx - C) \left[ \frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial u_x} u_{xx} \right] - K \frac{u_{xx}^2}{u_{xx}} \left[ \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial u_x \partial x} u_{xx} + \frac{\partial^2 \xi}{\partial u_{xx} \partial x} u_{xx} \right] + r\xi = 0.
\] (34)
This may be separated by powers of \(u_{xx}\) (since \(\xi\) is independent of \(u_{xx}\)) to give
\[
\frac{\partial \xi}{\partial t} + (2K - r) u_x \frac{\partial \xi}{\partial u_x} + Ku_x^2 \frac{\partial^2 \xi}{\partial u_x^2} = r\xi - (rx - C) \frac{\partial \xi}{\partial x},
\] (35)
\[
\frac{\partial^2 \xi}{\partial u_x \partial x} = 0,
\] (36)
\[
\frac{\partial^2 \xi}{\partial u_x^2} = 0.
\] (37)
We solve (36) and (37) to obtain
\[
\xi = ax + b
\] (38)
where \(a = a(t)\) and \(b = b(t, u_x)\). Substituting (38) into (35) and separating by powers of \(x\) we have
\[
\frac{\partial b}{\partial t} + (2K - r) u_x \frac{\partial b}{\partial u_x} + Ku_x^2 \frac{\partial^2 b}{\partial u_x^2} = rb - Ca.
\] (39)
and
\[ \frac{\partial a}{\partial t} = 0. \] (40)

Finally
\[ \frac{\partial b}{\partial t} + (2K - r) u_x \frac{\partial b}{\partial u_x} + Ku^2_x \frac{\partial^2 b}{\partial u^2_x} = rb - a_0 C, \] (41)

where
\[ a = a_0 \] (42)

for some constant \( a_0 \). We have
\[ \xi = a_0 x + b(t, u_x), \] (43)

so that the generalized symmetry is
\[ \nu = a_0 x \partial_x + b(t, u_x) \partial_x. \] (44)

This we write as
\[ V_1 = x \partial_x \] (45)

and
\[ V_2 = b(t, u_x) \partial_x, \] (46)

where without loss of generality \( a_0 = 1 \).

The information contained in the symmetry can be interpreted from a knowledge of standard properties of symmetry. The occurrence of the linear partial differential equation in the function \( b \) suggests that a mapping from \( E \) to (41) is possible. Since (44) is a generalized symmetry and depends upon \( u_x \), its structure is related to \( D_x E \) rather than to \( E \) itself, where \( E \) is (21) likewise a generalized symmetry as a function containing \( u_{xx} \) would depend on \( D^2_x E \) and so forth [8]. This points to the differentiation, (12), in Section 3. The change of variables (16) may be seen from using a comparison with the symmetries of the standard one-dimensional heat equation [8]. A feature of the point symmetries of a linear partial differential equation such as the one-dimensional heat equation
\[ \frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial y^2} \] (47)

is the existence of the generators
\[ G_1 = p \partial_p, \] (48)

and
\[ G_2 = f(t, y) \partial_p, \] (49)

where \( f(t, x) \) again satisfies the heat equation
\[ \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial y^2}. \] (50)

These reflect the linearity and homogeneity of the equation and \( G_2 \) in particular is related to the superposition of solutions for linear partial differential equations [6,8]. Comparing the generators \( G_1 \) and \( G_2 \) with \( V_1 \) and \( V_2 \) one may see that the roles of dependent and independent variables are to be reversed in order to have the symmetries \( G_2 \) and \( V_2 \) in the same form. This is exactly (16) purely on a Lie group theoretic basis. Further details on the various properties of \((1 + 1)\) scalar parabolic equations may be found in [19,8].

5. Conclusion and remarks

In this paper we have presented a transformation which maps the nonlinear partial differential equation for the classical two-asset Merton portfolio selection model [4,5] to a linear partial differential equation. Moreover this linear differential equation is identical to the corresponding differential equation in the Cox–Huang model [2] which approaches the classical Merton problem using martingales and stochastic integrals. Our result demonstrates that the integrability of the model is in fact independent of the approach taken and rather has to do with the structure of the differential equation and underlying relationships between contingent claims and portfolio selection. Evidence for this is that the generalized symmetries contain the linear equation derived in [2]. A further point of interest is that the use of symmetry methods and transformations on the evolution equations such as (1) are, from the nature of our result, analogous to the manipulations of martingales and stochastic integrals performed in the approach of Cox and Huang [2], before obtaining the linear differential equation in their work. We anticipate that a deeper understanding of Lie symmetry groups and the relationship to martingales and stochastic integrals is a promising direction for future research in both finance and differential equations.
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