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On the Poset of Partitions of an Integer

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We study the posets (partially ordered sets) P_n of partitions of an integer n, ordered by refinement, as defined by G. Birkhoff, "Lattice Theory" (3rd ed.) Colloq. Publ. Vol. 25, 1967, Amer. Math. Soc. Providence, R.I. In particular we disprove the conjecture that the posets P_n are Cohen–Macaulay for all n, and show that even the Möbius function on the intervals does not alternate in sign in general.

INTRODUCTION

Let P_n for $n \geqslant 1$ denote the poset of (unordered) partitions of the integer n, ordered by refinement, as introduced by Birkhoff [2, pp. 16, 104].

We write partitions as $x = (a_1, a_2, ..., a_k)$, $y = (b_1, b_2, ..., b_l)$ etc., where we assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > 0$, $a_1 + a_2 + \cdots + a_k = n$, and similarly for y. Then $x \leqslant y$ is defined to mean that there is a partition $\{1, ..., k\} = J_1 \cup J_2 \cup \cdots \cup J_l$ of the index set of x into l disjoint, nonempty subsets, such that $b_i = \sum_{j \in J_i} a_j$ for all $1 \leqslant i \leqslant l$ (compare Fig. 1).

The posets P_n have been discussed by Björner [3, p. 176], who raised the question about their topological properties.

This discussion is organized as follows: After some preliminary remarks in Section 1, we interpret in Section 2 the structure of intervals in P_n in terms of associated "puzzles." This technique is applied in Section 3 to disprove the shellability and Cohen–Macaulay property for large n. In Section 4 we study the Möbius function on P_n, and elaborate on the possible structure of intervals of P_n.

(1) GENERAL STRUCTURE

We first discuss the general structure of the posets P_n, numerical invariants, the natural embeddings and the connection to the Young lattice

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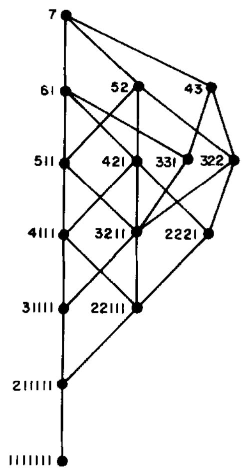

FIG. 1. Poset P_7

of all partitions, ordered by containment of their Young diagrams (see, e.g., [1, p. 17]).

For general poset notations as well as for the notions of shellability, Cohen–Macaulay poset and related concepts, the reader is referred to [3, 5].

Fix $n \geqslant 1$, then P_n is a graded modular poset of rank $n-1$, with maximal element $\hat{1} = (n)$ and minimal element $\hat{0} = (1,\dots, 1)$. Its rank function is given by $r((a_1,\dots, a_k)) = n-k$.

We note that for $m \leqslant n$, P_m has a natural order preserving embedding $i_{n,m}\colon P_m \to P_n$ given by $(a_1,\dots, a_k) \to (a_1,\dots, a_k, 1,\dots, 1)$. These embeddings are full and faithful in the sense that for $x \leqslant y$ in P_m, we have $[i_{n,m}(x), i_{n,m}(y)] \cong i_{n,m}([x, y])$ isomorphic. As we obviously have $i_{k,n} \circ i_{n,m} = i_{k,m}$ for $m \leqslant n \leqslant k$, the direct limit of the sequence $(P_n)_{n \geqslant 1}$ of posets is an infinite poset:

$$\begin{aligned} P_\infty &= \{(a_1, a_2,\dots) \mid a_1 \geqslant a_2 \geqslant \cdots;\ a_i \in \mathbb{N},\ a_n = 1 \text{ for all } n \gg 1\} \\ &\cong \{(a_1, a_2,\dots, a_N) \mid a_1 \geqslant a_2 \geqslant \cdots \geqslant a_N > 1 \text{ for some } N \geqslant 0\}, \end{aligned}$$

endowed with the obvious (induced) order-relation (See Fig. 2).) This poset does not seem to have been studied before. We disregard the infinite sequence of parts of size one in every element of P_∞. P_∞ inherits its rank function and its local properties (structure of intervals) from the posets P_n, has however no maximal element.

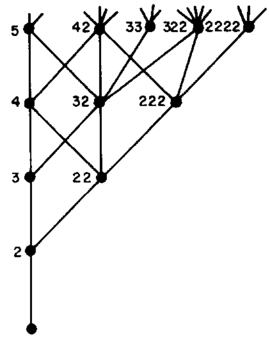

FIG. 2. Poset P_∞

The Whitney numbers of the second kind (cardinalities of the rank levels) are

for P_n: $W_k = p(n, n-k) =$ number of partitions of n into $n-k$ parts,
for P_∞: $W_k = p(k) =$ number of partitions of k.

This suggests a relation between P_∞ and the Young lattice Y of all partitions, ordered by containment of their Young diagrams, which has the same Whitney numbers W_k. Indeed, there is the following order preserving, bijective map:

$$\phi\colon Y \to P_\infty$$
$$(a_1,\dots, a_k) \to (a_1+1, a_2+1,\dots, a_k+1).$$

Now Y is a distributive lattice, and as such even EL-shellable (see, e.g., [4]), it is Cohen-Macaulay and has all the related "nice" combinatorial properties. We suggest as a partial explanation of the "bad" behavior of P_n and P_∞ (as discussed in Sects. 3, 4) the fact that P_∞ can be thought of as an extension of the well-behaved lattice Y, where the additional order-relations (respectively the additional faces in the corresponding complexes) spoil the topological properties of Y. For example, it is easy to see that for $x \leqslant y$ in Y with $r(x, y) > 2$, $[\phi(x), \phi(y)]$ *has* connected proper part in P_∞, contrary to the behavior observed in Section 3.

(2) PUZZLE INTERPRETATION

Let n be fixed, $x, y \in P_n$, $x \leqslant y$. To study the structure of the interval $[x, y]$, we observe that it can be visualized as a puzzle, where the "board" is given as the multiset Y of parts of y, the "pieces" as the multiset X of parts of x. (Depicting y as its Young diagram, X as a collection of rectangles, the connection to the notion of a puzzle as described by Rota and Joni [7] becomes obvious.)

Now a "solution" of the puzzle $[x, y]$ corresponds to a subposet of the interval $[x, y]$ with connected proper part (for $r(x, y) > 2$).

EXAMPLES. (a) The puzzle corresponding to $[5, 4, 3, 2, 1), (8, 6, 1)]$ in P_{15} has a unique solution (in the obvious sense) given by $8 = 5 + 3$, $6 = 4 + 2$, $1 = 1$. It is easy to see that this interval is isomorphic to the Boolean algebra B_2.

(b) The puzzle $[(5, 5, 5), (15)]$ in P_{15} has the unique solution $15 = 5 + 5 + 5$. The interval is a chain, because all possible ways to "split 15" are here essentially equivalent.

(3) SHELLABILITY OF P_n

The posets P_n can be viewed as quotients of the (geometric) partition lattices Π_n under the natural action of the symmetric group S_n. The posets P_n are semimodular, however not locally semimodular for $n \geqslant 8$, as first pointed out by A. Björner [3] in view of the not-semimodular interval $[(3, 2, 1, 1, 1); (5, 3)]$ in P_8. Similarly P_n^* is semimodular, but not locally semimodular for $n \geqslant 8$. Local semimodularity would imply that the posets are even CL-shellable [4]. Björner remarks that P_8 is nevertheless shellable. Indeed, a shelling of P_n for $n \leqslant 9$ is given by the reverse lexicographic order of the maximal chains of P_n, as induced by the lexicographic order of the partitions in P_n themselves. This method however breaks down in the interval $[(3, 2, 2, 2, 1), (6, 4)]$ in P_{10}. But P_{10} can still be checked to be shellable.

We use now the technique developed in Section 2 to show that P_n does not have these nice topological properties for larger n. In particular we give a negative answer to the question raised by Björner in [3, p. 177]:

THEOREM. *For $n \geqslant 19$ the posets P_n contain an interval of rank 3 with disconnected proper part. The P_n are therefore not Cohen–Macaulay and (a fortiori) not shellable for $n \geqslant 19$.*

Proof. Consider the interval $J_1 = [(6, 5, 3, 2, 2, 1), (8, 7, 4)]$ in P_{19}. The corresponding puzzle has two distinct solutions, given by

	8	7	4
1st Solution:	$6+2$	$5+2$	$3+1$
2nd Solution:	$5+3$	$6+1$	$2+2$

which are totally disjoint in the sense that they do not allow any "common split".

Thus the maximal chains in $\bar{J}_1$ are split into two disjoint classes, which do not have any point in common, i.e., J_1 is an interval of rank 3 with disconnected proper part, which contradicts Cohen–Macaulay type of J_1, P_{19} and (via the embedding in Sect. 1) of P_n for all $n \geqslant 19$. (The interval J_1 has actually the structure of two Boolean algebras of rank 3, identified at their maximal and minimal elements: $\bar{J}_1 = \bar{B}_3 + \bar{B}_3$, $|J_1| = 14$, $\tilde{H}_0(\bar{J}_1) \cong \mathbb{Z}$, $H_1(\bar{J}_1) \cong \mathbb{Z}^2$.)

We remark that the interval J_1 in P_{19} is not a singular "bad" incident, as can be seen from the intervals $[(6, 4, 4, 3, 2, 1); (8, 7, 5)]$ in P_{20} or $[(5, 4, 4, 3, 3, 2); (8, 7, 6)]$ in P_{21}, which have the same structure as the interval J_1 just discussed. In fact there are infinitely many intervals isomorphic to J_1 in P_{∞}, even if intervals obtained by scalar multiplication or addition of constants are not counted as different. This can be seen from the study of the four-parameter set of intervals $[(a, b, c, d, e, f); (a+b, c+d, e+f)]$, where $a, b, c, d, e = a+b-d$ and $f = c+d-a$ are positive integer coordinates. The intervals failing to have the proper structure will lie on a finite number of hyperplanes in four-space.

Furthermore it is easy to construct product intervals $J_1 \times B_k$: the puzzle $[(2^{2k}, 2^{2k-1}, ..., 2^1), (2^k + 2^{2k}, 2^{k-1} + 2^{2k-1}, ..., 2^1 + 2^{k+1})]$ is uniquely solvable because binary representation is unique. Thus this interval as well as any scalar multiple corresponds to B_k. To get an interval $J_1 \times B_k$, we multiply "board-parts" and "pieces" of this puzzle by $l \geqslant 5$ and adjoin them to those of the puzzle J_1. Similarly we can construct intervals of the form $J_1^k = J_1 \times \cdots \times J_1$ by duplicating J_1 with parts and pieces of larger size, e.g., $[(8+6, 8+5, 8+3, 8+2, 8+2, 8+1, 6, 5, 3, 2, 2, 1), (16+8, 16+7, 16+4, 8, 7, 4)] = [(14, 13, 11, 10, 10, 9, 6, 5, 3, 2, 2, 1), (24, 23, 20, 8, 7, 4)] \cong J_1 \times J_1$. Now by [6, Theorem 4.3] and [8, Theorem 62.5] we compute the homology of $\overline{J_1 \times B_k}$ to be $\tilde{H}_p(\overline{J_1 \times B_k}) = \tilde{H}_p(S(\bar{J}_1 * \bar{B}_k)) = \tilde{H}_{p-1}(\bar{J}_1 * S^{k-2}) = \tilde{H}_{p-k}(\bar{J}_1)$, which shows that the Cohen-Macaulay property is violated in arbitrarily high homology groups.

On the other hand standard arguments in homology theory (Eilenberg–Zilber theorem, Künneth formula and Mayer–Vietoris sequence, see [8]) allow to compute that the Betti numbers of J_1^k satisfy the recursion

$$\tilde{\beta}_{p+1}(\overline{J_1^{k+1}}) = \tilde{\beta}_{p-1}(\overline{J_1^k}) + 2\tilde{\beta}_{p-2}(\overline{J_1^k}),$$

hence

$$\tilde{\beta}_p(\overline{J_1^k}) = \binom{k}{p-2k+2} 2^{p-2k+2}, \qquad k \geqslant 1.$$

This shows that the Betti numbers below the top-dimension become arbitrarily large, and the number of nonvanishing homology groups is not limited either.

Thus, in a certain sense, the Cohen–Macaulay property fails "to unbounded extent" on the intervals of P_∞.

(4) Möbius Function

As P_n is Cohen–Macaulay for $n \leqslant 10$, its Möbius function will alternate is sign, i.e.,

$$\mu(x; y) \cdot (-1)^{r(x,y)} \geqslant 0 \tag{1}$$

for $x \leqslant y$. The counterexample in Section 3 has $\mu(J) = -1$, which does not violate this condition (as $r(J_1) = 3$). However, we construct:

Theorem. *The Möbius function does not alternate in sign on P_n for $n \geqslant 111$. For sufficiently large n, the property* (1) *fails on intervals of arbitrary rank $r \geqslant 7$.*

Proof. We study the following interval of rank 7 in P_{111}:

$$J_2 = [(21, 20, 11, 11, 8, 8, 6, 6, 6, 5, 5, 5); (27, 26, 25, 18, 15)].$$

The corresponding puzzle has only the two following solutions.

	27	26	25	18	15
Solution 1:	$21+6$	$20+6$	$11+8+6$	$11+7$	$5+5+5$
Solution 2:	$11+11+5$	$21+5$	$20+5$	$6+6+6$	$8+7$

To check that these solutions are actually disjoint, first observe that the corresponding two parts of J_2 have no atom in common, as no pair of numbers that occur in the same column in Solution 1 also occur in the same column in Solution 2. Second, the two parts of J_2 have no coatom in common, as no column can be split into two parts in the same way in both solutions. For example, Solution 1 allows 25 to be written as $11 + 14$, or $8 + 17$, or $6 + 19$, whereas Solution 2 splits 25 as $20 + 5$. Now if the two solutions had any proper element in common or any relation, then the interval J_2 would contain a maximal chain that contains an atom of one and a coatom of the other. This maximal chain determines a third solution of the puzzle, which does not exist. Thus J_2 has a disconnected proper part and is especially not shellable. Let C_1 and C_2 be the connected components of $\bar{J}_2$. Then from the equivalence of the "cuts" in $15=5+5+5$ and $18=6+6+6$ we see that $\mathbf{3}$ is a factor of both $\hat{C}_1$ and $\hat{C}_2$. Hence $\mu(\hat{C}_1)=\mu(\hat{C}_2)=0$, and $\mu(J_2)=+1$, violating (1). (The structure of J_2 can be seen to be $\bar{J}_2=\bar{C}_1+\bar{C}_2$, where $C_1 \cong \mathbf{3}\times B_5$, $C_2 \cong \mathbf{3}\times B_3 \times M_5$, where M_5 is the lattice of rank 2 and five elements corresponding to "$25=11+8+6$". We have $\tilde{H}_0(\overline{J_2})=\mathbb{Z}$, $H_p(\overline{J_2})=0$ for $p>0$, $p\neq 5$ as $\hat{C}_1$ and $\hat{C}_2$ are Cohen-Macaulay, and $H_5(\overline{J_2})=0$ can be read from the structure of C_1 and C_2, as well as $|J_2|=3\cdot 32+3\cdot 8\cdot 5-2=214$).

Adding different pieces and boards as in Section 3, all of sizes larger than 27 and yielding a uniquely solvable puzzle, it is easy to construct intervals isomorphic to $J_2\times B_k$ of rank $7+k$ in P_∞, which still violate (1), as $\mu(J_2\times B_k)=\mu(J_2)\,\mu(B_k)=(-1)^k$.

In fact the complicated structure of μ on P_n (or: P_∞) reflects the variety of patterns that can arise in puzzles as described in Section 2. On the other hand, we can note that

— the number $i(r)$ of nonisomorphic intervals of given rank r in P_∞ is finite, e.g., $i(1)=1$, $i(2)=6$,

— the Möbius function on intervals of rank 3 is indeed never positive.

The first assertion follows by induction on r, observing that each coatom in $[x, y]$ corresponds to splitting a part of y into two. Now the multisets of parts X, Y (as in Sect. 2) satisfy $|Y\backslash X|\leqslant r$, $|X\backslash Y|\leqslant 2r$, and the part in Y split to get a coatom has to be a sum of elements in $X\backslash Y$, i.e., there are less than 2^{2r} coatoms in $[x, y]$, and the number of nonisomorphic intervals of rank $r-1$ is finite by induction hypothesis. (The maximum value of six elements in the proper part of an interval of rank 2 is, e.g., reached in $[(6, 5, 4, 3, 2, 1);\ (7, 6, 5, 3)]$ of P_{21}.) The second assertion is readily established by case-by-case analysis of the possible situations that can yield a poset of rank 3 with disconnected proper part. In the connected case, the interval is Cohen-Macaulay and has therefore never positive Möbius function.

Finally we note the following extension (and correction) of the result in [2, p. 104]:

THEOREM. *In* P_n *let* $x_1(r):=(r+1, 1,..., 1)$, $x_2(r)=(r, 2, 1,..., 1)$, *and* $S_1:=\{x_1(r)\mid 1\leqslant r\leqslant n-1\}$, $S_2:=\{x_2(r)\mid 2\leqslant r\leqslant n-2\}$. *Then for* $x\leqslant y$, $x\in S_1$:

$$\mu(x, y)=\begin{cases}(-1)^{r(x,y)} & \textit{for } \ y\in S_1\cap S_2,\\ 0 & \textit{otherwise},\end{cases}$$

$$\mu(\hat{0}, y)=0 \qquad \textit{for all} \qquad y\in P_n \textit{ with } r(y)\geqslant 2.$$

Proof. We use induction over $r(x, y)$, $x = y$ being trivial. The Möbius function satisfies $\mu(x, y) = -\sum_{x \leqslant z < y} \mu(x, z)$ for $x < y$, where $x \leqslant z < y$ implies $r(x, z) < r(x, y)$.

Now the theorem follows from the observation that $(S_1 \cup S_2) \cap [x, y]$ is an interval in $S_1 \cup S_2$, with minimal element x, and maximal element y_0, where $y = y_0$ if $y \in S_1 \cup S_2$, $y_0 < y$ otherwise (in this case $y_0 \in S_2$, as $y_0 \geqslant x_1(r)$ implies $y_0 \geqslant x_2(r+1) > x_1(r)$).

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