On the Poset of Partitions of an Integer

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We study the posets (partially ordered sets) $P_n$ of partitions of an integer $n$, ordered by refinement, as defined by G. Birkhoff, "Lattice Theory" (3rd ed.) Colloq. Publ. Vol. 25, 1967, Amer. Math. Soc. Providence, R.I. In particular we disprove the conjecture that the posets $P_n$ are Cohen-Macaulay for all $n$, and show that even the Möbius function on the intervals does not alternate in sign in general.

Let $P_n$ for $n \geq 1$ denote the poset of (unordered) partitions of the integer $n$, ordered by refinement, as introduced by Birkhoff [2, pp. 16, 104].

We write partitions as $x = (a_1, a_2, \ldots, a_k), y = (b_1, b_2, \ldots, b_l)$ etc., where we assume that $a_1 \geq a_2 \geq \cdots \geq a_k > 0$, $a_1 + a_2 + \cdots + a_k = n$, and similarly for $y$. Then $x \leq y$ is defined to mean that there is a partition $\{1, \ldots, k\} = J_1 \cup J_2 \cup \cdots \cup J_l$ of the index set of $x$ into $l$ disjoint, nonempty subsets, such that $b_i = \sum_{j \in J_i} a_j$ for all $1 \leq i \leq l$ (compare Fig. 1).

The posets $P_n$ have been discussed by Björner [3, p. 176], who raised the question about their topological properties.

This discussion is organized as follows: After some preliminary remarks in Section 1, we interpret in Section 2 the structure of intervals in $P_n$ in terms of associated "puzzles." This technique is applied in Section 3 to disprove the shellability and Cohen-Macaulay property for large $n$. In Section 4 we study the Möbius function on $P_n$, and elaborate on the possible structure of intervals of $P_n$.

(1) General Structure

We first discuss the general structure of the posets $P_n$, numerical invariants, the natural embeddings and the connection to the Young lattice.
of all partitions, ordered by containment of their Young diagrams (see, e.g., [1, p. 17]).

For general poset notations as well as for the notions of shellability, Cohen–Macaulay poset and related concepts, the reader is referred to [3, 5].

Fix $n \geq 1$, then $P_n$ is a graded modular poset of rank $n - 1$, with maximal element $\hat{1} = (n)$ and minimal element $\hat{0} = (1, \ldots, 1)$. Its rank function is given by $r((a_1, \ldots, a_k)) = n - k$.

We note that for $m \leq n$, $P_m$ has a natural order preserving embedding $i_{n,m}: P_m \to P_n$ given by $(a_1, \ldots, a_k) \mapsto (a_1, \ldots, a_k, 1, \ldots, 1)$. These embeddings are full and faithful in the sense that for $x \preceq y$ in $P_m$, we have $[i_{n,m}(x), i_{n,m}(y)] \cong i_{n,m}([x, y])$ isomorphic. As we obviously have $i_{k,n} \circ i_{n,m} = i_{k,m}$ for $m \leq n \leq k$, the direct limit of the sequence $(P_n)_{n \geq 1}$ of posets is an infinite poset:

$$P_\infty = \{(a_1, a_2, \ldots) | a_1 \geq a_2 \geq \cdots; a_i \in \mathbb{N}, a_n = 1 \text{ for all } n \geq 1\}$$

$$\cong \{(a_1, a_2, \ldots, a_N) | a_1 \geq a_2 \geq \cdots \geq a_N > 1 \text{ for some } N \geq 0\},$$

endowed with the obvious (induced) order-relation (See Fig. 2.). This poset does not seem to have been studied before. We disregard the infinite sequence of parts of size one in every element of $P_\infty$. $P_\infty$ inherits its rank function and its local properties (structure of intervals) from the posets $P_n$, has however no maximal element.
The Whitney numbers of the second kind (cardinalities of the rank levels) are

- for $P_n$: $W_k = p(n, n-k) =$ number of partitions of $n$ into $n-k$ parts,
- for $P_\infty$: $W_k = p(k) =$ number of partitions of $k$.

This suggests a relation between $P_\infty$ and the Young lattice $Y$ of all partitions, ordered by containment of their Young diagrams, which has the same Whitney numbers $W_k$. Indeed, there is the following order preserving, bijective map:

$$\phi: Y \to P_\infty$$

$$(a_1, \ldots, a_k) \to (a_1+1, a_2+1, \ldots, a_k+1).$$

Now $Y$ is a distributive lattice, and as such even EL-shellable (see, e.g., [4]), it is Cohen-Macaulay and has all the related "nice" combinatorial properties. We suggest as a partial explanation of the "bad" behavior of $P_n$ and $P_\infty$ (as discussed in Sects. 3, 4) the fact that $P_\infty$ can be thought of as an extension of the well-behaved lattice $Y$, where the additional order-relations (respectively the additional faces in the corresponding complexes) spoil the topological properties of $Y$. For example, it is easy to see that for $x \leq y$ in $Y$ with $r(x, y) > 2$, $[\phi(x), \phi(y)]$ has connected proper part in $P_\infty$, contrary to the behavior observed in Section 3.

(2) **Puzzle Interpretation**

Let $n$ be fixed, $x, y \in P_n$, $x \leq y$. To study the structure of the interval $[x, y]$, we observe that it can be visualized as a puzzle, where the "board" is given as the multiset $Y$ of parts of $y$, the "pieces" as the multiset $X$ of...
parts of \( x \). (Depicting \( y \) as its Young diagram, \( X \) as a collection of rectangles, the connection to the notion of a puzzle as described by Rota and Joni [7] becomes obvious.)

Now a "solution" of the puzzle \([x, y]\) corresponds to a subposet of the interval \([x, y]\) with connected proper part (for \( r(x, y) > 2 \)).

**Examples.** (a) The puzzle corresponding to \([5, 4, 3, 2, 1), (8, 6, 1)\] in \( P_{15} \) has a unique solution (in the obvious sense) given by \( 8 = 5 + 3, 6 = 4 + 2, 1 = 1 \). It is easy to see that this interval is isomorphic to the Boolean algebra \( B_2 \).

(b) The puzzle \([(5, 5, 5), (15)]\) in \( P_{15} \) has the unique solution \( 15 = 5 + 5 + 5 \). The interval is a chain, because all possible ways to "split 15" are here essentially equivalent.

(3) **Shellability of \( P_n \)**

The posets \( P_n \) can be viewed as quotients of the (geometric) partition lattices \( II_n \) under the natural action of the symmetric group \( S_n \). The posets \( P_n \) are semimodular, however not locally semimodular for \( n \geq 8 \), as first pointed out by A. Björner [3] in view of the not-semimodular interval \([(3, 2, 1, 1, 1); (5, 3)]\) in \( P_8 \). Similarly \( P_n^* \) is semimodular, but not locally semimodular for \( n \geq 8 \). Local semimodularity would imply that the posets are even CL-shellable [4]. Björner remarks that \( P_8 \) is nevertheless shellable. Indeed, a shelling of \( P_n \) for \( n \leq 9 \) is given by the reverse lexicographic order of the maximal chains of \( P_n \), as induced by the lexicographic order of the partitions in \( P_n \) themselves. This method however breaks down in the interval \([(3, 2, 2, 2, 1), (6, 4)]\) in \( P_{10} \). But \( P_{10} \) can still be checked to be shellable.

We use now the technique developed in Section 2 to show that \( P_n \) does not have these nice topological properties for larger \( n \). In particular we give a negative answer to the question raised by Björner in [3, p. 177]:

**Theorem.** For \( n \geq 19 \) the posets \( P_n \) contain an interval of rank 3 with disconnected proper part. The \( P_n \) are therefore not Cohen–Macaulay and (a fortiori) not shellable for \( n \geq 19 \).

**Proof.** Consider the interval \( J_1 = [(6, 5, 3, 2, 2, 1), (8, 7, 4)] \) in \( P_{19} \). The corresponding puzzle has two distinct solutions, given by

\[
\begin{array}{ccc}
8 & 7 & 4 \\
\text{1st Solution:} & 6 + 2 & 5 + 2 & 3 + 1 \\
\text{2nd Solution:} & 5 + 3 & 6 + 1 & 2 + 2 \\
\end{array}
\]
which are totally disjoint in the sense that they do not allow any "common split".

Thus the maximal chains in $J_1$ are split into two disjoint classes, which do not have any point in common, i.e., $J_1$ is an interval of rank 3 with disconnected proper part, which contradicts Cohen-Macaulay type of $J_1$, $P_{19}$ and (via the embedding in Sect. 1) of $P_n$ for all $n \geq 19$. (The interval $J_1$ has actually the structure of two Boolean algebras of rank 3, identified at their maximal and minimal elements: $J_1 = B_3 + B_3$, $|J_1| = 14$, $\tilde{H}_0(J_1) \cong \mathbb{Z}$, $H_1(J_1) \cong \mathbb{Z}^2$.)

We remark that the interval $J_1$ in $P_{19}$ is not a singular "bad" incident, as can be seen from the intervals $[(6, 4, 4, 3, 2, 1); (8, 7, 5)]$ in $P_{20}$ or $[(5, 4, 4, 3, 2, 1); (8, 7, 5)]$ in $P_{21}$, which have the same structure as the interval $J_1$ just discussed. In fact there are infinitely many intervals isomorphic to $J_1$ in $P_{\infty}$, even if intervals obtained by scalar multiplication or addition of constants are not counted as different. This can be seen from the study of the four-parameter set of intervals $[(a, b, c, d, e, f); (a + b, c + d, e + f)]$, where $a, b, c, d, e = a + b - d$ and $f = c + d - a$ are positive integer coordinates. The intervals failing to have the proper structure will lie on a finite number of hyperplanes in four-space.

Furthermore it is easy to construct product intervals $J_1 \times B_k$: the puzzle $[(2^{2k}, 2^{2k-1}, \ldots, 2^1), (2^k + 2^{2k}, 2^{k-1} + 2^{2k-1}, \ldots, 2^1 + 2^{k+1})]$ is uniquely solvable because binary representation is unique. Thus this interval as well as any scalar multiple corresponds to $B_k$. To get an interval $J_1 \times B_k$, we multiply "board-parts" and "pieces" of this puzzle by $l \geq 5$ and adjoin them to those of the puzzle $J_1$. Similarly we can construct intervals of the form $J_1^k = J_1 \times \cdots \times J_1$ by duplicating $J_1$ with parts and pieces of larger size, e.g., $[(8 + 6, 8 + 5, 8 + 3, 8 + 2, 8 + 2, 8 + 1, 6, 5, 3, 2, 2, 1), (16 + 8, 16 + 7, 16 + 4, 8, 7, 4)] = [(14, 13, 11, 10, 9, 6, 5, 3, 2, 2, 1), (24, 23, 20, 8, 7, 4)] \cong J_1 \times J_1$. Now by [6, Theorem 4.3] and [8, Theorem 62.5] we compute the homology of $J_1 \times B_k$ to be $\tilde{H}_p(J_1 \times B_k) = \tilde{H}_p(S(J_1 * B_k)) = H_{p-1}(J_1 \times S^{k-2}) = H_{p-1}(J_1)$, which shows that the Cohen-Macaulay property is violated in arbitrarily high homology groups.

On the other hand standard arguments in homology theory (Eilenberg-Zilber theorem, Künneth formula and Mayer-Vietoris sequence, see [8]) allow to compute that the Betti numbers of $J_1^k$ satisfy the recursion

$$\beta_{p+1}(J_1^{k+1}) = \beta_{p-1}(J_1^k) + 2\beta_{p-2}(J_1^k),$$

hence

$$\beta_{p}(J_1^k) = \binom{k}{p-2k+2} 2^p 2^{k-2}, \quad k \geq 1.$$ 

This shows that the Betti numbers below the top-dimension become
arbitrarily large, and the number of nonvanishing homology groups is not limited either.

Thus, in a certain sense, the Cohen–Macaulay property fails “to unbounded extent” on the intervals of $P_\infty$.

(4) MÖBIUS FUNCTION

As $P_n$ is Cohen–Macaulay for $n \leq 10$, its Mönbius function will alternate is sign, i.e.,

$$\mu(x; y) \cdot (-1)^{r(x, y)} \geq 0$$

for $x \leq y$. The counterexample in Section 3 has $\mu(J) = -1$, which does not violate this condition (as $r(J_1) = 3$). However, we construct:

**Theorem.** The Mönbius function does not alternate in sign on $P_n$ for $n \geq 111$. For sufficiently large $n$, the property (1) fails on intervals of arbitrary rank $r \geq 7$.

**Proof.** We study the following interval of rank 7 in $P_{111}$:

$$J_2 = [(21, 20, 11, 11, 8, 8, 6, 6, 5, 5, 5); (27, 26, 25, 18, 15)].$$

The corresponding puzzle has only the two following solutions.

<table>
<thead>
<tr>
<th>27</th>
<th>26</th>
<th>25</th>
<th>18</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution 1:</td>
<td>21+6</td>
<td>20+6</td>
<td>11+8+6</td>
<td>11+7</td>
</tr>
<tr>
<td>Solution 2:</td>
<td>11+11+5</td>
<td>21+5</td>
<td>20+5</td>
<td>6+6+6</td>
</tr>
</tbody>
</table>

To check that these solutions are actually disjoint, first observe that the corresponding two parts of $J_2$ have no atom in common, as no pair of numbers that occur in the same column in Solution 1 also occur in the same column in Solution 2. Second, the two parts of $J_2$ have no coatom in common, as no column can be split into two parts in the same way in both solutions. For example, Solution 1 allows 25 to be written as 11+14, or 8+17, or 6+19, whereas Solution 2 splits 25 as 20+5. Now if the two solutions had any proper element in common or any relation, then the interval $J_2$ would contain a maximal chain that contains an atom of one and a coatom of the other. This maximal chain determines a third solution of the puzzle, which does not exist. Thus $J_2$ has a disconnected proper part and is especially not shellable. Let $C_1$ and $C_2$ be the connected components
of \( J_2 \). Then from the equivalence of the "cuts" in \( 15 = 5 + 5 + 5 \) and \( 18 = 6 + 6 + 6 \) we see that 3 is a factor of both \( \hat{C}_1 \) and \( \hat{C}_2 \). Hence \( \mu(\hat{C}_1) = \mu(\hat{C}_2) = 0 \), and \( \mu(J_2) = +1 \), violating (1). (The structure of \( J_2 \) can be seen to be \( J_2 = \hat{C}_1 + \hat{C}_2 \), where \( C_1 \cong 3 \times B_5 \), \( C_2 \cong 3 \times B_3 \times M_5 \), where \( M_5 \) is the lattice of rank 2 and five elements corresponding to \( "25 = 11 + 8 + 6" \). We have \( H_0(J_2) = \mathbb{Z} \), \( H_p(J_2) = 0 \) for \( p > 0 \), \( p \neq 5 \) as \( \hat{C}_1 \) and \( \hat{C}_2 \) are Cohen-Macaulay, and \( H_5(J_2) = 0 \) can be read from the structure of \( C_1 \) and \( C_2 \), as well as \( |J_2| = 3 \cdot 32 + 3 \cdot 8 \cdot 5 - 2 = 214 \).

Adding different pieces and boards as in Section 3, all of sizes larger than 27 and yielding a uniquely solvable puzzle, it is easy to construct intervals isomorphic to \( J_2 \times B_k \) of rank \( 7 + k \) in \( P_\infty \), which still violate (1), as \( \mu(J_2 \times B_k) = \mu(J_2) \mu(B_k) = (-1)^k \).

In fact the complicated structure of \( \mu \) on \( P_n \) (or: \( P_\infty \)) reflects the variety of patterns that can arise in puzzles as described in Section 2. On the other hand, we can note that

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- the number \( i(r) \) of nonisomorphic intervals of given rank \( r \) in \( P_\infty \) is finite, e.g., \( i(1) = 1 \), \( i(2) = 6 \),

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- the Möbius function on intervals of rank 3 is indeed never positive.

The first assertion follows by induction on \( r \), observing that each coatom in \([x, y]\) corresponds to splitting a part of \( y \) into two. Now the multisets of parts \( X, Y \) (as in Sect. 2) satisfy \(|Y \setminus X| \leq r\), \(|X \setminus Y| \leq 2r\), and the part in \( Y \) split to get a coatom has to be a sum of elements in \( X \setminus Y \), i.e., there are less than \( 2^{2r} \) coatoms in \([x, y]\), and the number of nonisomorphic intervals of rank \( r - 1 \) is finite by induction hypothesis. (The maximum value of six elements in the proper part of an interval of rank 2 is, e.g., reached in \([6, 5, 4, 3, 2, 1] \); \((7, 6, 5, 3)\) of \( P_{21}\).) The second assertion is readily established by case-by-case analysis of the possible situations that can yield a poset of rank 3 with disconnected proper part. In the connected case, the interval is Cohen-Macaulay and has therefore never positive Möbius function.

Finally we note the following extension (and correction) of the result in [2, p. 1043]:

**THEOREM.** In \( P_n \) let \( x_1(r) := (r + 1, 1, \ldots, 1) \), \( x_2(r) := (r, 2, 1, \ldots, 1) \), and \( S_1 := \{x_1(r) | 1 \leq r \leq n - 1\} \), \( S_2 := \{x_2(r) | 2 \leq r \leq n - 2\} \). Then for \( x \leq y \), \( x \in S_1 \):

\[
\mu(x, y) = \begin{cases} 
(-1)^{r(x, y)} & \text{for } y \in S_1 \cap S_2, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\mu(\hat{0}, y) = 0 \quad \text{for all} \quad y \in P_n \text{ with } r(y) \geq 2.
\]
Proof. We use induction over $r(x, y)$, $x = y$ being trivial. The Möbius function satisfies $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$ for $x < y$, where $x \leq z < y$ implies $r(x, z) < r(x, y)$.

Now the theorem follows from the observation that $(S_1 \cup S_2) \cap [x, y]$ is an interval in $S_1 \cup S_2$, with minimal element $x$, and maximal element $y_0$, where $y = y_0$ if $y \in S_1 \cup S_2$. $y_0 < y$ otherwise (in this case $y_0 \in S_2$, as $y_0 \geq x_1(r)$ implies $y_0 \geq x_2(r + 1) > x_1(r)$).

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