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Pseudo and strongly pseudo 2-factor isomorphic regular graphs and digraphs

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ABSTRACT

A graph G is pseudo 2-factor isomorphic if the parity of the number of cycles in a 2-factor is the same for all 2-factors of G. In Abreu et al. (2008) [3] we proved that pseudo 2-factor isomorphic k-regular bipartite graphs exist only for $k \leq 3$. In this paper we generalize this result for regular graphs which are not necessarily bipartite. We also introduce strongly pseudo 2-factor isomorphic graphs and we prove that pseudo and strongly pseudo 2-factor isomorphic $2k$-regular graphs and k-regular digraphs do not exist for $k \geq 4$. Moreover, we present constructions of infinite families of regular graphs in these classes. In particular we show that the family of Flower snarks is strongly pseudo 2-factor isomorphic but not 2-factor isomorphic and we conjecture that, together with the Petersen and the Blanuša2 graphs, they are the only cyclically 4-edge-connected snarks for which each 2-factor contains only cycles of odd length.

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1. Introduction

All graphs considered are finite and simple (without loops or multiple edges). We shall use the term multigraph when multiple edges are permitted. For further definitions and notation not explicitely stated here, please refer to [5].

A graph with a 2-factor is said to be 2-*factor Hamiltonian* if all its 2-factors are Hamilton cycles, and, more generally, 2-*factor isomorphic* if all its 2-factors are isomorphic. Examples of such graphs are K_4, K_5, $K_{3,3}$, the Heawood graph (which are all 2-factor Hamiltonian) and the Petersen graph (which is 2-factor isomorphic).

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Several recent papers have addressed the problem of characterizing families of graphs (particularly regular graphs) which have these properties. It is shown in [4,8] that k-regular 2-factor isomorphic bipartite graphs exist only when $k \in \{2, 3\}$ and an infinite family of 3-regular 2-factor Hamiltonian bipartite graphs, based on $K_{3,3}$ and the Heawood graph, is constructed in [8]. It is conjectured in [8] that every 3-regular 2-factor Hamiltonian bipartite graph belongs to this family. Faudree, Gould and Jacobsen in [7] determine the maximum number of edges in both 2-factor Hamiltonian graphs and 2-factor Hamiltonian bipartite graphs. In addition, Diwan [6] has shown that K_4 is the only 3-regular 2-factor Hamiltonian planar graph.

In [3] the above mentioned results on regular 2-factor isomorphic bipartite graphs are extended to the more general family of *pseudo 2-factor isomorphic graphs* i.e. graphs G with the property that the parity of the number of cycles in a 2-factor is the same for all 2-factors of G. Example of these graphs are $K_{3,3}$, the Heawood graph H_0 and the Pappus graph P_0. In particular, it is proven that pseudo 2-factor isomorphic k-regular bipartite graphs exist only when $k \in \{2, 3\}$ and that there are no planar pseudo 2-factor isomorphic cubic bipartite graphs. Moreover, it is conjectured in [3] that $K_{3,3}$, the Heawood graph H_0 and the Pappus graph P_0 are are the only 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graphs together with their repeated star products and some partial results towards this conjecture are obtained.

In this paper, we extend the above mentioned results on regular pseudo 2-factor isomorphic bipartite graphs to the not necessarily bipartite case (cf. Section 3). We introduce strongly pseudo 2-factor isomorphic graphs (Definition 2.4(ii)) and we prove that pseudo and strongly pseudo 2-factor isomorphic k-regular digraphs and $2k$-regular graphs only exist for $k \leq 3$ (Theorems 3.1, 3.3 and Corollaries 3.2, 3.4). Moreover, we present four different constructions of infinite classes of regular graphs in these classes (cf. Appendix). Finally, we deal with snarks and we show that the family of Flower snarks $J(t)$ is strongly pseudo 2-factor isomorphic but not 2-factor isomorphic (Proposition 4.2) and we conjecture that they are, together with the Petersen and the Blanuša 2 graphs, the only cyclically 4-edge-connected snarks for which each 2-factor contains only cycles of odd length (Conjecture 4.3).

2. Preliminaries

Let G be a bipartite graph with bipartition (X, Y) such that $|X| = |Y|$, and A be its bipartite adjacency matrix. In general $|\det(A)| \leq \operatorname{per}(A)$. We say that G is *det-extremal* if G has a 1-factor and $|\det(A)| = \operatorname{per}(A)$. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of G. For L a 1-factor of G define the *sign* of L, $\operatorname{sgn}(L)$, to be the sign of the permutation of $\{1, 2, \ldots, n\}$ corresponding to L. Thus G is det-extremal if and only if all 1-factors of G have the same sign.

Lemma 2.1. *Let L_1, L_2 be 1-factors in a bipartite graph G and t be the number of cycles in $L_1 \cup L_2$ of length congruent to zero modulo 4. Then* $\operatorname{sgn}(L_1)\operatorname{sgn}(L_2) = (-1)^t$.

Proof. This is a special case of [11, Lemma 8.3.1]. The proof is simple. □

A result of Thomassen [14, Theorem 5.4] implies:

Theorem 2.2. *Let G be a 1-extendable det-extremal bipartite graph. Then G has a vertex of degree at most three.* □

Another result of Thomassen [13, Theorem 3.2] implies:

Theorem 2.3. *Let G be a det-extremal bipartite graph with bipartition A, B and $|A| = |B| = n$. Then G has a vertex of degree at most $\lfloor \log_2 n \rfloor + 1$.* □

Definition 2.4. (i) Let G be a graph which contains a 2-factor. Then G is said to be *pseudo 2-factor isomorphic* if the parity of the number of cycles in a 2-factor is the same for all the 2-factors of G.

(ii) Let G be a graph which has a 2-factor. For each 2-factor F of G, let $t_i^*(F)$ be the number of cycles of F of length $2i$ modulo 4. Set t_i to be the function defined on the set of 2-factors F of G by:

$$t_i(F) = \begin{cases} 0 & \text{if } t_i^*(F) \text{ is even} \\ 1 & \text{if } t_i^*(F) \text{ is odd} \end{cases} \quad (i = 0, 1).$$

Then G is said to be *strongly pseudo 2-factor isomorphic* if both t_0 and t_1 are constant functions. Moreover, if in addition $t_0 = t_1$, set $t(G) := t_i(F),\ i = 0, 1$.

By definition, *if G is strongly pseudo 2-factor isomorphic then G is pseudo 2-factor isomorphic*. On the other hand there exist graphs such as the Dodecahedron which are pseudo 2-factor isomorphic but not strongly pseudo 2-factor isomorphic: the 2-factors of the Dodecahedron consist either of a cycle of length 20 or of three cycles: one of length 10 and the other two of length 5.

In [3] we studied pseudo 2-factor isomorphic regular bipartite graphs. In the bipartite case, pseudo 2-factor isomorphic and strongly pseudo 2-factor isomorphic are equivalent.

Theorem 2.5. *Let G be a pseudo 2-factor isomorphic bipartite graph with bipartition A, B and $|A| = |B| = n$. Then G has a vertex of degree at most $\lfloor \log_2 n \rfloor + 2$.*

Proof. Since G is pseudo 2-factor isomorphic, it has a 2-factor X. Since G is bipartite, X can be partitioned into disjoint 1-factors L_0, L_1. Let L be a 1-factor of G disjoint from L_0. Then $Y = L \cup L_0$ is a 2-factor in G. Let t be the number of cycles of length congruent to zero modulo four in Y. By Lemma 2.1, $\text{sgn}(L)\text{sgn}(L_0) = (-1)^t$. Since G is pseudo 2-factor isomorphic, t is constant for all choices of L. Thus all 1-factors of G, disjoint from L_0, have the same sign. Hence $G - L_0$ is det-extremal. So by Theorem 2.3, $G - L_0$ has minimum degree at most $\lfloor \log_2 n \rfloor + 1$. Hence G has minimum degree at most $\lfloor \log_2 n \rfloor + 2$. □

In what follows we will denote by HU, U, SPU and PU the sets of 2-factor Hamiltonian, 2-factor isomorphic, strongly pseudo 2-factor isomorphic and pseudo 2-factor isomorphic graphs, respectively. Similarly, HU(k), $U(k)$, SPU(k), PU(k) respectively denote the k-regular graphs in HU, U, SPU and PU.

3. Existence theorems

In this section we generalize the results obtained in [3] for bipartite graphs proving results that extend those obtained in [1,2].

For v a vertex of a digraph D, let $d^+(v)$ and $d^-(v)$ denote the out-degree and in-degree of v respectively. We say that D is k-diregular if for all vertices v of D, $d^+(v) = k = d^-(v)$.

Theorem 3.1. *Let D be a digraph with n vertices and X be a directed 2-factor of D. Suppose that either*

(a) $d^+(v) \geq \lfloor \log_2 n \rfloor + 2$ *for all $v \in V(D)$, or*
(b) $d^+(v) = d^-(v) = k$ *for all $v \in V(D)$ and some integer $k \geq 4$.*

Then D has a directed 2-factor Y with a different parity of number of cycles from X.

Proof. Suppose that all directed 2-factors Y of D have the same parity of number of cycles. Let $t = 0$ if such a number is even, and $t = 1$ if such a number is odd. Construct the associated bipartite graph G for the digraph D in the following way. For each vertex $u \in V(D)$ make two copies u' and u'' in $V(G)$. Each directed $(u, v) \in E(D)$ becomes the undirected edge $(u', v'') \in E(G)$. Additionally we add the edges (u', u'') to $E(G)$ for all $u \in V(D)$. Note that $L_0 = \{(u', u'') : u \in V(D)\}$ is a 1-factor of G, and that $\{(u', v'') : (u, v) \in X\}$ is a 1-factor of $G - L_0$.

Let L be a 1-factor of G disjoint from L_0. Then $Y' := L \cup L_0$ is a 2-factor in G in which each cycle has alternately edges of L and edges of L_0. This 2-factor gives rise to a directed 2-factor Y of D when we contract each edge of L_0. Now each cycle of Y' corresponds to exactly one cycle of Y but with twice the length. This implies that for any 1-factor L of G disjoint from L_0, the number of cycles in $L \cup L_0$ of length congruent to 0 modulo 4 is equal to the number of even cycles in Y, i.e. it is congruent to t modulo 2.

Using Lemma 2.1, we deduce that for any 1-factor L of G, disjoint from L_0, $\text{sgn}(L)\text{sgn}(L_0) = (-1)^t$. Since t is a constant, we conclude that all 1-factors of G, disjoint from L_0 have the same sign. Hence $G - L_0$ is det-extremal.

Now (a) and (b) follow directly using Theorems 2.5 and 2.2 respectively. Notice here that in case (b), because of regularity, G is 1-extendable. □

Let DSPU and DPU be the sets of digraphs in SPU and PU, i.e. strongly pseudo and pseudo 2-factor isomorphic digraphs, respectively. Similarly, DSPU(k) and DPU(k) respectively denote the k-diregular digraphs in DSPU and DPU.

Corollary 3.2.

(i) $\text{DSPU}(k) = \text{DPU}(k) = \emptyset$ *for* $k \geq 4$*;*
(ii) *If* $D \in \text{DPU}$ *then* D *has a vertex of out-degree at most* $\lfloor \log_2 n \rfloor + 1$. □

Theorem 3.3. *Let G be a graph with n vertices and X be a 2-factor of G. Suppose that either*

(a) $d(v) \geq 2(\lfloor \log_2 n \rfloor + 2)$ *for all* $v \in V(G)$*, or*
(b) *G is a $2k$-regular graph for some $k \geq 4$.*

Then G has a 2-factor Y with a different parity of number of cycles from X.

Proof. Let $G_1 = G - X$ and U be the set of vertices of odd degree in G_1. Let M be a matching between the vertices of U. Let G_2 be the multigraph obtained by adding the edges of M to G_1. Each vertex of G_2 has even degree, and hence each component of G_2 has an Euler tour. Thus we can construct a digraph D_2 by orientating the edges of G_2 in such a way that $d^+_{D_2}(v) = d^-_{D_2}(v)$ for all $v \in V(D_2)$. Let D_1 be the digraph obtained from D_2 by deleting the arcs corresponding to edges in M. Thus either

(i) $d^+_{D_1}(v) \geq \lfloor \log_2 n \rfloor + 1$, $d^-_{D_1}(v) \geq \lfloor \log_2 n \rfloor + 1$ for all $v \in V(D_1)$, or
(ii) $d^+_{D_1}(v) = d^-_{D_1}(v) = k - 1 \geq 3$ for all $v \in V(D_1)$.
Let X_1 be a 1-diregular digraph obtained by directing the edges of X and D be the digraph obtained from D_1 by adding the arcs of X_1. Then either
(iii) $d^+_D(v) \geq \lfloor \log_2 n \rfloor + 2$, $d^-_D(v) \geq \lfloor \log_2 n \rfloor + 2$ for all $v \in V(D)$, or
(iv) $d^+_D(v) = d^-_D(v) = k \geq 4$ for all $v \in V(D)$.

The result now follows from (iii), (iv) and Theorem 3.1. □

Corollary 3.4.

(i) *If* $G \in \text{PU}$ *then* G *contains a vertex of degree at most* $2\lfloor \log_2 n \rfloor + 3$*;*
(ii) $\text{PU}(2k) = \text{SPU}(2k) = \emptyset$ *for* $k \geq 4$. □

We know that PU(3), SPU(3), PU(4) and SPU(4) are not empty (cf. table in Appendix) and we conjectured in [1] that $\text{HU}(4) = \{K_5\}$.
There are many gaps in our knowledge even when we restrict attention to regular graphs. Some questions arise naturally.

Problem 3.5. Is $\text{PU}(2k + 1) = \emptyset$ for $k \geq 2$?

In particular we wonder if PU(7) and PU(5) are empty.

Problem 3.6. Is PU(6) empty?

Problem 3.7. Is K_5 the only 4-edge-connected graph in PU(4)?

In Appendix A we present examples of 2-edge-connected graphs in PU(4).

Of course a major problem is to find some sort of classification of the elements of PU(3). A general resolution of this problem is unlikely since we have no classification of the bipartite elements of PU(3).

A first step might be to attempt to classify the *near bipartite* elements of PU(3) (a non-bipartite graph is near bipartite if it can be made bipartite by the deletion of exactly two edges). The cubic near bipartite graph obtained from the Petersen graph by adding an edge joining two new vertices in two edges at maximum distance apart is not in PU(3). On the other hand, if a vertex of $K_{3,3}$ is inflated to a triangle the resulting graph is near bipartite and belongs to PU(3).

Problem 3.8. Do there exist near bipartite graphs of girth at least four in PU(3)?

In Section 4 we have taken a different direction in examining elements of PU(3) which contain only 'odd 2-factors'.

We close this section with some remarks on the operation of star products of cubic graphs.

Let G, G_1, G_2 be graphs such that $G_1 \cap G_2 = \emptyset$. Let $y \in V(G_1)$ and $x \in V(G_2)$ such that $d_{G_1}(y) = 3 = d_{G_2}(x)$. Let x_1, x_2, x_3 be the neighbors of y in G_1 and y_1, y_2, y_3 be the neighbors of x in G_2. If $G = (G_1 - y) \cup (G_2 - x) \cup \{x_1y_1, x_2y_2, x_3y_3\}$, then we say that G is a *star product* of G_1 and G_2 and write $G = (G_1, y) * (G_2, x)$, or $G = G_1 * G_2$ for short, when we are not concerned which vertices are used in the star product. The set $\{x_1y_1, x_2y_2, x_3y_3\}$ is a 3-edge cut of G and we shall also say that G_1 and G_2 are 3-*cut reductions* of G.

Star products preserve the property of being 2-factor Hamiltonian, 2-factor isomorphic, pseudo 2-factor isomorphic and, obviously, strongly pseudo 2-factor isomorphic in the family of cubic bipartite graphs (cf. [8,4,3]). Note that the converse is not true for 2-connected pseudo 2-factor isomorphic bipartite graphs [3].

In general for graphs not necessarily bipartite, star products do not preserve the property of being 2-factor Hamiltonian graphs, since it is easy to check that $K_4 * K_4$ is not 2-factor Hamiltonian. Hence, 2-factor isomorphic, pseudo 2-factor isomorphic and strongly pseudo 2-factor isomorphic non-bipartite graphs are also not preserved under star products.

Still, it is easily proved that the cubic graph $G := (G_1, x) * (G_1, y)$ is 2-factor Hamiltonian if and only if G_1 and G_2 are 2-factor Hamiltonian and the 3-edge cut $E_1(x, y) = \{x_1y_1, x_2y_2, x_3y_3\}$ is *tight* (i.e. every 1-factor of G contains exactly one edge of $E_1(x, y)$, c.f. e.g [11, p. 295]).

However, if G_1, G_2 and $G := (G_1, x) * (G_2, y)$ are pseudo 2-factor isomorphic graphs for some $x \in V(G_1)$ and $y \in V(G_2)$, then $E_1(x, y)$ is *not necessarily tight*. For example, if $G_1 = K_4$ and G_2 is the Petersen graph, they are both pseudo 2-factor isomorphic, and so is their star product which contains 2-factors of type $(3, 9)$ and $(5, 7)$, but the 3-edge cut is not tight, since the 2-factor of type $(3, 9)$ contains no edges of the 3-edge cut.

4. Snarks

A *snark* (cf. e.g. [9]) is a bridgeless cubic graph with edge chromatic number four. (By Vizing's theorem the edge chromatic number of every cubic graph is either three or four so a snark corresponds to the special case of four.) In order to avoid trivial cases, snarks are usually assumed to have girth at least five and not to contain a non-trivial 3-edge cut. The Petersen graph P is the smallest snark and Tutte conjectured that all snarks have Petersen graph minors. This conjecture was confirmed by Robertson, Seymour and Thomas (unpublished, see [12]). Necessarily, snarks are non-Hamiltonian.

We say that a graph G is *odd* 2-*factored* if for each 2-factor F of G each cycle of F is odd. By definition, *an odd* 2-*factored graph G is strongly pseudo* 2-*factor isomorphic*.

Lemma 4.1. *Let G be a cubic* 3-*connected odd* 2-*factored graph then G is a snark.*

Proof. Since G is odd 2-factored, the chromatic index of G is at least four. Hence, by Vizing's Theorem, G has chromatic index 4. □

Question: Which snarks are odd 2-factored?

Let $t \geq 5$ be an odd integer. The *Flower snark* (cf. [10]) $J(t)$ is defined in much the same way as the graph $A(t)$ described in [1]. The graph $J(t)$ has vertex set

$$V(t) = \{h_i, u_i, v_i, w_i : i = 1, 2, \ldots, t\}$$

and edge set

$$E(t) = \{h_iu_i, h_iv_i, h_iw_i, u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}, : i = 1, 2, \ldots, t-1\} \cup \{u_tv_1, v_tu_1, w_1w_t\}.$$

For $i = 1, 2, \ldots, t$ we call the subgraph IC_i of $J(t)$ induced by the vertices $\{h_i, u_i, v_i, w_i\}$ the ith *interchange* of $J(t)$. The vertices h_i and the edges $\{h_iu_i, h_iv_i, h_iw_i\}$ are called respectively the *hub* and the *spokes* of IC_i. The set of edges $\{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}\}$ linking IC_i to IC_{i+1} are said to be the ith *link* L_i of $J(t)$. The edge $u_iu_{i+1} \in L_i$ is called the *u-channel of the link*. The subgraph of $J(t)$ induced by the vertices $\{u_i, v_i : i = 1, 2, \ldots, t\}$ and $\{w_i : i = 1, 2, \ldots, t\}$ are respectively cycles of length $2t$ and t and are said to be the *base cycles* of $J(t)$.

Recall that in a cubic graph G, a 2-factor, F, determines a corresponding 1-factor, namely $E(G) - F$. In studying 2-factors in $J(t)$ it is more convenient to consider the structure of 1-factors.

Proposition 4.2. *Let $t \geq 5$ be an odd integer. Then $J(t)$ is odd 2-factored. Moreover, $J(t)$ is strongly pseudo 2-factor isomorphic but not 2-factor isomorphic.*

Proof. If L is a 1-factor of $J(t)$ each of the t links of $J(t)$ contain precisely one edge from L. This follows from the argument in [1, Lemma 4.7]. Then, a 1-factor L may be completely specified by the ordered t-tuple $(a_1, a_2, \ldots, a_t)$ where $a_i \in \{u_i, v_i, w_i\}$ for each $i = 1, 2, \ldots, t$ and indicates which edge in L_i belongs to L. Together these edges leave a unique spoke in each IC_i to cover its hub. Note that $a_i \neq a_{i+1}, i = 1, 2, \ldots, t$. To read off the corresponding 2-factor F simply start at a vertex in a base cycle at the first interchange. If the corresponding channel to the next interchange is not banned by L, proceed along the channel to the next interchange. If the channel is banned, proceed via a spoke to the hub (this spoke cannot be in L) and then along the remaining unbanned spoke and continue along the now unbanned channel ahead. Continue until reaching a vertex already encountered, so completing a cycle C_1. At each interchange C_1 contains either 1 or 3 vertices. Furthermore as C_1 is constructed iteratively, the cycle C_1 is only completed when the first interchange is revisited. Since C_1 uses either 1 or 3 vertices from IC_1 it can revisit either once or twice. If C_1 revisits twice then C_1 is a Hamiltonian cycle which is not the case. Hence it follows that F consists of two cycles C_1 and C_2. Let k_1 and k_3 be respectively the number of interchanges which contain 1 and 3 vertices of C_1. Then the length of C_1 is $k_1 + 3k_3$. Since C_1 visits iteratively each of the t interchanges, $k_1 + k_3$ is odd. Thus, the length of C_1 is odd and so is the length of C_2. Hence $J(t)$ is odd 2-factored and $J(t) \in SPU(3)$.

Finally, $J(t) \notin U(3)$ since it has 2-factors of types $(t, 3t)$ and $(t+4, 3t-4)$. Indeed, if $(a_1, a_2, \ldots, a_t)$ is such that $a_i \in \{u_i, v_i\}$, we obtain a 2-factor of type $(t, 3t)$ in $J(t)$. On the other hand, if $(a_1, a_2, \ldots, a_t)$ is such that $a_j = w_j$, for some $j \in \{1, \ldots, t\}$, and $a_i \in \{u_i, v_i\}$, for all $i \neq j$, we obtain a 2-factor of type $(t+4, 3t-4)$ in $J(t)$. □

A set S of edges of a graph G is a *cyclic edge cut* if $G - S$ has two components each of which contains a cycle. We say that a graph G is *cyclically m-edge-connected* if each cyclic edge cut of G has size at least m. We consider graphs without cyclic edge cuts to be cyclically m-edge-connected for all $m \geq 1$. Thus, for instance K_4 and $K_{3,3}$ are cyclically m-edge-connected for all $m \geq 1$.

We have the following information about some well-known snarks

	Odd 2-factored	2-factor types
Blanuša snark 1	No	(5, 5, 8) et al.
Blanuša snark 2	Yes	(5, 13) and (9, 9)
Loupekine snark 1	No	(5, 8, 9) et al.
Loupekine snark 2	No	(5, 8, 9) et al.
Celmins–Swart snark 1	No	(5, 5, 8, 8) et al.
Double star snark	No	(7, 7, 16) et al.
Szekeres snark	No	(5, 5, 40) et al.

We have also checked all known snarks up to 22 vertices and all the named snarks up to 50 vertices and they are all not odd 2-factored, except for the Petersen graph, Blanuša 2, and the Flower snark $J(t)$. We tentatively and possibly wildly suggest the following:

Conjecture 4.3. *A cyclically* 4*-edge-connected snark is odd* 2*-factored if and only if* G *is the Petersen graph, Blanuša* 2*, or a Flower snark* $J(t)$*, with* $t \geq 5$ *and odd.*

Appendix. 2-edge-connected constructions

In this section we present some sporadic examples and some constructions for graphs in $\mathrm{HU}(k)$, $U(k)$, $\mathrm{SPU}(k)$ and $\mathrm{PU}(k)$, for $k = 3, 4$. The sporadic examples will be presented in a table, and since some platonic solids belong to some of these classes we have included them all (even those that do not belong to any of these sets). Lists of numbers (if present), in the last column of the table, represent the types of 2-factors of the corresponding graph.

	HU(3)	$U(3)$	SPU(3)	PU(3)	Bipartite	2-factor types
Tetrahedron $= K_4$	✓	✓	✓	✓	No	(4)
$K_{3,3}$	✓	✓	✓	✓	Yes	(6)
Heawood	✓	✓	✓	✓	Yes	(14)
Petersen	×	✓	✓	✓	No	(5, 5)
Coxeter	×	✓	✓	✓	No	(14, 14)
Pappus	×	×	✓	✓	Yes	(18)(6, 6, 6)
Dodecahedron	×	×	×	✓	No	(5, 5, 10)(20)
Octahedron	×	×	×	×	No	(3, 3)(6)
Cube	×	×	×	×	No	(4, 4)(8)
	HU(4)	$U(4)$	SPU(4)	PU(4)	Bipartite	2-factor types
K_5	✓	✓	✓	✓	No	(5)
	HU(5)	$U(5)$	SPU(5)	PU(5)	Bipartite	2-factor types
Icosahedron	×	×	×	×	No	(3, 3, 3, 3)(12) et al.

Some of these sporadic examples will be used as seeds for the following 2-edge-connected constructions. Firstly we describe a family of pseudo 2-factor isomorphic cubic graphs based on a construction used in [3] for 2-factor isomorphic *bipartite* graphs. Here we show that this construction preserves pseudo 2-factor isomorphic *not necessarily bipartite* graphs but not strongly pseudo 2-factor isomorphic ones. Then we present a specific construction of strongly pseudo 2-factor isomorphic cubic graphs which are not 2-factor isomorphic. Finally we present two infinite families of 2-edge-connected 4-regular graphs which are strongly pseudo 2-factor isomorphic.

(1) *We construct an infinite family of graphs in* PU(3).

Let G_i be a cubic graph and $e_i = (x_i, y_i) \in E(G_i),\ i = 1, 2, 3$. Let $G^* = (G_1, e_1) \circ (G_2, e_2) \circ (G_3, e_3)$ be the 3-regular graph called 3-*joins* (cf. [3, p. 440]) defined as follows:

$$V(G^*) = \left(\bigcup_{i=1}^{3} V(G_i)\right) \cup \{u, v\}$$

$$E(G^*) = \left(\bigcup_{i=1}^{3} (E(G_i) - \{e_i\})\right) \cup \left(\bigcup_{i=1}^{3} \{(x_i, u), (y_i, v)\}\right),$$

G^* is 2-edge-connected but not 3-edge connected. In [3, Proposition 3.18] we proved that *if* G_i *are* 2*-factor Hamiltonian cubic bipartite graphs, then* G^* *is* 2*-factor isomorphic.*

Proposition A.1. *Let* G_i $(i = 1, 2, 3)$ *be pseudo* 2*-factor isomorphic cubic graphs. Then* G^* *is a cubic pseudo* 2*-factor isomorphic graph.*

Proof. All the 2-factors F in G^* are composed from 2-factors F_1, F_2, F_3 of G_1, G_2, G_3 such that, for some $\{i, j, k\} = \{1, 2, 3\}$, we have $e_i \notin F_i$, $e_j \in F_j$ and $e_k \in F_k$. Let C_j and C_k be the cycles of F_j, F_k, containing the edges e_j, e_k respectively. Then the cycles of F are all the cycles from F_1, F_2 and F_3, except for C_j and C_k, and the cycle $C = (C_j \cup C_k) - \{e_j, e_k\} \cup \{x_j u, y_j v, x_k u, y_k v\}$. Therefore, the parity of the number of cycles in a 2-factor F of G^* is $t(F) = t(F_1) + t(F_2) + t(F_3) - 1 \pmod 2$. Since $t(F_i)$ is constant for each $i = 1, 2, 3$, then $t(F)$ is also constant and G^* is pseudo 2-factor isomorphic. □

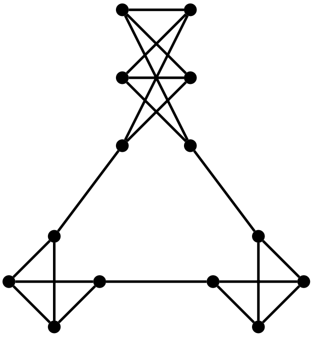
A brief analysis of the values of t_0 and t_1 over all 2-factors of G^*, with respect to the values of t_0 and t_1 in G_i, for $i = 1, 2, 3$, gives rise to the following proposition.

Proposition A.2. *Let G_i be strongly pseudo 2-factor isomorphic graphs such that in any 2-factor of G_i all cycles have even length, $i = 1, 2, 3$. Then G^* is strongly pseudo 2-factor isomorphic.* □

However, in general, strongly pseudo 2-factor isomorphism is not preserved under this construction. A counterexample can be built from the Flower snark $J(5)$ (cf. Section 4). In fact, the graph $J(5)^*$, obtained as a 3-join of $G_i := J(5)$ and $e_i := v_5 u_1$, $i = 1, 2, 3$, is not strongly pseudo 2-factor isomorphic since it contains 2-factors of types (5, 5, 5, 15, 32) and (5, 5, 11, 15, 26).

(2) *We construct an infinite family of graphs $H(n)$ in* SPU(3).

Let $H(n)$, be the family of cubic graphs on $n \geq 14$ vertices, n even, defined as follows. Let $K^*_{3,3}$ and K^*_4 be the graphs obtained by deleting exactly one edge from $K_{3,3}$ and K_4 respectively. Set $n \equiv 2j \pmod 8$ $j = 0, 1, 2, 3$. Set $\theta :\equiv j + 2 \pmod 4$ where $0 \leq \theta \leq 3$. Then $H(n)$ is an infinite family of cubic graphs on $n \geq 14$ vertices, n even, obtained from a cycle of length $(n - 2\theta)/4$ by "*inflating*" θ of the vertices of the cycle into copies of $K^*_{3,3}$ and $(n - 6\theta)/4$ of the vertices of the cycle into copies of K^*_4 (cf. e.g. picture below for $H(14)$).



$H(14)$

Proposition A.3. *The family of cubic graphs $H(n)$ is strongly pseudo 2-factor isomorphic but not 2-factor isomorphic.*

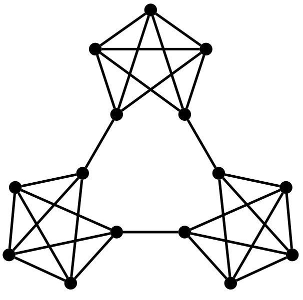
Proof. By construction $H(n)$ has 2-factors $F_1 := F_1(n)$, where F_1 consists of θ cycles of length 6 and $(n - 6\theta)/4$ cycles of length 4, and $F_2 := F_2(n)$, where F_2 consists of a cycle of length n (i.e. it is Hamiltonian). Hence $H(n)$ is not 2-factor isomorphic.

First suppose $n \equiv 0 \pmod 4$. Then $j = 0$ or 2 and $\theta = 2$ or 0, respectively. Therefore, θ is even and $(n - 6\theta)/4$ is odd. Thus, the number of cycles in a 2-factor of $H(n)$ is odd, and all such cycles have even length. Thus $H(n) \in PU(3)$. Moreover, it is easy to check that t_0 and t_1 are constant. Hence $H(n) \in$ SPU(3).

Now suppose $n \equiv 2 \pmod 4$. Then $j = 1$ or 3 and $\theta = 3$ or 1, respectively. Therefore, θ is odd and $(n - 6\theta)/4$ is even. Thus, the number of cycles in a 2-factor of $H(n)$ is odd, and all such cycles have even length. Thus $H(n) \in PU(3)$. Again it is easily checked that t_0 and t_1 are constant. Hence $H(n) \in$ SPU(3). □

(3) *We construct an infinite family of graphs* $H^*(5(2k+1))$ *in* SPU(4).

Let $K_5^* = K_5 - e$. Take an odd cycle C_{2k+1}. Let $H^*(5(2k+1)),\ k \geq 1$ be the graph of degree 4 obtained by inflating each vertex of C_{2k+1} to a graph isomorphic to K_5^*. The 2-factors of $H^*(5(2k+1))$ are $F_1 = (5(2k+1))$ and $F_2 = (5, 5, \ldots, 5)$ with $2k+1$ cycles of size 5. Therefore, $t^*(H^*(5(2k+1))) = 0$ and $H^*(5(2k+1))$ is a 4-regular 2-edge-connected *strongly pseudo* 2*-factor isomorphic but not* 2*-factor isomorphic* (cf. e.g. picture below for $H^*(15)$). Notice that adding any edge to $H^*(5(2k+1))$ results in a graph which is not pseudo 2-factor isomorphic.



$H^*(15)$

(4) *We construct a second infinite family of graphs in* SPU(4).

In [1, p. 400] we defined an edge e belonging to a 2-factor of a graph G to be *loyal* if for each 2-factor F containing e, the cycle to which e belongs had constant length, independently of the choice of F. We used graphs containing a loyal edge to define an infinite family of 2-connected 4-regular 2-factor isomorphic graphs [1, Construction (1), p. 400]. We extend this construction to the strongly pseudo 2-factor isomorphic case.

Let G be a graph and let e be one of its edges such that there are 2-factors F, F' of G containing and avoiding e respectively. We now define e to be *pseudo loyal* if for each 2-factor F containing e, the cycle to which e belongs has constant length modulo 4, independently of the choice of F.

Let $G \in$ SPU(4) and let e be a pseudo loyal edge in G, and let c be the length (modulo 4) of the cycle containing e in a 2-factor of G containing e. Let G_1, G_2, G_3, G_4 be four isomorphic copies of G and $e_i = x_i y_i$ be the loyal edge in G_i corresponding to e. We construct a 4-regular graph G' called a 4-*seed graft* of G by taking

$$V(G') = \left(\bigcup_{i=1}^{4} V(G_i)\right) \cup \{u, v\}$$

and

$$E(G') = \left(\bigcup_{i=1}^{4} (E(G_i) - \{e_i\})\right) \cup \left(\bigcup_{i=1}^{4} \{(x_i, u), (y_i, v)\}\right).$$

We call the new vertices u, v *clips* and we refer to G as a *seed* for G'.

Proposition A.4. *Let $G \in$ SPU(4) and let e be a pseudo loyal edge in G. Then the* 4*-regular seed graft G' of G is strongly pseudo* 2*-factor isomorphic, has connectivity* 2 *and each edge of G' which is adjacent to a clip is pseudo loyal.*

Proof. By construction G' is not 3-edge connected thus G' has connectivity 2. Let F be a 2-factor of G'. Relabeling if necessary, we may suppose that $\{ux_1, ux_2, vy_1, vy_2\} \subseteq F$. Then $(F \cap G_i) + e_i$ are 2-factors of G_i containing e_i for $i = 1, 2$, and $F \cap G_j$ is a 2-factor of G_j avoiding e_j for $j = 3, 4$. The cycle of F containing the clips is $C = (C_1 - e_1) \cup (C_2 - e_2) \cup \{x_1u, y_1v, x_2u, x_2v\}$ and it has constant length $2c + 2 \pmod 4$, independently of the choice of F, where c is the length (modulo 4) of the cycle containing e in a 2-factor of G containing e. Then, each edge of G' adjacent to a clip is pseudo loyal. This also implies that the values t_0 and t_1 are constant over all 2-factors of G', independently of the choice of F. Hence, $G' \in$ SPU(4). $\square$

Note: In [1, p. 400] the only seed we had for the family of graphs with loyal edges was $K_5 \in U(4)$, in which each edge is loyal. In the family $H^*(5(2k+1))$ the edges of the cycle C_{2k+1} are pseudo loyal, and if k is even, then all edges of the graph are pseudo loyal. Therefore, Proposition A.4 gives rise to an infinite family of 2-connected graphs in SPU(4) starting from $H^*(5(2k+1))$ for each value of k.

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