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Exact solutions of the coupled Higgs equation and the Maccari system using He's semi-inverse method and $\left(\frac{G'}{G}\right)$-expansion method

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ABSTRACT

In this paper, we establish exact solutions for complex nonlinear equations. The He's semi-inverse and the $\left(\frac{G'}{G}\right)$-expansion methods are used to construct exact solutions of these equations. We apply He's semi-inverse method to establish a variational theory for the coupled Higgs equation and Maccari system. Based on this formulation, a solitary solution can be easily obtained using the Ritz method. The $\left(\frac{G'}{G}\right)$-expansion method is used to seek more general exact solutions of the coupled Higgs equation and the Maccari system. As a result, hyperbolic function solutions, trigonometric function solutions and rational function solutions with free parameters are obtained. When the parameters are taken as special values the solitary wave solutions are also derived from the traveling wave solutions. Moreover, it is observed that the suggested technique is compatible with the physical nature of such problems.

1. Introduction

There have been various approaches to search for soliton solutions for nonlinear wave equations. These methods include the inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation method [3], algebra method [4], sine–cosine method [5], tanh–coth method [6], Jacobi elliptic function method [7], Homotopy perturbation method (HPM) [8,9], Luapanov's artificial small parameter method, δ-expansion method, Adomian decomposition method, variational iterative method, Homotopy analysis method (HAM), Homotopy Padé method (HPadéM) [10–13] and so on.

In the past few decades, qualitative analysis together with ingenious mathematical techniques for handling various nonlinear problems has been studied. Among them, variational approaches, such as He's semi-inverse method is a powerful tool to the search for variational principles for nonlinear physical problems directly from field equations without using the Lagrange multiplier and provides physical insight into the nature of the solution of the problem. Based on this formulation, a solitary solution can be obtained using the Ritz method. Variational principles have been studied widely in physics and mathematics [14–18].

Unlike some known methods such as the $\left(\frac{G'}{G}\right)$-expansion method, He's semi- inverse method is a powerful mathematical tool to the construction of variational formulations for physical problems.

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Recently, Wang et al. [19] proposed the $\left(\frac{G'}{G}\right)$-expansion method to find traveling wave solutions of NLEEs. Next, this method was applied to obtain traveling wave solutions of some NLEEs [20–22]. Zhang generalized the $\left(\frac{G'}{G}\right)$-expansion method [23–25].

The $\left(\frac{G'}{G}\right)$-expansion method is based on the explicit linearization of nonlinear differential equations for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Computations are performed with a computer algebra system such as Maple to deduce the solutions of the nonlinear equations in an explicit form. The solution process of the method is direct, effective and convenient due to solving the auxiliary equation of second-order differential equation with constant coefficients. The main merits of the $\left(\frac{G'}{G}\right)$-expansion method over the other methods are that it gives more general solutions with some free parameters which, by a suitable choice of parameters, turn out to be some known solutions gained by the existing methods. As a result, hyperbolic function solutions, trigonometric function solutions and rational function solutions with free parameters are obtained. When the parameters are taken as special values, some solitary wave solutions are also derived from the hyperbolic function solutions. Whereas only solitary wave solutions are obtained by using He's semi-inverse method. Besides, the $\left(\frac{G'}{G}\right)$-expansion method handles NLEEs in a direct manner with no requirement for initial/boundary conditions or initial trial functions at the outset. But, He's semi-inverse method depends on the initial trial function.

Consider the following coupled Higgs equations

$$\begin{aligned} &u_{tt} - u_{xx} + |u|^2 u - 2uv = 0, \\ &v_{tt} + v_{xx} - (|u|^2)_{xx} = 0. \end{aligned} \tag{1}$$

Tajiri obtained N-soliton solutions to Eq. (1) in [26]. Zhao constructed more general traveling wave solutions of Eq. (1) in [27].

Recently, Attilio Maccari derived a new integrable $(2 + 1)$-dimensional nonlinear system [28]

$$\begin{aligned} &iu_t + u_{xx} + uv = 0, \\ &v_t + v_y + (|u|^2)_x = 0. \end{aligned} \tag{2}$$

The integrability property was explicitly demonstrated and the Lax pairs were also obtained. Zhao also constructed more general traveling wave solutions of system Eq. (2) in [27].

In this paper we will use He's semi-inverse and $\left(\frac{G'}{G}\right)$-expansion methods to the coupled Higgs equation and Maccari system.

2. Description of He's semi-inverse method

We suppose that the given nonlinear partial differential equation for $u(x, t)$ to be in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{3}$$

where P is a polynomial in its arguments. The essence of He's semi-inverse method can be presented in the following steps:

Step 1. Seek solitary wave solutions of Eq. (3) by taking $u(x, t) = U(\xi)$, $\xi = x - ct$, and transform Eq. (3) to the ordinary differential equation

$$Q(U, U', U'', \ldots) = 0, \tag{4}$$

where prime denotes the derivative with respect to ξ.

Step 2. If possible, integrate Eq. (4) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3. According to He's semi-inverse method, we construct the following trial-functional

$$J(U) = \int L \, \mathrm{d}\xi, \tag{5}$$

where L is an unknown function of U and its derivatives.

There exist alternative approaches to the construction of the trial-functionals, see Refs. [29–33].

Step 4. By the Ritz method, we can obtain different forms of solitary wave solutions, such as $U(\xi) = A\ \mathrm{sech}(B\xi)$, $U(\xi) = A\,\mathrm{csch}(B\xi)$, $U(\xi) = A\ \tanh(B\xi)$, $U(\xi) = A\ \coth(B\xi)$ and so on. For example in this paper we search a solitary wave solution in the form

$$U(\xi) = A\ \mathrm{sech}(B\xi), \tag{6}$$

where A and B are constants to be further determined.

Substituting Eq. (6) into Eq. (5) and making J stationary with respect to A and B results in

$$\frac{\partial J}{\partial A} = 0, \tag{7}$$

$$\frac{\partial J}{\partial B} = 0. \tag{8}$$

Solving simultaneously Eqs. (7) and (8) we obtain A and B. Hence, the solitary wave solution (6) is well determined.

3. Description of the (G'/G)-expansion method

We suppose that the given nonlinear partial differential equation for $u(x, t)$ to be in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{9}$$

where P is a polynomial in its arguments. The essence of the (G'/G)-expansion method can be presented in the following steps:

Step 1. Seek traveling wave solutions of Eq. (9) by taking $u(x, t) = U(\xi)$, $\xi = x - ct$, and transform Eq. (9) to the ordinary differential equation

$$Q(U, U', U'', \ldots) = 0, \tag{10}$$

where prime denotes the derivative with respect to ξ.

Step 2. If possible, integrate Eq. (10) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3. Introduce the solution $U(\xi)$ of Eq. (10) in the finite series form

$$U(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{11}$$

where a_i are real constants with $a_m \neq 0$ to be determined, m is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{12}$$

where λ and μ are real constants to be determined.

Step 4. Determine m. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in Eq. (10).

Step 5. Substituting (11) together with (12) into Eq. (10) yields an algebraic equation involving powers of (G'/G). Equating the coefficients of each power of (G'/G) to zero gives a system of algebraic equations for a_i, λ, μ and c. Then, we solve the system with the aid of a computer algebra system, such as Maple, to determine these constants. On the other hand, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the solutions of Eq. (12) are well known to us. So, as a final step, we can obtain exact solutions of the given Eq. (9).

4. Using He's semi-inverse method

In this section, we apply He's semi-inverse method to solve the coupled Higgs equation and the Maccari system.

4.1. *Coupled Higgs equation*

We first consider the coupled Higgs equation (1). Using the wave variables

$$u = e^{i\theta} U(\xi), \qquad v = V(\xi), \quad \theta = px + rt, \ \xi = x + ct. \tag{13}$$

Substituting (13) into (1), we have

$$\begin{aligned} &(c^2 - 1)U'' + (p^2 - r^2)U - 2UV + U^3 = 0, \\ &(c^2 + 1)V'' - 2(U')^2 - 2UU'' = 0. \end{aligned} \tag{14}$$

Integrating the second equation in (14) and neglecting the constant of integration we find

$$(c^2 + 1)V = U^2. \tag{15}$$

Substituting (15) into the first equation of the system and integrating we find

$$(c^4-1)U''+(c^2+1)(p^2-r^2)U+(c^2-1)U^3=0, \tag{16}$$

where prime denotes differentiation with respect to ξ.

By He's semi-inverse method [14], we can obtain the following variational formulation

$$J=\int_0^\infty\left[-\frac{(c^4-1)}{2}(U')^2+\frac{(c^2+1)(p^2-r^2)}{2}U^2+\frac{(c^2-1)}{4}U^4\right]\mathrm{d}\xi. \tag{17}$$

By a Ritz-like method, we search a solitary wave solution in the form

$$U(\xi)=A\ \mathrm{sech}(B\xi), \tag{18}$$

where A and B are unknown constants to be further determined. Substituting Eq. (18) into Eq. (17), we have

$$\begin{aligned} J &= \int_0^\infty\left[-\frac{A^2B^2(c^4-1)}{2}\,\mathrm{sech}^2(B\xi)\tanh^2(B\xi)\right.\\ &\quad\left.+\frac{(c^2+1)(p^2-r^2)A^2}{2}\,\mathrm{sech}^2(B\xi)+\frac{(c^2-1)A^4}{4}\,\mathrm{sech}^4(B\xi)\right]\mathrm{d}\xi\\ &= -\frac{A^2B(c^4-1)}{6}+\frac{(c^2+1)(p^2-r^2)A^2}{2B}+\frac{(c^2-1)A^4}{6B}. \end{aligned} \tag{19}$$

Making J stationary with A and B results in

$$\frac{\partial J}{\partial A}=-\frac{AB(c^4-1)}{3}+\frac{(c^2+1)(p^2-r^2)A}{B}+\frac{2(c^2-1)A^3}{3B}=0, \tag{20}$$

$$\frac{\partial J}{\partial B}=-\frac{A^2(c^4-1)}{6}-\frac{(c^2+1)(p^2-r^2)A^2}{2B^2}-\frac{(c^2-1)A^4}{6B^2}=0. \tag{21}$$

From Eqs. (20) and (21), we get

$$A=\mathrm{i}\sqrt{\frac{2(c^2+1)(r^2-p^2)}{c^2-1}} \tag{22}$$

and

$$B=\sqrt{\frac{r^2-p^2}{c^2-1}},\qquad \frac{(p^2-r^2)}{c^2-1}<0. \tag{23}$$

The soliton solutions are, therefore, obtained as follows

$$\begin{aligned} U(x,t)&=\mathrm{i}\sqrt{\frac{2(c^2+1)(r^2-p^2)}{c^2-1}}\ \mathrm{sech}\left[\sqrt{\frac{r^2-p^2}{c^2-1}}(x+ct)\right],\\ V(x,t)&=\frac{2(r^2-p^2)}{c^2-1}\ \mathrm{sech}^2\left[\sqrt{\frac{r^2-p^2}{c^2-1}}(x+ct)\right],\\ u(x,t)&=\left(\mathrm{i}\sqrt{\frac{2(c^2+1)(r^2-p^2)}{c^2-1}}\ \mathrm{sech}\left[\sqrt{\frac{r^2-p^2}{c^2-1}}(x+ct)\right]\right)\exp[\mathrm{i}(px+rt)],\\ v(x,t)&=\frac{2(r^2-p^2)}{c^2-1}\ \mathrm{sech}^2\left[\sqrt{\frac{r^2-p^2}{c^2-1}}(x+ct)\right]. \end{aligned} \tag{24}$$

The solutions (24) are same Eq. (4.11) in [34] respectively. If we take the solution in the form $U(\xi)=A\,\mathrm{csch}(B\xi)$, $U(\xi)=A\ \tanh(B\xi)$ and $U(\xi)=A\ \coth(B\xi)$, the other solutions in [34] can be derived.

4.2. Maccari system

We next consider the Maccari system (2). Let us assume the traveling wave solution of (2) has the form

$$u = e^{i\theta} U(\xi), \qquad v = V(\xi), \quad \theta = px + qy + rt,\ \xi = x + y + ct. \tag{25}$$

Substituting (25) into (2), we have

$$\begin{aligned} &U'' - (r + p^2)U + UV = 0, \\ &(c+1)V' + 2UU'' = 0. \end{aligned} \tag{26}$$

Integrating the second equation in the system and neglecting the constant of integration we find

$$-(c+1)V = U^2. \tag{27}$$

Substituting (27) into the first equation of the system and integrating we find

$$(c+1)U'' - (c+1)(r - p^2)U - U^3 = 0, \tag{28}$$

where prime denotes differentiation with respect to ξ.

By He's semi-inverse method [14], we can arrive at the following variational formulation

$$J = \int_0^\infty \left[-\frac{(c+1)}{2}(U')^2 - \frac{(c+1)(r-p^2)}{2}U^2 - \frac{1}{4}U^4 \right] d\xi. \tag{29}$$

By a Ritz-like method, we search a solitary wave solution in the form

$$U(\xi) = A\ \mathrm{sech}(B\xi), \tag{30}$$

where A and B are unknown constants to be further determined.

Substituting Eq. (30) into Eq. (29), we have

$$\begin{aligned} J &= \int_0^\infty \left[-\frac{A^2B^2(c+1)}{2}\mathrm{sech}^2(B\xi)\tanh^2(B\xi) - \frac{(c+1)(r-p^2)A^2}{2}\mathrm{sech}^2(B\xi) - \frac{A^4}{4}\mathrm{sech}^4(B\xi) \right] d\xi \\ &= -\frac{A^2B(c+1)}{6} - \frac{(c+1)(r-p^2)A^2}{2B} - \frac{A^4}{6B}. \end{aligned} \tag{31}$$

Making J stationary with A and B results in

$$\frac{\partial J}{\partial A} = -\frac{AB(c+1)}{3} - \frac{(c+1)(r-p^2)A}{B} - \frac{2A^3}{3B} = 0, \tag{32}$$

$$\frac{\partial J}{\partial B} = -\frac{A^2(c+1)}{6} + \frac{(c+1)(r-p^2)A^2}{2B^2} + \frac{A^4}{6B^2} = 0. \tag{33}$$

From Eqs. (32) and (33), we get

$$A = i\sqrt{2(c+1)(p^2 - r)} \tag{34}$$

and

$$B = \sqrt{r - p^2}, \qquad p^2 - r < 0. \tag{35}$$

The soliton solutions are, therefore, obtained as follows

$$\begin{aligned} &U(x,y,t) = i\sqrt{2(c+1)(p^2-r)}\ \mathrm{sech}\left[\sqrt{r-p^2}(x+y+ct)\right], \\ &V(x,y,t) = 2(p^2-r)\ \mathrm{sech}^2\left[\sqrt{r-p^2}(x+y+ct)\right], \\ &u(x,y,t) = \left(i\sqrt{2(c+1)(p^2-r)}\ \mathrm{sech}\left[\sqrt{r-p^2}(x+y+ct)\right]\right)\exp[i(px+qy+rt)], \\ &v(x,y,t) = 2(p^2-r)\ \mathrm{sech}^2\left[\sqrt{r-p^2}(x+y+ct)\right]. \end{aligned} \tag{36}$$

The solutions (36) are the same as Eq. (5.11) in [34] respectively. If we take the solution in the form $U(\xi) = A\,\mathrm{csch}(B\xi)$, $U(\xi) = A\ \tanh(B\xi)$ and $U(\xi) = A\ \coth(B\xi)$, the other solutions in [34] can be derived.

5. Using the (G'/G)-expansion method

In this section, we apply the $\left(\frac{G'}{G}\right)$-expansion method to solve the coupled Higgs equation and the Maccari system.

5.1. Coupled Higgs equation

We begin first with the coupled Higgs equation (1). Using the wave variables

$$u = \mathrm{e}^{\mathrm{i}\theta} U(\xi), \qquad v = V(\xi), \quad \theta = px + rt,\ \xi = x + ct. \tag{37}$$

Substituting (37) into (1), we have

$$\begin{aligned} &(c^2 - 1)U'' + (p^2 - r^2)U - 2UV + U^3 = 0, \\ &(c^2 + 1)V'' - 2(U')^2 - 2UU'' = 0. \end{aligned} \tag{38}$$

Integrating the second equation in the system and neglecting the constant of integration we find

$$(c^2 + 1)V = U^2. \tag{39}$$

Substituting (39) into the first equation of the system and integrating we find

$$(c^4 - 1)U'' + (c^2 + 1)(p^2 - r^2)U + (c^2 - 1)U^3 = 0, \tag{40}$$

where prime denotes differentiation with respect to ξ. By using (11) and balancing U'' terms with U^3 in (40) gives

$$m + 2 = 3m, \tag{41}$$

so that

$$m = 1. \tag{42}$$

The (G'/G)-expansion method (11) admits the use of the finite expansion

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{43}$$

Substituting Eq. (43) into (40), collecting all terms with the powers in $\left(\frac{G'}{G}\right)^i$ $(i = 0, 1, \dots, 3)$ and setting each of the obtained coefficients for $\left(\frac{G'}{G}\right)^i$ to zero yields the following set of algebraic equations with respect to a_0, a_1 and r:

$$\left(\frac{G'}{G}\right)^0 : -a_1\lambda\mu + c^4 a_1\lambda\mu + p^2 a_0 - r^2 a_0 + c^2 a_0^3 - a_0^3 + c^2 p^2 a_0 - c^2 r^2 a_0 = 0,$$

$$\left(\frac{G'}{G}\right)^1 : -a_1\lambda^2 - 2a_1\mu - 3a_1 a_0^2 + p^2 a_1 - r^2 a_1 + c^4 a_1\lambda^2 + 2c^4 a_1\mu + c^2 p^2 a_1 - c^2 r^2 a_1 + 3c^2 a_1 a_0^2 = 0,$$

$$\left(\frac{G'}{G}\right)^2 : -3a_1\lambda - 3a_1^2 a_0 + 3c^4 a_1\lambda + 3c^2 a_1^2 a_0 = 0,$$

$$\left(\frac{G'}{G}\right)^3 : -2a_1 - a_1^3 + 2c^4 a_1 + c^2 a_1^3 = 0.$$

Solving this system by Maple gives

$$a_0 = \pm\mathrm{i}\sqrt{1 + \frac{p^2 - r^2}{\lambda^2 - 4\mu}}\lambda, \qquad a_1 = \pm 2\mathrm{i}\sqrt{1 + \frac{p^2 - r^2}{\lambda^2 - 4\mu}}, \qquad c = \pm\sqrt{1 + 2\frac{p^2 - r^2}{\lambda^2 - 4\mu}}, \tag{44}$$

where λ, μ, p and r are arbitrary constants. Substituting Eq. (44) into Eq. (43) yields

$$U(\xi) = \pm\mathrm{i}\sqrt{1 + \frac{p^2 - r^2}{\lambda^2 - 4\mu}}\lambda \pm 2\mathrm{i}\sqrt{1 + \frac{p^2 - r^2}{\lambda^2 - 4\mu}} \left(\frac{G'}{G}\right). \tag{45}$$

Substituting general solutions of Eq. (12) into (45) we have three types of traveling wave solutions of the coupled Higgs equation as follows:

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solutions

$$\begin{aligned}
U_1(\xi) &= \pm i\sqrt{(\lambda^2-4\mu)\left[1+\frac{p^2-r^2}{\lambda^2-4\mu}\right]}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right),\\
V_1(\xi) &= -\frac{\lambda^2-4\mu}{2}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2,\\
u_1(\xi) &= \pm i\sqrt{(\lambda^2-4\mu)\left[1+\frac{p^2-r^2}{\lambda^2-4\mu}\right]}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)\exp[i(px+rt)],\\
v_1(\xi) &= -\frac{\lambda^2-4\mu}{2}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2.
\end{aligned}\tag{46}$$

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solutions

$$\begin{aligned}
U_2(\xi) &= \pm i\sqrt{(4\mu-\lambda^2)\left[1+\frac{p^2-r^2}{\lambda^2-4\mu}\right]}\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right),\\
V_2(\xi) &= -\frac{4\mu-\lambda^2}{2}\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2,\\
u_2(\xi) &= \pm i\sqrt{(4\mu-\lambda^2)\left[1+\frac{p^2-r^2}{\lambda^2-4\mu}\right]}\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)\exp[i(px+rt)],\\
v_2(\xi) &= -\frac{4\mu-\lambda^2}{2}\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2,
\end{aligned}\tag{47}$$

where $\xi = x + \left(\pm\sqrt{1+2\frac{p^2-r^2}{\lambda^2-4\mu}}\right)t$, for (46) and (47). In solutions (46) and (47), C_1 and C_2 are left as free parameters. In particular, if in (46), we take $C_1 \neq 0$, $C_2 = \mu = 0$ and $\lambda = 2$, then we obtain

$$\begin{aligned}
U_1(x,y,t) &= \pm i\sqrt{2(c^2+1)}\tanh\left(x+\sqrt{\frac{p^2-r^2}{2}+1t}\right),\\
V_1(x,y,t) &= -2\tanh^2\left(x+\sqrt{\frac{p^2-r^2}{2}+1t}\right),\\
u_1(x,y,t) &= \pm i\sqrt{2(c^2+1)}\tanh\left(x+\sqrt{\frac{p^2-r^2}{2}+1t}\right)\exp[i(px+rt)],\\
v_1(x,y,t) &= -2\tanh^2\left(x+\sqrt{\frac{p^2-r^2}{2}+1t}\right).
\end{aligned}\tag{48}$$

Also, if we take $C_1 = 0$ and $C_2 \neq 0$, the solutions in terms of coth can be derived. which are the solitary wave solutions of the coupled Higgs equation. The solutions (48) are the same as Eq. (4.21) in [34] respectively. Therefore the solutions in [34] are only a special case of the our solutions.

5.2. Maccari system

We next consider the Maccari system (2). Let us assume the traveling wave solution of (2) has the form

$$u = e^{i\theta}U(\xi), \qquad v = V(\xi), \quad \theta = px + qy + rt,\ \xi = x + y + ct. \tag{49}$$

Substituting (49) into (2), we have

$$\begin{aligned}
&U'' - (r+p^2)U + UV = 0,\\
&(c+1)V' + 2UU'' = 0.
\end{aligned}\tag{50}$$

Integrating the second equation in the system and neglecting the constant of integration we find

$$-(c+1)V = U^2. \tag{51}$$

Substituting (51) into the first equation of the system and integrating we find

$$(c+1)U'' - (c+1)(r-p^2)U - U^3 = 0, \tag{52}$$

where prime denotes differentiation with respect to ξ. By using (11) and balancing U'' terms with U^3 in (52) gives

$$m+2 = 3m, \tag{53}$$

so that

$$m = 1. \tag{54}$$

The (G'/G)-expansion method (11) admits the use of the finite expansion

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{55}$$

Substituting Eq. (55) into (52), collecting all terms with the powers in $\left(\frac{G'}{G}\right)^i$ $(i = 0, 1, \ldots, 3)$ and setting each of the obtained coefficients for $\left(\frac{G'}{G}\right)^i$ to zero yields the following set of algebraic equations with respect to a_0, a_1 and r:

$$\left(\frac{G'}{G}\right)^0 : ca_1\lambda\mu + a_1\lambda\mu - cra_0 + cp^2a_0 - ra_0 + p^2a_0 - {a_0}^3 = 0,$$

$$\left(\frac{G'}{G}\right)^1 : ca_1\lambda^2 + 2ca_1\mu + a_1\lambda^2 + 2a_1\mu - cra_1 + cp^2a_1 - ra_1 + p^2a_1 - 3a_1a_0^2 = 0,$$

$$\left(\frac{G'}{G}\right)^2 : 3ca_1\lambda + 3a_1\lambda - 3a_1^2a_0 = 0,$$

$$\left(\frac{G'}{G}\right)^3 : 2ca_1 + 2a_1 - a_1^3 = 0.$$

Solving this system by Maple gives

$$a_0 = \pm\frac{\lambda}{2}\sqrt{2(c+1)}, \qquad a_1 = \pm\sqrt{2(c+1)}, \qquad r = -\frac{\lambda^2}{2} + 2\mu + p^2, \tag{56}$$

where λ, μ, p and r are arbitrary constants. Substituting Eq. (56) into Eq. (55) yields

$$U(\xi) = \pm\frac{\lambda}{2}\sqrt{2(c+1)} \pm \sqrt{2(c+1)}\left(\frac{G'}{G}\right). \tag{57}$$

Substituting general solutions of Eq. (12) into (57) we have three types of traveling wave solutions of the Maccari system as follows:

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solutions

$$\begin{aligned}
U_1(\xi) &= \pm\frac{1}{2}\sqrt{2(c+1)(\lambda^2-4\mu)}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right),\\
V_1(\xi) &= -(\lambda^2-4\mu)\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2,\\
u_1(\xi) &= \pm\frac{1}{2}\sqrt{2(c+1)(\lambda^2-4\mu)}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)\\
&\quad\times\exp\left[\mathrm{i}\left(px + qy + \left(-\frac{\lambda^2}{2} + 2\mu + p^2\right)t\right)\right],\\
v_1(\xi) &= -(\lambda^2-4\mu)\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2.
\end{aligned} \tag{58}$$

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solutions

$$
\begin{aligned}
U_2(\xi) &= \pm\frac{1}{2}\sqrt{2(c+1)(4\mu-\lambda^2)}\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right),\\
V_2(\xi) &= -(\lambda^2-4\mu)\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2,\\
u_2(\xi) &= \pm\frac{1}{2}\sqrt{2(c+1)(4\mu-\lambda^2)}\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)\\
&\quad\times\exp\left[\mathrm{i}\left(px+qy+\left(-\frac{\lambda^2}{2}+2\mu+p^2\right)t\right)\right],\\
v_2(\xi) &= -(\lambda^2-4\mu)\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2.
\end{aligned}
\tag{59}
$$

When $\lambda^2 - 4\mu = 0$, we obtain the rational function traveling wave solutions

$$
\begin{aligned}
U_3(\xi) &= \frac{\sqrt{2(c+1)}C_2}{C_1+C_2\xi},\\
V_3(\xi) &= \frac{2{C_2}^2}{(C_1+C_2\xi)^2},\\
u_3(\xi) &= \frac{\sqrt{2(c+1)}C_2}{C_1+C_2\xi}\exp\left[\mathrm{i}(px+qy+\left(-\frac{\lambda^2}{2}+2\mu+p^2\right)t)\right],\\
v_3(\xi) &= \frac{2{C_2}^2}{(C_1+C_2\xi)^2},
\end{aligned}
\tag{60}
$$

where $\xi = x + y + ct$, for (58)–(60).

In solutions (58)–(60), C_1 and C_2 are left as free parameters. It is obvious that hyperbolic, trigonometric and rational solutions were obtained by using the $\left(\frac{G'}{G}\right)$-expansion method, whereas only hyperbolic and trigonometric solutions were obtained in [34] by using the sine–cosine method and the tanh–coth method. In particular, if in (58), we take $C_1 \neq 0$, $C_2 = \mu = 0$ and $\lambda = 2$, then we obtain

$$
\begin{aligned}
U_1(x,y,t) &= \pm\sqrt{2(c+1)}\tanh(x+y+ct),\\
V_1(x,y,t) &= -2\tanh^2(x+y+ct),\\
u_1(x,y,t) &= \left(\pm\sqrt{2(c+1)}\tanh(x+y+ct)\right)\exp[\mathrm{i}(px+qy+(p^2-2)t)],\\
v_1(x,y,t) &= -2\tanh^2(x+y+ct).
\end{aligned}
\tag{61}
$$

Also, if we take $C_1 = 0$ and $C_2 \neq 0$, the solutions in terms of coth can be derived, which are the solitary wave solutions of the Maccari system. The solutions (61) are the same as Eq. (5.21) in [34] respectively. Therefore the solutions in [34] are only a special case of the our solutions.

6. Conclusions

We established variational formulations for the coupled Higgs equation and the Maccari system by He's semi-inverse method. It is obvious that the employed approach is useful and manageable and remarkably simple to find various kinds of solitary solutions. Also the (G'/G)-expansion method was used to conduct an analytic study on the coupled Higgs equation and Maccari system. The exact traveling wave solutions being determined in this study are more general, and it is not difficult to arrive at some known analytic solutions for certain choices of the parameters C_1 and C_2. Compared with the methods used in [27,34], one can see that the (G'/G)-expansion method is not only simple and straightforward, but also avoids tedious calculations. Moreover, the methods are capable of greatly minimizing the size of computational work compared to other existing techniques.

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