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Discrete Mathematics 243 (2002) 195–200

DISCRETE MATHEMATICS

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Note

Path factors in claw-free graphs

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Received 14 June 2000; revised 21 February 2001; accepted 5 March 2001

Abstract

A graph G is called claw-free if G has no induced subgraph isomorphic to $K_{1,3}$. We prove that if G is a claw-free graph with minimum degree at least d, then G has a path factor such that the order of each path is at least $d+1$.

Keywords: Path factor; Claw-free

1. Introduction

In this paper, we consider finite undirected graphs G without loops or multiple edges. The complete bipartite graph $K_{1,3}$ is called a *claw*, and G is said to be *claw-free* if G has no induced subgraph isomorphic to $K_{1,3}$. A *path factor* is a spanning subgraph whose components are paths. For a positive integer k, $P_{\geqslant k}$-factor means a path factor such that each component has at least k vertices.

Our main result is the following theorem.

Theorem 1. *Let G be a claw-free graph with $\delta(G) \geqslant d$. Then G has a $P_{\geqslant d+1}$-factor.*

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0012-365X/02/$ - see front matter
PII: S0012-365X(01)00214-X

We now list some of the known results concerning a $P_{\geqslant k}$-factor. Let $i(G)$ denote the number of isolated vertices in a graph G. Let $w(G)$ denote the number of components of a graph G.

The following is a classical result proved by Akiyama et al. [1].

Theorem 2. *A graph G has a $P_{\geqslant 2}$-factor if and only if $i(G-S)$ is at most $2|S|$ for every subset S of $V(G)$.*

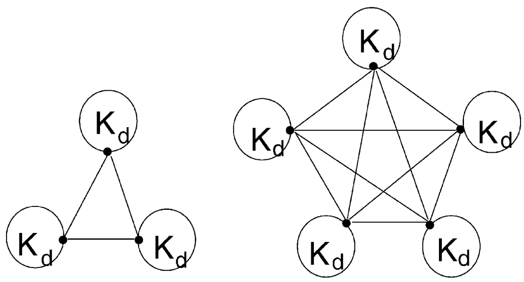
Recently, Kaneko [5] proved the following theorem.

Theorem 3. *A graph G has a $P_{\geqslant 3}$-factor if and only if $C_S(G-T)$ is at most $2|T|$ for every subset T of $V(G)$, where $C_S(G)$ denotes the number of so-called sun components of a graph G (see in [5]).*

Very recently, Hanazawa et al. [4] proved the following theorem.

Theorem 4. *Let G be a connected bipartite graph of order at least 4. If $w(G-S) < \frac{4}{3}|S|$ for every subset S of $V(G)$ with $|S| \geqslant 2$, then G has a $P_{\geqslant 4}$-factor.*

The bound "$d+1$" in Theorem 1 is sharp in the sense that we cannot replace $d+1$ by $d+2$. If we allow G to be disconnected, this can be seen by $K_{d+1} \cup K_{d+1} \cup \cdots$. Even if we require G to be connected, the following examples show the sharpness of the bound $d+1$.

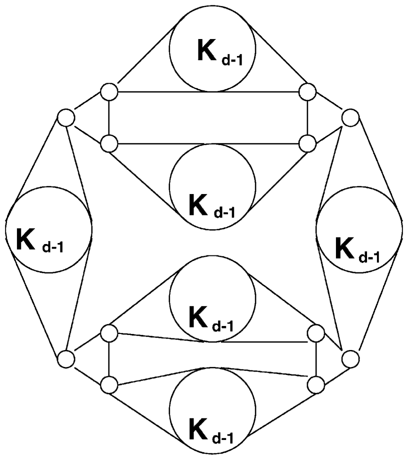


It is easy to see that the graphs above are claw-free graphs with minimum degree at least $d-1$, but have no path-factor not containing a path of order less than or equal to $d+1$.

If we add the assumption "2-connected", the bound "$d+1$" may not be best possible. We conjecture the following.

Conjecture 1. Let G be a 2-connected claw-free graph with $\delta(G) \geqslant d$. Then G has a $P_{\geqslant 3d+3}$-factor.

If Conjecture 1 is true, the assumption that the bound "$3d + 3$" is best possible, is shown by the following example:



It is easy to see that the graph above is a claw-free graph with minimum degree at least d and there is no path factor not containing a path of order less than or equal to $3d + 3$.

For graph theoretic notation not defined in this paper, we refer the reader to [2]. We denote by $\delta(G)$ the minimum degree of a graph G. Let $N_G(x)$ denote the set of vertices adjacent to x in G. With a slight abuse of notation, for a subgraph H of G and a vertex $x \in V(G) - V(H)$, $N_H(x) = N_G(x) \cap V(H)$.

Given a subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. For two disjoint subsets A and B of $V(G)$, we denote by $E_G(A, B)$ the set of edges of G joining A to B. The number of vertices in a maximum independent set of vertices in G is denoted by $\alpha(G)$.

2. Proof of Theorem 1

In order to prove Theorem 1, we need the following.

Theorem A (Chvátal and Erdős [3]). *Let $k \geqslant 1$ be an integer and G a k-connected graph. If $\alpha(G) \leqslant k + 1$, then G has a hamiltonian path.*

We now prove Theorem 1. Let G be as in Theorem 1. Let P_1 be a longest path in G and let P_2 be a longest path in $G - V(P_1)$. By repeating this procedure, we obtain paths $P_1, \ldots, P_r$ such that $V(P_1 \cup \cdots \cup P_r) = V(G)$ and $P_1 \cap \cdots \cap P_r = \emptyset$. Possibly, for some q, $|V(P_i)| = 1$ for $q \leqslant i \leqslant r$. Note that $|V(P_1)| \geqslant \cdots \geqslant |V(P_r)|$. If $|V(P_r)| \geqslant d + 1$, then the proof is complete. Thus we may assume that $|V(P_r)| \leqslant d$.

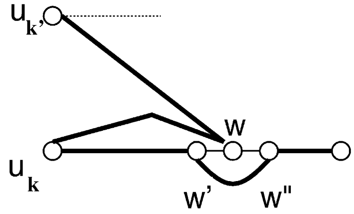


Fig. 1. Proof of Claim 2.

Let u_i be an endvertex of P_i for $1 \leqslant i \leqslant r-1$. The following claim immediately follows from the maximality of $|V(P_i)|$.

Claim 1. *If* $\{w\} \in N_G(u_k) \cap V(P_i)$ *for* $1 \leqslant i < k \leqslant r$, *then* w *is not an endvertex of* P_i, *and* $w'u_k, w''u_k \notin E(G)$ *and* $w'w'' \in E(G)$, *where* w' *and* w'' *are the neighbors of* w *in* P_i.

Proof. By the maximality of $|V(P_i)|$, w is not an endvertex of P_i and $w', w'' \notin N_G(u_k)$. Since $G[\{w, u_k, w', w''\}]$ does not induce a claw, this implies $w'w'' \in E(G)$. □

Next, we prove the following claims.

Claim 2. $N_G(u_k) \cap N_G(u_{k'}) = \emptyset$ *for any two endvertices* $u_k \in P_k$ *and* $u_{k'} \in P_{k'}$, $1 \leqslant k < k' \leqslant r$.

Proof. By way of contradiction, suppose that for some k and k' with $1 \leqslant k < k' \leqslant r$, $N_G(u_k) \cap N_G(u_{k'}) \neq \emptyset$. By the maximality of $|V(P_j)|$, $N_G(u_k) \cap V(P_j) = \emptyset$ for any j with $j > k$. Let $w \in N_G(u_k) \cap N_G(u_{k'})$. Then $w \in V(P_l)$ for some l with $1 \leqslant l \leqslant k$. Assume $l < k$. By Claim 1, w is not an endvertex of P_l. Let w' be a neighbor of w in P_l. Again, by Claim 1, $u_k w', u_{k'} w' \notin E(G)$. But $G[\{u_k, u_{k'}, w, w'\}]$ induces a claw, which contradicts the fact that G is claw-free.

Assume now that $l = k$. By Claim 1, w is not an endvertex of P_k. Let w' and w'' be the neighbors of w in P_l. By Claim 1, $w'w'' \in E(G)$. But we can find a longer path containing $V(P_k) \cup \{u_{k'}\} \cup V(P_{k'})$. See Fig. 1. This completes the proof. □

Claim 3. *Let* i *be an integer with* $1 \leqslant i < j \leqslant r$, *and write* $V(P_i) - (\bigcup_{j \leqslant k \leqslant r} N_G(u_k)) = \{z_1, \ldots, z_m\}$ *so that* $z_1, \ldots, z_m$ *occur on* P_i *in this order. Then, for each* t *with* $1 \leqslant t \leqslant m-1$, $z_t z_{t+1} \in E(G)$.

Proof. We proceed by backward induction on j. If $j = r$, then the result immediately follows from Claim 1. Assume $j < r$. Write $V(P_i) - (\bigcup_{j+1 \leqslant k \leqslant r} N_G(u_k)) = \{y_1, \ldots, y_l\}$ so that $y_1, \ldots, y_l$ occur on P_i in this order. By the induction hypothesis, we know that, for each t' with $1 \leqslant t' \leqslant l-1$, $y_{t'} y_{t'+1} \in E(G)$. Take a vertex $y_s \in N_G(u_j)$. By Claim 1, we know that y_1 and y_l are ends of P_i. Hence $s \neq 1, l$. It suffices to prove that

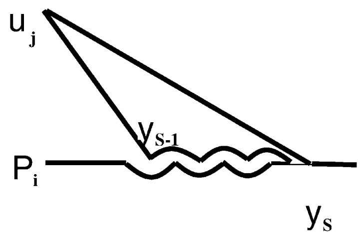


Fig. 2. Proof of Claim 3; the case where n is odd.

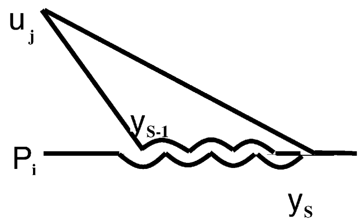


Fig. 3. Proof of Claim 3; the case where n is even.

$y_{s-1}, y_{s+1} \notin N_G(u_j)$ because if $y_{s-1}, y_{s+1} \notin N_G(u_j)$, then, since $G[\{y_{s-1}, y_s, y_{s+1}, u_j\}]$ does not induce a claw, we have $y_{s-1}y_{s+1} \in E(G)$, which implies the conclusion of the claim. Assume $y_{s-1} \in N_G(u_j)$ or $y_{s+1} \in N_G(u_j)$. Without loss of generality, we may assume $y_{s-1}u_j \in E(G)$. Let $(p_2, \ldots, p_n)$ be the segment of P_i between y_{s-1} and y_s with $p_2 = y_{s-1}$ and $p_n = y_s$ and let p_1 be the predecessor of p_2 on P_i. Since $\{p_2, \ldots, p_{n-1}\} \in \bigcup_{j \leqslant k \leqslant r} N_G(u_k)$, it follows from Claim 2 that, for any q with $1 \leqslant q \leqslant n-2$, $p_q p_{q+2} \in E(G)$. But, then we can find a longer path P_i' such that $V(P_i') = V(P_i) \cup \{u_j\}$, which contradicts the choice of P_i, see Figs. 2 and 3. □

Claim 4. *Let i be an integer with $1 \leqslant i \leqslant r$. Then there exists a path P_i' such that $V(P_i') = (V(P_i) - \bigcup_{i+1 \leqslant k \leqslant r} N_G(u_k)) \cup N_G(u_i)$.*

Proof. Write $V(P_i) - (\bigcup_{i+1 \leqslant k \leqslant r} N_G(u_k)) = \{z_1, \ldots, z_m\}$ so that $z_1, \ldots, z_m$ occur on P_i in this order. Then, by Claim 3, for each t with $1 \leqslant t \leqslant m-1$, $z_t z_{t+1} \in E(G)$. Define a path P' by $P' = (z_1, \ldots, z_m)$. By Claim 1, we may assume $u_i = z_1$. Since G is claw-free, $\alpha(G[N_{G-P'}(u_i)]) \leqslant 2$ holds. If $N_{G-P'}(u_i) = \emptyset$, then we are done. Thus, we may assume $N_{G-P'}(u_i) \neq \emptyset$. If $N_{G-P'}(u_i)$ is connected, then $N_{G-P'}(u_i)$ has a hamiltonian path Q by Theorem A, and hence we obtain a desired path P_i' by adding Q to P' at u_i. Thus, we may assume that $N_{G-P'}(u_i)$ is not connected. Since $\alpha(N_{G-P'}(u_i)) \leqslant 2$, $N_{G-P'}(u_i)$ consists of two components A and B, where A and B are complete. Let $V(A) = \{a_1, \ldots, a_s\}$ and $V(B) = \{b_1, \ldots, b_t\}$. If $P' = \{u_i\}$, then we can simply let $P_i' = a_1, \ldots, a_s, u_i, b_1, \ldots, b_t$. Thus we may assume $|P_i| \geqslant 2$. Since G is claw-free and $E_G(A, B) = \emptyset$, we obtain either $A \subseteq N_G(z_2)$ or $B \subseteq N_G(z_2)$. Without loss of generality, we may assume $B \subseteq N_G(z_2)$.

We now obtain a desired path P_i' by letting $P_i' = a_1, \ldots, a_t, u_i, b_1, \ldots, b_s, z_2, z_3, \ldots, z_m$. □

Note that $|P_i'| \geqslant d+1$ for any i with $1 \leqslant i \leqslant r$ since $N_G(u_i) \subseteq V(P_i')$ by Claim 4 and since $u_i \in V(P_i')$ by Claim 1. Further by Claim 2, $V(P_i') \cap V(P_j') = \emptyset$ for any i, j with $1 \leqslant i < j \leqslant k$. Since it immediately follows from Claim 4 that $V(G) = V(P_1' \cup P_2' \cup \cdots \cup P_r')$, this means that $P_1' \cup P_2' \cup \cdots \cup P_r'$ form a $P_{\geqslant d+1}$-factor. This completes the proof. □

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