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# The moment problem associated with the Stieltjes–Wigert polynomials

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**Abstract**

We consider the indeterminate Stieltjes moment problem associated with the Stieltjes–Wigert polynomials. After a presentation of the well-known solutions, we study a transformation $T$ of the set of solutions. All the classical solutions turn out to be fixed under this transformation but this is not the case for the so-called canonical solutions. Based on generating functions for the Stieltjes–Wigert polynomials, expressions for the entire functions $A$, $B$, $C$, and $D$ from the Nevanlinna parametrization are obtained. We describe $T^{(n)}(\mu)$ for $n \in \mathbb{N}$ when $\mu = \mu_0$ is a particular $N$-extremal solution and explain in detail what happens when $n \to \infty$.


*Keywords:* Indeterminate moment problems; Stieltjes–Wigert polynomials; Nevanlinna parametrization

## 1. Introduction

T.J. Stieltjes was the first to give examples of indeterminate moment problems. In [18] he pointed out that if $f$ is an odd function satisfying $f(u + 1/2) = \pm f(u)$, then

$$\int_0^\infty u^n u^{-\log u} f(\log u)\, du = 0$$

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for all $n \in \mathbb{Z}$. In particular,

$$\int_0^\infty u^n u^{-\log u} \sin(2\pi \log u)\, du = 0, \quad n \in \mathbb{Z},$$

so independent of $\lambda$ we have

$$\int_0^\infty \frac{1}{\sqrt{\pi}} u^n u^{-\log u}\big(1 + \lambda \sin(2\pi \log u)\big)\, du = \frac{1}{\sqrt{\pi}} \int_0^\infty u^n u^{-\log u}\, du = e^{(n+1)^2/4}.$$

In other words, for $\lambda \in [-1, 1]$ the densities

$$w_\lambda(u) = \frac{1}{\sqrt{\pi}} u^{-\log u}\big(1 + \lambda \sin(2\pi \log u)\big), \quad u > 0,$$

have the same moments.

More generally, one could consider the weight function[1]

$$w(x) = \frac{1}{\sqrt{\pi}} k x^{-k^2 \log x}, \quad x > 0, \tag{1.1}$$

which has the moments

$$s_n = \int_0^\infty x^n w(x)\, dx = e^{(n+1)^2/4k^2}. \tag{1.2}$$

Here $k > 0$ is a constant (and $k = 1$ corresponds to Stieltjes' example). This was done by Wigert in [20]. He succeeded in finding the orthonormal polynomials $(P_n)$ corresponding to $w(x)$ using the general formula

$$P_n(x) = \frac{1}{\sqrt{D_{n-1} D_n}} \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}, \quad n \geqslant 1, \tag{1.3}$$

where $(s_n)$ denotes the moment sequence and $D_n = \det((s_{i+j})_{0 \leqslant i,j \leqslant n})$ denotes the Hankel determinant. If we set $q = e^{-1/2k^2}$, the moment sequence (1.2) has the form $s_n = q^{-(n+1)^2/2}$ and it is readily seen that all the determinants in (1.3) are of the Vandermonde type. Following the notation of Gasper and Rahman [13] for basic hypergeometric series, Wigert's expressions are

$$P_n(x) = (-1)^n \frac{q^{n/2+1/4}}{\sqrt{(q;q)_n}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2+k/2} x^k, \tag{1.4}$$

[1] Note that $w(x)/x$ is the density of the log-normal distribution with parameter $\sigma^2 = 1/2k^2 > 0$.

cf. Szegö [19] and Chihara [9], where these polynomials are called the Stieltjes–Wigert polynomials. Wigert also considered the behaviour of $P_n(x)$ when $n \to \infty$ and proved that

$$(-1)^n q^{-n/2} P_n(x) \to \frac{q^{1/4}}{\sqrt{(q;q)_\infty}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k^2+k/2}}{(q;q)_k} x^k \quad \text{for } n \to \infty. \tag{1.5}$$

The convergence is uniform on compact subsets of $\mathbb{C}$.

Later, Chihara [10] pointed out that the weight function $w(x)$ satisfies the functional equation

$$w(xq) = \sqrt{q}\, x\, w(x), \quad x > 0, \tag{1.6}$$

and this observation led to the discovery of a family of discrete measures with the same moments as $w(x)$. The discrete version of the functional equation (1.6) is the following. Suppose that $\mu$ is a discrete measure. Then $c > 0$ is a mass point of $\mu$ exactly if $cq$ likewise is a mass point of $\mu$ and $\mu(\{cq\}) = cq\sqrt{q}\,\mu(\{c\})$. This property is certainly satisfied by the measures

$$\mu_c = \frac{1}{\sqrt{q}M(c)} \sum_{n=-\infty}^{\infty} c^n q^{n+n^2/2} \varepsilon_{cq^n}, \quad c > 0, \tag{1.7}$$

where $M(c)$ is some constant depending on $c$ and $\varepsilon_x$ denotes the Dirac measure at the point $x$. Setting $M(c) = (-cq\sqrt{q}, -1/c\sqrt{q}, q; q)_\infty$, it follows by the Jacobi triple product identity [2, p. 497]

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = (x, q/x, q; q)_\infty, \quad x \neq 0, \tag{1.8}$$

and the translation invariance of $\sum_{-\infty}^{\infty}$ that each $\mu_c$ has the moments $q^{-(n+1)^2/2}$.

In [5] Askey and Roy presented a symmetric $q$-analogue of the usual beta integral. With $a$ and $b$ instead of $q^{a+c}$ and $q^{b-c}$, their formula reads

$$\int_0^\infty t^{c-1} \frac{(-at, -bq/t; q)_\infty}{(-t, -q/t; q)_\infty}\, dt = \frac{(ab, q^c, q^{1-c}; q)_\infty}{(q, aq^{-c}, bq^c; q)_\infty} \frac{\pi}{\sin \pi c},$$
$$c > 0,\ |a| < q^c,\ |b| < q^{-c}. \tag{1.9}$$

When $a = b = 0$, (1.9) simplifies to

$$\int_0^\infty \frac{t^{c-1}}{(-t, -q/t; q)_\infty}\, dt = \frac{(q^c, q^{1-c}; q)_\infty}{(q;q)_\infty} \frac{\pi}{\sin \pi c}, \quad c > 0,$$

and we have

$$\int_0^\infty t^n \frac{t^{c-1}}{(-t, -q/t; q)_\infty}\, dt = q^{-cn-\binom{n}{2}} \frac{(q^c, q^{1-c}; q)_\infty}{(q;q)_\infty} \frac{\pi}{\sin \pi c}, \quad c > 0. \tag{1.10}$$

Setting $c = 3/2$, the right-hand side in (1.10) becomes

$$q^{-n-n^2/2}\frac{(q^{3/2}, q^{-1/2}; q)_\infty}{(q; q)_\infty}(-\pi) = q^{-(n+1)^2/2}\frac{\pi(\sqrt{q}; q)_\infty^2}{(q; q)_\infty}$$

so the weight function

$$\widetilde{w}(x) = \frac{(q; q)_\infty}{\pi(\sqrt{q}; q)_\infty^2}\frac{\sqrt{x}}{(-x, -q/x; q)_\infty}, \quad x > 0, \tag{1.11}$$

has the moments $q^{-(n+1)^2/2}$. This observation was made by Askey in [4] and introduces a new weight function for the polynomials (1.4).

As a basic knowledge of the theory of the moment problem we shall refer to Akhiezer [1]. Recall that the Nevanlinna parametrization gives a one-to-one correspondence between the set of Pick functions (including $\infty$) and the set of solutions to an indeterminate Hamburger moment problem. If $\mu_\varphi$ is the solution corresponding to the Pick function $\varphi$, then the Stieltjes transform of $\mu_\varphi$ is given by

$$\int_{\mathbb{R}} \frac{1}{t-x}\, d\mu_\varphi(t) = -\frac{A(x)\varphi(x) - C(x)}{B(x)\varphi(x) - D(x)}, \quad x \in \mathbb{C} \setminus \mathbb{R}, \tag{1.12}$$

where $A$, $B$, $C$, and $D$ are certain entire functions defined in terms of the orthonormal polynomials $(P_n)$ and $(Q_n)$ by

$$A(x) = x\sum_{n=0}^{\infty} Q_n(0)Q_n(x), \qquad C(x) = 1 + x\sum_{n=0}^{\infty} P_n(0)Q_n(x),$$

$$B(x) = -1 + x\sum_{n=0}^{\infty} Q_n(0)P_n(x), \qquad D(x) = x\sum_{n=0}^{\infty} P_n(0)P_n(x).$$

According to the Stieltjes–Perron inversion formula, the measure $\mu_\varphi$ is uniquely determined by its Stieltjes transform.

The solutions corresponding to the Pick function being a real constant (or $\infty$) are called $N$-extremal and the solutions corresponding to the Pick function being a real rational function are called canonical. To be precise, the solutions are called $n$-canonical or canonical of order $n$ if the Pick function is a real rational function of degree $n$. Thus, canonical of order 0 is the same as $N$-extremal. It is well-known that canonical solutions are discrete. If $\varphi = P/Q$ (assuming that $P$ and $Q$ are polynomials with real coefficients and no common zeros), then $\mu_\varphi$ is supported on the zeros of the entire function $B(x)P(x) - D(x)Q(x)$. In particular, the $N$-extremal solution $\mu_t$ is supported on the zeros of $B(x)t - D(x)$ (or $B(x)$ when $t = \infty$).

Considering a Stieltjes moment problem, of course not every Pick function gives rise to a Stieltjes solution. In this connection the quantity $\alpha \leqslant 0$ defined by

$$\alpha = \lim_{n\to\infty} \frac{P_n(0)}{Q_n(0)} \tag{1.13}$$

plays an important part. As Pedersen proved in [17], the measure $\mu_\varphi$ corresponding to the Pick function $\varphi$ is supported within $[0, \infty)$ precisely if $\varphi$ has an analytic continuation to

$\mathbb{C}\setminus[0,\infty)$ such that $\alpha\leqslant\varphi(x)\leqslant 0$ for $x<0$. In particular, the only $N$-extremal Stieltjes solutions are $\mu_t$ with $\alpha\leqslant t\leqslant 0$. Furthermore, it is well-known that the moment problem is determinate in the sense of Stieltjes exactly if $\alpha=0$.

This paper is organized as follows. In Section 2 we start by adjusting the normalization in order to follow the normalization in Koekoek and Swarttouw [14]. Then we present the well-known solutions to the moment problem and explain how to obtain them. These solutions can also be found in Berg [6,7]. The functional equation $f(xq)=xf(x)$ is of great importance both in connection with absolutely continuous and discrete solutions. A transformation $T$ of the set of solutions is established and we classify the absolutely continuous and discrete fixed points. These include all the well-known absolutely continuous solutions and a wide class of the well-known discrete solutions. However, some of the well-known discrete solutions are only fixed under $T^{(2)}$. A method to construct continuous singular solutions to the moment problem concludes the section. In Section 3 we introduce the Stieltjes–Wigert polynomials. These polynomials are proportional to the orthonormal polynomials and converge uniformly on compact subsets of $\mathbb{C}$ when $n\to\infty$. We show that the zeros of the Stieltjes–Wigert polynomials are very well separated, that is, the ratio between two consecutive zeros is strictly greater than $q^{-2}$. Based on generating functions for the Stieltjes–Wigert polynomials, expressions for the four entire functions from the Nevanlinna parametrization are obtained in terms of their power series expansions. Concerning the canonical solutions to the moment problem an entire function $\Phi$ becomes important. The zeros of $\Phi$ turn out to be closely related to the supports of certain $N$-extremal and canonical solutions. However, the zeros of $\Phi$ cannot be found explicitly but since $\Phi$ is proportional to the limit of the Stieltjes–Wigert polynomials when $n\to\infty$, these zeros are very well separated. Moreover, in the end of the section we get as a corollary that the ratio between two consecutive zeros of $\Phi$ actually converges to $q^{-2}$. The canonical solutions are not fixed points of the transformation $T$ defined in Section 2. We describe $T$ at the level of Pick functions and show that $T$ maps a canonical solution into another canonical solution. For the particular $N$-extremal solution $\mu_0$ we are able to describe $T^{(n)}(\mu_0)$ for each $n\in\mathbb{N}$. There is a difference between $n$ odd and $n$ even. We show that the limits of $T^{(2n+1)}(\mu_0)$ and $T^{(2n+2)}(\mu_0)$ exist when $n\to\infty$ and coincide with already known solutions to the moment problem.

## 2. The classical solutions

Let us start by adjusting the normalization in order to follow the standard reference, Koekoek and Swarttouw [14]. So instead of $w(x)$ we consider the weight function

$$v(x)=\frac{w(x\sqrt{q}\,)}{x},\quad x>0,$$

that is, explicitly we have

$$v(x)=\frac{q^{1/8}}{\sqrt{2\pi\log q^{-1}}}\frac{1}{\sqrt{x}}e^{\frac{1}{2}\frac{(\log x)^2}{\log q}},\quad x>0. \tag{2.1}$$

Note that $v$ satisfies the functional equation

$$v(xq) = xv(x), \quad x > 0 \tag{2.2}$$

and is the density of a probability measure $v$ on $(0, \infty)$ with the moments

$$\int_0^\infty x^n v(x)\,dx = q^{-\binom{n+1}{2}}. \tag{2.3}$$

Using the same procedure as Wigert in [20], we find that the orthonormal polynomials $(P_n)$ associated with the moment sequence (2.3) are given by

$$P_n(x) = (-1)^n \sqrt{\frac{q^n}{(q;q)_n}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2} x^k, \quad n \geqslant 0. \tag{2.4}$$

We stress that

$$P_n(x) = (-1)^n \sqrt{q^n (q;q)_n}\, S_n(x;q),$$

where $S_n(x;q)$ denotes the Stieltjes–Wigert polynomials given by

$$S_n(x;q) = \frac{1}{(q;q)_n} {}_1\varphi_1\left( \begin{matrix} q^{-n} \\ 0 \end{matrix}; q, -q^{n+1}x \right), \quad n \geqslant 0,$$

see Koekoek and Swarttouw [14].

The functional equation (2.2) is important due to the following observation which is also contained in Chihara's paper [11].

**Proposition 2.1.** *Let $f$ be a positive measurable function defined on the interval $(0, \infty)$. If $f$ satisfies the functional equation $f(xq) = xf(x)$ and*

$$\int_0^\infty f(x)\,dx = c \in (0, \infty),$$

*then the absolutely continuous measure with density $\frac{1}{c} f$ has the moments $q^{-\binom{n+1}{2}}$.*

**Remark 2.2.** The conditions in Proposition 2.1 are sufficient but not necessary.

**Proof.** Without loss of generality we can assume that $\int_0^\infty f(x)\,dx = 1$. For if this is not the case, one can simply replace $f$ by $\frac{1}{c} f$. If $f$ satisfies the functional equation $xf(x) = f(xq)$, it is seen by induction that $f$ satisfies the functional equation

$$q^{\binom{n}{2}} x^n f(x) = f\left(xq^n\right) \tag{2.5}$$

for each $n \in \mathbb{Z}$ and, consequently,

$$\int_0^\infty x^n f(x)\,dx = q^{-\binom{n}{2}} \int_0^\infty f\left(xq^n\right) dx = q^{-\binom{n}{2}} q^{-n} \int_0^\infty f(x)\,dx = q^{-\binom{n+1}{2}}.$$

□

So the question is whether we know of any positive and integrable functions on $(0,\infty)$, which satisfy the functional equation (2.2)—besides $v$ of course. At this point the functions $f_c$ given by

$$f_c(x)=\frac{x^{c-1}}{(-q^{1-c}x,-q^c/x;q)_\infty},\quad x>0,$$

become relevant. They certainly satisfy the functional equation (2.2) and by the Askey–Roy $q$-beta integral (1.9), we have

$$\int_0^\infty f_c(x)\,dx=q^{c(c-1)}\frac{(q^c,q^{1-c};q)_\infty}{(q;q)_\infty}\frac{\pi}{\sin\pi c}.$$

Therefore, by Proposition 2.1 the absolutely continuous measures $v_c$ with densities

$$v_c(x)=q^{c(1-c)}\frac{\sin\pi c}{\pi}\frac{(q;q)_\infty}{(q^c,q^{1-c};q)_\infty}\frac{x^{c-1}}{(-q^{1-c}x,-q^c/x;q)_\infty},\quad x>0, \tag{2.6}$$

have the moments (2.3). Since $v_{c+1}=v_c$, it suffices to consider $v_c$ for $c\in(0,1]$.

As Askey stated in [3] (but only for $c=1$), the densities $v_c(x)$ appear to be certain (normalized) accumulation points of the weight function

$$v^{(\alpha)}(x)=\frac{x^\alpha}{(-x;q)_\infty},\quad x>0,$$

for the $q$-Laguerre polynomials when $\alpha\to\infty$. It is well known, see [14], that the $q$-Laguerre polynomials given by

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\varphi_1\left(\begin{matrix}q^{-n}\\ q^{\alpha+1}\end{matrix};q,-q^{n+\alpha+1}x\right),\quad n\geqslant 0,$$

in a suitable way converge to the Stieltjes–Wigert polynomials when $\alpha\to\infty$ and results on convergence at the level of orthogonality measures can be worked out as well. For the precise statements and computations, the reader is referred to [12].

If one should be tempted to look at the graphs of the densities $v$ and $v_c$ for some fixed value of $q$, say $q=1/2$, the variation turns out to be surprisingly small. For a minute one might be afraid that the measures are not different at all. However, the measures cannot coincide because $v_c$ can be considered as a meromorphic function in $\mathbb{C}\setminus\{i\beta\mid\beta\geqslant 0\}$ with simple poles at $-q^{c+n}$ for $n\in\mathbb{Z}$, whereas $v$ can be considered as a holomorphic function in $\mathbb{C}\setminus\{i\beta\mid\beta\geqslant 0\}$.

Let us now return to the functional equation (2.2) and suppose that $f_1$ and $f_2$ are two functions satisfying this equation. If $f_2$ is strictly positive, then the quotient $g=f_1/f_2$ is well defined and it satisfies the simple functional equation

$$g(x)=g(xq),\quad x>0.$$

So the two functions differ at the most by a factor which in a certain sense is periodic—what we shall call $q$-periodic. In other words, if we know one strictly positive solution to the functional equation (2.2), we can get all the others by multiplying with $q$-periodic functions. Therefore, whenever $g$ is a positive, measurable and $q$-periodic function such that

$$\int_0^\infty v(x)g(x)\,dx = c \in (0,\infty),$$

the absolutely continuous measure with density $\frac{1}{c}v(x)g(x)$, $x>0$, has the moments (2.3). This is exactly Stieltjes' observation in full generality—he only considered the case $q=1/2$. Since the sine function is periodic with period $2\pi$, it can be made $q$-periodic by changing the argument to $2\pi\log x/\log q$. In order to get a positive function, just add the constant 1 and obviously the function remains positive and $q$-periodic if the sine term is multiplied by any constant $\lambda$ between $-1$ and 1. It is easily verified that

$$\int_0^\infty v(x)\sin\left(2\pi\frac{\log x}{\log q}\right)dx = 0$$

so for $\lambda\in[-1,1]$, the densities

$$\tilde{v}_\lambda(x) = v(x)\left(1+\lambda\sin\left(2\pi\frac{\log x}{\log q}\right)\right), \quad x>0, \tag{2.7}$$

have the same moments. Note that each $\tilde{v}_\lambda(x)$ is a convex combination of the end points $\tilde{v}_{-1}(x)$ and $\tilde{v}_1(x)$, to be precise

$$\tilde{v}_\lambda(x) = \frac{1-\lambda}{2}\tilde{v}_{-1}(x) + \frac{1+\lambda}{2}\tilde{v}_1(x).$$

After this, let us turn the attention to discrete solutions to the moment problem. Suppose that $f$ is a strictly positive function satisfying the functional equation (2.2) and consider for $c>0$ the discrete measure $\lambda_c$ supported on $\{cq^n \mid n\in\mathbb{Z}\}$ and given by

$$\lambda_c\big(\{cq^n\}\big) = \frac{1}{f(c)L(c)}q^n f\big(cq^n\big), \quad n\in\mathbb{Z}.$$

Here $L(c)$ is a constant which ensures that $\lambda_c$ is a probability measure. Recall from (2.5) that

$$f\big(cq^n\big) = q^{\binom{n}{2}}c^n f(c), \quad n\in\mathbb{Z},$$

so independent of $f$, the measure $\lambda_c$ is given by

$$\lambda_c = \frac{1}{L(c)}\sum_{n=-\infty}^{\infty}(cq)^n q^{\binom{n}{2}}\varepsilon_{cq^n}. \tag{2.8}$$

According to the Jacobi triple product identity (1.8), we have $L(c) = (-cq,-1/c,q;q)_\infty$ and using the translation invariance of $\sum_{-\infty}^{\infty}$, we see that

$$\int_0^\infty x^n\,d\lambda_c(x) = q^{-\binom{n+1}{2}}.$$

Since $\lambda_{c/q} = \lambda_c$, it suffices to consider $\lambda_c$ for $c \in (q, 1]$ and this perfectly agrees with the fact that a function satisfying the functional equation (2.2) is uniquely determined by its values on the interval $(q, 1]$.

The particular solution $\lambda_1$ is supported on the geometric progression $\{q^n \mid n \in \mathbb{Z}\}$ and one could ask if this is the only solution supported within this special set. The answer is in the negative, see [6], where Berg pointed out that for $s \in [-1, 1]$, the measures

$$\kappa_s = \frac{1}{L(1)} \sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}} \bigl(1 + s(-1)^n\bigr) \varepsilon_{q^n} \tag{2.9}$$

have the same moments. To justify this, one has to realize that

$$\sum_{n=-\infty}^{\infty} \bigl(q^n\bigr)^k q^{\binom{n+1}{2}} (-1)^n = 0$$

which is a consequence of the Jacobi triple product identity (1.8). The end points $\kappa_{-1}$ and $\kappa_1$ are supported on $\{q^{2n+1} \mid n \in \mathbb{Z}\}$ and $\{q^{2n} \mid n \in \mathbb{Z}\}$, respectively, and we stress that each $\kappa_s$ can be thought of as a convex combination of $\kappa_{-1}$ and $\kappa_1$, to be precise

$$\kappa_s = \frac{1-s}{2} \kappa_{-1} + \frac{1+s}{2} \kappa_1.$$

On the previous pages we have given a survey of the well-known solutions to the moment problem. To learn even more about the structure of these solutions and to obtain further insight, we shall now introduce a transformation of the set $V$ of solutions. But first some notation. For $a > 0$, let $\tau_a$ denote the map given by $\tau_a(x) = ax$ and recall that the image measure $\tau_a(\mu)$ of a measure $\mu$ on $[0, \infty)$ under $\tau_a$ is defined by

$$\tau_a(\mu)(B) = \mu\bigl(a^{-1}B\bigr)$$

for all Borel sets $B \subset [0, \infty)$.

**Proposition 2.3.** *Suppose that $\mu$ is a measure on $[0, \infty)$ with moments $q^{-\binom{n+1}{2}}$. Then the support of $\nu = \tau_q(qx\,d\mu(x))$ is contained in $[0, \infty)$ and $\nu$ has the moments $q^{-\binom{n+1}{2}}$.*

**Proof.** The proof is straightforward. The support of $\nu$ is certainly contained in $[0, \infty)$ and

$$\int_0^\infty x^n\,d\nu(x) = \int_0^\infty (qx)^n qx\,d\mu(x) = q^{n+1} \int_0^\infty x^{n+1}\,d\mu(x) = q^{-\binom{n+1}{2}}.$$

□

The above proposition gives rise to the following definition.

**Definition 2.4.** We denote by $T : V \mapsto V$ the map given by $T(\mu) = \tau_q(qx\,d\mu(x))$.

A probability measure $\mu$ is a fixed point of $T$ if and only if it satisfies the equation

$$\tau_{q^{-1}}(\mu) = qx\,d\mu(x). \tag{2.10}$$

When $\mu$ is absolutely continuous with density, say $f(x)$, this equation exactly corresponds to the functional equation $f(xq) = xf(x)$, $x > 0$ and when $\mu$ is a discrete measure, the equation tells us that $c > 0$ is a mass point of $\mu$ exactly when $cq$ likewise is a mass point of $\mu$ and $\mu(\{cq\}) = qc\mu(\{c\})$. The latter property is satisfied by the measures $\lambda_c$ in (2.8).

As a matter of fact, we can classify all the absolutely continuous and all the discrete fixed points of $T$. Whenever $g$ is a positive, measurable and $q$-periodic function on $(0, \infty)$ such that

$$\int_0^\infty v(x)g(x)\,dx = 1,$$

the absolutely continuous measure with density $v(x)g(x)$, $x > 0$ is a fixed point of $T$ and every absolutely continuous fixed point of $T$ has this form (for some $g$). The discrete fixed points of $T$ are precisely the countable convex combinations of the measures $\lambda_c$.

So nearly all the solutions presented till now are fixed points of $T$. The only exception is the measures $\kappa_s$ in (2.9) when $s \neq 0$. For $-1 < s < 1$, the support of $\kappa_s$ is the geometric progression $\{q^n \mid n \in \mathbb{Z}\}$ and $T$ has at most one fixed point with this support. However, we know that $\kappa_0 = \lambda_1$ is a fixed point of $T$. In general, it turns out that $T(\kappa_s) = \kappa_{-s}$ so all the measures $\kappa_s$ are fixed points of $T^{(2)}$.

It is worth while dwelling somewhat on Eq. (2.10) since this is the full generalization of the functional equation (2.2). Suppose that $\mu$ is a finite measure on $(0, \infty)$ which satisfies this equation or, equivalently,

$$\mu(qB) = q\int_B x\,d\mu(x)$$

for all Borel sets $B \subset (0, \infty)$. By induction, we have

$$\tau_{q^{-n}}(\mu) = q^{\binom{n+1}{2}} x^n\,d\mu(x), \quad n \in \mathbb{Z},$$

and this means that

$$\int_0^\infty x^n\,d\mu(x) = q^{-\binom{n+1}{2}} \int_0^\infty d\tau_{q^{-n}}(\mu)(x).$$

So if $\mu$ is a probability measure, it surely has the moments (2.3). But furthermore, we see that $\mu$ is uniquely determined by its restriction $\mu|_{(q,1]}$ to the interval $(q, 1]$ or any other interval of the form $(q^{n+1}, q^n]$ for some $n \in \mathbb{Z}$. For if $\mu|_{(q,1]} = \nu$, then

$$\mu|_{(q^{n+1},q^n]} = \tau_{q^n}\bigl(q^{\binom{n+1}{2}} x^n\,d\nu(x)\bigr)$$

for each $n \in \mathbb{Z}$ and $\bigcup_{n=-\infty}^{\infty}(q^{n+1}, q^n] = (0, \infty)$.

On the other hand, suppose that $\nu$ is any finite measure on $(q, 1]$. Then there is exactly one way to extend $\nu$ to a finite measure $\mu$ on $(0, \infty)$ such that $\mu$ satisfies Eq. (2.10). Simply define

$$\mu|_{(q^{n+1},q^n]} = \tau_{q^n}\bigl(q^{\binom{n+1}{2}} x^n\,d\nu(x)\bigr), \quad n \in \mathbb{Z},$$

that is,

$$\mu\big(q^n B\big) = q^{\binom{n+1}{2}} \int_B x^n \, d\nu(x)$$

for all Borel sets $B \subset (q, 1]$. In this way,

$$\tau_{q^{-1}}\big(\mu|_{(q^{n+1}, q^n]}\big) = qx \, d\mu|_{(q^n, q^{n-1}]}(x), \quad n \in \mathbb{Z},$$

so the measure $\mu$ satisfies the desired equation and it is a finite measure since

$$\begin{aligned}\mu\big((0,\infty)\big) &= \sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}} \int_{(q,1]} x^n \, d\nu(x) \\ &\leqslant \nu\big((q,1]\big)\left(1/q \sum_{n=0}^{\infty} q^{\binom{n}{2}} + \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}\right) < \infty.\end{aligned}$$

Starting from a finite measure $\nu$ on the interval $(q, 1]$, we can thus construct a solution to the moment problem by, if necessary, normalizing the extension $\mu$. The solution obtained from $\nu$ is of the same type as $\nu$. So if $\nu$ is a continuous singular measure, we end up with a continuous singular solution to the moment problem.

Similar observations was made by Pakes in [15]. Using a slightly different notation, he proved that a measure $\mu$ is solution to (2.10) if and only if $\mu$ has the form

$$\mu = K \sum_{n=-\infty}^{\infty} \tau_{q^n}\big(q^{\binom{n+1}{2}} x^n \, d\nu(x)\big),$$

where $K$ is some constant and $\nu$ is a finite measure supported within the interval $(q, 1]$.

## 3. The $N$-extremal solutions and canonical solutions

The orthonormal polynomials $(P_n)$ associated with the moment sequence (2.3) are given explicitly in (2.4). Recall that the polynomials $(Q_n)$ of the second kind are defined by

$$Q_n(x) = \int \frac{P_n(x) - P_n(y)}{x - y} \, d\mu(y), \quad n \geqslant 0,$$

where $\mu$ is any measure with the moments $s_n$ $(= q^{-\binom{n+1}{2}}$ in our case). Obviously, $Q_0(x) = 0$ and when $P_n(x) = \sum_{k=0}^{n} c_k x^k$, we have

$$Q_n(x) = \sum_{m=0}^{n-1} \left( \sum_{k=m+1}^{n} c_k s_{k-m-1} \right) x^m, \quad n \geqslant 1.$$

Consequently, the polynomials $(Q_n)$ of the second kind associated with the moment sequence (2.3) are given by

$$Q_n(x) = (-1)^n \sqrt{\frac{q^n}{(q;q)_n}} \sum_{m=0}^{n-1} q^{-\binom{m+1}{2}} \left( \sum_{k=m+1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2} + (m+1)k} \right) x^m, \quad n \geqslant 1. \tag{3.1}$$

**Remark 3.1.** The inner sum $\sum_{k=m+1}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q (-1)^k q^{\binom{k}{2}+(m+1)k}$ is the tail in the finite version of the $q$-binomial theorem [2, p. 490]

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k = (x;q)_n. \tag{3.2}$$

Therefore, we could also write this sum as

$$\left(q^{m+1};q\right)_n - \sum_{k=0}^{m} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(m+1)k}.$$

From time to time we shall be dealing with the Stieltjes–Wigert polynomials of the first and second kind given by

$$S_n(x;q) = \frac{1}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2} x^k, \quad n \geqslant 0,$$

and

$$\widetilde{S}_n(x;q) = \frac{1}{(q;q)_n} \sum_{m=0}^{n-1} q^{-\binom{m+1}{2}} \left( \sum_{k=m+1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(m+1)k} \right) x^m,$$
$$n \geqslant 1,$$

that is, $P_n(x) = (-1)^n \sqrt{q^n (q;q)_n} S_n(x;q)$ and $Q_n(x) = (-1)^n \sqrt{q^n (q;q)_n} \widetilde{S}_n(x;q)$.

It is essential that $S_n(x;q)$ and $\widetilde{S}_n(x;q)$ converge uniformly on compact subsets of $\mathbb{C}$ when $n \to \infty$. In fact, $S_n(x;q) \to \Phi(x)/(q;q)_\infty$ and $\widetilde{S}_n(x;q) \to \Psi(x)/(q;q)_\infty$ for $n \to \infty$, where $\Phi$ and $\Psi$ denote the entire functions

$$\Phi(x) = \sum_{k=0}^{\infty} (-1)^k \frac{q^{k^2}}{(q;q)_k} x^k \tag{3.3}$$

and

$$\Psi(x) = \sum_{m=0}^{\infty} q^{-\binom{m+1}{2}} \left( \sum_{k=m+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(m+1)k}}{(q;q)_k} \right) x^m. \tag{3.4}$$

From the general theory of orthogonal polynomials it is well known that $S_n(x;q)$ has $n$ simple positive zeros and that the polynomials $S_{n-1}(x;q)$ and $S_n(x;q)$ have no common zeros. Moreover, the zeros of $S_{n-1}(x;q)$ and $S_n(x;q)$ interlace, that is, $S_{n-1}(x;q)$ has exactly one zero between two consecutive zeros of $S_n(x;q)$.

Since the Stieltjes–Wigert polynomials are orthogonal with respect to the discrete measures $\lambda_c$ in (2.8), it follows that $S_n(x;q)$ has at most one zero in the open interval $(cq, c)$ for each $c > 0$. In other words, the $n$ zeros of $S_n(x;q)$, say $0 < x_{n,1} < \cdots < x_{n,n}$, are separated and this was mentioned by Chihara in [10]. Using the identity

$$S_{n-1}(x;q) = \left(1 - q^n\right) S_n(x;q) + x q^n S_{n-1}(xq;q), \tag{3.5}$$

which can be verified by direct computations, Chihara proved in [11] that

$$x_{n,m} < x_{n-1,m} < q x_{n,m+1}.$$

So in a sense, the $m$th zero of $S_{n-1}(x;q)$ lies in the first part of the interval from the $m$th to the $(m+1)$th zero of $S_n(x;q)$ and we have

$$\frac{x_{n,m+1}}{x_{n,m}} > q^{-1}. \tag{3.6}$$

Referring to (3.6), we say that the zeros of $S_n(x;q)$ are well separated. Using the identity

$$S_n(x;q) = \bigl(1 + xq^{n+1}\bigr)S_n(xq;q) - qxS_n\bigl(xq^2;q\bigr), \tag{3.7}$$

which can also be verified by direct computations, we shall give a refinement of the separation property (3.6). Assume that $S_n(x;q) > 0$ for $x_{n,m} < x < x_{n,m+1}$. The case $S_n(x;q) < 0$ can be handled in a completely similar way. Since $x_{n,m} < qx_{n,m+1} < x_{n,m+1}$, this in particular means that $S_n(qx_{n,m+1};q) > 0$. The open interval $(qx_{n,m}, x_{n,m})$ contains no zero of $S_n(x;q)$ and, consequently, $S_n(x;q) < 0$ for $qx_{n,m} < x < x_{n,m}$. Suppose now that $q^2x_{n,m+1} \leqslant x_{n,m}$. Since $qx_{n,m} < q^2x_{n,m+1}$, this results in $S_n(q^2x_{n,m+1};q) \leqslant 0$ which clearly contradicts the identity (3.7). Therefore, we have $q^2x_{n,m+1} > x_{n,m}$ or, equivalently,

$$\frac{x_{n,m+1}}{x_{n,m}} > q^{-2} \tag{3.8}$$

and we say that the zeros of $S_n(x;q)$ are very well separated.

**Remark 3.2.** One should not expect to find a stronger separation property than (3.8) after looking at the zeros of $S_2(x;q)$. For instance, $x_{2,2}/x_{2,1} < q^{-3}$ when $q = 1/2$.

In some sense, to solve an indeterminate moment problem means to find the four entire functions $A$, $B$, $C$, and $D$ from the Nevanlinna parametrization. Based on generating functions for the Stieltjes–Wigert polynomials, we shall give expressions for these functions. The generating function for $S_n(x;q)$ is also stated in Koekoek and Swarttouw [14].

**Proposition 3.3.** *For $\gamma \in \mathbb{C}$ and $|t| < 1$, we have*

$$\sum_{n=0}^{\infty}(\gamma;q)_n t^n S_n(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{(\gamma;q)_n}{(\gamma t,q;q)_n}q^{n^2}(xt)^n,$$

$$\sum_{n=0}^{\infty}(\gamma;q)_n t^n \widetilde{S}_n(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty}\sum_{n=0}^{\infty}q^{-\binom{n+1}{2}} \times\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{(\gamma;q)_k}{(\gamma t,q;q)_k}q^{\binom{k}{2}+(n+1)k}t^k\right)x^n.$$

*In particular, with $\gamma = 0$ and $t = q$ we have*

$$\sum_{n=0}^{\infty}q^n S_n(x;q) = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+1)}}{(q;q)_n}x^n,$$

$$\sum_{n=0}^{\infty}q^n \widetilde{S}_n(x;q) = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{-\binom{n+1}{2}}\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{q^{\binom{k}{2}+(n+2)k}}{(q;q)_k}\right)x^n,$$

*and with $\gamma = t = q$ we have*

$$\sum_{n=0}^{\infty}(q;q)_n q^n S_n(x;q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}}{(q;q)_{n+1}} x^n,$$

$$\sum_{n=0}^{\infty}(q;q)_n q^n \widetilde{S}_n(x;q) = \sum_{n=0}^{\infty} q^{-\binom{n+1}{2}} \left( \sum_{k=n+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+2)k}}{(q;q)_{k+1}} \right) x^n.$$

**Remark 3.4.** The inner sum $\sum_{k=n+1}^{\infty}(-1)^k q^{\binom{k}{2}+(n+2)k}/(q;q)_k$ is the tail in Euler's formula [2, p. 490]

$$\sum_{n=0}^{\infty}(-1)^n \frac{q^{\binom{n}{2}}}{(q;q)_n} x^n = (x;q)_\infty. \tag{3.9}$$

So this sum can also be written as

$$\left(q^{n+2};q\right)_\infty - \sum_{k=0}^{n}(-1)^k \frac{q^{\binom{k}{2}+(n+2)k}}{(q;q)_k}.$$

Concerning the inner sum $\sum_{k=n+1}^{\infty}(-1)^k q^{\binom{k}{2}+(n+2)k}/(q;q)_{k+1}$, we can say almost the same.

**Proof.** The point of the proof is to interchange the order of summation and use the $q$-binomial theorem [2, p. 488]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{3.10}$$

Absolute convergence assures that we can change the summation. Hence

$$\begin{aligned}\sum_{n=0}^{\infty}(\gamma;q)_n t^n S_n(x;q) &= \sum_{n=0}^{\infty}(\gamma;q)_n t^n \sum_{k=0}^{n}(-1)^k \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} x^k \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{q^{k^2}}{(q;q)_k} x^k \sum_{n=k}^{\infty} \frac{(\gamma;q)_n}{(q;q)_{n-k}} t^n \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{(\gamma;q)_k}{(q;q)_k} q^{k^2} t^k x^k \sum_{n=0}^{\infty} \frac{(\gamma q^k;q)_n}{(q;q)_n} t^n\end{aligned}$$

and similarly

$$\begin{aligned}&\sum_{n=0}^{\infty}(\gamma;q)_n t^n \widetilde{S}_n(x;q) \\ &\quad = \sum_{n=0}^{\infty}(\gamma;q)_n t^n \sum_{m=0}^{n-1} q^{-\binom{m+1}{2}} \left( \sum_{k=m+1}^{n} (-1)^k \frac{q^{\binom{k}{2}+(m+1)k}}{(q;q)_k (q;q)_{n-k}} \right) x^m\end{aligned}$$

$$= \sum_{m=0}^{\infty} q^{-\binom{m+1}{2}} \left( \sum_{n=m+1}^{\infty} (\gamma;q)_n t^n \sum_{k=m+1}^{n} (-1)^k \frac{q^{\binom{k}{2}+(m+1)k}}{(q;q)_k (q;q)_{n-k}} \right) x^m$$

$$= \sum_{m=0}^{\infty} q^{-\binom{m+1}{2}} \left( \sum_{k=m+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(m+1)k}}{(q;q)_k} \sum_{n=k}^{\infty} \frac{(\gamma;q)_n}{(q;q)_{n-k}} t^n \right) x^m$$

$$= \sum_{m=0}^{\infty} q^{-\binom{m+1}{2}} \left( \sum_{k=m+1}^{\infty} (-1)^k \frac{(\gamma;q)_k}{(q;q)_k} q^{\binom{k}{2}+(m+1)k} t^k \sum_{n=0}^{\infty} \frac{(\gamma q^k;q)_n}{(q;q)_n} t^n \right) x^m.$$

By the $q$-binomial theorem (3.10), we have

$$\sum_{n=0}^{\infty} \frac{(\gamma q^k;q)_n}{(q;q)_n} t^n = \frac{(\gamma t q^k;q)_\infty}{(t;q)_\infty}$$

so it follows that

$$\sum_{n=0}^{\infty} (\gamma;q)_n t^n S_n(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{(\gamma;q)_k}{(\gamma t,q;q)_k} q^{k^2} (xt)^k$$

and

$$\sum_{n=0}^{\infty} (\gamma;q)_n t^n \widetilde{S}_n(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty} \sum_{m=0}^{\infty} q^{-\binom{m+1}{2}} \times \left( \sum_{k=m+1}^{\infty} (-1)^k \frac{(\gamma;q)_k}{(\gamma t,q;q)_k} q^{\binom{k}{2}+(m+1)k} t^k \right) x^m.$$

□

The special cases from Proposition 3.3 leads to the following result.

**Theorem 3.5.** *The four entire functions $A$, $B$, $C$, and $D$ from the Nevanlinna parametrization are given by*

$$A(x) = -\sum_{n=0}^{\infty} q^{-\binom{n+1}{2}} \left( \sum_{k=n+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+nk}}{(q;q)_k} \right) x^n - \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{-\binom{n}{2}} \left( \sum_{k=n}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+1)k}}{(q;q)_k} \right) x^n,$$

$$B(x) = -\sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1)}}{(q;q)_n} x^n - \frac{x}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q;q)_n} x^n,$$

$$C(x) = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{-\binom{n}{2}} \left( \sum_{k=n}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+1)k}}{(q;q)_k} \right) x^n,$$

$$D(x) = \frac{x}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q;q)_n} x^n.$$

**Proof.** From (2.4) we see that

$$P_n(0) = (-1)^n \sqrt{\frac{q^n}{(q;q)_n}} \tag{3.11}$$

and using the finite version of the $q$-binomial theorem (3.2), we get from (3.1) that

$$Q_n(0) = \big((q;q)_n - 1\big) P_n(0). \tag{3.12}$$

Recalling that $P_n(x) = (-1)^n \sqrt{q^n (q;q)_n} S_n(x;q)$ and $Q_n(x) = (-1)^n \sqrt{q^n (q;q)_n} \times \widetilde{S}_n(x;q)$, we thus obtain

$$\begin{aligned}
D(x) &= x \sum_{n=0}^{\infty} P_n(0) P_n(x) = x \sum_{n=0}^{\infty} q^n S_n(x;q) \\
&= \frac{x}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q;q)_n} x^n, \\
B(x) &= -1 + x \sum_{n=0}^{\infty} Q_n(0) P_n(x) = -1 + x \sum_{n=0}^{\infty} \big((q;q)_n - 1\big) q^n S_n(x;q) \\
&= -1 - \sum_{n=0}^{\infty} (-1)^{n+1} \frac{q^{n(n+1)}}{(q;q)_{n+1}} x^{n+1} - D(x) \\
&= -\sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1)}}{(q;q)_n} x^n - D(x), \\
C(x) &= 1 + x \sum_{n=0}^{\infty} P_n(0) Q_n(x) = 1 + x \sum_{n=0}^{\infty} q^n \widetilde{S}_n(x;q) \\
&= 1 + \frac{x}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{-\binom{n+1}{2}} \left( \sum_{k=n+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+2)k}}{(q;q)_k} \right) x^n \\
&= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{-\binom{n}{2}} \left( \sum_{k=n}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+1)k}}{(q;q)_k} \right) x^n, \\
A(x) &= x \sum_{n=0}^{\infty} Q_n(0) Q_n(x) = x \sum_{n=0}^{\infty} \big((q;q)_n - 1\big) q^n \widetilde{S}_n(x;q) \\
&= 1 + \sum_{n=0}^{\infty} q^{-\binom{n+1}{2}} \left( \sum_{k=n+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+2)k}}{(q;q)_{k+1}} \right) x^{n+1} - C(x) \\
&= -\sum_{n=0}^{\infty} q^{-\binom{n+1}{2}} \left( \sum_{k=n+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+nk}}{(q;q)_k} \right) x^n - C(x).
\end{aligned}$$

In the computations of $C$ and $A$, we have used Euler's formula (3.9) in the last steps. □

The expressions for $A$ and $B$ are more complicated than the expressions for $C$ and $D$. However, we obviously have

$$B(x)+D(x)=-\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n-1)}}{(q;q)_n}x^n$$

and

$$A(x)+C(x)=-\sum_{n=0}^{\infty}q^{-\binom{n+1}{2}}\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{q^{\binom{k}{2}+nk}}{(q;q)_k}\right)x^n.$$

The quantity $\alpha$ in (1.13) is explicitly given by

$$\alpha=\lim_{n\to\infty}\frac{1}{(q;q)_n-1}=\frac{1}{(q;q)_\infty-1} \tag{3.13}$$

since $Q_n(0)=((q;q)_n-1)P_n(0)$, see (3.12). Due to the fact that $0<(q;q)_\infty<1$, this in particular means that $\alpha<-1$. Realizing that $-1/\alpha=1-(q;q)_\infty$, simple computations give that

$$B(x)-\frac{1}{\alpha}D(x)=-\sum_{n=0}^{\infty}(-1)^n\frac{q^{n^2}}{(q;q)_n}x^n$$

and

$$A(x)-\frac{1}{\alpha}C(x)=-\sum_{n=0}^{\infty}q^{-\binom{n+1}{2}}\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{q^{\binom{k}{2}+(n+1)k}}{(q;q)_k}\right)x^n.$$

In the light of Theorem 3.5, we have thus established the power series expansions of the entire functions $C$, $D$, $A+C$, $B+D$, $A-\frac{1}{\alpha}C$, and $B-\frac{1}{\alpha}D$. One should note that

$$D(x)=\frac{x}{(q;q)_\infty}\Phi(xq),\quad B(x)+D(x)=-\Phi(x/q)\quad\text{and}$$
$$B(x)-\frac{1}{\alpha}D(x)=-\Phi(x),$$

whereas

$$A(x)-\frac{1}{\alpha}C(x)=-\Psi(x),$$

cf. (3.3) and (3.4). In particular, we have

$$\lim_{n\to\infty}\frac{\widetilde{S}_n(x;q)}{S_n(x;q)}=\frac{\Psi(x)}{\Phi(x)}=\frac{A(x)-\frac{1}{\alpha}C(x)}{B(x)-\frac{1}{\alpha}D(x)}=\frac{A(x)\alpha-C(x)}{B(x)\alpha-D(x)}$$
$$\text{for } x\in\mathbb{C}\setminus[0,\infty). \tag{3.14}$$

We will now focus on the canonical solutions to the moment problem and especially on the $N$-extremal solutions. Since a canonical solution is discrete and supported on the zeros of an entire function, these solutions cannot be convex combinations of the measures $\lambda_c$ in (2.8). For 0 is an accumulation point of the set $\{cq^n \mid n\in\mathbb{Z}\}$ and the zeros of an entire function cannot have an accumulation point without the function being identically zero. Compare with [10], where Chihara made it clear that the measures $\lambda_c$ are not $N$-extremal. Consequently, the canonical solutions are not fixed points of the transformation $T$ in Definition 2.4.

Recall that the only $N$-extremal solutions supported within $[0, \infty)$ are $\mu_t$ when $\alpha \leqslant t \leqslant 0$. In our case, three of these solutions are leaping to the eye, namely $\mu_t$ when $t \in \{0, -1, \alpha\}$. In order to find these solutions explicitly, one needs to know the zeros of $\Phi$ since $\mu_0$ is supported on the zeros of $\Phi(xq)$ (plus 0), $\mu_\alpha$ is supported on the zeros of $\Phi(x)$ and $\mu_{-1}$ is supported on the zeros of $\Phi(x/q)$. However, the zeros of $\Phi$ cannot be found explicitly.

Since the zeros of $S_n(x; q)$ in a certain sense converge to the zeros of $\Phi$, we are able to show that the zeros of $\Phi$ are very well separated. For each $m \in \mathbb{N}$, the sequence $(x_{n,m})$ is decreasing in $n$ and thus convergent, say $x_{n,m} \to x_m$ for $n \to \infty$. Since $S_n(x; q)$ converge uniformly to $\Phi(x)/(q; q)_\infty$ on compact subsets of $\mathbb{C}$, the limit points $x_m$ are zeros of $\Phi$ and since $\Phi(0) = 1$, we have $x_1 > 0$. Recalling that the zeros of $S_n(x; q)$ are very well separated, the points $x_m$ are surely well separated, at the worst $x_{m+1}/x_m \geqslant q^{-2}$. According to Rouché's theorem, the points $x_m$ are the only zeros of $\Phi$. For if $x_m < y < x_{m+1}$, then the closed ball with center at $y$ and radius $r < \min(y - x_m, x_{m+1} - y)$ contains no zero of $S_n(x; q)$ for $n$ sufficiently large. Due to the uniform convergence, this is also the case for $\Phi$ and, in particular, $y$ is not a zero of $\Phi$. It is easy to see from (3.7) by letting $n \to \infty$ that

$$\Phi(x) = \Phi(xq) - qx\Phi\left(xq^2\right) \tag{3.15}$$

and with a similar argumentation as for $S_n(x; q)$, it therefore follows that the zeros of $\Phi$ are very well separated, that is, $x_{m+1}/x_m > q^{-2}$.

It is straightforward to see that $\Phi$ is a $q$-analogue of the exponential function and an entire function of order 0. The latter implies that $A$, $B$, $C$, and $D$ from Theorem 3.5 also are entire functions of order 0 since these functions are known to have the same order, see [8].

To underline the fact that $\Phi$ is a very interesting and complicated function, we point out that

$$\Phi(-1) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \prod_{n=0}^{\infty} \left(1 - q^{5n+1}\right)^{-1} \left(1 - q^{5n+4}\right)^{-1}$$

and

$$\Phi(-q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n} = \prod_{n=0}^{\infty} \left(1 - q^{5n+2}\right)^{-1} \left(1 - q^{5n+3}\right)^{-1}.$$

These are the famous Rogers–Ramanujan identities, cf. [2, p. 565].

We shall now make the preparations for describing the transformation $T$ at the level of Pick functions. If $\mu$ is a measure on $[0, \infty)$ with moments (2.3), then the moments of $\tilde{\mu} = qx\,d\mu(x)$ are

$$\int_0^\infty x^n\,d\tilde{\mu}(x) = q\int_0^\infty x^{n+1}\,d\mu(x) = q^{-\binom{n+1}{2}-n}. \tag{3.16}$$

The key is to look at the connection between the moment problems associated with the moment sequences (2.3) and (3.16). Suppose that $\mu$ is a probability measure on $(0,\infty)$ satisfying Eq. (2.10). Since $\mu$ has the moments (2.3), we know that

$$\int_0^\infty P_m(x)P_n(x)\,d\mu(x)=\delta_{mn}$$

and, equivalently,

$$\int_0^\infty P_m(xq)P_n(xq)\,d\tau_{q^{-1}}(\mu)(x)=\delta_{mn}.$$

This means that the orthonormal polynomials $(\widetilde{P}_n)$ associated with the moment sequence (3.16) are given by $\widetilde{P}_n(x)=P_n(xq)$. Moreover, the polynomials $(\widetilde{Q}_n)$ of the second kind are given by $\widetilde{Q}_n(x)=qQ_n(xq)$ since

$$\begin{aligned}\int_0^\infty \frac{\widetilde{P}_n(x)-\widetilde{P}_n(y)}{x-y}\,d\tau_{q^{-1}}(\mu)(y)&=\int_0^\infty \frac{P_n(xq)-P_n(y)}{x-y/q}\,d\mu(y)\\&=q\int_0^\infty \frac{P_n(xq)-P_n(y)}{xq-y}\,d\mu(y).\end{aligned}$$

In this way, we see that the entire functions from the Nevanlinna parametrization for the two moment problems are related by

$$\begin{aligned}\widetilde{A}(x)&=x\sum_{n=0}^\infty \widetilde{Q}_n(0)\widetilde{Q}_n(x)=q^2x\sum_{n=0}^\infty Q_n(0)Q_n(xq)=qA(xq),\\\widetilde{B}(x)&=-1+x\sum_{n=0}^\infty \widetilde{Q}_n(0)\widetilde{P}_n(x)=-1+qx\sum_{n=0}^\infty Q_n(0)P_n(xq)=B(xq),\\\widetilde{C}(x)&=1+x\sum_{n=0}^\infty \widetilde{P}_n(0)\widetilde{Q}_n(x)=1+qx\sum_{n=0}^\infty P_n(0)Q_n(xq)=C(xq),\\\widetilde{D}(x)&=x\sum_{n=0}^\infty \widetilde{P}_n(0)\widetilde{P}_n(x)=x\sum_{n=0}^\infty P_n(0)P_n(xq)=D(xq)/q.\end{aligned}$$

On the other hand, a general result given by Pedersen in [16, Proposition 6.3] tells us that

$$\begin{pmatrix}\widetilde{A}(x)\\\widetilde{B}(x)\\\widetilde{C}(x)\\\widetilde{D}(x)\end{pmatrix}=M(x)\begin{pmatrix}A(x)\\B(x)\\C(x)\\D(x)\end{pmatrix},$$

where $M(x)$ denotes the matrix

$$\begin{pmatrix} qx(1-D'(0)) & -q(1-D'(0)) & -\frac{qx}{\alpha}(1-D'(0))-q & \frac{q}{\alpha}(1-D'(0))+\frac{q}{x} \\ 0 & 1-D'(0) & 0 & -\frac{1}{\alpha}(1-D'(0))-\frac{1}{x} \\ xD'(0) & -D'(0) & -\frac{x}{\alpha}D'(0)+1 & \frac{1}{\alpha}D'(0)-\frac{1}{x} \\ 0 & \frac{1}{q}D'(0) & 0 & -\frac{1}{q\alpha}D'(0)+\frac{1}{qx} \end{pmatrix}$$

and since $D'(0)=1/(q;q)_\infty$, we have

$$\begin{pmatrix} A(xq) \\ B(xq) \\ C(xq) \\ D(xq) \end{pmatrix} = \begin{pmatrix} \frac{x((q;q)_\infty-1)}{(q;q)_\infty} & -1+\frac{1}{(q;q)_\infty} & -\frac{x((q;q)_\infty-1)}{\alpha(q;q)_\infty}-1 & \frac{((q;q)_\infty-1)}{\alpha(q;q)_\infty}+\frac{1}{x} \\ 0 & 1-\frac{1}{(q;q)_\infty} & 0 & -\frac{((q;q)_\infty-1)}{\alpha(q;q)_\infty}-\frac{1}{x} \\ \frac{x}{(q;q)_\infty} & -\frac{1}{(q;q)_\infty} & -\frac{x}{\alpha(q;q)_\infty}+1 & \frac{1}{\alpha(q;q)_\infty}-\frac{1}{x} \\ 0 & \frac{1}{(q;q)_\infty} & 0 & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \times \begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix}.$$

This can also be written as

$$\begin{pmatrix} A(xq)+B(xq) \\ B(xq) \\ C(xq)+D(xq) \\ D(xq) \end{pmatrix} = \begin{pmatrix} \frac{x((q;q)_\infty-1)}{(q;q)_\infty} & 0 & -\frac{x((q;q)_\infty-1)}{\alpha(q;q)_\infty}-1 & 0 \\ 0 & 1-\frac{1}{(q;q)_\infty} & 0 & -\frac{((q;q)_\infty-1)}{\alpha(q;q)_\infty}-\frac{1}{x} \\ \frac{x}{(q;q)_\infty} & 0 & -\frac{x}{\alpha(q;q)_\infty}+1 & 0 \\ 0 & \frac{1}{(q;q)_\infty} & 0 & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix}$$

or

$$\begin{pmatrix} A(xq)+C(xq) \\ B(xq)+D(xq) \\ C(xq) \\ D(xq) \end{pmatrix} = \begin{pmatrix} x & -1 & -\frac{x}{\alpha} & \frac{1}{\alpha} \\ 0 & 1 & 0 & -\frac{1}{\alpha} \\ \frac{x}{(q;q)_\infty} & -\frac{1}{(q;q)_\infty} & -\frac{x}{\alpha(q;q)_\infty}+1 & \frac{1}{\alpha(q;q)_\infty}-\frac{1}{x} \\ 0 & \frac{1}{(q;q)_\infty} & 0 & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix}$$

or even

$$\begin{pmatrix} A(xq)+B(xq)+C(xq)+D(xq) \\ B(xq)+D(xq) \\ C(xq)+D(xq) \\ D(xq) \end{pmatrix} = \begin{pmatrix} x & 0 & -\frac{x}{\alpha} & 0 \\ 0 & 1 & 0 & -\frac{1}{\alpha} \\ \frac{x}{(q;q)_\infty} & 0 & -\frac{x}{\alpha(q;q)_\infty}+1 & 0 \\ 0 & \frac{1}{(q;q)_\infty} & 0 & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix}.$$

The last expression is equivalent to

$$\begin{pmatrix} B(xq)+D(xq) \\ D(xq) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{\alpha} \\ \frac{1}{(q;q)_\infty} & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \begin{pmatrix} B(x) \\ D(x) \end{pmatrix} \tag{3.17}$$

and

$$\begin{pmatrix} A(xq)+B(xq)+C(xq)+D(xq) \\ C(xq)+D(xq) \end{pmatrix}$$
$$= x \begin{pmatrix} 1 & -\frac{1}{\alpha} \\ \frac{1}{(q;q)_\infty} & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \begin{pmatrix} A(x) \\ C(x) \end{pmatrix}. \tag{3.18}$$

We are now ready to describe the transformation $T$ at the level of Pick functions.

**Theorem 3.6.** *Suppose that $\mu \in V$ and let $\varphi$ be the Pick function corresponding to $\mu$. Then $\nu = \tau_q(qx\,d\mu(x)) \in V$ and the Pick function $\psi$ corresponding to $\nu$ is given by*

$$\psi(x) = \frac{\frac{x}{(q;q)_\infty}\bigl(1-\frac{\varphi(x/q)}{\alpha}\bigr)+q\varphi(x/q)}{\frac{x}{(q;q)_\infty}((q;q)_\infty-1)\bigl(1-\frac{\varphi(x/q)}{\alpha}\bigr)-q\varphi(x/q)}.$$

**Proof.** The conclusion of Proposition 2.3 is that $\nu \in V$. Since

$$\int_0^\infty \frac{1}{qx-t}\,d\nu(t) = \int_0^\infty \frac{1}{qx-qt}qt\,d\mu(t) = \int_0^\infty \frac{t}{x-t}\,d\mu(t)$$
$$= -1 + x\int_0^\infty \frac{1}{x-t}\,d\mu(t),$$

we have to show that

$$\frac{A(xq)\psi(xq)-C(xq)}{B(xq)\psi(xq)-D(xq)} = -1 + x\frac{A(x)\varphi(x)-C(x)}{B(x)\varphi(x)-D(x)}$$

and this is done by direct computations. With

$$\zeta(x) = \frac{x}{(q;q)_\infty}\left(1-\frac{\varphi(x)}{\alpha}\right)+\varphi(x) \quad \text{and} \quad \eta(x) = x\left(1-\frac{\varphi(x)}{\alpha}\right),$$

we have

$$\frac{A(xq)\psi(xq)-C(xq)}{B(xq)\psi(xq)-D(xq)} = \frac{\zeta(x)A(xq)+(\zeta(x)-\eta(x))C(xq)}{\zeta(x)B(xq)+(\zeta(x)-\eta(x))D(xq)}$$

and by (3.18) and (3.17), it follows that

$$\frac{\zeta(x)(A(xq)+C(xq))-\eta(x)C(xq)}{\zeta(x)(B(xq)+D(xq))-\eta(x)D(xq)}$$
$$= -1 + \frac{\zeta(x)x\bigl(A(x)-\frac{1}{\alpha}C(x)\bigr)-\eta(x)x\bigl(\frac{1}{(q;q)_\infty}A(x)+\bigl(\frac{1}{x}-\frac{1}{\alpha(q;q)_\infty}\bigr)C(x)\bigr)}{\zeta(x)(B(xq)+D(xq))-\eta(x)D(xq)}$$

$$= -1 + \frac{\zeta(x)x\big(A(x) - \frac{1}{\alpha}C(x)\big) - \eta(x)x\big(\frac{1}{(q;q)_\infty}A(x) + \big(\frac{1}{x} - \frac{1}{\alpha(q;q)_\infty}\big)C(x)\big)}{\zeta(x)\big(B(x) - \frac{1}{\alpha}D(x)\big) - \eta(x)\big(\frac{1}{(q;q)_\infty}B(x) + \big(\frac{1}{x} - \frac{1}{\alpha(q;q)_\infty}\big)D(x)\big)}$$
$$= -1 + \frac{x\varphi(x)A(x) - \big(x\frac{\varphi(x)}{\alpha} + x\big(1 - \frac{\varphi(x)}{\alpha}\big)\big)C(x)}{\varphi(x)B(x) - \big(\frac{\varphi(x)}{\alpha} + \big(1 - \frac{\varphi(x)}{\alpha}\big)\big)D(x)}$$
$$= -1 + x\frac{A(x)\varphi(x) - C(x)}{B(x)\varphi(x) - D(x)}.$$

□

Let us list some consequences of Theorem 3.6. First of all, we see that $T$ maps a $N$-extremal solution into another $N$-extremal solution or into a canonical solution of order 1. In general, $T$ maps a canonical solution of order $n$ into another canonical solution of order $\leqslant n+1$.

It is straightforward to verify that $T(\mu_0) = \mu_\alpha$ and $T(\mu_\alpha) = \mu_{-1}$. Actually, we can describe $T^{(n)}(\mu_0)$ for each $n \in \mathbb{N}$.

**Theorem 3.7.** *Let $T : V \mapsto V$ denote the map given by $T(\mu) = \tau_q(qx\,d\mu(x))$. For $n = 0, 1, \ldots,$ we have*

$$T^{(2n+1)}(\mu_0) = \mu_{R_n} \quad \text{and} \quad T^{(2n+2)}(\mu_0) = \mu_{\widetilde{R}_n},$$

*where $R_n$ and $\widetilde{R}_n$ are real rational functions of order $\leqslant n$ given by*

$$R_n(x) = \frac{\sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} 2n-k \\ k \end{bmatrix}_q q^{(n-k)^2} x^k}{\sum_{k=0}^n (-1)^{n-k}\big((q;q)_\infty \begin{bmatrix} 2n-k-1 \\ k-1 \end{bmatrix}_q q^{(n-k+1)^2-1} - \begin{bmatrix} 2n-k \\ k \end{bmatrix}_q q^{(n-k)^2}\big)x^k}$$

*and*

$$\widetilde{R}_n(x) = \left(\sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} 2n-k+1 \\ k \end{bmatrix}_q q^{(n-k)(n-k+1)} x^k\right) \Bigg/ \left(\sum_{k=0}^n (-1)^{n-k} \left((q;q)_\infty \begin{bmatrix} 2n-k \\ k-1 \end{bmatrix}_q q^{(n-k+1)(n-k+2)-1} - \begin{bmatrix} 2n-k+1 \\ k \end{bmatrix}_q q^{(n-k)(n-k+1)}\right) x^k\right).$$

**Proof.** The proof is by induction. Start by noting that $R_0(x) = \alpha$ and $\widetilde{R}_0(x) = -1$. Suppose next that $T^{(2n+1)}(\mu_0) = \mu_{R_n}$ for some $n > 0$ and let $T^{(2n+2)}(\mu_0) = T(\mu_{R_n}) = \mu_\psi$, where $\psi$ is a certain Pick function. The real rational function $R_n$ has the form

$$R_n(x) = \frac{S_n(x)}{(q;q)_\infty T_n(x) - S_n(x)}$$

with

$$S_n(x) = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} 2n-k \\ k \end{bmatrix}_q q^{(n-k)^2} x^k$$

and

$$T_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} 2n-k-1 \\ k-1 \end{bmatrix}_q q^{(n-k+1)^2-1} x^k.$$

So according to Theorem 3.6, we have

$$\begin{aligned}
\psi(x) &= \frac{\frac{x}{(q;q)_\infty}\bigl(1 - \frac{R_n(x/q)}{\alpha}\bigr) + qR_n(x/q)}{\frac{x}{(q;q)_\infty}((q;q)_\infty - 1)\bigl(1 - \frac{R_n(x/q)}{\alpha}\bigr) - qR_n(x/q)} \\
&= \frac{x(T_n(x/q) - S_n(x/q)) + qS_n(x/q)}{x((q;q)_\infty - 1)(T_n(x/q) - S_n(x/q)) - qS_n(x/q)} \\
&= \frac{U_n(x)}{(q;q)_\infty V_n(x) - U_n(x)},
\end{aligned}$$

where

$$V_n(x) = U_n(x) - qS_n(x/q) = x\bigl(T_n(x/q) - S_n(x/q)\bigr).$$

By collecting the terms, it follows that

$$\begin{aligned}
V_n(x) &= \sum_{k=0}^{n} (-1)^{n-k} \left( \begin{bmatrix} 2n-k-1 \\ k-1 \end{bmatrix}_q q^{(n-k+1)^2-1} \right. \\
&\qquad \left. - \begin{bmatrix} 2n-k \\ k \end{bmatrix}_q q^{(n-k)^2} \right) q^{-k} x^{k+1} \\
&= \sum_{k=0}^{n-1} (-1)^{n-k} \begin{bmatrix} 2n-k-1 \\ k \end{bmatrix}_q q^{(n-k)^2-k} \\
&\qquad \times \left( q^{2(n-k)} \frac{1-q^k}{1-q^{2(n-k)}} - \frac{1-q^{2n-k}}{1-q^{2(n-k)}} \right) x^{k+1} \\
&= \sum_{k=0}^{n-1} (-1)^{n-k+1} \begin{bmatrix} 2n-k-1 \\ k \end{bmatrix}_q q^{(n-k)^2-k} x^{k+1} \\
&= \sum_{k=1}^{n} (-1)^{n-k} \begin{bmatrix} 2n-k \\ k-1 \end{bmatrix}_q q^{(n-k+1)^2-k+1} x^k
\end{aligned}$$

and

$$\begin{aligned}
U_n(x) &= \sum_{k=0}^{n} (-1)^{n-k} \left( \begin{bmatrix} 2n-k \\ k-1 \end{bmatrix}_q q^{(n-k+1)^2} + \begin{bmatrix} 2n-k \\ k \end{bmatrix}_q q^{(n-k)^2} \right) q^{-k+1} x^k \\
&= \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} 2n-k+1 \\ k \end{bmatrix}_q q^{(n-k)(n-k+1)}
\end{aligned}$$

$$\times\left(q^{n-2k+2}\frac{1-q^k}{1-q^{2n-k+1}}+q^{-n+1}\frac{1-q^{2(n-k)+1}}{1-q^{2n-k+1}}\right)x^k$$
$$=q^{-n+1}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}2n-k+1\\k\end{bmatrix}_q q^{(n-k)(n-k+1)}x^k.$$

Hence

$$\psi(x)=\frac{q^{n-1}U_n(x)}{q^{n-1}(q;q)_\infty V_n(x)-q^{n-1}U_n(x)}=\widetilde{R}_n(x)$$

and this means that $T(\mu_{R_n})=\mu_{\widetilde{R}_n}$. In a similar way, one can prove that $T(\mu_{\widetilde{R}_n})=\mu_{R_{n+1}}$ and this completes the proof. □

An interesting question is what may happen when $n\to\infty$. In the light of Theorem 3.7, one should not expect $T^{(n)}(\mu_0)$ to converge. More likely $T^{(2n+1)}(\mu_0)$ and $T^{(2n+2)}(\mu_0)$ would converge and if so, the limit points would be fixed points of $T^{(2)}$ and possibly fit into the measures $\kappa_s$ from (2.9). Since

$$S_n(x)=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}2n-k\\k\end{bmatrix}_q q^{(n-k)^2}x^k=\sum_{j=0}^{n}(-1)^j\begin{bmatrix}n+j\\n-j\end{bmatrix}_q q^{j^2}x^{n-j}$$

and

$$T_n(x)=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}2n-k-1\\k-1\end{bmatrix}_q q^{(n-k+1)^2-1}x^k$$
$$=\sum_{j=0}^{n}(-1)^j\begin{bmatrix}n+j-1\\n-j-1\end{bmatrix}_q q^{(j+1)^2-1}x^{n-j},$$

we see that $x^{-n}S_n(x)\to S(x)$ and $x^{-n}T_n(x)\to T(x)$ for $n\to\infty$, where

$$S(x)=\sum_{j=0}^{\infty}(-1)^j\frac{q^{j^2}}{(q;q)_{2j}}(1/x)^j\quad\text{and}$$
$$T(x)=\sum_{j=0}^{\infty}(-1)^j\frac{q^{(j+1)^2-1}}{(q;q)_{2j}}(1/x)^j.$$

Seeing that $T(x)=S(x/q^2)$, we thus find that

$$R_n(x)\to R_\infty(x)=\frac{S(x)}{(q;q)_\infty S(x/q^2)-S(x)}\quad\text{for } n\to\infty$$

and similarly

$$\widetilde{R}_n(x)\to\widetilde{R}_\infty(x)=\frac{\widetilde{S}(x)}{q(q;q)_\infty\widetilde{S}(x/q^2)-\widetilde{S}(x)}\quad\text{for } n\to\infty,$$

where

$$\widetilde{S}(x)=\sum_{j=0}^{\infty}(-1)^j\frac{q^{j(j+1)}}{(q;q)_{2j+1}}(1/x)^j.$$

Since the above convergence is uniform on compact subsets of $\mathbb{C} \setminus \{0\}$, it follows that $R_\infty$ and $\widetilde{R}_\infty$ are Pick functions corresponding to solutions to the moment problem. In order to find the solutions $\mu_{R_\infty}$ and $\mu_{\widetilde{R}_\infty}$ explicitly, we need the following result containing useful information about the supports of the measures $T^{(n)}(\mu_0)$.

**Theorem 3.8.** *Let $T : V \mapsto V$ denote the map given by $T(\mu) = \tau_q(qx\,d\mu(x))$. For each $n \in \mathbb{N}$, the canonical solution $T^{(n)}(\mu_0)$ is supported on the zeros of $\Phi(x/q^{n-1})$.*

**Proof.** The proof is by induction. Start by noting that $T(\mu_0) = \mu_\alpha$ and recall that $\mu_\alpha$ is supported on the zeros of $\Phi(x)$. As a matter of fact, by (3.14) we have

$$\int_0^\infty \frac{1}{x-t}\,d\mu_\alpha(t) = \frac{\Psi(x)}{\Phi(x)}.$$

Suppose next that

$$\int_0^\infty \frac{1}{x-t}\,dT^{(n)}(\mu_0)(t) = \frac{\Psi_n(x)}{\Phi(x/q^{n-1})}$$

for some entire function $\Psi_n(x)$ having no common zeros with $\Phi(x/q^{n-1})$. With $\sigma = T^{(n)}(\mu_0)$, we then have

$$\begin{aligned}\int_0^\infty \frac{1}{x-t}\,dT^{(n+1)}(\mu_0)(t) &= \int_0^\infty \frac{1}{x-t}\,dT(\sigma)(t) = \int_0^\infty \frac{1}{x-qt} qt\,d\sigma(t)\\ &= -1 + \frac{x}{q}\int_0^\infty \frac{1}{x/q-t}\,d\sigma(t) = -1 + \frac{x}{q}\frac{\Psi_n(x/q)}{\Phi(x/q^n)}\\ &= \frac{\frac{x}{q}\Psi_n(x/q) - \Phi(x/q^n)}{\Phi(x/q^n)}.\end{aligned}$$

Since $\Phi(x/q^{n-1})$ and $\Psi_n(x)$ are without common zeros, neither $\Phi(x/q^n)$ and

$$\Psi_{n+1}(x) = \frac{x}{q}\Psi_n\left(\frac{x}{q}\right) - \Phi\left(\frac{x}{q^n}\right)$$

have common zeros. For if $\Psi_{n+1}(y) = \Phi(y/q^n) = 0$ for some $y > 0$, then $\Psi_n(z) = \Phi(z/q^{n-1}) = 0$ with $z = y/q$. Consequently, $T^{(n+1)}(\mu_0)$ is supported on the zeros of $\Phi(x/q^n)$ and this proves the assertion. $\square$

Since $R_\infty$ and $\widetilde{R}_\infty$ are meromorphic functions in $\mathbb{C} \setminus \{0\}$, the solutions $\mu_{R_\infty}$ and $\mu_{\widetilde{R}_\infty}$ are discrete and supported on the zeros of

$$B(x)R_\infty(x) - D(x) \quad \text{and} \quad B(x)\widetilde{R}_\infty(x) - D(x),$$

respectively. Being a discrete fixed point of $T^{(2)}$ means that $c > 0$ is a mass point of, say $\mu$, exactly if $cq^2$ likewise is a mass point of $\mu$ and $\mu(\{cq^2\}) = q^3c^2\mu(\{c\})$. Recalling that the zeros of $\Phi$ are very well separated, Theorem 3.7 implies that if $c$ and $c'$ belong to the support of $\mu_{R_\infty}$ (or $\mu_{\widetilde{R}_\infty}$) and $c > c'$, then $c/c' \geqslant q^{-2}$. Consequently, the supports of $\mu_{R_\infty}$ and $\mu_{\widetilde{R}_\infty}$ have the form

$$\{cq^{2n} \mid n \in \mathbb{Z}\} \quad \text{and} \quad \{\tilde{c}q^{2n} \mid n \in \mathbb{Z}\}$$

for some $c, \tilde{c} > 0$. It is a natural conclusion that there may be a connection with the measures $\kappa_{-1}$ and $\kappa_1$. To show that $c = q$, it suffices to prove that

$$B(q)R_\infty(q) - D(q) = 0 \tag{3.19}$$

and multiplying with $S(q) - (q;q)_\infty S(1/q) \neq 0$, it comes to prove that

$$\begin{aligned} 0 &= D(q)\big((q;q)_\infty S(1/q) - S(q)\big) - B(q)S(q) \\ &= q\Phi\big(q^2\big)S(1/q) - D(q)S(q) + \big(\Phi(1) + D(q)\big)S(q) \\ &= \Phi(1)S(q) + q\Phi\big(q^2\big)S(1/q). \end{aligned}$$

At this point, the identity

$$\begin{aligned} \big(-aq;q^2\big)_\infty \sum_{n=0}^\infty \frac{q^{n(n+1)}}{(-aq,q^2;q^2)_n} a^n &= \big(-aq^2;q^2\big)_\infty \sum_{n=0}^\infty \frac{q^{n^2}}{(-aq^2,q^2;q^2)_n} a^n \\ &= \sum_{n=0}^\infty \frac{q^{n^2}}{(q;q)_n} a^n \end{aligned} \tag{3.20}$$

due to Rogers [19] becomes useful. See also [2]. With $a = -1$ and $a = -1/q$ in (3.20), we get

$$\Phi(1) = \big(q;q^2\big)_\infty S(1/q) \quad \text{and} \quad \Phi(1/q) = \big(q;q^2\big)_\infty S(q)$$

which means that

$$\begin{aligned} \Phi(1)S(q) + q\Phi\big(q^2\big)S(1/q) &= S(1/q)\big(\big(q;q^2\big)_\infty S(q) + q\Phi\big(q^2\big)\big) \\ &= S(1/q)\big(\Phi(1/q) + q\Phi\big(q^2\big)\big). \end{aligned}$$

According to (3.15), we have

$$\Phi(1/q) + q\Phi\big(q^2\big) = \Phi(1/q) + \Phi(q) - \Phi(1) = 0$$

and this proves (3.19). Consequently, $\mu_{R_\infty}$ is supported on $\{q^{2n+1} \mid n \in \mathbb{Z}\}$ and being a fixed point of $T^{(2)}$, it must coincide with $\kappa_{-1}$. In a similar way, we can prove that

$$B(1)\widetilde{R}_\infty(1) - D(1) = 0 \tag{3.21}$$

which implies that $\tilde{c} = 1$ and $\mu_{\widetilde{R}_\infty} = \kappa_1$. To sum up, we have established the following result.

**Theorem 3.9.** *Let $R_\infty$ and $\widetilde{R}_\infty$ denote the Pick functions*

$$R_\infty(x) = \frac{\sum_{j=0}^\infty (-1)^j \frac{q^{j^2}}{(q;q)_{2j}} (1/x)^j}{(q;q)_\infty \sum_{j=0}^\infty (-1)^j \frac{q^{j(j+2)}}{(q;q)_{2j}} (1/x)^j - \sum_{j=0}^\infty (-1)^j \frac{q^{j^2}}{(q;q)_{2j}} (1/x)^j}$$

*and*

$$\widetilde{R}_\infty(x) = \frac{\sum_{j=0}^\infty (-1)^j \frac{q^{j(j+1)}}{(q;q)_{2j+1}} (1/x)^j}{q(q;q)_\infty \sum_{j=0}^\infty (-1)^j \frac{q^{j(j+3)}}{(q;q)_{2j+1}} (1/x)^j - \sum_{j=0}^\infty (-1)^j \frac{q^{j(j+1)}}{(q;q)_{2j+1}} (1/x)^j}.$$

*The measures $\mu_{R_\infty}$ and $\mu_{\widetilde{R}_\infty}$ are explicitly given by*

$$\mu_{R_\infty} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{\binom{2n+2}{2}} \varepsilon_{q^{2n+1}}$$

*and*

$$\mu_{\widetilde{R}_\infty} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{\binom{2n+1}{2}} \varepsilon_{q^{2n}}.$$

Theorem 3.9 really brings the Nevanlinna parametrization into focus. As we have seen, finding the $N$-extremal solutions explicitly is out of reach and it is hardly possible to find the Pick functions corresponding to, for instance, the solutions $v_c$ in (2.6). But for $\kappa_{-1}$ and $\kappa_1$ we can determine the corresponding Pick function explicitly.

As a corollary, we can say somewhat about the asymptotic behaviour of the very well separated zeros of $\Phi$.

**Corollary 3.10.** *Let $0 < x_1 < \cdots < x_m < x_{m+1} < \cdots$ denote the zeros of $\Phi$. When $m \to \infty$, we have $x_{m+1}/x_m \to q^{-2}$.*

## References


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