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On stochastic differential equations and semigroups of probability operators in quantum probability

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Abstract

Some "classical" stochastic differential equations have been used in the theory of measurements continuous in time in quantum mechanics and, more generally, in quantum open system theory. In this paper, we introduce and study a class of such equations which allow us to achieve the same level of generality as the one obtained by the approach to continuous measurements based on semigroups of operators. To this aim, we have to study some linear and non-linear stochastic differential equations for processes in Hilbert spaces and in some related Banach spaces. By this stochastic approach we can also obtain new results on the evolution systems which substitute the semigroups of probability operators in the time inhomogeneous case. © 1998 Elsevier Science B.V. All rights reserved

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1. Stochastic evolution equations in Hilbert spaces

The theory of measurements continuous in time in quantum mechanics gave rise to an interesting connection between quantum probability and "classical" stochastic processes (Davies, 1976; Barchielli et al., 1983). This theory had a mathematical development essentially in three different directions: connections with quantum stochastic calculus (Barchielli and Lupieri, 1985), with semigroups of operators (Barchielli and Lupieri, 1991; Barchielli et al., 1993), with stochastic differential equations (SDEs) and filtering theory (Belavkin, 1988, 1989a,b, 1992; Diósi, 1988a,b, Barchielli and Holevo, 1995). In this paper we shall consider the two last approaches.

By using the theory of semigroups it has been possible to find and characterize the most general quantum continuous measurement process satisfying some technical requirements (Barchielli et al., 1993). On the other side the approach based on SDEs give us some other advantages (for instance, it allows to introduce "memory", it is

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suitable for numerical simulations,...), but up to now it was not developed up to the same level of generality as the formulation based on semigroups. The aim of the present paper is to fill this gap. We shall be able to obtain a stochastic representation of the most general semigroup of probability operators studied in Barchielli et al. (1993); moreover, this stochastic approach gives us the way to treat the non-autonomous case and to obtain new results on the evolution systems which take place of the semigroups in the time inhomogeneous case. The comparison between the SDE approach studied in Barchielli and Holevo (1995) and the analysis of the infinitesimal generator of the involved semigroups done in Barchielli and Paganoni (1996) suggests us the form of the SDE we have to take as a starting point.

Let us consider the following linear SDE for a process $\psi \equiv \{\psi_t,\ t \in \mathbb{R}_+\}$ with values in a Hilbert space $\mathscr{H}$:

$$\mathrm{d}\psi_t = -K_t\psi_{t^-}\,\mathrm{d}t + \sum_{k=1}^{\infty} L_{kt}\psi_{t^-}\,\mathrm{d}W_{kt} + \int_{\mathscr{Y}} (J_t\psi_{t^-})(y)\widetilde{\Pi}(\mathrm{d}y,\mathrm{d}t), \tag{1.1}$$

$$\psi_0 = \xi, \tag{1.2}$$

the stochastic integrals are in Itô's sense. The following assumptions give us the meaning of the objects appearing in Eq. (1.1).

Assumption 1.1. *$\mathscr{H}$ is a separable complex Hilbert space and $\mathscr{Y}$ a locally compact Hausdorff space with a topology with a countable basis. Let $\mathscr{B}(\mathscr{Y})$ be the Borel σ-algebra of $\mathscr{Y}$ and v be a Radon measure on $(\mathscr{Y},\mathscr{B}(\mathscr{Y}))$. Finally, let $\gamma_t(y)$ be a non-negative measurable function on $\mathscr{Y}\times\mathbb{R}_+$, bounded on $\mathscr{Y}\times[0,T]$, $\forall T \geqslant 0$.*

Let us introduce now the complex Hilbert spaces $L^2(\mathscr{Y},v)$ and $L^2(\mathscr{Y},v;\mathscr{H}) \simeq \mathscr{H}\otimes L^2(\mathscr{Y},v)$. We can consider γ_t also as a bounded multiplication operator in the space $L^2(\mathscr{Y},v)$ with norm $\|\gamma_t\| := \operatorname{ess\,sup}_{y\in\mathscr{Y}} \gamma_t(y)$; we have also $\sup_{t\leqslant T}\|\gamma_t\| < +\infty$, $\forall T \in \mathbb{R}_+$.

Assumption 1.2. *For every $t \in \mathbb{R}_+$, let us have three bounded operators $L_t \in \mathscr{L}(\mathscr{H};\mathscr{H}\otimes l^2)$, $J_t \in \mathscr{L}(\mathscr{H};L^2(\mathscr{Y},v;\mathscr{H}))$, $K_t \in \mathscr{L}(\mathscr{H})$. The functions $t\mapsto L_t$, $t\mapsto J_t$, $t\mapsto K_t$ are required to be strongly measurable and to satisfy $\sup_{t\leqslant T}\|K_t\| < +\infty$, $\sup_{t\leqslant T}\|L_t\| < +\infty$, $\sup_{t\leqslant T}\|J_t\| < +\infty$, $\forall T \in \mathbb{R}_+$.*

Let us denote by $\{e_k\}$ the canonical c.o.n.s. in l^2, i.e. $(e_k)_l = \delta_{kl}$, and define $L_{kt} \in \mathscr{L}(\mathscr{H})$, $k = 1,2,\dots$, by $\langle y|L_{kt}x\rangle_{\mathscr{H}} = \langle y\otimes e_k|L_t x\rangle_{\mathscr{H}\otimes l^2}$, $\forall x,y \in \mathscr{H}$. It is easy to see that $\|L_t x\|^2_{\mathscr{H}\otimes l^2} = \sum_{k=1}^{\infty}\|L_{kt}x\|^2_{\mathscr{H}}$; but, for an increasing sequence of positive operators, weak convergence implies strong convergence, so we can write $\sum_{k=1}^{\infty} L_{kt}^* L_{kt} = L_t^* L_t$, where the series is strongly convergent. In the following, we shall omit the subscripts from norms and inner products.

Assumption 1.3. *Let $(\Omega,(\mathscr{F}_t),\mathscr{F},P)$ be a stochastic basis satisfying the usual hypotheses (Métivier, 1982, pp. 2–3). The initial condition ξ is an $\mathscr{F}_0$-measurable $\mathscr{H}$-valued random variable and the W_{kt} are continuous versions of adapted, standard, independent Wiener processes with increments independent of the past. Finally, we have an adapted Poisson point process $\Pi(\mathrm{d}y,\mathrm{d}t)$ on $\mathscr{Y}\times\mathbb{R}_+$ of intensity $\gamma_t(y)v(\mathrm{d}y)\,\mathrm{d}t$; Π is independent of the Wiener processes and with increments independent of the past.*

According to Ikeda and Watanabe (1981, Definition 3.1, p. 59), Π is a "quasi-left-continuous" (QL) point process; moreover, we denote by $\widetilde{\Pi}$ the compensated process

$$\widetilde{\Pi}(\mathrm{d}y,\mathrm{d}t) := \Pi(\mathrm{d}y,\mathrm{d}t) - \gamma_t(y)\nu(\mathrm{d}y)\,\mathrm{d}t. \tag{1.3}$$

Let us recall that an $\mathscr{H}$-valued RRC process ψ_t is said to be a *strong solution* of Eqs. (1.1) and (1.2) (Da Prato and Zabczyk, 1992, p. 118; Métivier, 1982, p. 224) if it satisfies P-a.s. the integral equation

$$\psi_t = \xi - \int_0^t K_s\psi_{s-}\,\mathrm{d}s + \sum_{k=1}^{\infty}\int_0^t L_{ks}\psi_{s-}\,\mathrm{d}W_{ks} + \int_{\mathscr{Y}\times(0,t]} (J_s\,\psi_{s-})(y)\widetilde{\Pi}(\mathrm{d}y,\mathrm{d}s). \tag{1.4}$$

Eq. (1.1) and its consequences have been studied in Barchielli and Holevo (1995) in the case in which the operators involved are permitted to be random, the W_{kt} are replaced by a continuous finite-dimensional martingale and Π by a more general point process. In that paper, however, there is the important restriction $(J_t f)(y) = \widetilde{J}_t(y)f$, where $\widetilde{J}_t(y)$ is a bounded operator on $\mathscr{H}$ satisfying some further assumptions on the y-dependence; it is this restriction which prevents to obtain sufficiently general results. The results of this section can be proved or by standard means of the theory of SDEs or, by slight modifications of the proofs given in that article. For this reason, no proof is given in this section, but we limit ourselves to indicate the relevant references.

The first result is about existence and uniqueness of the solution of Eq. (1.1); the proof can be obtained by the same technique used in Theorem 7.4 of da Prato and Zabczyk (1992).

Theorem 1.1. *Under Assumptions* 1.1, 1.2, 1.3, *Eqs.* (1.1) *and* (1.2) *admit a unique* (*up to P-equivalence*) *strong solution* ψ_t. *Moreover,*

$$\mathbb{E}_P[\|\xi\|^2] < \infty \;\Rightarrow\; \sup_{t\leqslant T}\mathbb{E}_P[\|\psi_t\|^2] < +\infty, \quad \forall T>0. \tag{1.5}$$

In the construction of next sections we need that the mean value of $\|\psi_t\|^2$ be a constant; for this aim we add the following assumption.

Assumption 1.4. *For all* $t\geqslant 0$ *we have*

$$K_t + K_t^* = L_t^* L_t + J_t^*(\mathbb{1}\otimes\gamma_t)J_t. \tag{1.6}$$

It is useful to introduce the self-adjoint operator

$$H_t := \frac{1}{2\mathrm{i}}(K_t - K_t^*), \tag{1.7}$$

so that we can write

$$K_t = \mathrm{i}H_t + \tfrac{1}{2}(L_t^* L_t + J_t^*(\mathbb{1}\otimes\gamma_t)J_t). \tag{1.8}$$

By applying Itô's formula to $\langle\psi_t|a\psi_t\rangle$ (Métivier and Pellaumail, 1980, Remark 3.9 (2), p. 50; Métivier, 1982, Theorem 27.2, p. 190), we obtain

$$\begin{aligned}\langle\psi_t|a\psi_t\rangle &= \langle\xi|a\xi\rangle + \int_0^t \langle\psi_{s^-}|\mathscr{L}_s[a]\psi_{s^-}\rangle\,\mathrm{d}s + \sum_{k=1}^{\infty}\int_0^t \langle\psi_{s^-}|(L_{ks}^*a + aL_{ks})\psi_{s^-}\rangle\,\mathrm{d}W_{ks}\\ &\quad + \int_{\mathscr{Y}\times(0,t]} [\langle(J_s\,\psi_{s^-})(y)+\psi_{s^-}|a((J_s\,\psi_{s^-})(y)+\psi_{s^-})\rangle\\ &\qquad -\langle\psi_{s^-}|a\psi_{s^-}\rangle]\widetilde{\Pi}(\mathrm{d}y,\mathrm{d}s), \end{aligned}\tag{1.9}$$

where $\mathscr{L}_t\in\mathscr{L}(\mathscr{L}(\mathscr{H}))$ and $\forall a\in\mathscr{L}(\mathscr{H})$

$$\begin{aligned}\mathscr{L}_t[a] &:= \mathrm{i}\,[H_t,a] - \tfrac{1}{2}L_t^*L_ta - \tfrac{1}{2}aL_t^*L_t + L_t^*(a\otimes\mathbb{1})L_t\\ &\quad -\tfrac{1}{2}J_t^*(\mathbb{1}\otimes\gamma_t)J_ta - \tfrac{1}{2}a\,J_t^*(\mathbb{1}\otimes\gamma_t)J_t + J_t^*(a\otimes\gamma_t)J_t\end{aligned}\tag{1.10}$$

with $[a,b]:=ab-ba$. Let us observe that for $a=\mathbb{1}$ the time integral vanishes and so $\mathbb{E}_P[\|\psi\|^2]$ is a constant, as we needed.

Let v be a non-random unit vector in $\mathscr{H}$; we define

$$\widehat{m}_{kt} := \langle v|L_{kt}v\rangle, \qquad \widehat{I}_t(y) := \|(J_tv)(y)+v\|^2 \quad \text{if } \psi_{t^-}=0, \tag{1.11a}$$

$$\widehat{m}_{kt} := \frac{\langle\psi_{t^-}|L_{kt}\psi_{t^-}\rangle}{\|\psi_{t^-}\|^2}, \qquad \widehat{I}_t(y) := \frac{\|(J_t\psi_{t^-})(y)+\psi_{t^-}\|^2}{\|\psi_{t^-}\|^2} \quad \text{if } \psi_{t^-}\neq 0, \tag{1.11b}$$

$$Z_t := 2\sum_{k=1}^{\infty}\int_0^t(\operatorname{Re}\widehat{m}_{ks})\,\mathrm{d}W_{ks} + \int_{\mathscr{Y}\times(0,t]}[\widehat{I}_s(y)-1]\widetilde{\Pi}(\mathrm{d}y,\,\mathrm{d}s). \tag{1.12}$$

In particular the process Z is a martingale with jumps given by

$$\Delta Z_t := Z_t - Z_{t^-} = \int_{\mathscr{Y}}[\widehat{I}_t(y)-1]\,\Pi(\mathrm{d}y,\{t\}). \tag{1.13}$$

Theorem 1.2. *Under Assumptions* 1.1–1.4, *if* $\xi\in L^2(\Omega,\mathscr{F}_0,P;\mathscr{H})$, *then* $\psi_t\in L^2(\Omega,\mathscr{F}_t,P;\mathscr{H})$ *and the process* $(\|\psi_t\|^2)$ *is a positive martingale satisfying the equation*

$$\|\psi_t\|^2 = \|\xi\|^2 + \int_{(0,t]}\|\psi_{s^-}\|^2\,\mathrm{d}Z_s, \tag{1.14}$$

where Z *is defined by Eq.* (1.12). *Moreover, we have also*

$$\|\psi_t\|^2 = \|\xi\|^2\exp\left\{Z_t - 2\sum_{k=1}^{\infty}\int_0^t(\operatorname{Re}\widehat{m}_{ks})^2\,\mathrm{d}s\right\}\prod_{s\leqslant t}[(1+\Delta Z_s)\exp(-\Delta Z_s)], \tag{1.15}$$

where the infinite product is a.s. absolutely convergent. Finally, in Eq. (1.9) *the two stochastic integrals* (*with respect to* W *and* $\widetilde{\Pi}$) *are martingales and*

$$\mathbb{E}_P[\langle\psi_t|a\psi_t\rangle] = \mathbb{E}_P[\langle\xi|a\xi\rangle] + \int_0^t\mathbb{E}_P[\langle\psi_s|\mathscr{L}_s[a]\psi_s\rangle]\,\mathrm{d}s. \tag{1.16}$$

The representation (1.15) is a consequence of Theorem 29.2, p. 203, of Métivier (1982). The proof of the other statements is essentially equal to the proof of Theorem 1.4 of Barchielli and Holevo (1995). Now, in order $\|\psi_t\|^2$ be a family of local probability densities, we introduce a further assumption (cf. Ikeda and Watanabe, 1981, p. 176).

Assumption 1.5. *We assume that* $(\Omega, \mathscr{F}_\infty)$, $\mathscr{F}_\infty := \bigvee_{t\geq 0} \mathscr{F}_t$, *is a standard measurable space and that* $\mathbb{E}_P[\|\xi\|^2] = 1$.

Proposition 1.1. *Under Assumptions* 1.1–1.5, *the formula*

$$\forall t \geq 0, \qquad \forall F \in \mathscr{F}_t, \quad \widehat{P}_\xi(F) := \mathbb{E}_P[\|\psi_t\|^2 1_F] \tag{1.17}$$

defines a new probability law $\widehat{P}_\xi$ *on* $(\Omega, \mathscr{F}_\infty)$. *Moreover, under the law* $\widehat{P}_\xi$, $\Pi(\mathrm{d}y, \mathrm{d}t)$ *is a point process with stochastic intensity* $\widehat{I}_t(y)\gamma_t(y)v(\mathrm{d}y)\,\mathrm{d}t$ *and the processes* $\widehat{W}_{kt}$, *defined by*

$$\widehat{W}_{kt} := W_{kt} - 2\int_0^t (\mathrm{Re}\,\widehat{m}_{ks})\,\mathrm{d}s, \quad t \geq 0,\ k = 1, 2, \ldots, \tag{1.18}$$

are independent $(\mathscr{F}_t)$*-adapted standard Wiener processes.*

The proof is again the same as those of Proposition 2.5 and Remark 2.6 of Barchielli and Holevo (1995). We shall write

$$\widehat{\Pi}(\mathrm{d}y, \mathrm{d}t) := \Pi(\mathrm{d}y, \mathrm{d}t) - \widehat{I}_t(y)\gamma_t(y)v(\mathrm{d}y)\,\mathrm{d}t \tag{1.19}$$

for the compensated point process in $(\Omega, (\mathscr{F}_t), \mathscr{F}_\infty, \widehat{P}_\xi)$.

In the definitions of $\widehat{m}_{kt}$ and $\widehat{I}_t(y)$ the quantity $\psi_t/\|\psi_t\|$ appears and it is interesting to have an equation for it. However, because a change of phase in ψ_t does not matter in the construction of the following sections, we can modify it in order to obtain a more symmetric equation. So, let us define $\widehat{\psi}_t := v$ if $\psi_t = 0$ and, when $\psi_t \neq 0$,

$$\widehat{\psi}_t := \frac{1}{\|\psi_t\|} \exp\left\{ -\mathrm{i} \sum_{k=1}^\infty \int_0^t (\mathrm{Im}\,\widehat{m}_{ks})(\mathrm{d}W_{ks} - \mathrm{Re}\,\widehat{m}_{ks}\,\mathrm{d}s) \right\} \psi_t. \tag{1.20}$$

Theorem 1.3. *Under Assumptions* 1.1–1.5, $\widehat{\psi}_t$ *satisfies the equation*

$$\mathrm{d}\widehat{\psi}_t = -\widehat{K}(t, \widehat{\psi}_{t-})\,\mathrm{d}t + \sum_{k=1}^\infty \widehat{L}_k(t, \widehat{\psi}_{t-})\,\mathrm{d}\widehat{W}_{kt} + \int_{\mathscr{Y}} \widehat{J}(t, \widehat{\psi}_{t-}, y)\widehat{\Pi}(\mathrm{d}y, \mathrm{d}t), \tag{1.21}$$

in the stochastic basis $(\Omega, (\mathscr{F}_t), \mathscr{F}_\infty, \widehat{P}_\xi)$. *The coefficients in Eq.* (1.21) *are defined by*

$$\widehat{K}(t,f) := 0, \quad \widehat{L}_k(t,f) := 0, \quad \text{if } f = 0, \tag{1.22a}$$

$$\widehat{J}(t,f,y) := 0 \quad \text{if } (J_t f)(y) + f = 0, \tag{1.22b}$$

and, otherwise, by

$$\begin{aligned}\widehat{K}(t,f) &:= \mathrm{i}H_t f + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{\langle f|L_{kt}f\rangle}{\|f\|^2}L_{kt}^* - \frac{\langle f|L_{kt}^*f\rangle}{\|f\|^2}L_{kt}\right)f\\ &\quad + \frac{1}{2}\sum_{k=1}^{\infty}\left(L_{kt}^* - \frac{\langle f|L_{kt}^*f\rangle}{\|f\|^2}\right)\left(L_{kt} - \frac{\langle f|L_{kt}f\rangle}{\|f\|^2}\right)f + \frac{1}{2}J_t^*(\mathbb{1}\otimes\gamma_t)J_t f\\ &\quad + \int_{\mathscr{Y}}\left[\left(1 - \frac{\|(J_t f)(y)+f\|}{\|f\|}\right)(J_t f)(y)\right.\\ &\quad \left. + \frac{1}{2}\left(1 - \frac{\|(J_t f)(y)+f\|}{\|f\|}\right)^2 f\right]\gamma_t(y)\nu(\mathrm{d}y), \end{aligned} \tag{1.23}$$

$$\widehat{L}_k(t,f) := L_{kt}f - \frac{\langle f|L_{kt}f\rangle}{\|f\|^2}f, \tag{1.24}$$

$$\widehat{J}(t,f,y) := \frac{\|f\|}{\|(J_t f)(y)+f\|}[(J_t f)(y)+f] - f. \tag{1.25}$$

The structure of the stochastic differential of $\widehat{\psi}_t$ can be found by the formal rules of stochastic calculus; then the theorem can be proved similarly to Theorem 2.7 of Barchielli and Holevo (1995). Some particular cases of Eq. (1.21) have been introduced in the physical literature either by theoretical motivations (Gisin, 1984; Ghirardi et al., 1990) either to use it for numerical simulation in quantum optics (Gisin and Percival, 1992; Wiseman and Milburn, 1993). The idea of connecting equations of the type of Eq. (1.21) to some quantum analogue of filtering theory and to measurement continuous in time is due to Belavkin and for more particular cases to Diósi (Belavkin, 1988, 1989a, b; Diósi, 1988a, b). For recent applications see Carmichael (1996). Eq. (1.1) has been studied also in Holevo (1996), where the case is considered when $J_t \equiv 0$, $L_{kt} \equiv L_k$, but L_k unbounded.

2. The continuous measurement

Let us introduce now an $\mathbb{R}^d$-valued process $X(t)$, which will represent, under the law $\widehat{P}_\xi$, the output of the continuous measurement.

Assumption 2.1. *The functions* $c : \mathbb{R}_+ \to \mathbb{R}^d$, $a_{ik} : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1,\dots,d,\ k = 1,2,\dots)$, $g : \mathscr{Y} \times \mathbb{R}_+ \to \mathbb{R}^d$ *are measurable and* $\sum_{k=1}^{\infty}[a_{ik}(t)]^2 < +\infty$ $(\forall t,\ \forall i)$; *c and*

$$b_{ij}(t) := \sum_{k=1}^{\infty} a_{ik}(t)a_{jk}(t), \quad i,j = 1,\dots,d, \tag{2.1}$$

are Lebesgue integrable on every compact interval. The function g and the intensity of the Poisson process satisfy

$$\int_{\mathscr{Y}\times(0,t]} \varphi(g(y;s))\nu(\mathrm{d}y)\,\mathrm{d}s < +\infty, \quad \forall t > 0, \tag{2.2}$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}$ *is defined by*

$$\varphi(x) := \frac{|x|^2}{1+|x|^2}, \quad x \in \mathbb{R}^d. \tag{2.3}$$

For $i = 1, \ldots, d$ we define

$$X_i(t) := \int_0^t c_i(s)\,\mathrm{d}s + \sum_{k=1}^{\infty} \int_0^t a_{ik}(s)\,\mathrm{d}W_{ks} + \int_{\mathscr{Y}\times(0,t]} \varphi(g(y;s))g_i(y;s)\Pi(\mathrm{d}y,\mathrm{d}s)$$
$$+ \int_{\mathscr{Y}\times(0,t]} \frac{g_i(y;s)}{1+|g(y;s)|^2}\widetilde{\Pi}(\mathrm{d}y,\mathrm{d}s). \tag{2.4}$$

By Assumption 2.1 the integrals with respect to W and $\widetilde{\Pi}$ are L^2–P-martingales; see Ikeda and Watanabe (1981), pp. 59–63, about the integrals with respect to point processes.

Let us note that, under the law P, X is a process with independent increments; however, this is not true in general under the law $\widehat{P}_\xi$, because W and Π are no more a Wiener and a Poisson process (see Proposition 1.1). Finally, we denote by $\mathscr{E}_t^s$, $0 \leqslant s \leqslant t$, the σ-algebra generated by $X(r) - X(s)$, $r \in [s,t]$, and we set $\mathscr{E}_t := \mathscr{E}_t^0$, $\mathscr{E} := \mathscr{E}_\infty^0$.

Let us denote by $\Psi(t,s;\omega;u)$ a solution of Eq. (1.1) with a non-random initial condition $u \in \mathscr{H}$ at time s. If $\xi \in L^2(\Omega,\mathscr{F}_s,P;\mathscr{H})$, $\Psi(t,s;\omega;\xi(\omega))$ is a solution of Eq. (1.1) with random initial condition ξ at time s. By the results of Section 1 and the P-uniqueness of the strong solution of Eq. (1.1), we have for $t \geqslant \tau \geqslant s$, $u,w \in \mathscr{H}$, $\alpha,\beta \in \mathbb{C}$

$$\Psi(t,s;\cdot;u) \in L^2(\Omega,\mathscr{F}_t,P;\mathscr{H}), \qquad \mathbb{E}_P[\|\Psi(t,s;\cdot;u)\|^2] = \|u\|^2, \tag{2.5}$$

$$\Psi(t,s;\omega;\alpha u + \beta w) = \alpha\Psi(t,s;\omega;u) + \beta\Psi(t,s;\omega;w), \quad P\text{-a.s.} \tag{2.6}$$

$$\Psi(t,\tau;\omega;\Psi(\tau,s;\omega;u)) = \Psi(t,s;\omega;u), \quad P\text{-a.s.} \tag{2.7}$$

Let us recall that in quantum mechanics measurements (observations) are represented by instruments (Davies and Lewis, 1970; Ozawa, 1984). Given a measurable space (Ω,Σ), an *instrument* $\mathscr{I}$ is a map from Σ into $\mathscr{L}(\mathscr{L}(\mathscr{H}))$ such that

(i) $\forall B \in \Sigma$, $\mathscr{I}(B)$ is a normal, completely positive map, i.e. $\mathscr{I}(B)$ is continuous in the weak* topology (Davies, 1976, pp. 5–6) and for every choice of $n \in \mathbb{N}$, $\{u_j\} \subset \mathscr{H}$, $\{a_j\} \subset \mathscr{L}(\mathscr{H})$ the inequality $\sum_{i,j=1}^n \langle u_i|\mathscr{I}(B)[a_i^* a_j]u_j\rangle \geqslant 0$ is satisfied;
(ii) $\forall u \in \mathscr{H}$, $\forall a \in \mathscr{L}(\mathscr{H})$, $a \geqslant 0$, $\langle u|\mathscr{I}(\cdot)[a]u\rangle$ is a σ-additive measure;
(iii) $\mathscr{I}(\Omega)[\mathbb{1}] = \mathbb{1}$.

Proposition 2.1. *The equation*

$$\langle u|\mathscr{I}(s,t;E)[a]u\rangle := \mathbb{E}_P[1_E\langle\Psi(t,s;\cdot;u)|a\Psi(t,s;\cdot;u)\rangle], \tag{2.8}$$

$\forall u \in \mathscr{H}$, $\forall a \in \mathscr{L}(\mathscr{H})$, $a \geqslant 0$, $\forall s,t \in \mathbb{R}_+$, $t \geqslant s$, $\forall E \in \mathscr{E}_t^s$, *defines a family of instruments such that*

$$\mathscr{I}(s,\tau;E)\circ\mathscr{I}(\tau,t;F) = \mathscr{I}(s,t;E\cap F) \quad \text{for } s \leqslant \tau \leqslant t, \quad E \in \mathscr{E}_\tau^s,\ F \in \mathscr{E}_t^\tau. \tag{2.9}$$

To show that $\mathscr{I}(s,t)$ is an instrument we need some results in the theory of W^*-algebras; but apart from this, the proof is standard and we omit it. To prove Eq. (2.9) one needs to exploit the Markov properties of the solutions of our SDE (essentially Eq. (2.7)).

Families of instruments with property (2.9) have been introduced by Barchielli et al. (1983) in order to formalize the idea of measurements continuous in time in quantum mechanics; the physical probabilities are given by $\langle\xi|\mathscr{I}(0,t;\cdot)[\mathbb{1}]\xi\rangle$, where ξ is the initial state of the quantum system. Our construction is such that the physical law is nothing but $\widehat{P}_\xi$, defined in Proposition 1.1; indeed if ξ is a non-random unit vector in $\mathscr{H}$, from Eqs. (1.17) and (2.8) we have

$$\langle\xi|\mathscr{I}(0,t;E)[\mathbb{1}]\xi\rangle = \widehat{P}_\xi(E), \quad \forall t\geqslant 0, \forall E\in\mathscr{E}_t. \tag{2.10}$$

An evolution system of probability operators T_t^s, $0\leqslant s\leqslant t$, can be associated to the instruments $\mathscr{I}(s,t)$ (Barchielli and Lupieri, 1991). Let us introduce the von Neumann algebra $L^\infty(\mathbb{R}^d;\mathscr{L}(\mathscr{H}))\simeq\mathscr{L}(\mathscr{H})\otimes L^\infty(\mathbb{R}^d)$. A map $T_t^s\in\mathscr{L}(L^\infty(\mathbb{R}^d;\mathscr{L}(\mathscr{H})))$ is a *probability operator* if and only if, by definition,
(a) T_t^s is completely positive and normal,
(b) $T_t^s[\mathbb{1}]=\mathbb{1}$,
(c) T_t^s commutes with translations in $L^\infty(\mathbb{R}^d)$.

Let $C_0(\mathbb{R}^d_\infty)$ be the space of the continuous complex functions f on the one-point compactification of $\mathbb{R}^d$ with the supremum norm $\|f\| = \sup_x|f(x)|$. Moreover, let $C_2(\mathbb{R}^d_\infty)$ be the space of the complex functions f such that f and its first and second derivatives belong to $C_0(\mathbb{R}^d_\infty)$; $C_2(\mathbb{R}^d_\infty)$ becomes a Banach space with the norm

$$\|f\|_2 := \sup_x|f(x)| + \sum_{i=1}^d \sup_x\left|\frac{\partial f(x)}{\partial x_i}\right| + \sum_{i,j=1}^d \sup_x\left|\frac{\partial^2 f(x)}{\partial x_i\partial x_j}\right|. \tag{2.11}$$

We need also to introduce the analogous Banach spaces $C_0(\mathbb{R}^d_\infty;\mathscr{L}(\mathscr{H}))$ and $C_2(\mathbb{R}^d_\infty;\mathscr{L}(\mathscr{H}))$, where the norms are similar to the previous ones, with the moduli substituted by the operator norms. Let us set for short $L^\infty := L^\infty(\mathbb{R}^d;\mathscr{L}(\mathscr{H}))$, $C_0 := C_0(\mathbb{R}^d_\infty;\mathscr{L}(\mathscr{H}))$, $C_2 := C_2(\mathbb{R}^d_\infty;\mathscr{L}(\mathscr{H}))$. The space C_2 is $\|\cdot\|$-dense in C_0 and C_0 is weak* dense in L^∞.

Let us recall some important properties of a probability operator T_t^s:
(d) T_t^s is a norm-one contraction on L^∞,
(e) $T_t^s[C_0]\subset C_0$, the restriction of T_t^s to C_0 is a norm-one contraction (with respect to $\|\cdot\|$) and determines uniquely T_t^s by weak*-continuity,
(f) $T_t^s[C_2]\subset C_2$, the restriction of T_t^s to C_2 is a norm-one contraction (with respect to $\|\cdot\|_2$) and determines uniquely T_t^s on C_0 by $\|\cdot\|$-continuity.

Point (d) is a consequence of the positivity and the identity preserving property. The other properties follow from (d) and the fact that T_t^s commutes with translations.

Proposition 2.2. *The equation*

$$\begin{aligned}\langle u|T_t^s[a\otimes f](x)u\rangle &:= \int_\Omega f(x+X_t(\omega)-X_s(\omega))\langle u|\mathscr{I}(s,t;\,\mathrm{d}\omega)[a]u\rangle \\ &\equiv \mathbb{E}_P[f(x+X_t-X_s)\langle\Psi(t,s;\cdot;u)|a\Psi(t,s;\cdot;u)\rangle], \\ &\qquad \forall x\in\mathbb{R}^d,\quad \forall u\in\mathscr{H},\quad \forall a\in\mathscr{L}(\mathscr{H}),\quad \forall f\in C_0(\mathbb{R}^d_\infty),\end{aligned} \tag{2.12}$$

defines a probability operator. Moreover, T_t^t is the identity map and

$$T_\tau^s \circ T_t^\tau = T_t^s, \quad s \leqslant \tau \leqslant t. \tag{2.13}$$

The fact that Eq. (2.12) defines a probability operator on the whole L^∞ can be proved as in Barchielli and Lupieri (1991). The fact that the T_t^t is the identity is trivial; Eq. (2.13) can be proved similar to Eq. (2.9).

In order to arrive to an evolution equation for T_t^s, we define the map $\mathscr{K}_t$ from $\mathscr{L}(\mathscr{H}) \times C_2(\mathbb{R}_\infty^d)$ into C_0 by

$$\begin{aligned}\mathscr{K}_t[a \otimes f](x) := {} & f(x)\mathscr{L}_t[a] + \sum_{i=1}^d c_i(t) \frac{\partial f(x)}{\partial x_i} a \\ & + \mathscr{K}_t^1[a \otimes f](x) + \mathscr{K}_t^2[a \otimes f](x),\end{aligned} \tag{2.14}$$

$$\mathscr{K}_t^1[a \otimes f](x) := \frac{1}{2} \sum_{i,j=1}^d b_{ij}(t) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} a + \sum_{i=1}^d \sum_{k=1}^\infty \frac{\partial f(x)}{\partial x_i} a_{ik}(t)(L_{kt}^* a + a L_{kt}), \tag{2.15}$$

$$\begin{aligned}\langle u | \mathscr{K}_t^2[a \otimes f](x) w \rangle := \int_{\mathscr{Y}} \Bigg\{ & [f(x + g(y;t)) - f(x)] \, \langle (J_t u)(y) + u | a((J_t w)(y) + w) \rangle \\ & - \sum_{i=1}^d \frac{\partial f(x)}{\partial x_i} \frac{g_i(y;t)}{1 + |g(y;t)|^2} \langle u | a w \rangle \Bigg\} \gamma_t(y) \nu(\mathrm{d}y),\end{aligned} \tag{2.16}$$

where u and w are arbitrary vectors in $\mathscr{H}$. Let us recall that the objects which appear in the definition of $\mathscr{K}_t$ satisfy Assumptions 1.1, 1.2, 1.4 and 2.1. It is possible to prove that the series in Eq. (2.15) is strongly convergent and that the integral in Eq. (2.16) exists.

By comparing Eqs. (2.14)–(2.16) with the form of the quantum Lévy–Khinchin formula given in Theorem 3 of Barchielli and Paganoni (1996) it is easy to prove that $\mathscr{K}_t$ admits a *Lévy–Khinchin representation*, as defined in Barchielli et al. (1993).

Proposition 2.3. *The following statements hold*:

(i) *the map $\mathscr{K}_t$ can be extended to a unique continuous linear operator from C_2 into C_0, which we denote again by $\mathscr{K}_t$, and $\mathscr{K}_\cdot \in L^1([0,T]; \mathscr{L}(C_2; C_0))$, $\forall T > 0$;*

(ii) *the map $\mathscr{K}_t$ has a unique extension to the infinitesimal generator $\widehat{\mathscr{K}_t}$ (t is fixed) of a strongly continuous semigroup $S_t(s)$, $s \geqslant 0$, of probability operators on C_0;*

(iii) *the restriction $\widetilde{\mathscr{K}_t}$ of $\mathscr{K}_t$ to the C_2-elements which are mapped into C_2-elements is the inifinitesimal generator of a strongly continuous semigroup $\widetilde{S}_t(s)$, $s \geqslant 0$, of contractions on C_2, where $\widetilde{S}_t(s)$ is the restriction of $S_t(s)$ to C_2.*

Proof. By using the fact that $\mathscr{K}_t$ admits a Lévy–Khinchin representation and applying Proposition 1 of Barchielli and Paganoni (1996), we obtain the extension of $\mathscr{K}_t$ to a continuous operator from C_2 into C_0. Moreover, from Eqs. (2.14)–(2.16) it is possible to obtain the estimate

$$\begin{aligned}\|\mathscr{K}_t\|_{2\to 0} \leqslant {} & \|\mathscr{L}_t\| + \max_{1 \leqslant i \leqslant d} |c_i(t)| + \tfrac{1}{2} \max_{1 \leqslant i,j \leqslant d} |b_{ij}(t)| \\ & + 2 \max_{1 \leqslant i \leqslant d} \sqrt{b_{ii}(t)} \|L_t\| + 2\|\gamma_t\| \left(\|J_t\| + 2\sqrt{\int_{\mathscr{Y}} \varphi(g(y;t)) \nu(\mathrm{d}y)} \right)^2 .\end{aligned}$$

Because of the assumptions on the time dependence, this norm turns out to be time integrable on every compact interval.

By the fact that $\mathscr{K}_t$ has a Lévy–Khinchin representation, we have that $\mathscr{K}_t$ determines uniquely the generator $\widetilde{\mathscr{K}_t}$ of a strongly continuous semigroup of probability operators on C_0 (Barchielli et al. 1993). The statement about $\widetilde{\mathscr{K}_t}$ follows from the fact that $\widetilde{\mathscr{K}_t}$ and $S_t(s)$ commute with translations. □

Let us recall that a two-parameter family of bounded operators $U(s,t)$, $0\leqslant s\leqslant t$, on some Banach space X, is called an *evolution system* if the following conditions are satisfied:

(i) $U(s,s)=\mathbb{1}$, $U(s,r)U(r,t)=U(s,t)$ for $0\leqslant s\leqslant r\leqslant t$;

(ii) $(s,t)\mapsto U(s,t)$ is strongly continuous for $0\leqslant s\leqslant t$.

With respect to standard definitions (cf. Pazy, 1983, Definition 5.3, p. 129) we have changed the time ordering, because we are interested in T_t^s, which satisfies Eq. (2.13). Consistently with this, we shall have to consider final value problems for differential equations, instead of the usual initial value problems; however, the passage from one case to the other one is trivially obtained by changing sign to time.

We need to recall also that the predual of the space $\mathscr{L}(\mathscr{H})$ is $\mathscr{T}(\mathscr{H})$: trace-class operators on $\mathscr{H}$, i.e. $\rho\in\mathscr{T}(\mathscr{H})$ if and only if $\rho\in\mathscr{L}(\mathscr{H})$ and $\|\rho\|_1:=\mathrm{Tr}\{\sqrt{\rho^*\rho}\} < +\infty$. Similarly, the predual of the space L^∞ can be identified with $L^1(\mathbb{R}^d;\mathscr{T}(\mathscr{H}))=:L^1$. We shall use the notations $\langle\rho,a\rangle:=\mathrm{Tr}\{\rho a\}$ $(\rho\in\mathscr{T}(\mathscr{H}),\ a\in\mathscr{L}(\mathscr{H}))$ and $\langle\Phi,A\rangle:=\int_{\mathbb{R}^d}\mathrm{Tr}\{\Phi(x)A(x)\}\,\mathrm{d}x$ $(\Phi\in L^1, A\in L^\infty)$.

Theorem 2.2. *The family* $\{T_t^s, 0\leqslant s\leqslant t\}$ *is an evolution system in* C_0 *satisfying,* $\forall A\in C_2$,

$$T_t^s[A]=A+\int_s^t T_\tau^s\circ\mathscr{K}_\tau[A]\,\mathrm{d}\tau, \tag{2.17}$$

$$T_t^s[A]=A+\int_s^t \mathscr{K}_\tau\circ T_t^\tau[A]\,\mathrm{d}\tau, \tag{2.18}$$

where the integrals are Bochner integrals in C_0.

Moreover, for $A\in C_2$, $0\leqslant s\leqslant t$, $u(s):=T_t^s[A]$ *is the unique* C_2*-valued solution of the final value problem*

$$u(s)=A+\int_s^t \mathscr{K}_\tau[u(\tau)]\,\mathrm{d}\tau. \tag{2.19}$$

Proof. First, let us apply Itô's formula to $f(x+X_t-X_s)\langle\Psi(t,s;\cdot;u)|a\Psi(t,s;\cdot;u)\rangle$; the stochastic differential of the second factor is already given by Eq. (1.9) with the substitution $\xi\to u$. In the resulting formula one can check that all the integrals with respect to W and $\widetilde{\Pi}$ are martingales and not only local martingales, so that they disappear by taking the expectation value. By this we obtain

$$\langle u|T_t^s[a\otimes f](x)u\rangle=f(x)\langle u|au\rangle+\int_s^t\langle u|(T_\tau^s\circ\mathscr{K}_\tau[a\otimes f])(x)u\rangle\,\mathrm{d}\tau \tag{2.20}$$

which holds for every $x\in\mathbb{R}^d$, $u\in\mathscr{H}$, $a\in\mathscr{L}(\mathscr{H})$ and $f\in C_2(\mathbb{R}^d_\infty)$.

Let us recall that the linear subspace of $\mathscr{T}(\mathscr{H})$ generated by the projectors on the one-dimensional subspaces of $\mathscr{H}$ is dense and also the linear span of $C_2(\mathbb{R}^d_\infty)\times\mathscr{L}(\mathscr{H})$ is a dense subspace of C_2. So, by applying the dominated convergence theorem, we obtain from Eq. (2.20) the weak form of Eq. (2.17). Moreover, because of the property (i) of Proposition 2.3 we have

$$\int_s^t \|T_\tau^s \circ \mathscr{K}_\tau[A]\|\,\mathrm{d}\tau \leqslant \|A\|_2 \int_s^t \|\mathscr{K}_\tau\|_{2\to 0}\,\mathrm{d}\tau < \infty;$$

so the r.h.s. of Eq. (2.17) is a Bochner integral and the weak form of Eq. (2.17) implies the strong one.

By using properties (e) and (f) of T_t^s, property (i) of $\mathscr{K}_t$ (Proposition 2.3) and Eq. (2.17), we can show that T_t^s is strongly continuous in (s,t). Together with Eq. (2.13), this gives that $\{T_t^s\}$ is a norm-one evolution system.

By using again property (i) of Proposition 2.3, we can show that the integral in the r.h.s. of Eq. (2.18) is well defined as a Bochner integral. Moreover, for $A\in C_2$ we have the estimate

$$\frac{1}{\varepsilon}\left\| T_s^{s-\varepsilon}[A] - A - \int_{s-\varepsilon}^s \mathscr{K}_\tau \circ T_t^\tau[A]\,\mathrm{d}\tau \right\| \leqslant \frac{2}{\varepsilon}\|A\|_2 \int_{s-\varepsilon}^s \|\mathscr{K}_\tau - \mathscr{K}_s\|_{2\to 0}\,\mathrm{d}\tau$$
$$+ \sup_{s-\varepsilon\leqslant\tau\leqslant s} \|(T_\tau^{s-\varepsilon} - \mathbb{1})\circ \mathscr{K}_s[A]\| + \|\mathscr{K}_s\|_{2\to 0} \sup_{s-\varepsilon\leqslant\tau\leqslant s} \|(\mathbb{1} - T_s^\tau)[A]\|.$$

The r.h.s. of this inequality goes to zero for $\varepsilon\downarrow 0$. In conclusion, $\{T_t^s\}$ is a norm-one evolution system satisfying Eq. (2.17) and

$$\frac{\partial^-}{\partial s} T_t^s[A]|_{t=s} = -\mathscr{K}_s[A] \quad \text{a.e. on} \quad s\geqslant 0,\ \forall A\in C_2. \tag{2.21}$$

Let us consider now Theorem 3.1, p. 135, of Pazy (1983) about evolution systems. Let us note that C_2 is densely and continuously imbedded in C_0, i.e. C_2 is a dense subspace of C_0 and $\|f\|\leqslant\|f\|_2$ for $f\in C_2$. Now we take the family $\widehat{\mathscr{K}_t}$ $(t\in[0,T])$ of infinitesimal generators of strongly continuous semigroups of probability operators $S_t(s)$ on C_0. The following conditions are satisfied:

(1) $\{\widehat{\mathscr{K}_t}\}$ is a stable family with stability constants 1 and 0, because the $S_t(s)$ are semigroups of contractions (Pazy, 1983, pp. 130–131);
(2) C_2 is $\widehat{\mathscr{K}_t}$-admissible for $t\in[0,T]$, i.e. C_2 is an invariant subspace of $S_t(s)$ (Pazy, 1983, Definition 5.3, p. 122), because the operators $\widehat{\mathscr{K}_t}$ commute with the translations and the restrictions $\widetilde{S}_t(s)$ of $S_t(s)$ to C_2 are again strongly continuous semigroups on C_2. Moreover, the operators $\mathscr{K}_t$ (which are called *parts* of $\widehat{\mathscr{K}_t}$ (cf. Pazy, 1983 Definition 10.3, p. 39) are a stable family in C_2 because they are again generators of strongly continuous semigroups of contractions on C_2 in Proposition 2.3);
(3) C_2 is a subspace of the domain of $\widehat{\mathscr{K}_t}$ and $\widehat{\mathscr{K}}_\cdot\in L^1([0,T],\mathscr{L}(C_2,C_0))$, $\forall T\geqslant 0$. So the requests H_1, H_2 of Pazy (1983), p. 135, and H_3' of Pazy (1983), Remark 3.2, p. 138, are satisfied in our case.

Theorem 3.1, p. 135, and the discussion at p. 139 of Pazy (1983) say that there exists a unique bounded evolution system satisfying Eqs. (2.17) and (2.21); by the previous discussion, this evolution system is $\{T_t^s\}$.

By considering the constructive part of the proof of the quoted theorem in Pazy (1983), where approximants $T_t^{s(n)}$ of T_t^s and $\mathscr{K}_t^{(n)}$ of $\mathscr{K}_t$ are introduced, it is possible to show that these approximants satisfy Eq. (2.18) and that it is possible to take the limit for $n \to \infty$. We conclude that T_t^s satisfies also Eq. (2.18) and this ends the proof of the first part of the theorem.

If we take a solution $u(s)$ of the integral equation (2.19), it is easy to prove that $\mathrm{d}/\mathrm{d}r(T_r^s[u(r)]) = 0$ a.e. Let us observe that $T_r^s[u(r)]|_{r=s} = u(s)$ and $T_r^s[u(r)]|_{r=t} = T_t^s[A]$. So if we prove that $T_\cdot^s[u(\cdot)]$ is an absolutely continuous function of time we can conclude that $T_r^s[u(r)]$ is a constant and so $T_t^s[A]$ is the unique solution of (2.19). Let us start by observing that it is sufficient to prove that $T_{\cdot *}^r[\Phi]$ is a strongly absolutely continuous function for every Φ in a suitable dense subset of L^1.

Let us denote by C_2^1 the subset of the elements of L^1 which are twice continuously differentiable and with first and second derivatives belonging to L^1. Then, the equation $\langle \mathscr{G}_t[\Phi], A\rangle = \langle \Phi, \mathscr{K}_t[A]\rangle, \Phi \in C_2^1, A \in C_2$, defines a linear map from C_2^1 into L^1 and by Eq. (2.17) we have

$$\langle T_{s*}^r[\Phi], A\rangle = \langle \Phi, A\rangle + \int_s^r \langle \mathscr{G}_\tau \circ T_{\tau *}^r[\Phi], A\rangle\, \mathrm{d}\tau, \quad \forall \Phi \in C_2^1, \ \forall A \in C_2. \tag{2.22}$$

As in Proposition 2.3 we have proved that $\mathscr{K}_\cdot \in L^1([0,T]; \mathscr{L}(C_2; C_0))$, now we can prove that $\mathscr{G}_\cdot \in L^1([0,T]; \mathscr{L}(C_2^1; L^1))$, $\forall T \geqslant 0$. Together with Eq. (2.22), this implies

$$T_{s*}^r[\Phi] = \Phi + \int_s^r \mathscr{G}_\tau \circ T_{\tau *}^r[\Phi]\, \mathrm{d}\tau, \quad \forall \Phi \in C_2^1,$$

and the strong absolute continuity of $T_{s*}^r[\Phi]$ follows. □

By the results of Barchielli et al. (1993) and Barchielli and Paganoni (1996) on the quantum Lévy–Khinchin formula one can check that our $\mathscr{K}_t$ is the most general map having a Lévy–Khinchin representation. Therefore, in the time homogeneous case T_t^s reduces to a semigroup which is the most general *semi-uniformly* continuous semigroup of probability operators (SCSPO). So, we have shown that the stochastic approach to continuous measurements in time and the whole construction we have done allows us to obtain the same generality as in the formulation based on instruments and SCSPO, followed in Barchielli et al. (1993). This gives us an answer to the open problem presented in Barchielli and Paganoni (1996) of bringing to the same level of generality these two formulations of the quantum measurements continuous in time; moreover, the semigroup approach is now generalized to the non-autonomous case.

3. A posteriori states

In the standard formulation of quantum mechanics the set $\mathscr{S}(\mathscr{H})$ of the possible states of a quantum system is taken to be

$$\mathscr{S}(\mathscr{H}) := \{\rho \in \mathscr{T}(\mathscr{H}) \colon \rho^* = \rho, \ \rho \geqslant 0, \langle \rho, \mathbb{1}\rangle = 1\}.$$

In the following, all the assumptions of the previous sections hold, in particular Assumption 1.5 about the initial condition: $\mathbb{E}_P[\|\xi\|^2]=1$. By using ψ_t we construct some families of states and of positive trace-class operators linked to the physical meaning of the whole construction; they will be denoted by ρ_t, η_t, σ_t. The aim of this section is to arrive, at the same level of generality of the rest of the paper, to the notion of a priori states (ρ_t) and a posteriori states (σ_t) for the continously observed quantum system and to a non-linear SDE for σ_t. We have some kind of "quantum" filtering problem; a related linear SDE (for η_t) analogous to Zakai equation is introduced (Belavkin, 1988, 1989a,b, 1992).

Let us start by defining $\rho_t \in \mathscr{S}(\mathscr{H})$ by

$$\langle \rho_t, a\rangle := \mathbb{E}_P[\langle \psi_t | a\psi_t\rangle], \quad \forall a \in \mathscr{L}(\mathscr{H}); \tag{3.1}$$

ρ_t represents the state of the system at time t when the results of the measurement are not taken into account. By setting $\rho := \rho_0$, we have in particular $\langle \rho, a\rangle = \mathbb{E}_P[\langle \xi | a\xi\rangle]$, $\forall a \in \mathscr{L}(\mathscr{H})$; by a suitable choice of $\Omega, \mathscr{F}_0, P$ and ξ, every initial quantum state ρ can be obtained in this way. By using Eq. (3.1), we can write Eq. (1.16) as

$$\langle \rho_t, a\rangle = \langle \rho, a\rangle + \int_0^t \langle \rho_s, \mathscr{L}_s[a]\rangle \, \mathrm{d}s. \tag{3.2}$$

The operator $\mathscr{L}_s$, given by Eq. (1.10) is not only bounded, but also weak* continuous; therefore it has a bounded preadjoint $\mathscr{L}_{s*}$ (Davies, 1976, pp. 5–6). By the estimate

$$\|\mathscr{L}_{s*}[\rho_s]\|_1 \leqslant \|\mathscr{L}_s\| \, \|\rho_s\|_1 \leqslant 2\|K_s\| + \|L_s\|^2 + \|\gamma_s\| \, \|J_s\|^2$$

and Assumptions 2.1 and 2.2, we have that $\mathscr{L}_{s*}[\rho_s]$ is Bochner integrable in every compact interval, so that Eq. (3.2) can be written also in the strong form

$$\rho_t = \rho + \int_0^t \mathscr{L}_{s*}[\rho_s] \, \mathrm{d}s. \tag{3.3}$$

Such an equation is known as *(quantum) master equation* and it is the typical form of an evolution equation for an open quantum system "without memory". The operator $\mathscr{L}_{s*}$ is called *Liouvillian* in the physical literature; at least in the time-independent case, our operator $\mathscr{L}_{s*}$ is the most general bounded Liouvillian (cf Lindblad, 1976). By construction ρ_t is a solution of Eq. (3.3) and, moreover, it is easy to prove the uniqueness of the solution because Eq. (3.3) is a linear evolution equation with a bounded generator.

Let us define the random trace-class operators η_t by

$$\langle \eta_t, a\rangle := \mathbb{E}_P[\langle \psi_t | a\psi_t\rangle | \mathscr{E}_t], \quad \forall a \in \mathscr{L}(\mathscr{H}); \tag{3.4}$$

then, $\eta_t \in L^1(\Omega, \mathscr{E}_t, P; \mathscr{T}(\mathscr{H}))$, $\eta_t \geqslant 0$, $\mathbb{E}_P[\langle \eta_t, \mathbb{1}\rangle] = 1$. Because ψ_t satisfies Eq. (1.1) and the processes W and Π have increments independent of the past, we have also

$$\langle \eta_s, a\rangle = \mathbb{E}_P[\langle \psi_s | a\psi_s\rangle | \mathscr{E}_t], \quad \forall t \geqslant s. \tag{3.5}$$

Obviously, $\mathbb{E}_P[\langle \eta_t, a\rangle] = \langle \rho_t, a\rangle$ and, since $\mathscr{E}_0$ is trivial, we have $\eta_0(\omega) = \rho$ (P-a.s.). It is easy to see that Eqs. (2.8) and (2.12) give, for $E \in \mathscr{E}_t$,

$$\langle \rho, \mathscr{I}(0,t;E)[a]\rangle = \mathbb{E}_P[1_E \langle \eta_t, a\rangle], \tag{3.6}$$

$$\langle \rho, T_t^0[a\otimes f](x)\rangle = \mathbb{E}_P[f(x+X_t)\langle \eta_t, a\rangle]. \tag{3.7}$$

An evolution equation for η_t can be obtained by conditioning Eq. (1.9). The problem is to compute the conditional expectations of the two stochastic integrals; to do this we need some new objects.

Let $\pi(t)$ be the orthogonal projection from l^2 into the kernel of $(a_{ik}(t))$ and let us set $\pi^\perp(t) := \mathbb{1} - \pi(t)$; let us denote by $\pi^\perp_{jk}(t)$ its matrix elements.

Moreover, for every Borel set A in $\mathbb{R}^d_* := \mathbb{R}^d \setminus \{0\}$, let us define

$$\mu_t(A) := \int_{\mathscr{Y}} 1_A(g(y;t))\gamma_t(y)\nu(\mathrm{d}y); \tag{3.8}$$

by Eq. (2.2), $\mu_t(\mathrm{d}x)\,\mathrm{d}t$ is a Radon measure on $\mathbb{R}^d_* \times \mathbb{R}_+$ and, up to sets of vanishing Lebesgue measure in $\mathbb{R}_+$, $\mu_t(\mathrm{d}x)$ is a Radon measure on $\mathbb{R}^d_*$ such that

$$\int_{\mathbb{R}^d_*} \varphi(x)\mu_t(\mathrm{d}x) < +\infty. \tag{3.9}$$

Eventually by modifying the function g on a set of times of vanishing measure, we can assume that Eq. (3.9) holds for every time. By setting, for every time t and every Borel set A in $\mathbb{R}^d_*$,

$$N(A;(0,t]) := \int_{\mathscr{Y}\times(0,t]} 1_A(g(y;s))\Pi(\mathrm{d}y,\mathrm{d}s) \tag{3.10}$$

we get, under the law P, a Poisson point process N over $\mathbb{R}^d_* \times \mathbb{R}_+$ of intensity $\mu_t(\mathrm{d}x)\,\mathrm{d}t$; we denote by

$$\widetilde{N}(A;(0,t]) := N(A;(0,t]) - \int_0^t \mu_s(A)\,\mathrm{d}s \tag{3.11}$$

its compensated form.

At this point, the observed process X, given by Eq. (2.4), can be written also as

$$\begin{aligned} X_i(t) = {}& \int_0^t c_i(s)\,\mathrm{d}s + \sum_{k=1}^{\infty} \int_0^t a_{ik}(s)(\pi^\perp(s)\,\mathrm{d}W_s)_k \\ & + \int_{\mathbb{R}^d_*\times(0,t]} \varphi(x)x_i N(\mathrm{d}x,\mathrm{d}s) + \int_{\mathbb{R}^d_*\times(0,t]} \frac{x_i}{1+|x|^2}\widetilde{N}(\mathrm{d}x,\mathrm{d}s). \end{aligned} \tag{3.12}$$

Let us note that, if $\pi^\perp(t) = \pi^\perp$ (a constant projection with, let us say, $\dim \pi^\perp = d'$), then $\int_0^t \pi^\perp(s)\,\mathrm{d}W_s \equiv \pi^\perp W_t$ is a d'-dimensional standard Wiener process. Eq. (3.12) shows that the filtration generated by X is the same as the filtration generated by $(\int_0^t \pi^\perp(s)\,\mathrm{d}W_s)$ and N. By this and Eq. (3.5), we have

$$\mathbb{E}_P\left[\int_0^t \langle \psi_{s-} | \mathscr{L}_s[a]\psi_{s-}\rangle\,\mathrm{d}s \middle| \mathscr{E}_t\right] = \int_0^t \langle \eta_{s-}, \mathscr{L}_s[a]\rangle\,\mathrm{d}s, \tag{3.13}$$

$$\mathbb{E}_P\left[\sum_{k=1}^{\infty}\int_0^t \langle\psi_{s-}|(L_{ks}^*a+aL_{ks})\psi_{s-}\rangle\,\mathrm{d}W_{ks}\middle|\mathscr{E}_t\right]$$
$$=\sum_{j,k=1}^{\infty}\int_0^t \langle\eta_{s-},L_{js}^*a+aL_{js}\rangle\pi_{jk}^{\perp}(s)\,\mathrm{d}W_{ks}. \tag{3.14}$$

In order to be able to evaluate the conditional expectation of the stochastic integral with respect to the Poisson process in Eq. (1.9), we need a final object, which is defined by the following proposition.

Proposition 3.1. *The formula*

$$\int_{\mathbb{R}_*^d}\left\langle\mathscr{R}_t\left[|u\rangle\langle w|\right](x),A(x)\right\rangle\varphi(x)\mu_t(\mathrm{d}x)$$
$$=\int_{\mathscr{Y}}\langle(J_tw)(y)+w|A(g(y;t))((J_tu)(y)+u)\rangle\varphi(g(y;t))\gamma_t(y)\nu(\mathrm{d}y), \tag{3.15}$$

$\forall A\in L^\infty(\mathbb{R}_*^d,\varphi(x)\mu_t(\mathrm{d}x);\mathscr{L}(\mathscr{H}))$, $\forall u,w\in\mathscr{H}$, *defines a bounded linear operator* $\mathscr{R}_t$ *from* $\mathscr{T}(\mathscr{H})$ *into* $L^1(\mathbb{R}_*^d,\varphi(x)\mu_t(\mathrm{d}x);\mathscr{T}(\mathscr{H}))$.

Proof. Let us start with the case $w=u$; we can observe that, by our assumptions, $u\mapsto(J_tu)(y)+u$ is a bounded linear operator from the Hilbert space $\mathscr{H}$ into $L^2(\mathscr{Y},\varphi(g(y;t))\gamma_t(y)\nu(\mathrm{d}y);\mathscr{H})$. Let us consider now the W*-algebra $L^\infty(\mathbb{R}_*^d,\varphi(x)\mu_t(\mathrm{d}x);\mathscr{L}(\mathscr{H}))$. The map

$$A\mapsto\int_{\mathscr{Y}}\langle(J_tu)(y)+u|A(g(y;t))((J_tu)(y)+u)\rangle\varphi(g(y;t))\gamma_t(y)\nu(\mathrm{d}y) \tag{3.16}$$

is a positive linear functional on $L^\infty(\mathbb{R}_*^d,\varphi(x)\mu_t(\mathrm{d}x);\mathscr{L}(\mathscr{H}))$; moreover it is not difficult to prove that this functional is normal (Sakai, 1971, Definition 1.13.1) by using some simple properties of the least upper bound in W*-algebras and by taking into account the monotone convergence theorem. So, by Theorem 1.13.2 of Sakai (1971), Eq. (3.16) defines a weak*-continuous functional on $L^\infty(\mathbb{R}_*^d,\varphi(x)\mu_t(\mathrm{d}x);\mathscr{L}(\mathscr{H}))$; Theorem IV.20 of Reed and Simon (1972) guarantees the existence of an element $\mathscr{R}_t[|u\rangle\langle u|]$ of the predual of $L^\infty(\mathbb{R}_*^d,\varphi(x)\mu_t(\mathrm{d}x);\mathscr{L}(\mathscr{H}))$ such that Eq. (3.15) holds. On the other hand we can identify the topological predual of $L^\infty(\mathbb{R}_*^d,\varphi(x)\mu_t(\mathrm{d}x);\mathscr{L}(\mathscr{H}))$ with $L^1(\mathbb{R}_*^d,\varphi(x)\mu_t(\mathrm{d}x);\mathscr{T}(\mathscr{H}))$; so, we can consider $\mathscr{R}_t[|u\rangle\langle u|]$ as an element in this last space. Obviously, $\mathscr{R}_t$ is linear in its argument and it is also possible to prove that

$$\int_{\mathscr{Y}}|\langle(J_tu)(y)+u|A(g(y;t))((J_tu)(y)+u)\rangle|\varphi(g(y;t))\gamma_t(y)\nu(\mathrm{d}y)$$
$$\leqslant C_t\|u\|^2=C_t\||u\rangle\langle u|\|_1, \tag{3.17}$$

for some positive time function C_t. Therefore, we can uniquely extend $\mathscr{R}_t$, first by polarization and then by continuity, to a bounded linear operator from $\mathscr{T}(\mathscr{H})$ into $L^1(\mathbb{R}_*^d,\varphi(x)\mu_t(\mathrm{d}x);\mathscr{T}(\mathscr{H}))$. □

Just to see an explicit example for $\mathscr{R}_t$, let us follow Barchielli and Holevo (1995) and take $\mathscr{Y}=(\mathbb{R}_*^d\cup\{\delta\})\times\mathbb{N}$, where δ is an extra point added to $\mathbb{R}_*^d$. Let us take

$g(x,n;s) = x$ and $g(\delta,n;s)=0$; we are asking that Assumption 2.1 hold with such a choice. Then Eqs. (3.8) and (3.10) become

$$\mu_t(\mathrm{d}x)=\sum_{n=1}^{\infty}\gamma_t(x,n)\nu(\mathrm{d}x\times\{n\}), \qquad N(\mathrm{d}x,\mathrm{d}t)=\sum_{n=1}^{\infty}\Pi(\mathrm{d}x\times\{n\},\mathrm{d}t), \tag{3.18}$$

while the definition 3.15 of $\mathscr{R}_t$ becomes

$$\langle\mathscr{R}_t\left[|u\rangle\langle w|\right],a\rangle = \sum_{n=1}^{\infty}\langle (J_t w)(x,n)+w|a((J_t u)(x,n)+u)\rangle\,\frac{\gamma_t(x,n)\nu(\mathrm{d}x\times\{n\})}{\mu_t(\mathrm{d}x)}; \tag{3.19}$$

the fraction at the end of the previous equation is a Radon–Nikodym derivative.

Let us go back to the general case. By means of $\mathscr{R}_t$ we can write

$$\mathbb{E}_P\left[\int_{\mathscr{Y}\times(0,t]}(\langle (J_t\psi_{s^-})(y)+\psi_{s^-}|a((J_t\psi_{s^-})(y)+\psi_{s^-})\rangle\right.$$
$$\left.-\langle\psi_{s^-}|a\psi_{s^-}\rangle)\widetilde{\Pi}(\mathrm{d}y,\mathrm{d}s)\middle|\mathscr{E}_t\right]=\int_{\mathbb{R}^d_*\times(0,t]}\left[\langle\mathscr{R}_s[\eta_{s^-}](x),a\rangle-\langle\eta_{s^-},a\rangle\right]\widetilde{N}(\mathrm{d}x,\mathrm{d}s). \tag{3.20}$$

Finally, Eqs. (3.13), (3.14), (3.20) and (1.9) give the evolution equation for η_t

$$\langle\eta_t,a\rangle=\langle\rho,a\rangle+\int_0^t\langle\eta_{s^-},\mathscr{L}_s[a]\rangle\,\mathrm{d}s+\sum_{j,k=1}^{\infty}\int_0^t\langle\eta_{s^-},L_{js}^*a+aL_{js}\rangle\pi_{jk}^{\perp}(s)\,\mathrm{d}W_{ks}$$
$$+\int_{\mathbb{R}^d_*\times(0,t]}[\langle\mathscr{R}_s[\eta_{s^-}](x),a\rangle-\langle\eta_{s^-},a\rangle]\widetilde{N}(\mathrm{d}x,\mathrm{d}s). \tag{3.21}$$

Let us consider now the probability $\widehat{P}_\xi$ (1.17) and the random unit vectors $\widehat{\psi}_t$ (1.20); by the definitions of $\widehat{P}_\xi$, $\widehat{\psi}_t$ and ρ_t we have

$$\langle\rho_t,a\rangle=\mathbb{E}_{\widehat{P}_\xi}[\langle\widehat{\psi}_t|a\widehat{\psi}_t\rangle],\quad\forall a\in\mathscr{L}(\mathscr{H}). \tag{3.22}$$

Moreover, like in Eq. (3.4), we can introduce the random states σ_t by

$$\langle\sigma_t,a\rangle:=\mathbb{E}_{\widehat{P}_\xi}[\langle\widehat{\psi}_t|a\widehat{\psi}_t\rangle|\mathscr{E}_t],\quad\forall a\in\mathscr{L}(\mathscr{H}); \tag{3.23}$$

note that $\sigma_t(\omega)\in\mathscr{S}(\mathscr{H})$, $\sigma_t\in L^1(\Omega,\mathscr{E}_t,\widehat{P}_\xi;\mathscr{T}(\mathscr{H}))$, $\langle\rho_t,a\rangle=\mathbb{E}_{\widehat{P}_\xi}[\langle\sigma_t,a\rangle]$, $\sigma_0(\omega)=\rho$. Moreover, $\forall E\in\mathscr{E}_t$ we have

$$\begin{aligned}\mathbb{E}_{\widehat{P}_\xi}[1_E\langle\sigma_t,a\rangle]&=\mathbb{E}_P[1_E\langle\widehat{\psi}_t|a\widehat{\psi}_t\rangle\|\psi_t\|^2]=\mathbb{E}_P[1_E\langle\psi_t|a\psi_t\rangle]\\&=\mathbb{E}_P[1_E\langle\eta_t,a\rangle]=\mathbb{E}_P\left[1_E\frac{\langle\eta_t,a\rangle}{\langle\eta_t,\mathbb{1}\rangle}\langle\eta_t,\mathbb{1}\rangle\right]\\&=\mathbb{E}_P\left[1_E\frac{\langle\eta_t,a\rangle}{\langle\eta_t,\mathbb{1}\rangle}\mathbb{E}_P[\langle\psi_t,\mathbb{1}\psi_t\rangle|\mathscr{E}_t]\right]\\&=\mathbb{E}_{\widehat{P}_\xi}\left[1_E\frac{\langle\eta_t,a\rangle}{\langle\eta_t,\mathbb{1}\rangle}\right],\end{aligned}$$

which means

$$\sigma_t = \frac{\eta_t}{\langle \eta_t, \mathbb{1} \rangle}, \qquad \widehat{P}_\xi\text{-a.s.} \tag{3.24}$$

By using Eq. (2.8) and the first two steps in the proof of the last equation, we have also

$$\langle \rho, \mathscr{I}(0,t;E)[a] \rangle = \mathbb{E}_{\widehat{P}_\xi}[1_E \langle \sigma_t, a \rangle], \quad \forall E \in \mathscr{E}_t, \ \forall a \in \mathscr{L}(\mathscr{H}). \tag{3.25}$$

This equation allows us to understand the physical meaning of σ_t: σ_t is the a posteriori state of the quantum system, i.e. the state we attribute to the system knowing the results of the measurement up to time t. The definition of a posteriori states for a generic instrument is given in Ozawa (1985), while this idea was introduced in the formalism of continuous measurements in Belavkin (1988). Eq. (3.25) says also that the knowledge of the laws $\widehat{P}_\xi$ and of the a posteriori states is equivalent to the one of the instruments $\mathscr{I}$.

It is interesting to obtain the non-linear SDE satisfied by the a posteriori states σ_t. Let us consider now the restrictions of P and $\widehat{P}_\xi$ to $\mathscr{E}$; from Proposition 1.1 and Eq. (3.5) we have that $\langle \eta_t, \mathbb{1} \rangle$ is the local density of $\widehat{P}_\xi$ with respect to P. In the stochastic basis $(\Omega, (\mathscr{E}_t), \mathscr{E}, \widehat{P}_\xi)$, let us introduce the Wiener processes

$$\check{W}_{jt} := \sum_{k=1}^{\infty} \int_0^t \pi_{jk}^{\perp}(s)\, \mathrm{d}W_{ks} - 2\int_0^t \mathrm{Re}\, m_j(s)\, \mathrm{d}s, \tag{3.26}$$

where $m_j(t) = \sum_{k=1}^{\infty} \pi_{jk}^{\perp}(t) \langle \sigma_{t^-}, L_{kt} \rangle$, and the compensated point processes

$$\check{N}(A,(0,t]) := N(A,(0,t]) - \int_{A \times (0,t]} I_s(x) \mu_s(\mathrm{d}x)\, \mathrm{d}s, \tag{3.27}$$

where $I_t(x) := \langle \mathscr{R}_t[\sigma_{t^-}](x), \mathbb{1} \rangle$. By applying Itô's formula to Eq. (3.24) and by using Eq. (3.21) and the definitions (3.26)–(3.27) it is possible to calculate the stochastic differential for the a posteriori states

$$\begin{aligned} \mathrm{d}\langle \sigma_t, a \rangle = {} & \langle \sigma_{t^-}, \mathscr{L}_t[a] \rangle\, \mathrm{d}t + \sum_{j=1}^{\infty} (\langle \sigma_{t^-}, L_{jt}^* a + a L_{jt} \rangle - 2\langle \sigma_{t^-}, a \rangle \mathrm{Re}\, m_j(t))\, \mathrm{d}\check{W}_{jt} \\ & + \int_{E_t} \left(\frac{1}{I_t(x)} \langle \mathscr{R}_t[\sigma_{t^-}](x), a \rangle - \langle \sigma_{t^-}, a \rangle \right) \check{N}(\mathrm{d}x, \mathrm{d}t), \end{aligned} \tag{3.28}$$

where $E_t(\omega) = \{ x \in \mathbb{R}^d \colon I_t(x, \omega) \neq 0 \}$.

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