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# Random attractors for quasi-continuous random dynamical systems and applications to stochastic reaction–diffusion equations ☆

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**Abstract**

This paper first introduces the so-called quasi-continuous random dynamical system (RDS) on a separable Banach space. The quasi-continuity is weaker than all the usual continuities and thus is easier to check in practice. We then establish a necessary and sufficient condition for the existence of random attractors for the quasi-continuous RDS. We also give a general method to obtain the random attractors for the RDS on the Banach space $L^q(D)$ for $q \geqslant 2$. As an application, it is shown that the RDS generated by the stochastic reaction–diffusion equation possesses a finite-dimensional random attractor in $L^q(D)$ for any $q \geqslant 2$, a comparison result of fractal dimensions under the different $L^q$-norms is also obtained.


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*Keywords:* Random dynamical systems; Random attractors; Quasi-continuity; Stochastic reaction–diffusion equations; Wiener processes; Fractal dimensions

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## 1. Introduction

Random attractors for random dynamical systems (RDS) were first introduced by Crauel and Flandoli [8] and Schmalfuss [19], with notable developments given in [2–9,11,12,17] and in the references there among many others. They are compact invariant random sets attracting all the orbits. To apply it to the certain stochastic partial differential equations (SPDE), one must rely on some sufficient or necessary and sufficient conditions for the existence of random attractors. For example, it is well known that there exists a random attractor if there is a compact absorbing random set (see [7,8]), while Crauel [6] has proved that there exists a random attractor if and only if there exists a compact attracting set.

As we know, the above general results have a basic assumption that the RDS is norm-to-norm continuous in some Banach spaces. However, there are some important RDS which are not norm-to-norm continuous. For example, consider the following stochastic reaction–diffusion equations (see [16,22])

$$du - \big(\Delta u - f(u)\big)\,dt = dW(t) \tag{1.1}$$

where $f$ is a polynomial of odd degree $2p-1$ and $W(t)$ is a Wiener process, the further assumptions and details will be placed in Section 6. Even in the deterministic case of Eq. (1.1), the norm-to-norm continuity of the solutions semigroup in $L^q(D)$ $(q \neq 2)$ or in $H_0^1(D)$ is unknown as pointed out by Robinson [18] and Zhong et al. [25], the norm-to-weak continuity of the semigroup in whole space $L^q(D)$ $(q \neq 2)$ is also not clear as pointed out by Zhong et al. [25], they have only proved the norm-to-weak continuity on some bounded sets of $L^q(D)$.

To overcome the continuity difficulty, we introduce in present paper the so-called quasi-continuity of a RDS. The quasi-continuity is weaker than all continuities mentioned above and thus is easier to check in practice. In fact, we will see that the RDS is always quasi-continuous in $L^q(D)$ $(q > 2)$ if it is (norm-to-norm or norm-to-weak) continuous in $L^2(D)$ (see Proposition 3.3).

We then establish a criterion of the existence of random attractors for the quasi-continuous RDS (see Theorem 4.1). Such a criterion in the deterministic case for the (norm-to-norm or norm-to-weak) continuous semigroup has been established by Zhong et al. [25] and Ma et al. [13], in which a very useful tool of the Kuratowski measure of non-compactness and the related notion of omega-compactness were involved. Of course, these notions have been used by other papers in the continuous cases when the phase space is the Hilbert space $L^2(D)$ (see [3,20,21, 23,24]). Recently, Kloeden and Langa [11] gave a generalization of the Ma et al. method to the RDS. In present paper, by using the above tool, we generalize their results to the quasi-continuous RDS case. Some difficulty arose in proving the compactness, but it is shown that an ergodicity argument can solve the problem.

In comparison with the criterion of random attractors obtained by Crauel [6] (as mentioned above), our criterion has the weaker continuity assumption and uses omega-compactness instead of the ordinary compactness. In particular, we can apply it to the wider range when the phase space is the Banach space $L^q(D)$ for $q > 2$ rather than the Hilbert space $L^2(D)$.

It is well known that many RDS generated by the concrete SPDEs possess a random attractor in the Hilbert space $L^2(D)$. Under the assumption of this fact, we will give in this paper a criterion for the existence of random attractors in $L^q(D)$ for $q > 2$ (see Theorem 5.3).

Our theoretical result can be applied to the stochastic reaction–diffusion equation (1.1) when the phase space is $L^q(D)$ for any $q > 2$. It is well known that the RDS generated by the above equation possesses a random attractor in $L^2(D)$ (see [4,6–9,12] for different noise assumptions). We will prove that the RDS has a random attractor in $L^q(D)$ for any $q > 2$ under some bounded noise assumptions (see Theorem 6.7). It is worth pointing out that, even in the non-random case, it seems to be the best result that there is an attractor in $L^q(D)$ for $q = 4p - 2$ and thus for all $2 \leqslant q \leqslant 4p - 2$ (see [25]).

We also prove that the random attractor for Eq. (1.1) has finite fractal dimensions in $L^q(D)$ (Theorem 6.9). We emphasize here that the fractal dimension is defined by the $L^q$-norm rather than the $L^2$-norm, and thus the same set may enjoy different fractal dimensions under different $L^q$-norms. A relationship of fractal dimensions under different $L^q$-norms will be given. This subject seems not to be discussed upon now even in the deterministic case.

## 2. Preliminaries

In this section, we first recall some basic notions of the theory of random dynamical systems (RDS) (see [1,6–9]). We then review briefly the Kuratowski measure of non-compactness (see [10]), which is a useful tool to study the attractor (see [13,20,21,23–25]).

### 2.1. Random attractors

Throughout this paper, $X$ is a separable Banach space equipped with the norm $\|\cdot\|$ and the Borel $\sigma$-algebra $\mathcal{B}(X)$. $(\Omega, \mathcal{F}, P)$ is a complete probability space and $\{\theta_t : \Omega \to \Omega,\ t \in \mathbb{R}\}$ a group of measure preserving transformations of $\Omega$. We assume that $\theta_t$ is ergodic.

**Definition 2.1.** A random dynamical system (RDS) $\varphi$ on $X$ over $(\Omega, \mathcal{F}, P, \theta_t)$ is a measurable map

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x$$

such that $P$-a.s. $\varphi(0, \omega) = \mathrm{id}$ on $X$ and the cocycle property holds, that is,

$$\varphi(t + s, \omega) = \varphi(t, \theta_s \omega)\varphi(s, \omega), \quad s, t \in \mathbb{R}^+. \tag{2.1}$$

A random set $\{K(\omega)\}_{\omega \in \Omega}$ is a family of subsets indexed by $\omega$ such that for every $x \in X$ the mapping $\omega \mapsto \mathrm{dist}(x, K(\omega))$ is measurable with respect to $\mathcal{F}$, where we denote $\mathrm{dist}(\cdot,\cdot)$ the Hausdorff semi-distance in $X$

$$\mathrm{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|, \quad A, B \subset X. \tag{2.2}$$

A random set $\mathcal{A}(\omega)$ is said to be an attracting set if $\mathcal{A}(\omega)$ attracts all deterministic bounded sets $B \subset X$, that is,

$$\lim_{t \to \infty} \mathrm{dist}\big(\varphi(t, \theta_{-t}\omega)B, \mathcal{A}(\omega)\big) = 0, \quad P\text{-a.s. } \omega \in \Omega. \tag{2.3}$$

A random set $\mathcal{A}(\omega)$ is said to be an absorbing set if for every deterministic bounded sets $B \subset X$ there exists a time $t_B(\omega)$ such that

$$\varphi(t, \theta_{-t}\omega)B \subset \mathcal{A}(\omega), \quad t \geqslant t_B(\omega),\ P\text{-a.s. } \omega \in \Omega. \tag{2.4}$$

**Definition 2.2.** A random set $\mathcal{A}(\omega)$ is said to be a random attractor for the RDS $\varphi$ if $\mathcal{A}(\omega)$ is compact, attracting (as (2.3)) and invariant, that is,

$$\varphi(t,\omega)\mathcal{A}(\omega)=\mathcal{A}(\theta_t\omega), \quad \forall t\geqslant 0,\ P\text{-a.s. } \omega\in\Omega. \tag{2.5}$$

The following result on the existence of random attractors can be found in [6–8].

**Theorem 2.3.** *A continuous RDS possesses a random attractor if there exists a compact absorbing set, while a continuous RDS possesses a random attractor if and only if there exists a compact attracting set.*

### *2.2. The Kuratowski measure of non-compactness*

**Definition 2.4.** Let $B$ be a bounded set in a Banach space $X$. The Kuratowski measure of non-compactness $\kappa(B)$ of $B$ is defined by

$$\kappa(B)=\inf\{d>0 \mid B \text{ admits a finite cover by sets of diameter} \leqslant d\}. \tag{2.6}$$

We define $\kappa(B)=\infty$ if $B$ is unbounded. The properties of $\kappa(B)$, which we need to use in this paper, are given in the following lemma (see [9], the conclusion (v) due to Zhong et al. [25]).

**Lemma 2.5.** (i) $\kappa(B)=0$ *iff* $\overline{B}$ *is compact, where* $\overline{B}$ *is the norm closure of* $B$.

(ii) $\kappa(\overline{co\,B})=\kappa(B)$, *where* $\overline{co\,B}$ *is the closed convex hull of* $B$.

(iii) $\kappa(B_1\cup B_2)\leqslant \max\{\kappa(B_1),\kappa(B_2)\}$ *and* $\kappa(B_1+B_2)\leqslant\kappa(B_1)+\kappa(B_2)$.

(iv) *For a decreasing family* $\{B_t\}_{t>r}$ *of nonempty, closed and bounded sets, if* $\kappa(B_t)\to 0$ *as* $t\to\infty$, *then* $\bigcap B_t$ *is a nonempty, compact set.*

(v) $\kappa(B)=\kappa(\overline{B}^{WS})$, *where* $\overline{B}^{WS}$ *is the weakly sequential closure of* $B$ *defined by*

$$\overline{B}^{WS}=\{x\in X \mid \exists x_n\in B,\ \textit{such that } x_n\rightharpoonup x\} \tag{2.7}$$

*where* $\rightharpoonup$ *means the weak convergence.*

## 3. Norm-to-weak continuity and quasi-continuity

**Definition 3.1.** A RDS $\varphi$ on a Banach space $X$ is said to be *norm-to-weak continuous* if $P$-a.s. $\omega\in\Omega$, $\varphi(t_n,\omega)x_n\rightharpoonup\varphi(t,\omega)x$ whenever $(t_n,x_n)\to(t,x)$ in $\mathbb{R}^+\times X$, where $\rightharpoonup$ means weak convergence. A RDS $\varphi$ is called to be *quasi-continuous* if $P$-a.s. $\omega\in\Omega$, $\varphi(t_n,\omega)x_n\rightharpoonup\varphi(t,\omega)x$ whenever $\{(t_n,x_n)\}$ is a sequence in $\mathbb{R}^+\times X$ such that $\{\varphi(t_n,\omega)x_n\}$ is bounded and $(t_n,x_n)\to(t,x)$ as $n\to\infty$.

Both kinds of continuity above are weaker than the usual one. In fact we have: Norm (resp. weak) continuity $\Rightarrow$ norm-to-weak continuity $\Rightarrow$ quasi-continuity.

As pointed out in Section 1, it is difficult to verify that the RDS is continuous (or weak continuous) in stronger norm spaces than $L^2(D)$, e.g. $L^q(D)$ for $q>2$. The following results present some easy-to-check criteria for norm-to-weak continuity and quasi-continuity in the stronger norm space.

Let $X,Y$ be two Banach spaces with the dual spaces $X^*,Y^*$, respectively, and assume also

(i) the embedding $i : X \to Y$ is densely continuous;
(ii) the adjoint operator $i^* : Y^* \to X^*$ is dense, i.e. $i^*(Y^*)$ is dense in $X^*$.

We remark here that $i^* : Y^* \to X^*$ is always injective and continuous if the assumption (i) holds (comparing this fact to the assumptions given in [25]). Indeed, if $i^*(y^*) = 0$ for $y^* \in Y^*$, then obviously $\langle i(X), y^*\rangle = 0$. Since $i(X)$ is dense in $Y$, it follows from the Hahn–Banach theorem that $y^* = 0$. Thus, $i^*$ is injective. The continuity of the linear operator $i^*$ follows from the continuity of $i$. We also note that the assumption (ii) holds if $X, Y$ satisfy the assumption (i) and, in addition, $X$ is reflexive.

**Proposition 3.2.** *Let $X, Y$ satisfy the assumptions above, and let $\varphi$ be a RDS on $X$, $Y$ respectively. Suppose $\varphi$ is continuous in $Y$, then $\varphi$ is norm-to-weak continuous in $X$ if and only if for $P$-a.s. $\omega \in \Omega$, $\varphi(\cdot, \omega)\cdot$ maps compact subsets of $\mathbb{R}^+ \times X$ into bounded sets of $X$.*

**Proof.** It is the same as in the non-random case (see [25, Theorem 3.2]). □

It is also difficult to verify the norm-to-weak continuity by using the above proposition. However, we will see from the following result that the quasi-continuity holds automatically in stronger norm spaces.

**Proposition 3.3.** *Let $X, Y$ satisfy the assumptions* (i) *and* (ii) *above. Let $\varphi$ be a RDS on $X$, $Y$ respectively. If $\varphi$ is continuous or norm-to-weak continuous in $Y$, then $\varphi$ is quasi-continuous in $X$.*

**Proof.** We work for every fixed $\omega \in \Omega_0$ with $P(\Omega_0) = 1$. Assume $\varphi$ is continuous or norm-to-weak continuous in $Y$, then $\varphi$ is quasi-continuous in $Y$. We must show that $\varphi$ is quasi-continuous in $X$. To do this, we take $(t_n, x_n) \in \mathbb{R}^+ \times X$ such that $\{\varphi(t_n, \omega)x_n\}$ is bounded in $X$ and $(t_n, x_n) \to (t, x)$. We need to show that $\varphi(t_n, \omega)x_n \rightharpoonup \varphi(t, \omega)x$, i.e.

$$\big\langle \varphi(t_n, \omega)x_n - \varphi(t, \omega)x,\ x^*\big\rangle_X \to 0, \quad \text{as } n \to \infty \text{ for every } x^* \in X^*. \tag{3.1}$$

Indeed, let $M$ be the positive constant such that

$$\big\|\varphi(t_n, \omega)x_n - \varphi(t, \omega)x\big\|_X \leqslant M. \tag{3.2}$$

For any $\varepsilon > 0$ and $x^* \in X^*$, by the assumption (ii) we can choose $y^* \in Y^*$ such that

$$\big\|i^*(y^*) - x^*\big\|_{X^*} < \frac{\varepsilon}{2M}. \tag{3.3}$$

Since $\varphi$ is norm or norm-to-weak continuous in $Y$, there exists $N > 0$ such that for all $n \geqslant N$,

$$\big|\big\langle i\big(\varphi(t_n, \omega)x_n - \varphi(t, \omega)x\big),\ y^*\big\rangle_Y\big| < \frac{\varepsilon}{2},$$

which, together with (3.1)–(3.2), implies that

$$\begin{aligned}&\left|\left\langle\varphi(t_n,\omega)x_n-\varphi(t,\omega)x,x^*\right\rangle_X\right|\\&\quad\leqslant\left|\left\langle\varphi(t_n,\omega)x_n-\varphi(t,\omega)x,\ i^*(y^*)-x^*\right\rangle_X\right|+\left|\left\langle i\big(\varphi(t_n,\omega)x_n-\varphi(t,\omega)x\big),\ y^*\right\rangle_Y\right|\\&\quad\leqslant M\cdot\frac{\varepsilon}{2M}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Therefore (3.3) holds and thus $\varphi$ is quasi-continuous in $X$. □

**Remark 3.4.** For the concrete SPDEs, one can choose $Y = L^2(D)$, $X = L^q(D)$ $(q > 2)$ or $H_0^m(D)$, and conclude that $\varphi$ is quasi-continuous in $L^q(D)$ or $H_0^m(D)$ if $\varphi$ is continuous in $L^2(D)$.

## 4. Random attractors for the quasi-continuous RDS

In this section, we present an existence criterion of random attractors.

**Theorem 4.1.** *Suppose $\varphi(t,\omega)$ be a quasi-continuous RDS on a separable Banach space $X$ over an ergodic system $(\Omega,\mathcal{F},P;\theta_t)$. Then $\varphi$ possesses a random attractor if and only if*

(i) *$\varphi$ has a bounded absorbing set $\mathcal{B}(\omega)$; and*
(ii) *$\varphi$ is omega-limit compact, that is, for every bounded non-random set $B\subset X$, we have*

$$\lim_{T\to+\infty}\kappa\left(\bigcup_{t\geqslant T}\varphi(t,\theta_{-t}\omega)B\right)=0,\quad P\text{-a.s. }\omega\in\Omega. \tag{4.1}$$

Under the assumptions in Theorem 4.1, we will prove one of the random attractors is

$$\mathcal{A}(\omega)=\bigcup_{B\subset X}A(B,\omega) \tag{4.2}$$

where the union is taken over all bounded subsets of $X$, and $A(B,\omega)$ is the weakly sequence omega-limit set, that is,

$$A(B,\omega)=\bigcap_{T\geqslant 0}\overline{\bigcup_{t\geqslant T}\varphi(t,\theta_{-t}\omega)B}^{WS} \tag{4.3}$$

for every bounded set $B\subset X$. Using the same method as in [25], one can prove that if $\varphi$ is omega-limit compact then $A(B,\omega)$ has the following character:

$$x\in A(B,\omega)\quad\text{iff}\quad\exists x_n\in B,\ t_n\to\infty\quad\text{such that}\quad\varphi(t_n,\theta_{-t_n}\omega)x_n\rightharpoonup x. \tag{4.4}$$

**Proof of Theorem 4.1.** *Necessity*. Suppose $\varphi$ has a random attractor $\tilde{\mathcal{A}}(\omega)$. Then it is easy to prove that the $\varepsilon$-neighborhood $N_\varepsilon(\tilde{\mathcal{A}}(\omega))$ is a bounded absorbing set. But $\tilde{\mathcal{A}}(\omega)$ is compact, we know that $\kappa(\tilde{\mathcal{A}}(\omega))=0$ and, thus, for large $T$,

$$\kappa\left(\bigcup_{t\geqslant T}\varphi(t,\theta_{-t}\omega)B\right)\leqslant\kappa\big(N_\varepsilon\big(\tilde{\mathcal{A}}(\omega)\big)\big)\leqslant 2\varepsilon,$$

which proves (4.1). Thus, $\varphi$ is omega-limit compact.

*Sufficiency*. We will show that $\mathcal{A}(\omega)$, defined by (4.2), is a random attractor. The proof can be accomplished by the following 4 steps.

*Step 1*. We show that $\mathcal{A}(\omega)$ is invariant. It is obvious that $\mathcal{A}(\omega)$ is invariant if every $A(B,\omega)$, defined by (4.3), is invariant. To prove $A(B,\omega)$ is invariant, we let $x \in A(B,\omega)$. Then, by the character (4.4), there exist $t_n \to \infty$ and $x_n \in B$ such that

$$\varphi(t_n, \theta_{-t_n}\omega)x_n \rightharpoonup x.$$

By the omega-limit compactness of $\varphi$, we can choose a subsequence $\{n_k\}$ such that

$$y_k := \varphi(t_{n_k}, \theta_{-t_{n_k}}\omega)x_{n_k} \to x. \tag{4.5}$$

Since, by the assumption (i), there exists a bounded absorbing set $\mathcal{B}(\omega)$, it follows from the cocycle property (2.1) that, for fixed $t > 0$ and large $k$,

$$\varphi(t,\omega)y_k = \varphi(t + t_{n_k}, \theta_{-(t+t_{n_k})}\theta_t\omega)x_{n_k} \in \mathcal{B}(\theta_t\omega).$$

In particular, $\{\varphi(t,\omega)y_k,\ k \in \mathbb{N}\}$ is bounded in $X$. Thus, by the quasi-continuity of $\varphi$, it follows from (4.5) that

$$\varphi(t,\omega)y_k \rightharpoonup \varphi(t,\omega)x.$$

Therefore, by the character (4.4) again, $\varphi(t,\omega)x \in A(B,\theta_t\omega)$, which shows that

$$\varphi(t,\omega)A(B,\omega) \subset A(B,\theta_t\omega), \quad t \geqslant 0.$$

Conversely, let $x \in A(B,\theta_t\omega)$, where $t$ is a fixed positive number. Then, by (4.4), there exist $x_n \in B$ and $t_n$ with $t < t_n \to \infty$ such that

$$\varphi(t_n, \theta_{-t_n}\theta_t\omega)x_n \rightharpoonup x.$$

Since $\varphi$ is omega-limit compact, there exist a $z \in X$ and a subsequence $\{n_k\}$ such that

$$z_k := \varphi(t_{n_k} - t, \theta_{-(t_{n_k}-t)}\omega)x_{n_k} \to z. \tag{4.6}$$

Thus, $z \in A(B,\omega)$ in view of (4.4). Using the cocycle property (2.1), we know that the sequence

$$\varphi(t,\omega)z_k = \varphi(t_{n_k}, \theta_{-t_{n_k}}\theta_t\omega)x_{n_k} \rightharpoonup x.$$

Noting that $\varphi(t,\omega)z_k$ is bounded in $X$ since a weakly convergent sequence is norm bounded, we see, by the quasi-continuity of $\varphi$ and by (4.6), that

$$\varphi(t,\omega)z_k \rightharpoonup \varphi(t,\omega)z.$$

Thus, $x = \varphi(t,\omega)z \in \varphi(t,\omega)A(B,\omega)$, which proves the converse inclusion and thus $A(B,\omega)$ is invariant as required.

*Step 2*. We show that there exists a bounded deterministic set $B_0 \subset X$ such that

$$\mathcal{A}(\omega) = A(B_0, \omega), \quad P\text{-a.s. } \omega \in \Omega. \tag{4.7}$$

This step is a key to prove the compactness of $\mathcal{A}(\omega)$ and needs the ergodicity assumption of $\theta_t$. Follows the idea of [6, Proposition 2.3], we set

$$R(\omega) = \inf\{r \geqslant 0 \mid \mathcal{A}(\omega) \subset B(O, r)\} \tag{4.8}$$

where $B(O, r)$ is the ball of center $O$ and radius $r$. We must verify that $R(\omega)$ is well defined. Indeed, for every bounded set $B \subset X$, there exists $t_B$ such that

$$\overline{co\left(\bigcup_{t \geqslant t_B} \varphi(t, \theta_{-t}\omega)B\right)}^{WS} = \overline{co\left(\bigcup_{t \geqslant t_B} \varphi(t, \theta_{-t}\omega)B\right)} \subset \overline{co\,\mathcal{B}(\omega)}.$$

Then it follows from (4.2) and (4.3) that $\mathcal{A}(\omega) \subset \overline{co\,\mathcal{B}(\omega)}$. Since the absorbing set $\mathcal{B}(\omega)$ is bounded, it follows that $\overline{co\,\mathcal{B}(\omega)}$ and thus $\mathcal{A}(\omega)$ is bounded. We conclude that $R(\omega)$ is finite for all $\omega \in \Omega_0$ with $P(\Omega_0) = 1$. We set also

$$R(\omega) = 0 \quad \text{if } \omega \in \Omega \setminus \Omega_0.$$

Then the function $R(\omega)$ is measurable, and it follows from the ergodic theorem (e.g. see [1]) that there exists a sequence $t_n \to \infty$ such that

$$R(\theta_{-t_n}\omega) \leqslant R_0 + 1, \quad P\text{-a.s. } \omega \in \Omega,$$

where

$$R_0 = \operatorname{ess\,inf}\big(R(\omega)\big)_{\omega \in \Omega}.$$

Let now $B_0 = B(O, R_0 + 1)$, we claim that $\mathcal{A}(\omega) = A(B_0, \omega)$. Indeed, let $x \in \mathcal{A}(\omega)$ and let $t_n$ be the sequence defined above. Then, by the invariant property (which has been proved in Step 1), there exists $x_n \in \mathcal{A}(\theta_{-t_n}\omega)$ such that

$$\varphi(t_n, \theta_{-t_n}\omega)x_n = x. \tag{4.9}$$

Since $\mathcal{A}(\theta_{-t_n}\omega) \subset B_0$, it follows from (4.9) and the character (4.4) that $x \in A(B_0, \omega)$. We have proved $\mathcal{A}(\omega) \subset A(B_0, \omega)$. Conversely, it is obvious that $A(B_0, \omega) \subset \mathcal{A}(\omega)$ and thus (4.7) holds.

*Step 3*. We show that $\mathcal{A}(\omega)$ is compact. By (4.7) in Step 2, we need only to prove that every $A(B, \omega)$ is compact. Indeed, by Lemma 2.5(v) and the omega-limit compactness of $\varphi$, we have

$$\kappa\left(\overline{\bigcup_{t \geqslant T} \varphi(t, \theta_{-t}\omega)B}^{WS}\right) = \kappa\left(\bigcup_{t \geqslant T} \varphi(t, \theta_{-t}\omega)B\right) \to 0 \quad \text{as } T \to \infty.$$

Since $\overline{\bigcup_{t \geqslant T} \varphi(t, \theta_{-t}\omega)B}^{WS}$ is norm-closed in $X$, thanks to the property (iv) in Lemma 2.5, we know that $A(B, \omega)$ is nonempty and compact as required.

*Step 4*. We show that $\mathcal{A}(\omega)$ is an attracting set. For every bounded set $B \subset X$, we claim that $A(B,\omega)$ attracts $B$, that is,

$$\operatorname{dist}\big(\varphi(t,\theta_{-t}\omega)B, A(B,\omega)\big) \to 0 \quad \text{as } t \to \infty,\ P\text{-a.s. } \omega \in \Omega. \tag{4.10}$$

Indeed, if not, then there exist $\delta > 0$, $t_n \to \infty$ and $x_n \in B$ such that

$$\operatorname{dist}\big(\varphi(t,\theta_{-t_n}\omega)x_n, A(B,\omega)\big) \geqslant \delta. \tag{4.11}$$

By the omega-limit compactness of $\varphi$, there exist a subsequence $\{n_k\}$ and an $x \in X$ such that

$$\varphi(t_{n_k},\theta_{-t_{n_k}}\omega)x_{n_k} \to x.$$

But by (4.4) $x \in A(B,\omega)$, which contradicts (4.11). Thus (4.10) holds true and implies that

$$\operatorname{dist}\big(\varphi(t,\theta_{-t}\omega)B, \mathcal{A}(\omega)\big) \leqslant \operatorname{dist}\big(\varphi(t,\theta_{-t}\omega)B, A(B,\omega)\big) \to 0$$

as $t \to \infty$. That is, $\mathcal{A}(\omega)$ attracts $B$, which completes the proof of Theorem 4.1. □

As a consequence, we obtain the sufficient condition for the existence of random attractor given in [7,8] under the weaker continuity assumption.

**Corollary 4.2.** *A quasi-continuous RDS $\varphi$ has a random attractor $\mathcal{A}(\omega)$ if $\varphi$ has a compact absorbing set $\mathcal{B}(\omega)$.*

**Proof.** For any bounded set $B \subset X$, there exists a $t_B$ such that

$$\bigcup_{t \geqslant t_B} \varphi(t,\theta_{-t}\omega)B \subset \mathcal{B}(\omega).$$

But $\mathcal{B}(\omega)$ is compact, $\kappa(\mathcal{B}(\omega)) = 0$, and so

$$\kappa\Bigg(\bigcup_{t \geqslant t_B} \varphi(t,\theta_{-t}\omega)B\Bigg) = 0, \tag{4.12}$$

which implies that $\varphi$ is omega-limit compact. □

**Remark 4.3.** The formulation (4.2) of the random attractor seems to differ from the random attractor obtained by the usual method in [7,8]. However, under the assumptions of Theorem 4.1, we can show that they are the same one. That is,

$$\mathcal{A}(\omega) = \bigcup_{B \subset X} A(B,\omega) = \overline{\bigcup_{B \subset X} A_B(\omega)} \tag{4.13}$$

where $A(B,\omega)$ is defined by (4.3), and $A_B(\omega)$ is defined by (see [7,8])

$$A_B(\omega) = \bigcap_{T\geqslant 0} \overline{\bigcup_{t\geqslant T} \varphi(t,\theta_{-t}\omega)B}. \tag{4.14}$$

Indeed, By (4.4) and the omega-limit compactness of $\varphi$, we can prove that $A(B,\omega) \subset A_B(\omega)$ and thus $A(B,\omega) = A_B(\omega)$. But by the proof of Theorem 4.1, $\bigcup_{B\subset X} A(B,\omega)$ is compact and thus norm closed particularly, which implies that

$$\bigcup_{B\subset X} A(B,\omega) = \overline{\bigcup_{B\subset X} A(B,\omega)} = \overline{\bigcup_{B\subset X} A_B(\omega)}.$$

**Remark 4.4.** Using the same argument given in [7,8], one can show that the random attractor $\mathcal{A}(\omega)$, defined by (4.2), is the largest compact invariant set and the minimal closed attracting set.

**Remark 4.5.** Of course, Theorem 4.1 still holds if quasi-continuity of $\varphi$ is replaced by the norm continuity, weak continuity or norm-to-weak continuity. Even in the deterministic case, Theorem 4.1 seems to be a slightly stronger result than [25, Theorem 4.2] since we need the weaker continuity assumption (than norm-to-weak continuity).

## 5. Random attractors on the $q$-times integrated spaces

It is well known that many RDS generated by the concrete SPDEs possess a random attractor on the Hilbert space $L^2(D)$, $D \subset \mathbb{R}^n$ is bounded. Assuming this fact, we will give in this section a general method to prove the existence of random attractors on the Banach space $L^q(D)$ for $q > 2$.

We denote by $\|\cdot\|_q$ the norm of $L^q(D)$, $|\cdot|$ the usual modular (or the absolute value), $m(\cdot)$ the Lebesgue measure and $D(|u| \geqslant M) = \{x \in D \mid |u(x)| \geqslant M\}$.

**Lemma 5.1.** *Suppose a RDS $\varphi$ has a bounded absorbing set $\mathcal{B}(\omega)$ in $L^q(D)$. Then for P-a.s. $\omega \in \Omega$, $\varepsilon > 0$ and bounded $B \subset L^q(D)$, there exist $T = T(B,\omega)$ and $M = M(\varepsilon,\omega)$ such that*

$$m\big(D\big(\big|\varphi(t,\theta_{-t}\omega)u\big| \geqslant M\big)\big) < \varepsilon, \quad \forall t \geqslant T,\ u \in B. \tag{5.1}$$

**Proof.** We let $\psi(t) := \varphi(t,\theta_{-t}\omega)$ for convenience and work for every fixed $\omega \in \Omega$. By the absorbing assumption, there exists a random variable $M_0 = M_0(\omega)$ such that, for bounded $B \subset L^q(D)$, we can find a constant $T = T(B,\omega)$ such that

$$\big\|\psi(t)u\big\|_q^q \leqslant M_0, \quad \text{whenever } t \geqslant T,\ u \in B.$$

On the other hand

$$\big\|\psi(t)u\big\|_q^q = \int_D \big|\psi(t)u\big|^q \geqslant \int_{D(|\psi(t)|\geqslant M)} \big|\psi(t)\big|^q \geqslant M^q \cdot m\big(D\big|\psi(t)u\big| \geqslant M\big).$$

Thus $m(D(|\psi(t)u| \geqslant M)) < \varepsilon$ if we choose $M$ large enough such that $M > (M_0/\varepsilon)^{1/q}$. $\square$

We also need the following basic lemma given in [25].

**Lemma 5.2.** *(See [25].) If a bounded set $B \subset L^q(D)$ has a finite $\varepsilon$-net in $L^q(D)$, then there is a number $M$ such that*

$$\sup_{u\in B} \int\limits_{(|u|\geqslant M)} |u|^q \leqslant 2^{q+1}\varepsilon^q. \tag{5.2}$$

**Theorem 5.3.** *Let $\varphi(t,\omega)$ be a RDS on $L^q(D)$ and a continuous RDS on $L^r(D)$ for some $1 < r \leqslant q$, where $D$ is bounded. Suppose $\varphi$ has a random attractor in $L^r(D)$. Then $\varphi$ has a random attractor in $L^q(D)$ if and only if*

(i) *$\varphi$ has a bounded absorbing set in $L^q(D)$;*
(ii) *for every bounded deterministic set $B \subset L^q(D)$, we have*

$$\lim_{t,M\to\infty} \sup_{u\in B} \int\limits_{D(|\varphi(t,\theta_{-t}\omega)u|\geqslant M)} \left|\varphi(t,\theta_{-t}\omega)u\right|^q = 0, \quad P\text{-a.s. } \omega \in \Omega. \tag{5.3}$$

**Proof.** Taking $X = L^q(D)$, $Y = L^r(D)$ in Proposition 3.3, we know that $\varphi$ is quasi-continuous in $L^q(D)$. Then by Theorem 4.1 we have only to consider the omega-limit compactness of $\varphi$ in $L^q(D)$. We work for every fixed $\omega \in \Omega_0$ with $P(\Omega_0) = 1$ and let $\psi(t) = \varphi(t,\theta_{-t}\omega)$ for convenience.

*Necessity*. Since $\varphi$ is by Theorem 4.1 omega-limit compact in $L^q(D)$, it follows that for every $\varepsilon > 0$ and bounded $B \subset L^q(D)$, there exists a $T = T(\varepsilon, B, \omega)$ such that the set

$$\tilde{B} = \bigcup_{t\geqslant T} \psi(t)B$$

admits a finite $\varepsilon$-net in $L^q(D)$. Applying Lemma 5.2 to the bounded set $\tilde{B}$, we can also choose an $M_0 = M_0(B,\varepsilon,\omega)$ such that if $M \geqslant M_0$ and $t \geqslant T$ then

$$\sup_{u\in B} \int\limits_{D(|\psi(t)u|\geqslant M)} \left|\psi(t)u\right|^q \leqslant 2^{q+1}\varepsilon^q,$$

which proves (5.3).

*Sufficiency*. For any $\varepsilon > 0$ and bounded $B \subset L^q(D)$ $(\subset L^r(D))$, by the assumption (5.3) there exist $t_1 = t_1(B,\varepsilon,\omega)$ and $M = M(B,\varepsilon,\omega)$ such that

$$\sup_{u\in B} \int\limits_{D(|\psi(t)u|\geqslant M)} \left|\psi(t)u\right|^q < \frac{\varepsilon^q}{2^{q+3}}, \quad \forall t \geqslant t_1. \tag{5.4}$$

Since by Theorem 4.1 $\varphi$ is omega-limit compact in $L^r(D)$, there exists a $t_2 \geqslant t_1$ such that the set

$$\hat{B} = \bigcup_{t\geqslant t_2} \psi(t)B$$

has a finite $(2M)^{(r-q)/r}(\varepsilon^q/2)^{1/r}$-net $\{u_k\} \subset \hat{B}$ in $L^r(D)$. Thus, for any $u \in \hat{B}$ we can choose some $u_k \in \hat{B}$ such that

$$\|u-u_k\|_r^r < (2M)^{r-q} \cdot \frac{\varepsilon^q}{2}. \tag{5.5}$$

It is obvious that $D = D_1 \cup D_2 \cup D_3 \cup D_4$, where

$$\begin{aligned} &D_1 = D\big(|u| \leqslant M,\ |u_k| \leqslant M\big), \qquad && D_2 = D\big(|u| \geqslant M,\ |u_k| \leqslant M\big),\\ &D_3 = D\big(|u| \leqslant M,\ |u_k| \geqslant M\big), \qquad && D_4 = D\big(|u| \geqslant M,\ |u_k| \geqslant M\big). \end{aligned} \tag{5.6}$$

Then (5.5) implies that

$$\int_{D_1} |u-u_k|^q \leqslant \int_{D(|u-u_k|\leqslant 2M)} |u-u_k|^q \leqslant (2M)^{q-r}\|u-u_k\|_r^r < \frac{\varepsilon^q}{2}. \tag{5.7}$$

Since $|u-u_k| \leqslant 2|u|$ in $D_2$, it follows from (5.4) that

$$\int_{D_2} |u-u_k|^q \leqslant 2^q \int_{D(|u|\geqslant M)} |u|^q < \frac{\varepsilon^q}{8}. \tag{5.8}$$

Similarly, since $|u-u_k| \leqslant 2|u_k|$ in $D_3$,

$$\int_{D_3} |u-u_k|^q \leqslant 2^q \int_{D(|u_k|\geqslant M)} |u_k|^q < \frac{\varepsilon^q}{8}. \tag{5.9}$$

Using the inequality $(a+b)^q \leqslant 2^q(|a|^q + |b|^q)$ and (5.4), we also find

$$\int_{D_4} |u-u_k|^q \leqslant 2^q\bigg(\int_{D(|u|\geqslant M)} |u|^q + \int_{D(|u_k|\geqslant M)} |u_k|^q\bigg) < \frac{\varepsilon^q}{4}. \tag{5.10}$$

It follows from (5.7)–(5.10) that

$$\|u-u_k\|_q^q \leqslant \bigg(\int_{D_1} + \int_{D_2} + \int_{D_3} + \int_{D_4}\bigg)|u-u_k|^q < \varepsilon^q, \tag{5.11}$$

which proved $\hat{B}$ has an $\varepsilon$-net in $L^q(D)$ and thus $\varphi$ is omega-limit compact in $L^q(D)$ as needed. □

## 6. Applications to the stochastic reaction–diffusion equation

### 6.1. Preliminaries for the equation

Let $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded open set with regular boundary $\partial D$. We consider the following stochastic reaction–diffusion equation

$$\begin{cases} du - \big(\Delta u - f(u)\big)\,dt = \sum_{j=1}^{m} g_j\,dW_j(t), \\ u = 0 \quad \text{on } \partial D. \end{cases} \tag{6.1}$$

The function $f$ is polynomial defined by

$$f(s) = \sum_{j=1}^{2p-1} a_j s^j, \quad a_{2p-1} > 0,\ p > 1. \tag{6.2}$$

We also assume that

$$g_j \in L^\infty(D), \quad j = 1, 2, \ldots, m. \tag{6.3}$$

The random functions $W_j$, $j = 1, \ldots, m$, are independent two-sided real-valued Wiener processes on a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^n) \mid \omega(0) = 0\}$, with $P$ being a product measure of two Wiener measures on the negative and the positive time parts of $\Omega$. We have

$$\big(W_1(t,\omega), W_2(t,\omega), \ldots, W_m(t,\omega)\big) = \omega(t), \quad t \in \mathbb{R}.$$

The time shift is simply defined by

$$\theta_s\omega(t) = \omega(t+s) - \omega(s), \quad t, s \in \mathbb{R}. \tag{6.4}$$

It is a family of ergodic transformations.

Let $A_q = -\Delta$ with $D(A_q) = W^{2,q}(D) \cap W_0^{1,q}(D)$ and write $A = A_2$. Then $A_q$ generates an analytic semigroup $e^{tA_q}$ on $L^q(D)$ (see Pazy [15]). The Ornstein–Uhlenbeck process defined by

$$z(t) = \sum_{j=1}^{m} \int_{-\infty}^{t} e^{A(t-s)} g_j\,dW_j(s) \tag{6.5}$$

is $P$-a.s. pathwise continuous and is the unique stationary solution of the linear equation

$$dz + Az\,dt = \sum_{j=1}^{m} g_j\,dW_j(t). \tag{6.6}$$

It is well known (see [8,9]) that for any $s \in \mathbb{R}$, $u_0 \in L^2(D)$ and $\omega \in \Omega$, there exists a unique solution

$$v \in C\big([s,t]; L^2(D)\big) \cup L^2\big(s,t; H_0^1(D)\big) \cup L^{2p}\big(s,t; L^{2p}(D)\big)$$

for any $t \geqslant s$ of the equation

$$\begin{cases} \dfrac{dv}{dt} + Av + f(v+z) = 0, \\ v(s) = u_0 - z(s) \end{cases} \tag{6.7}$$

and $u = v + z$ is a solution of (6.1) such that $u(s) = u_0$. Denote this solution by $u(t) = u(t; s, u_0; \omega)$ (denoted sometimes by $u(t; s, u_0)$, $u(t; s)$, $u(t)$ or even $u$ if no confusions). Then

$$\varphi(t,\omega)u_0 := u(t; 0, u_0; \omega), \quad t \geqslant 0, \tag{6.8}$$

is a continuous RDS on $L^2(D)$ with the following fact

$$\varphi(s, \theta_{-s}\omega)u_0 = u(0; -s, u_0; \omega), \quad s \geqslant 0. \tag{6.9}$$

The random attractor in $L^2(D)$ has been obtained by [8,9].

**Theorem 6.1.** *(See [8,9].) The RDS defined by* (6.8) *possesses a random attractor in* $L^2(D)$.

### *6.2. Absorption in* $L^q(D)$

In order to prove the RDS $\varphi$ defined by (6.8) has a random attractor in $L^q(D)$ $(q > 2)$, by Theorem 5.3, we must show that $\varphi$ has a bounded absorbing set and some asymptotic a priori estimates in $L^q(D)$ $(q > 2)$.

In this subsection, we will prove that the RDS $\varphi$ generated by Eq. (6.1) has a bounded absorbing set $\mathcal{B}(\omega) \subset L^q(D)$, $q > 2$, which absorbs in fact all bounded sets $B \subset L^2(D)$.

We need the properties of the polynomial $f$ defined by (6.2). The proof is elementary and so is omitted. In the sequel, the large positive constant $C$ may be different from line to line.

**Lemma 6.2.** *For the polynomial $f$ defined by* (6.2)*, there exist $c_1, c_2$ and large number $C$ such that*

(i) $f'(x) \geqslant -C$, *for all* $x \in \mathbb{R}$;
(ii) $f(x+y)x^{2k+1} \geqslant c_1 x^{2p+2k} - C(y^{2p+2k} + 1)$, *for all* $k \in \mathbb{N}$ *and* $x, y \in \mathbb{R}$;
(iii) *there exists an* $M > 0$ *such that* $f(x) \geqslant c_2 x^{2p-1}$ *if* $x \geqslant M$, *while* $f(x) \leqslant c_2 x^{2p-1}$ *if* $x \leqslant -M$.

We now prove the existence of bounded absorbing sets in $L^q(D)$. The idea of the induction proof follows from the proof of differentiability of the system in $L^2(D)$ given in [9, Lemma 4.3] or [14, Lemma 2.3]. Recall that $u(t) = u(t; s, u_0; \omega)$ is the solution of (6.1) with $u(s) = u_0$ and $v(t) = v(t; s, u_0 - z(s); \omega)$ is the solution of (6.7) with $v(s) = u_0 - z(s)$.

**Lemma 6.3.** *For every fixed $q > 2$, there exists a random variable $\gamma(\omega)$ such that, for any $R > 0$ there exists an $s_0 = s_0(R, \omega) < -1$ satisfying*

$$\left\|u(0; s, u_0; \omega)\right\|_q \leqslant \gamma(\omega), \quad \text{whenever } s \leqslant s_0,\ \|u_0\|_2 \leqslant R. \tag{6.10}$$

**Proof.** We have only to prove that (6.10) holds for all

$$q_k = 2k(p-1) + 2, \quad k \in \mathbb{N}, \tag{6.11}$$

since for every $q$, $L^q(D) \supset L^{q_k}(D)$ for some $q_k$. To do this, we shall prove, recursively in $k$, that there exist random numbers $\gamma_1^k(\omega), \gamma_2^k(\omega)$ such that, for any $R > 0$, there exists an $s_k = s_k(R, \omega) \leqslant -1$ satisfying

$$\int_{-\frac{1}{k}}^{0} \left\|v\big(\tau; s, u_0 - z(s); \omega\big)\right\|_{q_k}^{q_k} d\tau \leqslant \gamma_1^k(\omega), \tag{6.12}$$

$$\left\|u(t; s, u_0; \omega)\right\|_{q_k} \leqslant \gamma_2^k(\omega), \quad -\frac{1}{k+1} \leqslant t \leqslant 0, \tag{6.13}$$

whenever $s \leqslant s_k$ and $\|u_0\|_2 \leqslant R$.

We first prove (6.12)–(6.13) for $k = 1$, that is, $q_1 = 2p$. Taking the scalar product of (6.7) with $v$ on $L^2(D)$ and using Lemma 6.2(ii), we have

$$\frac{d}{dt}\|v\|_2^2 + \|v\|_{H^1}^2 + c_1\|v\|_{q_1}^{q_1} \leqslant C\|z\|_{q_1}^{q_1} + C. \tag{6.14}$$

Integrating (6.14) over $t \in [-1, 0]$ we find

$$\int_{-1}^{0} \left\|v(\tau)\right\|_{q_1}^{q_1} d\tau \leqslant C\int_{-1}^{0} \left\|z(\tau)\right\|_{q_1}^{q_1} d\tau + C + \left\|v(-1)\right\|_2^2. \tag{6.15}$$

But it has been proved in [7] that there exist $\gamma_0 = \gamma_0(\omega)$ and $s_0 = s_0(R, \omega)$ such that

$$\sup_{-1 \leqslant t \leqslant 0} \left\|v\big(t; s, u_0 - z(s)\big)\right\|_2 \leqslant \gamma_0 \tag{6.16}$$

whenever $s \leqslant s_0$ and $\|u_0\|_2 \leqslant R$. It note also that by the assumption (6.3) $g_j \in L^\infty(D)$ and thus $z$ has continuous path on $L^q(D)$ for any $q \geqslant 2$, which, together with (6.15)–(6.16), implies that

$$\int_{-1}^{0} \left\|v(\tau)\right\|_{q_1}^{q_1} d\tau \leqslant \gamma_1^1 := C\int_{-1}^{0} \left\|z(\tau)\right\|_{q_1}^{q_1} d\tau + C + \gamma_0^2 < +\infty, \tag{6.17}$$

which proved (6.12) for $k = 1$. Multiplying (6.7) by $v^{q_1 - 1}$ and using Lemma 6.2(ii) again we obtain

$$\frac{d}{dt}\|v\|_{q_1}^{q_1} + c_2\|v\|_{q_2}^{q_2} \leqslant C\|z\|_{q_2}^{q_2} + C. \tag{6.18}$$

Integrating (6.18) from $\tau$ $(-1 \leqslant \tau \leqslant -1/2)$ to $t$ $(-1/2 \leqslant t \leqslant 0)$ we get

$$\|v(t)\|_{q_1}^{q_1} \leqslant C \int_\tau^t \|z(\sigma)\|_{q_2}^{q_2}\, d\sigma + C(t-\tau) + \|v(\tau)\|_{q_1}^{q_1}, \quad -1 \leqslant \tau \leqslant -\frac{1}{2} \leqslant t \leqslant 0.$$

Integrating the above inequality from $\tau = -1$ to $\tau = -1/2$ and using (6.17), we have

$$\|v(t)\|_{q_1}^{q_1} \leqslant \gamma_2^1 := 2C \int_{-1}^0 \|z(\sigma)\|_{q_2}^{q_2}\, d\sigma + 2C + 2\gamma_1^1 < \infty, \quad -\frac{1}{2} \leqslant t \leqslant 0,$$

which proved (6.13) for $k = 1$.

Assume now (6.12)–(6.13) hold for $k$. Using a similar argument as in the derivation of (6.18), we can prove the existence of $c_1^k$ and $C$ such that

$$\frac{d}{dt}\|v\|_{q_k}^{q_k} + c_1^k \|v\|_{q_{k+1}}^{q_{k+1}} \leqslant C\big(\|z\|_{q_{k+1}}^{q_{k+1}} + 1\big). \tag{6.19}$$

Integrating (6.19) from $t = -\frac{1}{k+1}$ to $t = 0$ yields

$$c_1^k \int_{-\frac{1}{k+1}}^0 \|v(t)\|_{q_{k+1}}^{q_{k+1}}\, dt \leqslant C\left( \int_{-\frac{1}{k+1}}^0 \|z(t)\|_{q_{k+1}}^{q_{k+1}}\, dt + 1\right) + \left\|v\left(-\frac{1}{k+1}\right)\right\|_{q_k}^{q_k}.$$

Thanks to the induction assumption (6.13) for $k$, we see that

$$\left\|v\left(-\frac{1}{k+1}\right)\right\|_{q_k}^{q_k} \leqslant 2^{q_k}\left(\left\|u\left(-\frac{1}{k+1}\right)\right\|_{q_k}^{q_k} + \left\|z\left(-\frac{1}{k+1}\right)\right\|_{q_k}^{q_k}\right) \leqslant 2^{q_k}\big(\gamma_2^k + C\big).$$

Thus we have proved (6.12) for $k+1$. Integrating from $\tau$ $(-1 \leqslant \tau \leqslant -\frac{1}{k+2})$ to $t$ $(-\frac{1}{k+2} \leqslant t \leqslant 0)$ in (6.19) with $k$ replaced by $k+1$, we obtain

$$\|v(t)\|_{q_{k+1}}^{q_{k+1}} \leqslant C + \|v(\tau)\|_{q_{k+1}}^{q_{k+1}}, \quad -1 \leqslant \tau \leqslant -\frac{1}{k+2} \leqslant t \leqslant 0.$$

Integrating the above inequality from $\tau = -\frac{1}{k+1}$ to $\tau = -\frac{1}{k+2}$ we know

$$\left(\frac{1}{k+1} - \frac{1}{k+2}\right)\|v(t)\|_{q_{k+1}}^{q_{k+1}} \leqslant C + \int_{-\frac{1}{k+1}}^0 \|v(\tau)\|_{q_{k+1}}^{q_{k+1}}\, d\tau, \quad -\frac{1}{k+2} \leqslant t \leqslant 0,$$

which, together with (6.12) for $k+1$, implies that (6.13) holds for $k+1$ and completes the induction proof. □

*6.3. Asymptotic estimates*

This subsection presents some asymptotic a priori estimates for the unbounded part of the modular $|u|$ in $L^q(D)$ $(q>2)$. We need an auxiliary lemma which gives the asymptotic estimate in $L^2(D)$ at all time $t\in[-1,0]$. (The special case that $t=0$ is immediate from Theorems 5.3 and 6.1.)

**Lemma 6.4.** *For any $\varepsilon>0$ and bounded set $B\subset L^2(D)$, there exist $s_0=s_0(\varepsilon,B,\omega)$, $M_1=M_1(\varepsilon,B,\omega)$, $M_2=M_2(\varepsilon,B,\omega)$ such that, for all $s\leqslant s_0$, $u_0\in B$, $P$-a.s. $\omega\in\Omega$*

$$\int\limits_{D(|u(t)|\geqslant M_1)}\left|u(t;s,u_0;\omega)\right|^2<\varepsilon,\quad -1\leqslant t\leqslant 0, \tag{6.20}$$

$$\int\limits_{D(|v(t)|\geqslant M_2)}\left|v\big(t;s,u_0-z(s);\omega\big)\right|^2<\varepsilon,\quad -1\leqslant t\leqslant 0. \tag{6.21}$$

**Proof.** Using a similar argument as given in [7], one can prove that there exists a compact absorbing set $\mathcal{B}(\omega)\subset L^2(D)$, which absorbs every bounded set $B$ in $L^2(D)$ at all time $t\in[-1,0]$ (rather than $t=0$ only). More precisely, there exists $s_0\leqslant -1$ such that if $s\leqslant s_0$ then

$$u(t;s,B):=\bigcup_{u_0\in B}\big\{u(t;s,u_0)\big\}\subset\mathcal{B}(\omega),\quad -1\leqslant t\leqslant 0. \tag{6.22}$$

Since $\mathcal{B}(\omega)$ is compact, for any $\varepsilon>0$, the set

$$\bigcup_{-1\leqslant t\leqslant 0}\ \bigcup_{s\leqslant s_0}u(t;s,B)$$

has an $(\varepsilon/8)^{1/2}$-net. Thus by Lemma 5.2 there exists an $M_1$ such that

$$\sup_{-1\leqslant t\leqslant 0}\int\limits_{D(|u(t)|\geqslant M_1)}\left|u(t;s,u_0)\right|^2<2^{2+1}\varepsilon/8=\varepsilon,$$

whenever $u_0\in B$ and $s\leqslant s_0$, which proved (6.20).

To prove (6.21), we let

$$E=\sup_{-1\leqslant t\leqslant 0}\left\|z(t)\right\|_\infty, \tag{6.23}$$

which is finite by the bounded assumption (6.3) of $g_j$. By Lemma 5.1, we can choose $s_1\leqslant s_0\leqslant -1$ and $\tilde{M}\geqslant M_1$ such that, for all $s\leqslant s_1$ and $u_0\in B$,

$$m\big(D\big(\left|u(t;s,u_0)\right|\geqslant\tilde{M}\big)\big)\leqslant\frac{\varepsilon}{E^2},\quad -1\leqslant t\leqslant 0. \tag{6.24}$$

Taking now $M_2=\tilde{M}+E$ and noting that

$$D\big(|v(t)| \geqslant M_2\big) \subset D\big(|u(t)| \geqslant \tilde{M}\big) \subset D\big(|u(t)| \geqslant M_1\big),$$

we get from (6.20) and (6.23)–(6.24) that

$$\int\limits_{D(|v| \geqslant M_2)} |v(t)|^2 \leqslant 2 \int\limits_{D(|u| \geqslant M_1)} |u(t)|^2 + 2 \int\limits_{D(|u| \geqslant \tilde{M})} |z(t)|^2 \leqslant 4\varepsilon,$$

which completes the proof of (6.21). □

Using the auxiliary lemma above, we can now give the asymptotic estimates in $L^q(D)$ for $q = q_1 = 2p$.

**Lemma 6.5.** *For any bounded set $B \subset L^2(D)$ and $P$-a.s. $\omega \in \Omega$, we have*

$$\lim_{-s, M \to +\infty} \sup_{u_0 \in B} \int\limits_{D(|u(t;s,u_0;\omega)| \geqslant M)} |u(t; s, u_0; \omega)|^{2p} = 0, \quad t \in \left[-\frac{1}{2}, 0\right]. \tag{6.25}$$

**Proof.** For any $\varepsilon > 0$, let $s_0$, $M_1$, $M_2$ be the constants defined in Lemma 6.4. By Lemma 5.1 we choose $s_1 \leqslant s_0 \leqslant -1$ and $M_3$ such that, for all $s \leqslant s_1$, $u_0 \in B$,

$$m\big(D\big(|u(t; s, u_0)| \geqslant M_3\big)\big) \leqslant \frac{\varepsilon}{E^{2p}}, \quad -1 \leqslant t \leqslant 0, \tag{6.26}$$

where $E = \sup\{\|z(t)\|_\infty;\ -1 \leqslant t \leqslant 0\}$ defined as in (6.23). By Lemma 6.2(iii), we can choose $M_4$ such that

$$f(x) \geqslant 0 \quad \text{if } x \geqslant M_4. \tag{6.27}$$

Let now $M = E + \max\{M_1, M_2, M_3, M_4\}$. Multiplying Eq. (6.7) by $(v - M)_+$ and integrating over $D$ we find

$$\frac{d}{dt}\|(v - M)_+\|_2^2 + 2\int\limits_D |\nabla(v - M)_+|^2 + 2\int\limits_{D(v \geqslant M)} f(v + z)(v - M) = 0 \tag{6.28}$$

where

$$(v - M)_+ = \begin{cases} v - M & \text{if } v \geqslant M, \\ 0 & \text{if } v \leqslant M. \end{cases}$$

By (6.27) we know that $f(v + z)(v - M) \geqslant 0$ on $D(v \geqslant M)$ (since $v + z \geqslant M - E \geqslant M_4$). Integrating (6.28) from $t = -1$ to $t = 0$ and using (6.21) we obtain

$$2\int\limits_{-1}^{0} \int\limits_{D(v \geqslant M)} f(v + z)(v - M) \leqslant \big\|\big(v(-1) - M\big)_+\big\|_2^2 < \varepsilon. \tag{6.29}$$

Noting that

$$\int_{D(v\geqslant 2M)} f(v+z)v \leqslant 2 \int_{D(v\geqslant M)} f(v+z)(v-M)$$

and

$$v^{2p} \leqslant c_1 f(v+z)v + C\big(|z|^{2p}+1\big),$$

we get from (6.26) and (6.29) that

$$\int_{-1}^{0} \int_{D(v\geqslant 2M)} v^{2p} \leqslant C\varepsilon \tag{6.30}$$

where $C$ does not depend on $\varepsilon$ and $B$. We then take the product of (6.7) with $(v-2M)_+^{2p-1}$ to find

$$\frac{1}{2p}\frac{d}{dt}\big\|(v-2M)_+\big\|_{2p}^{2p} + \int_{D(v\geqslant 2M)} f(v+z)(v-2M)^{2p-1} \leqslant 0. \tag{6.31}$$

In particular

$$\frac{d}{dt}\big\|(v-2M)_+\big\|_{2p}^{2p} \leqslant 0, \tag{6.32}$$

which implies obviously that

$$\big\|\big(v(t)-2M\big)_+\big\|_{2p}^{2p} \leqslant \big\|\big(v(\tau)-2M\big)_+\big\|_{2p}^{2p}, \quad -1\leqslant \tau \leqslant -\frac{1}{2},\ -\frac{1}{2}\leqslant t\leqslant 0.$$

Integrating it from $\tau=-1$ to $\tau=-1/2$ and using (6.30) we get

$$\big\|\big(v(t)-2M\big)_+\big\|_{2p}^{2p} \leqslant 2\int_{-1}^{-1/2} \int_{D(v\geqslant 2M)} v^{2p} < C\varepsilon, \quad -\frac{1}{2}\leqslant t\leqslant 0,$$

which implies that

$$\int_{D(v\geqslant 4M)} v^{2p}(t) \leqslant 2^{2p+1} \int_{D(v\geqslant 2M)} \big(v(t)-2M\big)^{2p} \leqslant C\varepsilon, \quad -\frac{1}{2}\leqslant t\leqslant 0.$$

This, together with (6.26), implies that

$$\int_{D(u\geqslant 4M+E)} u^{2p}(t) \leqslant C\varepsilon, \quad -\frac{1}{2}\leqslant t\leqslant 0. \tag{6.33}$$

Taking $(v+M)_-$ and $(v+2M)_-^{2p-1}$ instead of $(v-M)_+$ and $(v-2M)_+^{2p-1}$ in the preceding proof, we deduce similarly that

$$\int\limits_{D(u\leqslant -4M-E)} u^{2p}(t) \leqslant C\varepsilon, \quad \frac{1}{2} \leqslant t \leqslant 0,$$

which completes the proof of (6.25). □

In what follows we give the asymptotic estimate in $L^q(D)$ for any $q > 2$.

**Lemma 6.6.** *Let $q > 2$ be fixed. For every bounded set $B \subset L^2(D)$, we have*

$$\lim_{-s,M\to+\infty} \sup_{u_0\in B} \int\limits_{D(|u|\geqslant M)} \left|u(0; s, u_0; \omega)\right|^q = 0, \quad P\text{-a.s. } \omega \in \Omega. \tag{6.34}$$

**Proof.** Let $q_k = 2k(p-1)+2$ defined as in (6.11). We will prove by induction that for any $\varepsilon > 0$ there exist $s_0^k = s_0^k(\varepsilon, B, \omega)$, $M_1^k = M_1^k(\varepsilon, B, \omega)$ and $M_2^k = M_2^k(\varepsilon, B, \omega)$ such that, for all $s < s_0^k$, $u_0 \in B$,

$$\int\limits_{-\frac{1}{k}}^{0} \Bigg( \int\limits_{D(|v|\geqslant M_1^k)} \left|v\big(\tau; s, u_0 - z(s)\big)\right|^{q_k} \Bigg) d\tau < C\varepsilon, \tag{6.35}$$

$$\int\limits_{D(|u|\geqslant M_2^k)} \left|u(t; s, u_0)\right|^{q_k} < C\varepsilon, \quad -\frac{1}{k+1} \leqslant t \leqslant 0. \tag{6.36}$$

The case that $k = 1$ has been proved by Lemma 6.5. Assume now that (6.35)–(6.36) hold for $k$. By Lemma 5.1 we can choose $s_1^k \leqslant s_0^k$ and $M_3^k$ such that, for $s \leqslant s_1^k$ and $u_0 \in B$,

$$m\big(D\big(\left|u(t; s, u_0)\right| \geqslant M_3^k\big)\big) \leqslant \frac{\varepsilon}{E^{q_{k+1}}}, \quad -1/k \leqslant t \leqslant 0. \tag{6.37}$$

Let $M = E + \max\{M_1^k, M_2^k, M_3^k, M_4\}$, where $M_4$ is the constant such that $f(x) \geqslant 0$ whenever $x \geqslant M_4$. Using similar arguments as in the derivation of (6.31) we can prove the existence of $c_k$ such that

$$\frac{d}{dt}\left\|(v-M)_+\right\|_{q_k}^{q_k} + c_k \int\limits_{D(v\geqslant M)} f(v+z)(v-M)^{q_k-1} \leqslant 0. \tag{6.38}$$

Using (6.37) and the induction assumption (6.36) we know that, for $-\frac{1}{k+1} \leqslant t \leqslant 0$,

$$\left\|\big(v(t) - M\big)_+\right\|_{q_k}^{q_k} \leqslant C\Bigg( \int\limits_{D(u\geqslant M_2^k)} u^{q_k}(t) + \int\limits_{D(u\geqslant M_3^k)} z^{q_k}(t) \Bigg) < C\varepsilon, \tag{6.39}$$

which implies by integrating (6.38) over $[-\frac{1}{k+1}, 0]$ that

$$\int_{-\frac{1}{k+1}}^{0} \int_{D(v \geqslant M)} f(v+z)(v-M)^{q_k-1} < C\varepsilon. \tag{6.40}$$

It is easy to prove that

$$\int_{D(v \geqslant 2M)} f(v+z)v^{q_k-1} \leqslant 2^{q_k} \int_{D(v \geqslant M)} f(v+z)(v-M)^{q_k-1}.$$

This, together with (6.40), (6.37), Lemma 6.2(ii), implies that

$$\int_{-\frac{1}{k+1}}^{0} \int_{D(v \geqslant 2M)} v^{q_{k+1}} < C\varepsilon. \tag{6.41}$$

We also multiply Eq. (6.7) by $(v-2M)_+^{q_{k+1}-1}$ to find

$$\frac{d}{dt}\left\|(v-2M)_+\right\|_{q_{k+1}}^{q_{k+1}} \leqslant 0. \tag{6.42}$$

Then

$$\left\|\big(v(t)-2M\big)_+\right\|_{q_{k+1}}^{q_{k+1}} \leqslant \left\|\big(v(\tau)-2M\big)_+\right\|_{q_{k+1}}^{q_{k+1}}, \quad -\frac{1}{k+1} \leqslant \tau \leqslant -\frac{1}{k+2} \leqslant t \leqslant 0.$$

Integrating it from $\tau = -\frac{1}{k+1}$ to $\tau = -\frac{1}{k+2}$ and using (6.41) we have

$$\left(\frac{1}{k+1} - \frac{1}{k+2}\right)\left\|\big(v(t)-2M\big)_+\right\|_{q_{k+1}}^{q_{k+1}} \leqslant \int_{-\frac{1}{k+1}}^{0} \int_{D(v \geqslant 2M)} (v-2M)^{q_{k+1}} < C\varepsilon, \tag{6.43}$$

whenever $-\frac{1}{k+2} \leqslant t \leqslant 0$, which implies easily that

$$\int_{D(v \geqslant 4M)} v^{q_{k+1}}(t) \leqslant C\varepsilon, \quad -\frac{1}{k+2} \leqslant t \leqslant 0,$$

and thus

$$\int_{D(u \geqslant 4M+E)} u^{q_{k+1}}(t) \leqslant C\varepsilon, \quad -\frac{1}{k+2} \leqslant t \leqslant 0. \tag{6.44}$$

On the other hand, we have the similarly asymptotic estimates for the negative part and thus (6.35)–(6.36) hold for $k+1$, which completes the induction proof of (6.35)–(6.36). In particular, (6.36) with $t=0$ implies that (6.34) holds for all $q_k$. Since for every $q>2$, $L^q(D)\supset L^{q_k}(D)$ for some large $q_k$, (6.34) holds for all $q>2$. □

*6.4. Existence and the fractal dimension of random attractors*

By using the preceding results, we can now state our conclusions about the existence of random attractors for Eq. (6.1) in $L^q(D)$ for any $q\geqslant 2$.

**Theorem 6.7.** *Assume that $D\subset\mathbb{R}^n$ is bounded and $g_j\in L^\infty(D)$, $j=1,\dots,m$. Then the RDS $\varphi$ generated by the stochastic reaction–diffusion equation* (6.1) *possesses a random attractor $\mathcal{A}_q(\omega)$ in $L^q(D)$ for any $q\geqslant 2$. $\mathcal{A}_q(\omega)$ is an invariant and compact random set which attracts bounded sets of $L^2(D)$ under the $L^q$-norm topology. Furthermore, $\mathcal{A}_q(\omega)=\mathcal{A}_2(\omega):=\mathcal{A}(\omega)$, for all $q\geqslant 2$, where $\mathcal{A}(\omega)$ is the usual attractor in $L^2(D)$.*

**Proof.** By Lemmas 6.3, 6.6 and Theorem 6.1, we know that the hypotheses of Theorem 5.3 are satisfied, which proved the RDS $\varphi$ has a random attractor $\mathcal{A}_q(\omega)$ in $L^q(D)$. It notes that by Lemma 6.3 $\mathcal{A}_q(\omega)$ attracts in fact all bounded sets in $L^2(D)$. By this fact and a similar argument as in the deterministic case, it is easy to prove that all $\mathcal{A}_q(\omega)$ $(q\geqslant 2)$ are the same set $\mathcal{A}(\omega)$. □

Finally we are concerned with the fractal dimension of $\mathcal{A}(\omega)$ under the $L^q$-norm topology. Recall that, for a bounded set $B\subset L^q(D)$, the *fractal dimension* of $B$ under $L^q$-norm is defined by (see [18,20])

$$\dim_q(B)=\limsup_{\varepsilon\to 0}\frac{\log M_q(\varepsilon)}{-\log\varepsilon}\tag{6.45}$$

where $M_q(\omega)$ denotes the minimum number of balls $B_q(u_i,\varepsilon):=\{u\in L^q(D)\mid \|u-u_i\|_q<\varepsilon\}$ of radius $\varepsilon$ required to cover $B$. By [12, Lemma 2.1], the centers $u_i$ can be taken in $B$, that is,

$$\dim_q(B)=\limsup_{\varepsilon\to 0}\frac{\log N_q(\varepsilon)}{-\log\varepsilon}\tag{6.46}$$

where $N_q(\varepsilon)$ denotes the minimum number of balls of radius $\varepsilon$ with centers in $B$ that required to cover $B$ under the $L^q$-norm.

In order to prove the finiteness of the fractal dimension of the random attractor $\mathcal{A}(\omega)$ under any $L^q$-norm $(q\geqslant 2)$, we need a key lemma, which gives a comparison result of fractal dimensions under the different $L^q$-norm.

**Lemma 6.8.** *Suppose a set $B$ is bounded in $L^\delta(D)$ for all $1<\delta<+\infty$, where $D$ is bounded. Then we have*

$$\dim_r(B)\leqslant\dim_q(B)\leqslant\frac{2q-r}{r}\dim_r(B),\quad \textit{for } q\geqslant r>1.\tag{6.47}$$

**Proof.** It is relatively easy to prove the first inequality. Indeed, by the Hölder inequality we know that

$$\|u\|_r \leqslant c_1 \|u\|_q, \quad \forall u \in L^q(D), \tag{6.48}$$

where $c_1 = |D|^{(q-r)/qr}$. Consider an $\varepsilon$-net $B_q(u_i, \varepsilon)$ in $L^q(D)$, $i = 1, 2, \dots, N_q(\varepsilon)$. By (6.48) we have

$$B_q(u_i, \varepsilon) \subset B_r(u_i, c_1\varepsilon),$$

it follows that $N_r(c_1\varepsilon) \leqslant N_q(\varepsilon)$, and so

$$\frac{\log N_r(c_1\varepsilon)}{-\log(c_1\varepsilon)} \leqslant \frac{\log N_q(\varepsilon)}{-\log c_1 - \log \varepsilon}$$

yields $\dim_r(B) \leqslant \dim_q(B)$.

In order to prove the second inequality in (6.47), we first show that for any fixed $\delta > 1$ and $q \geqslant r > 1$ there exists a number $C_1 = C_1(\delta, B, q)$ such that

$$\sup_{u \in B} \int\limits_{D(|u| \geqslant C_1 \varepsilon^{-\delta})} |u|^q < \varepsilon^q, \quad \forall \varepsilon > 0, \tag{6.49}$$

where $C_1$ does not depend on $\varepsilon$. Indeed, by the assumptions, $B$ is bounded in $L^{q\delta'}(D)$, where $\delta'$ satisfies $1/\delta + 1/\delta' = 1$, then

$$c_2 := \sup_{u \in B} \left\{ \|u\|_{q\delta'}, \|u\|_q \right\} < +\infty. \tag{6.50}$$

By the Hölder inequality we have

$$\int\limits_{D(|u| \geqslant C_1 \varepsilon^{-\delta})} |u|^q \leqslant \|u\|_{q\delta'}^q \cdot \left( m\left( D\left( |u| \geqslant C_1 \varepsilon^{-\delta} \right) \right) \right)^{\frac{1}{\delta}} \leqslant c_2^q \left( m\left( D\left( |u| \geqslant C_1 \varepsilon^{-\delta} \right) \right) \right)^{\frac{1}{\delta}}. \tag{6.51}$$

But by (6.50)

$$m\left( D\left( |u| \geqslant C_1 \varepsilon^{-\delta} \right) \right) \leqslant \frac{1}{(C_1 \varepsilon^{-\delta})^q} \int\limits_{D(|u| \geqslant C_1 \varepsilon^{-\delta})} |u|^q \leqslant \frac{c_2^q}{C_1^q} \varepsilon^{q\delta}, \tag{6.52}$$

which, together with (6.51), implies that

$$\int\limits_{D(|u| \geqslant C_1 \varepsilon^{-\delta})} |u|^q \leqslant \frac{c_2^{q+q/\delta}}{C_1^{q/\delta}} \varepsilon^q. \tag{6.53}$$

Thus (6.49) holds if we choose $C_1$ such that $C_1^{q/\delta} > c_2^{q+q/\delta}$.

We then show that for any $\delta > 1$ there exists a constant $C_2$ such that

$$N_q(C_2\varepsilon) \leqslant N_r\big(\varepsilon^{(\delta(q-r)+q)/r}\big), \quad \forall \varepsilon > 0. \tag{6.54}$$

To this end, let $B$ have a net $B_r(u_k, \varepsilon^{(\delta(q-r)+q)/r})$ in $L^r(D)$, $k = 1, 2, \dots, N_r(\varepsilon^{(\delta(q-r)+q)/r})$. Then for any $u \in B$ there exists some $u_k$ such that

$$\|u - u_k\|_r^r \leqslant \varepsilon^{\delta(q-r)+q}. \tag{6.55}$$

Let

$$D_1 = D\big(|u| \leqslant C_1\varepsilon^{-\delta},\ |u_k| \leqslant C_1\varepsilon^{-\delta}\big), \qquad D_2 = D\big(|u| \geqslant C_1\varepsilon^{-\delta},\ |u_k| \leqslant C_1\varepsilon^{-\delta}\big),$$
$$D_3 = D\big(|u| \leqslant C_1\varepsilon^{-\delta},\ |u_k| \geqslant C_1\varepsilon^{-\delta}\big), \qquad D_4 = D\big(|u| \geqslant C_1\varepsilon^{-\delta},\ |u_k| \geqslant C_1\varepsilon^{-\delta}\big).$$

By (6.55) we have

$$\int_{D_1} |u - u_k|^q \leqslant \big(2C_1\varepsilon^{-\delta}\big)^{q-r} \|u - u_k\|_r^r \leqslant (2C_1)^{q-r}\varepsilon^q.$$

Similar to the proof of (5.8)–(5.10), it follows from (6.49) that

$$\left(\int_{D_2} + \int_{D_3} + \int_{D_4}\right)|u - u_k|^q \leqslant 4 \cdot 2^q \varepsilon^q.$$

Thus $\|u - u_k\|_q \leqslant C_2\varepsilon$ if we choose $C_2$ such that $C_2^q \geqslant (2C_1)^{q-r} + 2^{q+2}$. That is, $B$ has a $C_2\varepsilon$-net in $L^q(D)$, where the number of balls equals to $N_r(\varepsilon^{(\delta(q-r)+q)/r})$, which proved (6.54).

Finally, it follows from (6.54) that

$$\frac{\log N_q(C_2\varepsilon)}{\log C_2 - \log(C_2\varepsilon)} \leqslant \frac{\log N_r(\varepsilon^{(\delta(q-r)+q)/r})}{-\log \varepsilon^{(\delta(q-r)+q)/r}} \cdot \frac{\delta(q-r)+q}{r}.$$

Taking the sup-limit as $\varepsilon \to 0$ we find

$$\dim_q(B) \leqslant \frac{\delta(q-r)+q}{r}\dim_r(B), \quad \forall \delta > 1,\ q \geqslant r > 1.$$

Letting $\delta \to 1$ we obtain

$$\dim_q(B) \leqslant \left(\frac{2q}{r} - 1\right)\dim_r(B), \quad q \geqslant r > 1,$$

which completes the proof of the lemma. □

**Theorem 6.9.** *Let $\mathcal{A}(\omega) = \mathcal{A}_q(\omega)$ be the same random attractor obtained in Theorem* 6.7*. Then $\mathcal{A}(\omega)$ has finite fractal dimensions under $L^q$-norm for any $q \geqslant 2$. Furthermore, we have*

$$\dim_2 \mathcal{A}(\omega) \leqslant \dim_q \mathcal{A}(\omega) \leqslant (q-1)\dim_2 \mathcal{A}(\omega), \quad q \geqslant 2, \tag{6.56}$$

*where $\dim_q \mathcal{A}(\omega)$ denotes the fractal dimension under the $L^q$-norm and $\dim_2 \mathcal{A}(\omega)$ is the usual fractal dimension under the $L^2$-norm.*

**Proof.** Taking $r = 2$ in Lemma 6.8 we obtain (6.56) immediately. But it has been proved by Langa and Robinson [12] that $\dim_2 \mathcal{A}(\omega) < +\infty$, then (6.56) implies that $\dim_q \mathcal{A}(\omega) < +\infty$ for all $q > 2$. $\square$

**Remark 6.10.** Since the Hausdorff dimension is less than the fractal dimension, the attractor $\mathcal{A}(\omega)$ has finite Hausdorff dimensions under any $L^q$-norm ($q \geqslant 2$). It is also possible to derive a similar estimate of (6.56) for the Hausdorff dimensions.

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