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Orders of Absolute Measurability

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A subset A of the torus $[0,1)^k$ is called absolute measurable if the value of $\mu(A)$ is the same for every finitely-additive translation-invariant probability measure μ defined on all subsets of $[0,1)^k$. We define four set functions (called orders) that measure how "strongly" a set A is absolute measurable. The order $o(A)$ equals $k - \dim_B(\partial A)$ and is connected to the Jordan measurability of A. The order $\delta(A)$ measures how small the oscillation of the average of n translates of χ_A can be. The order $\tau(A)$ is related to the absolute inner and outer measures defined by Tarski; finally, $\sigma(A)$ is defined by the oscillation of those functions that are "scissor-congruent" to χ_A. We prove that $o \ll \delta \ll \tau \ll \sigma$, that is, each of the orders $o,\ \delta,\ \tau,\ \sigma$ is "finer" than the previous one. We investigate the connection between the orders and questions of equidecomposability. We show that, under certain conditions, a set of large order is equidecomposable to a cube and present some results in the other direction as well. © 2000 Academic Press

1. ORDERS OF NATURAL INTEGRABILITY

We shall denote by I^k the unit cube

$$\{(t_1, \dots, t_k) : 0 \le t_i < 1\ (i = 1, \dots, k)\}.$$

In this paper we shall identify I^k with the torus $\mathbf{R}^k/\mathbf{Z}^k$; that is, addition will be meant mod 1 in each variable. Thus $I^1 = [0, 1)$ is the circle group. The translation operator will be denoted by T_a; that is, $T_a f(x) = f(x + a)$ for

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every $f : I^k \to \mathbf{R}$ and $a \in I^k$. The oscillation of a bounded function f on a set H is $\omega(f;H) = \sup f(H) - \inf f(H)$. Sometimes we shall write $\omega(f)$ instead of $\omega(f;I^k)$. For every positive integer n we shall denote by $\mathcal{Q}^n$ the set of cubes

$$Q^n_{i_1 \ldots i_k} = \left[\frac{i_1 - 1}{n}, \frac{i_1}{n}\right] \times \cdots \times \left[\frac{i_k - 1}{n}, \frac{i_k}{n}\right] \qquad (i_1, \ldots, i_k = 1, \ldots, n).$$

Let $f : I^k \to \mathbf{R}$ be bounded, and put $\Omega_f(n) = (1/n^k) \sum_Q \omega(f;Q)$, where Q runs through all elements of $\mathcal{Q}^n$. Then f is Riemann integrable over I^k if and only if $\lim_{n\to\infty} \Omega_f(n) = 0$. We shall define the *Riemann order* of f by

$$o(f) = \sup\{\varepsilon \geq 0 : \exists\, K > 0,\ \Omega_f(n) \leq K \cdot n^{-\varepsilon} \qquad (n = 1, 2, \ldots)\}.$$

Then $o(f) \geq 0$ for every bounded f. If $o(f) > 0$ then f is Riemann integrable, but the converse is not true. It is easy to see that $\sum_{Q \in \mathcal{Q}^n} \omega(f;Q) \geq \omega(f)$, and thus $\Omega_f(n) \geq \omega(f)/n^k$. Therefore $0 \leq o(f) \leq k$ holds for every nonconstant bounded f. If f is constant then $o(f) = \infty$.

By a *Banach measure* we shall mean a finitely additive and translation-invariant probability measure defined on $\mathcal{P}(I^k)$, the power set of I^k. Every bounded function can be integrated with respect to a Banach measure (see [12, p. 147]). A bounded function $f : I^k \to \mathbf{R}$ is called *naturally integrable* if the value of $\int_{I^k} f\, d\mu$ is the same for every Banach measure μ (see [1]). Clearly, every Riemann integrable function is naturally integrable. It is well-known that f is naturally integrable if and only if, for every $\varepsilon > 0$, there are vectors $a_1, \ldots, a_n \in I^k$ such that $\omega((T_{a_1} f + \cdots + T_{a_n} f)/n; I^k) < \varepsilon$. Let

$$\Delta_f(n) = \inf\left\{\omega\left(\frac{T_{a_1} f + \ldots + T_{a_n} f}{n}\right) : a_1, \ldots, a_n \in I^k\right\}$$

and

$$\underline{\Delta}_f(n) = \min\{\Delta_f(m) : m \leq n\} \qquad (n = 1, 2, \ldots).$$

Then $\underline{\Delta}_f(n)$ is a decreasing sequence and f is naturally integrable if and only if $\lim_{n\to\infty} \underline{\Delta}_f(n) = 0$. We define the *natural order* of the bounded function f by

$$\delta(f) = \sup\{\varepsilon \geq 0 : \exists\, K > 0,\ \underline{\Delta}_f(n) \leq K \cdot n^{-\varepsilon}\ (n = 1, 2, \ldots)\}.$$

Then $\delta(f) \geq 0$ for every bounded f. If $\delta(f) > 0$ then f is naturally integrable, but the converse is not true.

PROPOSITION 1. *For every bounded f we have $\delta(f) \geq o(f)/k$.*

Proof. Let $N = n^k$, and let $a_1, \dots, a_N$ be an enumeration of the vectors $(i_1/n, \dots, i_k/n)\,(i_1, \dots, i_k = 1, \dots, n)$. If $m_{i_1 \dots i_k}$ and $M_{i_1 \dots i_k}$ denote the infimum and supremum of f on the cube $Q^n_{i_1 \dots i_k}$ then

$$\sum_{i_1, \dots, i_k = 1}^{n} m_{i_1 \dots i_k} \le f(x + a_1) + \cdots + f(x + a_N) \le \sum_{i_1, \dots, i_k = 1}^{n} M_{i_1 \dots i_k}$$

for every $x \in I^k$. Thus

$$\omega\left(T_{a_1} f + \cdots + T_{a_N} f\right) \le \sum_{i_1, \dots, i_k = 1}^{n} (m_{i_1 \dots i_k} - M_{i_1 \dots i_k})$$
$$= \sum_{Q \in \mathbb{Q}^n} \omega(f; Q) = n^k \Omega_f(n)$$

and

$$\omega\left(\frac{T_{a_1} f + \cdots + T_{a_N} f}{N}\right) \le \Omega_f(n). \tag{1}$$

Therefore $\underline{\Delta}_f(n^k) \le \Omega_f(n)$ for every n, from which the statement $\delta(f) \ge o(f)/k$ is obvious. ∎

Let $B(I^k)$ denote the set of bounded, real-valued functions defined on I^k. The functions f and g are called *scissor-congruent* using the translations T_{a_i} $(i = 1, \dots, n)$ if there are functions $f_1, \dots, f_n \in B(I^k)$ such that $f = f_1 + \cdots + f_n$ and $g = T_{a_1} f_1 + \cdots + T_{a_n} f_n$. Dubins and Margolies proved that f is naturally integrable if and only if, for every $\varepsilon > 0$, f is scissor-congruent to a function of oscillation at most ε (see [1, Proposition 2]). If the functions f and g are scissor-congruent using n translations then we shall write $f \overset{s}{\sim}_n g$. Let

$$S_f(n) = \inf\left\{\omega(g) : f \overset{s}{\sim}_n g\right\}.$$

Then the sequence $S_f(n)$ is decreasing, since $f \overset{s}{\sim}_n g$ implies $f \overset{s}{\sim}_{n+1} g$ (add an identically zero extra summand). The bounded function f is naturally integrable if and only if $\lim_{n\to\infty} S_f(n) = 0$. We define the *scissor order* of the bounded function f by

$$\sigma(f) = \sup\{\varepsilon \ge 0 : \exists K > 0,\ S_f(n) \le K \cdot n^{-\varepsilon}\ (n = 1, 2, \dots)\}.$$

Then $\sigma(f) \ge 0$ for every bounded f. If $\sigma(f) > 0$ then f is naturally integrable, but the converse is not true. It is obvious that $S_f(n) \le \underline{\Delta}_f(n)$ for every n, and thus we obtain the following:

PROPOSITION 2. *For every bounded f we have $\sigma(f) \ge \delta(f)$.*

Our next aim is to show that $\sigma(f) = \infty$ for Riemann integrable functions.

LEMMA 1. *Suppose that f and g are scissor-congruent using the translations T_{a_i} $(i = 1, \ldots, n)$. Let G denote the additive group generated by $a_1, \ldots, a_n$, and let $h_1, \ldots, h_s$ be generators of G. Then f and g are also scissor-congruent using the identity and the translations T_{h_j} $(j = 1, \ldots, s)$.*

Proof. We shall write $f \equiv g$ if f and g are scissor-congruent using the identity and the translations T_{h_j} $(j = 1, \ldots, s)$. Obviously, if $f_1 \equiv g_1$ and $f_2 \equiv g_2$, then $f_1 + f_2 \equiv g_1 + g_2$.

Let $H = \{a \in G : f \equiv T_a f$ for every bounded $f\}$. We prove that H is a group. Let $a,\ b \in H$, and let f be a bounded function. Then the functions $f_1 = f,\ f_2 = T_{-b}f,\ f_3 = -T_{-b}f$ are bounded, $f = f_1 + f_2 + f_3$, and $T_{a-b}f = f_1 + T_a f_2 + T_b f_3$. Since $a, b \in H$, we have $f_1 \equiv f_1,\ f_2 \equiv T_a f_2$, and $f_3 \equiv T_b f_3$. Therefore $f \equiv T_{a-b}f$, proving $a - b \in H$. Thus H is a subgroup of G. It is obvious that $h_j \in H$ for every $j = 1, \ldots, s$, and thus $H = G$.

Now suppose that f and g are scissor-congruent using the translations T_{a_i} $(i = 1, \ldots, n)$, and let $f = \sum_{i=1}^n f_i$, $g = \sum_{i=1}^n T_{a_i} f_i$. As we have proved above, $f_i \equiv T_{a_i} f_i$ for every i, and thus $f \equiv g$. ∎

LEMMA 2. *For every bounded f we have $S_f(k+1) \leq \inf\{\Omega_f(n) : n = 1, 2, \ldots\}$.*

Proof. Let n be fixed. It follows from (1) that there is a function g such that $\omega(g) \leq \Omega_f(n)$ and f is scissor-congruent to g using translations T_{a_i}, where each a_i belongs to a group G that is generated by k vectors. By Lemma 1, $f \overset{s}{\sim}_{k+1} g$. Therefore $S_f(k+1) \leq \Omega_f(n)$ for every n, which proves the statement of the lemma. ∎

LEMMA 3. *If $f \overset{s}{\sim}_n g$ and $g \overset{s}{\sim}_d h$, then $f \overset{s}{\sim}_m h$, where $m = n + d + 1$.*

Proof. Suppose $f = \sum_{i=1}^n f_i$, $g = \sum_{i=1}^n T_{a_i} f_i$, $g = \sum_{j=1}^d g_j$, $h = \sum_{j=1}^d T_{b_j} g_i$. Then we have $f = \sum_{i=1}^n f_i + (-g) + \sum_{j=1}^d g_j$ and $h = \sum_{i=1}^n T_{a_i} f_i + (-g) + \sum_{j=1}^d T_{b_j} g_j$. ∎

PROPOSITION 3. (i) *If f is Riemann integrable then $\sigma(f) = \infty$; moreover, $S_f(n) = 0$ for every $n \geq k+1$.*

(ii) *If f and g are scissor-congruent then $\sigma(f) = \sigma(g)$.*

Proof. (i) is immediate from Lemma 2. If $f \overset{s}{\sim}_d g$, then it follows from Lemma 3 that $S_f(n+d+1) \leq S_g(n)$ and $S_g(n+d+1) \leq S_f(n)$ for every n, which easily implies (ii). ∎

2. ORDERS OF ABSOLUTE MEASURABILITY

A set $A \subset I^k$ is Jordan measurable if and only if χ_A is Riemann integrable. The number $o(A) \stackrel{\text{def}}{=} o(\chi_A)$ will be called the *Jordan order* of A. Then $o(A) \geq 0$ for every $A \subset I^k$. If $o(A) > 0$ then A is Jordan measurable, but the converse is not true. Since χ_A is constant if and only if $A = \emptyset$ or $A = I^k$, we find that $0 \leq o(A) \leq k$ unless $A = \emptyset$ or $A = I^k$. For these two exceptions we have $o(A) = \infty$. In the following we shall denote by $\dim_B(H)$ the box dimension of the set $H \subset I^k$.

PROPOSITION 4. *For every $A \subset I^k$, $A \neq \emptyset$, $A \neq I^k$ we have $o(A) = k - \dim_B(\partial A)$.*

Proof. Let $\eta < k - \dim_B(\partial A)$. Then $\dim_B(\partial A) < k - \eta$, and thus for every n large enough, the number of cubes $Q^n_{i_1 \ldots i_k}$ intersecting ∂A is less than $n^{k-\eta}$. If $Q \subset I^k$ is a cube and $Q \cap \partial A = \emptyset$ then either $Q \subset A$ or $Q \cap A = \emptyset$, and thus $\omega(\chi_A; Q) = 0$. Therefore we have $\Omega_{\chi_A}(n) \leq n^{k-\eta}/n^k = n^{-\eta}$ for every n large enough, and thus $o(A) = o(\chi_A) \geq \eta$. Since this is true for every $\eta < k - \dim_B(\partial A)$, we have $o(A) \geq k - \dim_B(\partial A)$.

To prove the other inequality, let $\eta < o(A)$. Since $A \neq \emptyset$ and $A \neq I^k$, we have $o(f) \leq k$ and thus $\eta < k$. There is a $K > 0$ such that $\Omega_{\chi_A}(n) \leq K \cdot n^{-\eta}$ for every n. Thus the number of cubes $Q \in \mathcal{Q}^n$ satisfying $\omega(\chi_A; Q) = 1$ is at most $K \cdot n^{k-\eta}$. It is clear that ∂A is covered by these cubes. Since every cube $Q \in \mathcal{Q}^n$ intersects at most 3^k elements of $\mathcal{Q}^n$, it follows that ∂A intersects at most $3^k K \cdot n^{k-\eta}$ elements of $\mathcal{Q}^n$. Therefore $\dim_B(\partial A) \leq k - \eta$ and $k - \dim_B(\partial A) \geq \eta$ for every $\eta < o(A)$; that is, $k - \dim_B(\partial A) \geq o(A)$. ∎

If μ is a Banach measure then $\int_{I^k} \chi_A d\mu = \mu(A)$. Thus χ_A is naturally integrable if and only if $\mu(A)$ has the same value for every Banach measure μ. These sets are called *absolute measurable* by Tarski [11].

For this reason we shall call the quantity $\delta(A) = \delta(\chi_A)$ the *absolute order* of the set A. Then $\delta(A) \geq 0$ for every $A \subset I^k$. If $\delta(A) > 0$ then A is absolute measurable, but the converse is not true. The set A is called *exceptionally absolute measurable* if $\delta(A) = \infty$.

PROPOSITION 5. *If $A \subset I^k$ is not exceptionally absolute measurable then* $0 \leq \delta(A) \leq 1$. *The set $A \subset I^k$ is exceptionally absolute measurable if and only if $T_{a_1}\chi_A + \cdots + T_{a_n}\chi_A$ is constant for some $a_1, \ldots, a_n \in I^k$. A closed set is exceptionally absolute measurable if and only if it is empty or equals I^k.*

Proof. Since $\Delta_{\chi_A}(n)$ is an integer multiple of $1/n$, it follows that either $\Delta_{\chi_A}(n) = 0$ for some n or $\Delta_{\chi_A}(n) \geq 1/n$ for every n. In the first case

$\omega(T_{a_1}\chi_A + \cdots + T_{a_n}\chi_A) = 0$ for some $a_1, \ldots, a_n \in I^k$, while in the second case we have $\underline{\Delta}_{\chi_A}(t) \geq 1/t$ for every $t \geq 1$. This proves the first two statements of the proposition.

Let A be a closed and exceptionally absolute measurable set, and let $T_{a_1}\chi_A + \cdots + T_{a_n}\chi_A$ be constant. Then the functions $T_{a_i}\chi_A$ are upper semicontinuous, and their sum is continuous. Therefore each function $T_{a_i}\chi_A$ is continuous, and thus either $A = \emptyset$ or $A = I^k$. ∎

We shall say that the sets $A, B \subset I^k$ are *equivalent* if they are equidecomposable using translations; that is, if there are finite decompositions $A = \cup_{j=1}^d A_j$, $B = \cup_{j=1}^d B_j$ and vectors $x_1, \ldots, x_d \in I^k$ such that $B_j = A_j + x_j$ $(j = 1, \ldots, d)$. We shall denote this fact by $A \overset{\mathrm{T}}{\sim} B$. If we want to indicate that A and B are equivalent using d pieces, then we write $A \overset{\mathrm{T}}{\sim}_d B$.

Tarski [11] introduced the notion of absolute inner and outer measures as follows. The *absolute inner measure*, $\underline{\mu}_{\mathrm{a}}(A)$, of a set $A \subset I^k$ is defined as the supremum of the Lebesgue measure of those cubes Q which are equivalent to a subset of A. The *absolute outer measure*, $\overline{\mu}_{\mathrm{a}}(A)$, of $A \subset I^k$ is defined as the infimum of the Lebesgue measure of those cubes Q that contain a subset equivalent to A. Then $\underline{\mu}_{\mathrm{a}}(A) \leq \overline{\mu}_{\mathrm{a}}(A)$ and, as Tarski proves in [11], A is absolute measurable if and only if $\underline{\mu}_{\mathrm{a}}(A) = \overline{\mu}_a(A)$. This fact motivates the following definition. Let

$$T_A(n) = \inf\{\lambda_k(Q_2) - \lambda_k(Q_1) : Q_1 \text{ and } Q_2 \text{ are cubes such that } Q_1 \overset{\mathrm{T}}{\sim}_n B \text{ and } A \overset{\mathrm{T}}{\sim}_n C \text{ for some } B \subset A \text{ and } C \subset Q_2\}.$$

Then the sequence $T_A(n)$ is decreasing, and A is absolute measurable if and only if $\lim_{n\to\infty} T_A(n) = 0$. We define the *Tarski order* of the set A by

$$\tau(A) = \sup\{\varepsilon \geq 0 : \exists K > 0, T_A(n) \leq K \cdot n^{-\varepsilon}\ (n = 1, 2, \ldots)\}.$$

We have $\tau(A) \geq 0$ for every $A \subset I^k$. If $\tau(A) > 0$ then A is absolute measurable, but the converse is not true. If A is equivalent to a cube using d pieces then $T_A(n) = 0$ for every $n \geq d$, and thus $\tau(A) = \infty$.

Finally, we put $\sigma(A) = \sigma(\chi_A)$ for every $A \subset I^k$. We have $\sigma(A) \geq 0$ for every $A \subset I^k$. If $\sigma(A) > 0$ then A is absolute measurable, but the converse is not true.

We regard the set functions o, δ, τ, σ as different orders of absolute measurability. The following theorem describes the relations among them. We shall write $\alpha \ll \beta$ if there is a positive constant c only depending on k such that $\alpha \leq c \cdot \beta$.

THEOREM 1. *For every $A \subset I^k$ we have $o(A) \ll \delta(A) \ll \tau(A) \ll \sigma(A)$.*

By Propositions 1 and 2, we only have to prove $\delta(A) \ll \tau(A)$ and $\tau(A) \ll \sigma(A)$. We shall prove these statements in Section 5.

Next we show by examples that the orders defined above are distinct.

1. EXAMPLES WITH $0 = o(A) < \delta(A)$. Let $k = 1$. If $H \subset [0, 1/2)$ is arbitrary then $A = ([0, 1/2) \setminus H) \cup (H + (1/2))$ is exceptionally absolute measurable by Proposition 5. If H is not Jordan measurable then we have $\delta(A) = \infty$ and $o(A) = 0$.

To take a simpler example, let A be a countable dense set. Since A has n disjoint translates for every n, it follows that $\delta(A) = 1$. On the other hand, as A is not Jordan measurable, we have $o(A) = 0$.

There are closed sets satisfying $\delta(A) > 0$ and $o(A) = 0$. For $k = 2$ the graph of a typical continuous function has this property. Indeed, the graph of any function has n disjoint (vertical) translates for every n, hence its absolute order is 1. On the other hand, by a theorem of Humke and Petruska [3], the box dimension of the graph of a typical continuous function is 2, and thus its Jordan order is zero by Proposition 4.

The graph of a continuous function is always Jordan measurable. But there are closed sets of positive absolute order which are *not* Jordan measurable: such a set is constructed in [7, Theorem 4.1].

EXAMPLE 2. If Q is a cube then $o(Q) = \delta(Q) < \tau(Q)$. Indeed, $o(Q) = 1$ by Proposition 4, and we shall see in the next section that $\delta(Q) = 1$ is also true. On the other hand, it is obvious that $\tau(A) = \infty$.

3. AN EXAMPLE WITH $\max(o(A), \delta(A), \tau(A)) < \sigma(A) = \infty$.

We shall construct a Jordan-measurable closed set $A \subset [0, 1)$ with $\tau(A) \le 1$. Then Proposition 3 gives $\sigma(A) = \infty$, while Propositions 1 and 5 yield $o(A) \le \delta(A) \le 1$.

We shall define A as $\cap_{i=1}^{\infty} P_i$, where each P_i is the union of finitely many closed intervals, $P_1 \supset P_2 \supset \cdots$, and $\lambda(P_i) \to 0$. Let P_1 be a closed interval, $F_1 = \emptyset$, and put $n_1 = 1$. Let $i > 1$ and suppose that for every $1 \le j < i$ we have defined the set P_j, the finite set $F_j \subset P_j$, and the positive integer n_j in such a way that $P_1 \supset \cdots \supset P_{i-1}$ and $\cup_{j=1}^{i-1} F_j \subset P_{i-1}$.

Let J_{i-1} be a closed interval contained in P_{i-1}. Let $n_i > |J_{i-1}|^{-i}$, and let $F_i \subset J_{i-1}$ be a finite set such that whenever we partition F_i into n_i subsets then one of these subsets have diameter $> |J_{i-1}|/(2n_i)$. Let P_i be the union of finitely many closed intervals such that $\cup_{j=1}^{i} F_j \subset P_i \subset P_{i-1}$ and $\lambda(P_i) \le 1/i$.

In this way we defined P_i, F_i, and n_i for every i. Let $A = \cap_{i=1}^{\infty} P_i$. Since $\lambda(P_i) \to 0$, it follows that A is a Jordan measurable closed set with $\lambda(A) = 0$. We prove that $\tau(A) \le 1$. Let $\eta < \tau(A)$. Then $T_A(n) \le K \cdot n^{-\eta}$ for every n. In particular, $T_A(n_i) \le K \cdot n_i^{-\eta}$ for every i, and thus A is equivalent to a subset of an interval of length $K \cdot n_i^{-\eta}$ using n_i pieces. However,

$A \supset F_i$, and thus F_i itself is equivalent to a subset of an interval of length $K \cdot n_i^{-\eta}$ using n_i pieces. By the choice of F_i, one of the pieces used in this decomposition has diameter $> |J_{i-1}|/(2n_i)$, and thus we have

$$|J_{i-1}|/(2n_i) < K \cdot n_i^{-\eta}$$

for every i. Since $n_i > |J_{i-1}|^{-i}$ we obtain

$$n_i^{-1/i} < |J_{i-1}| < 2K \cdot n_i^{1-\eta}$$

for every i. Therefore $-1/i \leq 1 - \eta$ for every i; that is, $\eta \leq 1$. Since this is true for every $\eta < \tau(A)$, we have $\tau(A) \leq 1$.

We remark that a similar but slightly more complicated construction gives a closed set A with $o(A) = \delta(A) = \tau(A) = 0$ and $\sigma(A) = \infty$.

3. DISCREPANCY AND THE ORDER δ

If $F \subset I^k$ is a finite set, $|F| = N$, and $H \subset I^k$ is Lebesgue measurable, then the discrepancy of F with respect to H is defined as

$$D(F; H) = \left| \frac{1}{N} |F \cap H| - \lambda_k(H) \right|.$$

The (absolute) discrepancy of the finite set $F \subset I^k$ is defined as

$$D(F) = \sup_J D(F; J),$$

where the sup is taken over all subintervals $J \subset I^k$. Let μ be a fixed Banach measure. We define

$$D_\mu(F; H) = \left| \frac{|F \cap H|}{|F|} - \mu(H) \right|$$

for every nonempty finite $F \subset I^k$ and for every $A \subset I^k$. Clearly, if H is absolute measurable then $D_\mu(F; H)$ is independent of μ, and if H is Jordan measurable then $D_\mu(F; H) = D(F; H)$. We shall also use the notation $D_\mu(F; H)$ when F is a finite *distribution* of points in I^k; that is, when F is a finite multiset. Let $F = \{a_1, \ldots, a_n\}$, and let $A \subset I^k$ be arbitrary. Then

$$\frac{1}{n}[\chi_A(x + a_1) + \cdots + \chi_A(x + a_n)] = \frac{1}{n} |F \cap (A - x)| \stackrel{\text{def}}{=} g(x). \qquad (2)$$

Therefore $|g(x) - \mu(A)| = D_\mu(F; A - x)$, and thus

$$\omega\left(\frac{T_{a_1}\chi_A + \ldots + T_{a_n}\chi_A}{n} \right) \leq 2 \cdot \sup_x D_\mu(F; A - x). \qquad (3)$$

On the other hand, it follows from (2) that

$$\int_{I^k} \frac{1}{n} |F \cap (A - x)| \, d\mu(x) = \mu(A),$$

and hence $m = \inf g \leq \mu(A) \leq \sup g = M$. Then

$$\frac{1}{n} |F \cap (A - x)| - \mu(A) = g(x) - \mu(A) \leq M - m$$

for every x and, similarly, $(1/n)|F \cap (A - x)| - \mu(A) \geq m - M$. Therefore $D_\mu(F; A - x) \leq M - m = \omega(g; I^k)$ for every $x \in I^k$, and we have, taking (3) into consideration,

$$\sup_x D_\mu(F; A - x) \leq \omega\left(\frac{T_{a_1}\chi_A + \ldots + T_{a_n}\chi_A}{n}\right) \leq 2 \cdot \sup_x D_\mu(F; A - x).$$

This gives the following description of $\delta(A)$.

Proposition 6. *For every Banach measure μ, $\delta(A)$ equals the supremum of those numbers $\varepsilon \geq 0$ for which there is a positive constant K such that*

$$\inf_{|F| \leq n} \sup_x D_\mu(F; A - x) \leq K \cdot n^{-\varepsilon} \qquad (n = 1, 2, \ldots).$$

It is well known that for every $\varepsilon > 0$ there exists a constant $K > 0$ depending only on ε and k such that for every N there is a finite set $F_N \subset I^k$ with $|F_N| = N$ and $D(F_N) \leq K \cdot N^{-1+\varepsilon}$ (see [4, Theorem 5.7, p. 154]). Combining this with (3) we obtain the following.

Lemma 4. *For every $\varepsilon > 0$ there exists a constant $K > 0$ depending only on ε and k such that the following statement is true: For every positive integer N there are points $a_1, \ldots, a_N \in I^k$ such that*

$$\omega\left(\frac{1}{N} \sum_{j=1}^{N} T(a_j)\chi_Q\right) \leq K \cdot N^{\varepsilon - 1}$$

for every cube $Q \subset I^k$.

As an immediate corollary we find that $\delta(Q) = 1$ for every cube $Q \subset I^k$, except $Q = I^k$.

4. EQUIDECOMPOSABILITY AND THE ORDER δ

In this section we shall investigate the relation between the condition $\delta(A) > 0$ and the equivalence of A with a cube. In [6] we proved that if $A \subset I^k$, $\lambda_k(A) > 0$, and $o(A) > 0$, then A is equivalent to a cube. The converse is not true, even if A is closed. It is shown in [7, Theorem 4.1] that there is a closed set $A \subset [0, 1)$ such that A is equivalent to an interval, but A is not Jordan measurable and, consequently, $o(A) = 0$. However, as the following theorem shows, equivalence with a cube implies $\delta(A) > 0$.

THEOREM 2. (i) *Suppose that* $\delta(A) < \infty$ *and* $\delta(B) < \infty$. *If* $A \overset{\mathrm{T}}{\sim}_d B$, *then* $\delta(A) \geq \delta(B)/(d+1)$ *and* $\delta(B) \geq \delta(A)/(d+1)$.

(ii) *If* $A \subset I^k$ *is equivalent to a cube, then* $\delta(A) > 0$. *More precisely, if* $A \subset I^k$ *is equivalent to a cube using d pieces, then* $\delta(A) \geq 1/(d+1)$.

Proof. Since $\delta(Q) \geq 1$ for every cube, it is enough to prove the first statement. Suppose that A is equivalent to B using d pieces. Then there are decompositions $A = \cup_{i=1}^d A_i$, $B = \cup_{i=1}^d B_i$, and vectors $x_1, \ldots, x_d \in I^k$ such that $B_i = A_i + x_i$ $(i = 1, \ldots, d)$.

In the following we shall write $T(a)$ instead of T_a. In the proof of $\delta(A) \geq \delta(B)/(d+1)$ we may assume $\delta(B) > 0$. Let $0 < \eta < \delta(B)$ be fixed, then we have

$$\underline{\Delta}_{\chi_B}(n) \leq K \cdot n^{-\eta}$$

for every n. We put $\alpha = \eta/(1 + d\eta)$ and $\beta = 1/(1 + d\eta)$. Let $n > 2^{1/\alpha}$ be a fixed integer. There is an $m \leq n^\beta$ such that

$$\Delta_{\chi_B}(m) = \underline{\Delta}_{\chi_B}([n^\beta]) \leq K \cdot [n^\beta]^{-\eta} < 2K \cdot n^{-\beta\eta}.$$

Then there are vectors $c_1, \ldots, c_m$ such that

$$\omega\left(\frac{1}{m}\sum_{i=1}^m T(c_i)\chi_B\right) < 2K \cdot n^{-\beta\eta}. \tag{4}$$

Let $p = [n^\alpha]$, and let $v_1, \ldots, v_q$ be an enumeration of the vectors $i_1 x_1 + \cdots + i_d x_d$, where $i_j = 0, 1, \ldots, p-1$ $(j = 1, \ldots, d)$. Then $q = p^d \leq n^{d\alpha}$ and hence the number of vectors $c_i + v_j$ $(i = 1, \ldots, m;\ j = 1, \ldots, q)$ is $mq \leq n^\beta \cdot n^{d\alpha} = n$. Therefore

$$\underline{\Delta}_{\chi_A}(n) \leq \Delta_{\chi_A}(mq) \leq \frac{1}{mq}\omega\left(\sum_{i=1}^m\sum_{j=1}^q T(c_i + v_j)\chi_A\right) = \frac{1}{mq}\omega\left(\sum_{i=1}^m T(c_i)f\right), \tag{5}$$

where $f = \sum_{j=1}^{q} T(v_j)\chi_A$. Let $g = \sum_{j=1}^{q} T(v_j)\chi_B$. Since $f = \sum_{j=1}^{q} \sum_{i=1}^{d} T(v_j)\chi_{A_i}$ and $g = \sum_{j=1}^{q} \sum_{i=1}^{d} T(v_j)\chi_{A_i+x_i}$, we have $|g - f| \le 2d \cdot p^{d-1}$, due to the cancellations of the common terms of f and g. Thus

$$\omega\left(\sum_{i=1}^{m} T(c_i)f\right) \le \omega\left(\sum_{i=1}^{m} T(c_i)g\right) + 4dp^{d-1}m. \tag{6}$$

On the other hand, we have

$$\omega\left(\sum_{i=1}^{m} T(c_i)g\right) = \omega\left(\sum_{j=1}^{q} T(v_j) \sum_{i=1}^{m} T(c_i)\chi_B\right) \le qm \cdot 2K \cdot n^{-\beta\eta}$$

by (4). Therefore (5) and (6) give

$$\underline{\Delta}_{\chi_A}(n) \le 2K \cdot n^{-\beta\eta} + (4d/p) < (2K + 8d) \cdot n^{-\alpha}.$$

Since n was arbitrary, we obtain

$$\delta(A) \ge \alpha = \eta/(1 + d\eta) \ge \eta/(d + 1)$$

for every $\eta < \delta(B)$. Then $\delta(A) \ge \delta(B)/(d + 1)$ and, by symmetry, we also have $\delta(B) \ge \delta(A)/(d + 1)$. ∎

Most probably, the converse of statement (ii) of Theorem 2 is not true; that is, $\delta(A) > 0$ does not imply that A is equivalent to a cube. Indeed, if A is equivalent to a cube then, by Theorem 2.1 of [7], there are vectors $x_1, \ldots, x_d \in I^k$ and there is a positive constant K such that

$$\omega\left(\frac{1}{N^d} \sum_{n_1,\ldots,n_d=0}^{N-1} T(n_1x_1 + \ldots + n_dx_d)\chi_A\right) \le \frac{K}{N}$$

for every N, and this condition is (probably) much stronger than $\delta(A) > 0$. We shall prove, however, that if $\delta(A) = \infty$ then A is equivalent to a cube.

LEMMA 5. *Let $A, B \subset I^k$, and suppose that there are vectors $a_1, \ldots, a_n$, $b_1, \ldots, b_m \in I^k$ and a positive real number c such that*

$$\frac{T_{a_1}\chi_A + \cdots + T_{a_n}\chi_A}{n} \le c \le \frac{T_{b_1}\chi_B + \cdots + T_{b_m}\chi_B}{m}$$

everywhere on I^k. Then A is equivalent to a subset of B using $n \cdot m$ pieces.

Proof. First we show that if $X \subset A$ is a finite set and $Y = \{x - a_i : x \in X,\ i = 1, \ldots, n\}$ then $|Y| \ge |X|/c$. Indeed, if $y \in Y$ then $y + a_i \in X \subset A$ for at least one i, and thus $\sum_{i=1}^{n} T_{a_i}\chi_A(y) > 0$. Since, by assumption, the value of this sum is at most cn, we can see that each element of Y is listed at most cn times in Y, and thus $|Y| \ge (|X| \cdot n)/(cn) = |X|/c$.

Next we show that if $Y \subset I^k$ is a finite set and $Z = B \cap \{y + b_j : y \in Y,\ j = 1, \dots, m\}$, then $|Z| \geq c \cdot |Y|$. Indeed, it follows from $\sum_{j=1}^{m} T_{b_j}\chi_B \geq c \cdot m$ that for every $y \in Y$ there are at least $c \cdot m$ indices j such that $y + b_j \in B$. Since every $z \in Z$ can be written in the form $y + b_j$ in at most m ways, we have $|Z| \geq (|Y| \cdot cm)/m = c \cdot |Y|$.

Now we define a bipartite graph on the sets A and B as follows. We join the points $x \in A$ and $y \in B$ by an edge if $x - a_i + b_j = y$ for some $1 \leq i \leq n,\ 1 \leq j \leq m$. If $X \subset A$ is a finite set and $Y = \{x - a_i : x \in X,\ i = 1, \dots, n\}$, then each point of $Z = B \cap \{y + b_j : y \in Y,\ j = 1, \dots, m\}$ is connected to at least one point of X by an edge. Since $|Z| \geq c \cdot |Y| \geq |X|$, it follows that the condition of the marriage lemma is satisfied (see [2, p. 248]). Therefore there is an injective function $f\colon A \to B$ such that for every $x \in A$ there are indices $1 \leq i \leq n,\ 1 \leq j \leq m$ with $f(x) = x - a_i + b_j$. We obtain that A is equidecomposable to a subset of B using the translations $x - a_i + b_j$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$. ∎

THEOREM 3. *If A is a nonempty exceptionally absolute measurable set, then A is equivalent to a cube having rational volume.*

Proof. Let A be exceptionally absolute measurable. By Proposition 5, there are vectors $a_1, \dots, a_n \in I^k$ such that $T_{a_1}\chi_A + \cdots + T_{a_n}\chi_A$ is constant. Obviously, the value of this constant must be a positive integer $m \leq n$. Let

$$J = \left[0, \frac{m}{n}\right) \times [0, 1) \times \cdots \times [0, 1).$$

If $b_j = (j/n, 0, \dots, 0)$ $(j = 1, \dots, n)$ then $T_{b_1}\chi_J + \cdots + T_{b_n}\chi_J \equiv m$. Therefore, by Lemma 5, A is equivalent to a subset of J and J is equivalent to a subset of A. Then A and J are equivalent to each other by [12, Theorem 3.5]. Since J is equivalent to a cube of the same volume by [6], the proof is finished. ∎

The converse of Theorem 3 is not true, as the following simple example shows. Let $k = 1$, let $u \in [0, 1/2)$ be an irrational number, and put $A = [0, u) \cup [1/2, 1 - u)$. Then $A \overset{\mathrm{T}}{\sim} [0, 1/2)$. The fact that A is not exceptionally absolute measurable is an easy consequence of the following theorem[1].

THEOREM 4. *Let $A \subset I^1$ be a Lebesgue measurable and exceptionally absolute measurable set. Then there is a positive integer N such that $\sum_{i=0}^{N-1} T_{i/N}(\chi_A)$ is constant almost everywhere.*

If the set $A = [0, u) \cup [1/2, 1 - u)$ was exceptionally absolute measurable then, by Theorem 4, the function $f = \sum_{i=0}^{N-1} T_{i/N}(\chi_A)$ would be constant a.e. for some N. However, from the irrationality of u it follows that each

[1] *Note added in proof.* Some closely related results can be found in the paper by M. N. Kolountzakis and J. C. Lagarias, Structure of tilings of the line by a function, *Duke Math. J.* **82** (1996), 653–678.

term of the sum $\sum_{i=1}^{N-1} T_{i/N}(\chi_A)$ is constant in a small neighbourhood of u. Therefore $f(x) = \chi_{[0,u)}(x) + c$ in a small neighbourhood of u, and thus f cannot be constant a.e.

Let μ be a Banach measure on I^1. If $f \in B(I^1)$ then we define the Fourier series of f with respect to μ as

$$\sum_{n\in\mathbf{Z}} c_n e^{2\pi inx}$$

where

$$c_n = \int_0^1 f(x)e^{-2\pi inx}\,d\mu(x).$$

It follows from the translation invariance of μ that if the Fourier series of f is $\sum_n c_n e^{2\pi inx}$ then the Fourier series of $T_a f$ is $\sum_n c_n e^{2\pi ina} e^{2\pi inx}$.

We shall say that the function $f \in B(I^1)$ is *virtually zero* if, for every Banach measure μ, the Fourier coefficients of f with respect to μ are all zero. We shall prove the following.

THEOREM 5. *Let f be a bounded function defined on I^1, and suppose that the sum of finitely many translates of f is virtually zero. Then there is a positive integer N such that $\sum_{i=0}^{N-1} T_{i/N} f$ is virtually zero.*

If A is Lebesgue measurable and exceptionally absolute measurable then, by Proposition 5, there are numbers $a_1, \dots, a_n \in [0,1)$ such that $T_{a_1}\chi_A + \cdots + T_{a_n}\chi_A \equiv c$. Then $T_{a_1} f + \cdots + T_{a_n} f = 0$, where $f = \chi_A - (c/n)$. By Theorem 5 this implies that $\sum_{i=0}^{N-1} T_{i/N} f$ is virtually zero for some positive integer N. It is clear that if a bounded Lebesgue measurable function is virtually zero then it is zero a.e. Thus $\sum_{i=0}^{N-1} T_{i/N} f$ is zero a.e. and $\sum_{i=0}^{N-1} \chi_A = c$ a.e., which proves Theorem 4.

Proof of Theorem 5. Let $f \in B(I^1)$ be bounded, and suppose that $\sum_{j=1}^{s} T_{a_j} f$ is virtually zero for some $a_1, \dots, a_s \in [0,1)$.

If some of the numbers $a_i - a_j$ ($1 \le i < j \le s$) are rational, then we shall denote by M the common denominator of these rational numbers. If each $a_i - a_j$ is irrational then we put $M = 1$. We claim that $g = \sum_{k=0}^{M-1} T_{k/M} f$ is a *virtual trigonometric polynomial* in the sense that all but finitely many of g's Fourier coefficients are zero with respect to any Banach measure.

If $a_i - a_j$ is rational for some $i \ne j$ then, as M is a multiple of the denominator of $a_i - a_j$, it follows that

$$\{a_i, a_i + (1/M), \dots, a_i + ((M-1)/M)\}$$
$$= \{a_j, a_j + (1/M), \dots, a_j + ((M-1)/M)\},$$

and thus

$$T_{a_i} g = \sum_{k=0}^{M-1} T_{k/M} T_{a_i} f = \sum_{k=0}^{M-1} T_{k/M} T_{a_j} f = T_{a_j} g.$$

Therefore $G = \sum_{j=1}^{s} T_{a_j} g$ can be written in the form $\sum_{k=1}^{t} m_k T_{b_k} g$, where m_k is a positive integer for every $k = 1, \dots, t$, and the difference of any two of the numbers $b_1, \dots, b_t$ is irrational.

Let $\sum_n d_n e^{2\pi i n x}$ be the Fourier series of g with respect to a Banach measure μ. Then the Fourier series of $G = \sum_{k=1}^{t} m_k T_{b_k} g$ is $\sum_n e_n e^{2\pi i n x}$, where

$$e_n = d_n \left[m_1 e^{2\pi i n b_1} + \dots + m_t e^{2\pi i n b_t} \right]$$

for every $n \in \mathbf{Z}$. Now G is virtually zero, since $G = \sum_{j=1}^{s} T_{a_j} g = \sum_{j=1}^{s} \sum_{k=1}^{M-1} T_{a_j} T_{k/M} f$. Therefore $e_n = 0$ for every n. Since $b_i - b_j$ is irrational if $i \neq j$, it follows that $e^{2\pi i n b_i} / e^{2\pi i n b_j}$ is not a root of unity for every $i \neq j$. Then, by the Lech–Mahler theorem, $m_1 e^{2\pi i n b_1} + \dots + m_t e^{2\pi i n b_t} \neq 0$ for every n such that $|n| > n_0$ (see [9] or [10]). Therefore $d_n = 0$ whenever $|n| > n_0$.

Let $P > n_0$ be an integer prime to M, and let $h = \sum_{j=0}^{P-1} T_{j/P} g$. It is easy to check that the Fourier series of h is $\sum_n u_n e^{2\pi i n x}$, where $u_n = 0$ if n is not a multiple of P and $u_n = P \cdot d_n$ if n is a multiple of P. Since $d_n = 0$ for every $|n| > n_0$, it follows that $u_n = 0$ for every $n \neq 0$. Let $N = PM$. It is easy to see that

$$h = \sum_{j=0}^{P-1} \sum_{i=0}^{M-1} T_{j/P} T_{i/M} f = \sum_{k=0}^{N-1} T_{k/N} f.$$

This representation shows that the constant term of the Fourier series of h is also zero, and thus h is virtually zero. ∎

5. EQUIDECOMPOSABILITY AND THE ORDERS τ AND σ

Our first aim is to prove Theorem 1.

THEOREM 6. *We have $\delta(A) \leq 2\tau(A)$ for every $A \subset I^k$.*

Proof. We may assume $\delta(A) > 0$. If $\delta(A) = \infty$ then, by Theorem 3, A is equivalent to a cube and thus $\tau(A) = \infty$. Therefore we may assume $0 < \delta(A) \leq 1$.

Let $0 < \eta < \delta(A)$ and $\varepsilon > 0$ be arbitrary. Let $K > 0$ be a constant such that $\underline{\Delta}_{\chi_A}(n) \leq K \cdot n^{-\eta}$ for every n. We may assume that K also satisfies the requirement of Lemma 4. Let n be fixed, and select an $m \leq n$ with

$$\Delta_{\chi_A}(m) = \underline{\Delta}_{\chi_A}(n) < K \cdot n^{-\eta}.$$

Then there are points $a_1, \dots, a_m \in I^k$ such that

$$\omega\left(\frac{T_{a_1} \chi_A + \dots + T_{a_m} \chi_A}{m} \right) < K \cdot n^{-\eta}.$$

It follows that if μ is a Banach measure, then

$$\mu(A) - K \cdot n^{-\eta} < \frac{T_{a_1}\chi_A + \cdots + T_{a_m}\chi_A}{m} < \mu(A) + K \cdot n^{-\eta}.$$

Let Q_1 be a cube with $\lambda_k(Q_1) = \mu(A) - 2K \cdot n^{\varepsilon-\eta}$. (If $\mu(A) - 2K \cdot n^{\varepsilon-\eta} \leq 0$ then let $Q_1 = \emptyset$.) By Lemma 4, there are vectors $b_1, \ldots, b_n$ such that

$$\omega\left(\frac{1}{n}\sum_{i=1}^{n} T_{b_i}\chi_{Q_1}\right) < K \cdot n^{\varepsilon-1} < K \cdot n^{\varepsilon-\eta}.$$

Then we have

$$\frac{1}{n}\sum_{i=1}^{n} T_{b_i}\chi_{Q_1} < \lambda_k(Q_1) + K \cdot n^{\varepsilon-\eta}$$
$$= \mu(A) - K \cdot n^{\varepsilon-\eta} < \mu(A) - K \cdot n^{-\eta} < \frac{1}{m}\sum_{i=1}^{m} T_{a_i}\chi_A.$$

Therefore, by Lemma 5, Q_1 is equivalent to a subset of A using $n \cdot m \leq n^2$ pieces. A similar argument shows that A is equivalent to a subset of a cube Q_2 using $n \cdot m \leq n^2$ pieces, where $\lambda_k(Q_2) = \mu(A) + 2K \cdot n^{\varepsilon-\eta}$. Thus $T_A(n^2) \leq 4K \cdot n^{\varepsilon-\eta}$ for every n. Since $\eta < \delta(A)$ and $\varepsilon > 0$ were arbitrary, it follows that $\tau(A) \geq \delta(A)/2$. ∎

LEMMA 6. *We have $S_{\chi_A}(3n + 3k + 4) \leq T_A(n)$ for every $A \subset I^k$ and for every positive integer n.*

Proof. Let $\varepsilon > 0$ be given, and let Q_1' and Q_2 be cubes such that $Q_1' \overset{\mathrm{T}}{\sim}_n B \subset A$, $A \overset{\mathrm{T}}{\sim}_n C \subset Q_2$, and

$$\lambda_k(Q_2) - \lambda_k(Q_1') < T_A(n) + \varepsilon.$$

We may assume that $Q_1' \subset Q_2$ and that the coordinates of the vertices of Q_2 and Q_1' are rational. We choose a cube Q_1 such that $Q_1 \subset Q_1'$, $\lambda_k(Q_1) < \lambda_k(Q_1')$, the coordinates of the vertices of Q_1 are rational, and

$$\lambda_k(Q_2) - \lambda_k(Q_1) < T_A(n) + \varepsilon.$$

There is a positive integer N such that each of the cubes Q_1, Q_2, and Q_1' is the union of a subset of $\mathcal{Q}^N$.

Suppose that the equivalence $Q_1' \overset{\mathrm{T}}{\sim}_n B$ involves the translations by $a_1, \ldots, a_n$, and the equivalence $A \overset{\mathrm{T}}{\sim}_n C$ involves the translations by $b_1, \ldots, b_n$. Let G denote the group generated by a_i, b_i $(i = 1, \ldots, n)$ and by the vectors $(1/N, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1/N)$. Then G can be generated by at most $2n + k$ elements. We shall write $H \overset{G}{\sim} K$ if the sets H and

K are equidecomposable using translations from the group G. Our first aim is to prove that

$$(A \setminus B) \overset{G}{\sim} E \text{ for an } E \subset Q_2 \setminus Q_1. \tag{7}$$

Let S denote the type semigroup of the relation $\overset{G}{\sim}$, and let $[H]$ denote the type of the set H (see [12, Chap. 8]). Let

$$[A] = a, [Q_1] = q_1, [Q_1'] = q_1', [Q_1' \setminus Q_1] = u,$$
$$[A \setminus B] = v, [Q_2 \setminus C] = w, [Q_2 \setminus Q_1'] = t.$$

Then $[B] = q_1'$ and $[C] = a$, therefore $a = q_1 + u + v$ and $q_1 + u + t = q_2 = a + w = q_1 + u + v + w$.

If $x, y \in S$ then we shall write $x \leq y$ if there is a $z \in S$ such that $x + z = y$. If $H, \ K \subset I^k$ then $[H] \leq [K]$ if and only if $H \overset{G}{\sim} H'$ for some $H' \subset K$.

We show that there is a positive integer M such that $q_1 \leq Mu$. Indeed, Q_1 and Q_1' are unions of some cubes from $\mathcal{Q}^N$. Suppose that Q_1 is the union of M such cubes. Since each of these cubes is congruent to a subset of $Q_1' \setminus Q_1$ using a traslation from G, it follows that $q_1 = [Q_1] \leq M \cdot [Q_1' \setminus Q_1] = Mu$.

Now we shall use the following simple lemma: if $x + z = y + z$ and $z \leq Mx, \ z \leq My$ for some positive integer M, then $x = y$ (see [5, Lemma 1 (ii)]). Since $q_1 + u + t = q_1 + u + v + w$ and $q_1 \leq Mu$, an application of this lemma gives $u + t = u + v + w$. Thus we have $v \leq u + t$; that is, $[A \setminus B] \leq [Q_2 \setminus Q_1]$, which proves (7).

Since $B \overset{\mathrm{T}}{\sim}_n Q_1'$, it follows that $\chi_B \overset{s}{\sim}_n \chi_{Q_1'}$. The function $f = \chi_{Q_1'}$ is Riemann integrable and thus, by Proposition 3, we obtain $S_f(k+1) = 0$. Thus there is a function g such that $\omega(g) < \varepsilon$ and $f \overset{s}{\sim}_{k+1} g$. Therefore, by Lemma 3 we have $\chi_B \overset{s}{\sim}_{n+k+2} g$.

It follows from (7) that $\chi_{A\setminus B}$ is scissor-congruent to χ_E using translations from G. Since G is generated by at most $2n + k$ elements, we have $\chi_{A\setminus B} \overset{s}{\sim}_{(2n+k+1)} \chi_E$ by Lemma 1. Now $E \subset Q_2 \setminus Q_1$ and $\lambda_k(Q_2 \setminus Q_1) < T_A(n) + \varepsilon$, therefore $\Omega_{\chi_E}(N) < T_A(n) + \varepsilon$. By Lemma 2, this implies $S_{\chi_E}(k+1) < T_A(n) + \varepsilon$; that is, there is a function h such that $\omega(h) < T_A(n) + \varepsilon$ and $\chi_E \overset{s}{\sim}_{k+1} h$. Then Lemma 3 gives $\chi_{A\setminus B} \overset{s}{\sim}_{(2n+2k+2)} h$.

Since $\chi_A = \chi_B + \chi_{A\setminus B}$, we find that $\chi_A \overset{s}{\sim}_{(3n+3k+4)} g + h$, where $\omega(g + h) < T_A(n) + 2\varepsilon$. Therefore $S_{\chi_A}(3n + 3k + 4) < T_A(n) + 2\varepsilon$ and, as ε was arbitrary, the proof is finished. ∎

As an immediate corollary we obtain the following statement which completes the proof of Theorem 1.

THEOREM 7. *We have $\tau(A) \leq \sigma(A)$ for every $A \subset I^k$.*

The following result is the analogue of Theorem 2.

PROPOSITION 7. (i) *If* $A \overset{T}{\sim} B$ *then* $\tau(A) = \tau(B)$.

(ii) *If* $A \subset I^k$ *is equivalent to a cube, then* $\tau(A) = \infty$. *More precisely, if* $A \subset I^k$ *is equivalent to a cube using d pieces, then* $T_A(n) = 0$ *for every* $n \geq d$.

Proof. Suppose $A \overset{T}{\sim}_d B$, and let $\eta < \tau(B)$ be fixed. Then there is a $K > 0$ such that $T_B(n) < K \cdot n^{-\eta}$ for every n. For a fixed n let Q_1 and Q_2 be cubes such that $\lambda_k(Q_2) - \lambda_k(Q_1) < K \cdot n^{-\eta}$, and $Q_1 \overset{T}{\sim}_n C$, $B \overset{T}{\sim}_n D$, where $C \subset B$ and $D \subset Q_2$. It is easy to see that the statements $Q_1 \overset{T}{\sim}_n C \subset B$ and $B \overset{T}{\sim}_d A$ imply that there is a set $E \subset A$ such that $Q_1 \overset{T}{\sim}_{dn} E$. Similarly, $A \overset{T}{\sim}_d B$ and $B \overset{T}{\sim}_n D \subset Q_2$ imply that $A \overset{T}{\sim}_{dn} D$. Therefore $T_A(dn) < K \cdot n^{-\eta}$ for every n. Thus $\tau(A) \geq \eta$ for every $\eta < \tau(B)$, which proves $\tau(A) \geq \tau(B)$. By symmetry we also have $\tau(B) \geq \tau(A)$, proving (i). The statement (ii) is obvious. ∎

By Lemma 3 and Proposition 3 we obtain the corresponding statement for σ.

PROPOSITION 8. (i) *If* $A \overset{T}{\sim} B$ *then* $\sigma(A) = \sigma(B)$.

(ii) *If* $A \subset I^k$ *is equivalent to a cube, then* $\sigma(A) = \infty$. *More precisely, if* $A \subset I^k$ *is equivalent to a cube using d pieces, then* $S_A(n) = 0$ *for every* $n \geq d + k + 2$.

The converse of (ii) of Proposition 7 is false. The following example shows that even the condition $T_3(n) = 0$ does not imply that A is equivalent to a cube.

A set $H \subset \mathbf{R}$ is said to be a *Vitali set* if H contains exactly one element of each coset of the subgroup $\mathbf{Q}$ of $\mathbf{R}$. Since $\mathbf{Q}$ is everywhere dense, it is easy to see that every interval contains a Vitali set. Let $H \subset (1/2, 1)$ be a Vitali set, and put $A = [0, 1/2) \cup H$. We claim that $T_A(3) = 0$. Let k be a positive integer, and define

$$B_k = \left\{u + \frac{i}{2k} : u \in H, i \in \mathbf{Z}\right\},$$

$$A_1 = ([0, 1/2) \setminus B_k) \cup \left(\left\{u + \frac{i}{2k} : u \in H, i < -k\right\} \cap [0, 1/2)\right),$$

$$A_2 = \left\{u + \frac{i}{2k} : u \in H, i \geq -k\right\} \cap [0, 1/2), \text{ and } A_3 = H.$$

Then $A_1 \cup A_2 \cup A_3$ is a decomposition of A. It is easy to check that A_1, $A_2 + (1/2k)$, and $H - (1/2)$ are pairwise disjoint and are contained in $[0, (1/2) + (1/2k))$. Therefore $T_A(3) \leq 1/2k$ for every k, and thus $T_A(3) = 0$.

On the other hand, one can show that there is a Vitali set $H \subset (1/2, 1)$ such that $A = [0, 1/2) \cup H$ is *not* equivalent to an interval. We omit the proof.

We shall prove that for a Lebesgue measurable set A the condition $T_A(n) = 0$ implies that A is "almost" equivalent to a cube.

We shall need some basic facts concerning weak* convergence in $L^\infty(I^k)$, the Banach space of bounded Lebesgue measurable functions defined on I^k. We say that the sequence of functions $f_i \in L^\infty(I^k)$ converges to $f \in L^\infty(I^k)$ in the weak* topology, and write $f_i \overset{w^*}{\to} f$ if $\int_{I^k} f_i g\, d\lambda_k \to \int_{I^k} fg\, d\lambda_k$ for every $g \in L^1(I^k)$. It follows from the Banach–Alaoglu theorem that in $L^\infty(I^k)$ every uniformly bounded sequence has a subsequence that converges in the weak* topology. If $0 \le f_i \le 1$ for every i and $f_i \overset{w^*}{\to} f$, then (i) $0 \le f \le 1$ a.e., and (ii) $T_{a_i} f_i \overset{w^*}{\to} T_a f$ whenever $a_i \to a$. Indeed, if $H = \{x : f(x) < 0\}$ then

$$0 \ge \int_H f\, d\lambda_k = \int_{I^k} f\chi_H\, d\lambda_k = \lim_{i\to\infty} \int_{I^k} f_i \chi_H\, d\lambda_k \ge 0.$$

Thus $\int_H f\, d\lambda_k = 0$ and $\lambda_k(H) = 0$; that is, $f \ge 0$ a.e. A similar argument proves $f \le 1$ a.e. In order to prove (ii) we may assume $a = 0$. If $g \in L^1(I^k)$ then $\|T_{a_i} g - g\|_1 \to 0$, and

$$\int_{I^k} (T_{a_i} f_i) g\, d\lambda_k = \int_{I^k} T_{a_i}(f_i g)\, d\lambda_k + \int_{I^k} (T_{a_i} f_i)(g - T_{a_i} g)\, d\lambda_k = A_i + B_i.$$

Here $A_i = \int_{I^k} f_i g\, d\lambda_k \to \int_{I^k} fg\, d\lambda_k$ and $|B_i| \le \|T_{a_i} g - g\|_1 \to 0$. Thus $\int_{I^k} (T_{a_i} f_i) g\, d\lambda_k \to \int_{I^k} (T_0 f) g\, d\lambda_k$, which proves (ii).

We shall say that the sets A and B are *almost equivalent,* if there are sets E and F of Lebesgue measure zero such that $A \setminus E \overset{\mathrm{T}}{\sim} B \setminus F$.

THEOREM 8. *Let A be Lebesgue measurable and suppose that $T_A(n) = 0$ for some n. Then A is almost equivalent to a cube.*

Proof. For every $i > 0$ we have $T_A(n) < 1/i$, and thus there are cubes Q_1^i and Q_2^i such that $\lambda_k(Q_2^i) - \lambda_k(Q_1^i) < 1/i$, $Q_1^i \overset{\mathrm{T}}{\sim}_n B^i \subset A$ and $A \overset{\mathrm{T}}{\sim}_n C^i \subset Q_2^i$. Since $\lambda_k(Q_1^i) \le \lambda_k(A) \le \lambda_k(Q_2^i)$, it follows that $\lambda_k(Q_2^i) - \lambda_k(A) < 1/i$.

There are decompositions $A = \cup_{j=1}^n A_j^i$, $C^i = \cup_{j=1}^n C_j^i$, and vectors a_j^i such that $A_j^i - a_j^i = C_j^i$ for every $j = 1, \ldots, n$. Let $f_j^i = \chi_{A_j^i}$ $(j = 1, \ldots, n)$. Then $0 \le f_j^i \le 1$ for every $j = 1, \ldots, n$, $\sum_{j=1}^n f_j^i = \chi_A$, and $\sum_{j=1}^n T(a_j^i) f_i^i = \chi_{C^i} \le \chi_{Q_2^i}$.

By Theorem 2 of [8], there exists a positive linear operator $\Phi : B(I^k) \to L^\infty(I^k)$ such that Φ commutes with translations and $\Phi(f) = f$ a.e. for every $f \in L^\infty(I^k)$. Let $g_j^i = \Phi(f_j^i)$ $(j = 1, \ldots, n,\ i = 1, 2, \ldots)$. Then $g_j^i \in L^\infty(I^k)$

and $0 \le g_j^i \le 1$ a.e. for every i and j (the latter follows from the positivity of Φ). Since $\chi_A, \chi_{Q_2^i} \in L^\infty(I^k)$, we have

$$\sum_{j=1}^{n} g_j^i = \sum_{j=1}^{n} \Phi(f_j^i) = \Phi\left(\sum_{j=1}^{n} f_j^i\right) = \Phi(\chi_A) = \chi_A \tag{8}$$

a.e. and

$$\begin{aligned}\sum_{j=1}^{n} T(a_j^i) g_j^i &= \sum_{j=1}^{n} T(a_j^i)\Phi(f_j^i) = \Phi\left(\sum_{j=1}^{n} T(a_j^i) f_j^i\right) \\ &= \Phi(\chi_{C^i}) \le \Phi(\chi_{Q_2^i}) = \chi_{Q_2^i}\end{aligned} \tag{9}$$

a.e. By selecting a subsequence we may assume that for every j the sequence a_j^i converges to a point a_j, and the sequence g_j^i converges in the weak* topology to the function g_j. Then $0 \le g_j \le 1$ a.e. for every $j = 1, \dots, n$, and $\sum_{j=1}^n g_j = \chi_A$ a.e. by (8).

Moving to another subsequence, if necessary, we may also assume that the sequence of the centers of the cubes Q_2^i is also convergent. Since $\lambda_k(A) \le \lambda_k(Q_2^i) < \lambda_k(A) + (1/i)$ for every i, the convergence of the centers implies that there is a cube Q such that $\lambda_k(Q) = \lambda_k(A)$ and that $\chi_{Q_2^i}$ converges to χ_Q a.e. Now $h_i = \chi_{Q_2^i} - \sum_{j=1}^n T(a_j^i) g_j^i \ge 0$ a.e. by (9) and $h_i \xrightarrow{w^*} \chi_Q - \sum_{j=1}^n T(a_j) g_j$, therefore $\chi_Q - \sum_{j=1}^n T(a_j) g_j \ge 0$ a.e. On the other hand,

$$\int_{I^k} \sum_{j=1}^{n} T(a_j) g_j \, d\lambda_k = \int_{I^k} \sum_{j=1}^{n} g_j \, d\lambda_k = \int_{I^k} \chi_A \, d\lambda_k = \lambda_k(A) = \lambda_k(Q),$$

and thus $\sum_{j=1}^n T(a_j) g_j = \chi_Q$ a.e. Let H denote the set of those points x where any of the statements $0 \le g_j(x) \le 1$ $(j = 1, \dots, n)$, $\sum_{j=1}^n g_j(x) = \chi_A(x)$, and $\sum_{j=1}^n T(a_j) g_j(x) = \chi_Q(x)$ is false. Then $\lambda_k(H) = 0$. Let G denote the group generated by the vectors $a_1, \dots, a_n$, and put $E = H + G = \{x + y : x \in H,\ y \in G\}$. Then $\lambda_k(E) = 0$, since G is countable. Now we define

$$g_j^*(x) = \begin{cases} g_j(x) & \text{if } x \notin E, \\ 0 & \text{if } x \in E, \end{cases}$$

for every $j = 1, \dots, n$. Then we have $0 \le g_j^* \le 1$ $(j = 1, \dots, n)$, $\sum_{j=1}^n g_j^* = \chi_{A \setminus E}$, and $\sum_{j=1}^n T(a_j) g_j^* = \chi_{Q \setminus E}$ everywhere on I^k. In other words, the sets $A \setminus E$ and $Q \setminus E$ are "continuously equidecomposable" using the translations a_j in the sense of [13]. By Lemma 1.3 of [14] this implies that $A \setminus E$ and $Q \setminus E$ are, in fact, equidecomposable using the translations a_j, and thus A is almost equivalent to the cube Q. ∎

We remark that Theorem 8 does not remain true if we replace the condition $T_A(n) = 0$ by $S_{\chi_A}(n) = 0$. Indeed, if A is Jordan measurable then $S_{\chi_A}(k+1) = 0$ by Proposition 3. On the other hand, a Jordan measurable set is not necessarily almost equivalent to a cube, as proved in Theorem 3.3 and Corollary 3.5 of [7].

We conclude by summarizing some of the results obtained concerning equivalence to cubes.

(i) If $\lambda_k(A) > 0$ and $o(A) > 0$ then A is equivalent to a cube.

(ii) If $\delta(A) = \infty$ then A is equivalent to a cube (of rational volume).

(iii) If A is Lebesgue measurable and $T_A(n) = 0$ for some n then A is almost equivalent to a cube.

(iv) If A is equivalent to a cube then $\delta(A) > 0$.

(v) If A is equivalent to a cube then $\tau(A) = \sigma(A) = \infty$.

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