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Some Stability Conditions for $y'' + qy = 0$

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Suppose that q is a real-valued continuous function defined on a ray $[a, \infty)$. We consider the second-order linear differential equation

$$y'' + qy = 0. \tag{1}$$

Three problems that have received much attention are: Find conditions on q such that if y is a solution of (1), then either (i) y is bounded on $[a, \infty)$, (ii) $y(x) \to 0$ as $x \to \infty$ or (iii) $\int_a^\infty y(x)^2\,dx < \infty$. For the literature on problems (i) and (ii) the reader may consult [1, Section 5]. Numerous contributions to problem (iii) are to be found in [2-6, 9].

A recent counter-example by Willet [10] gives the surprising result that condition (ii) above may not be satisfied even if $q'(x) \to \infty$ as $x \to \infty$. However Lazer [7], has given the simple conditions that $\int_a^\infty |(q^{-1/2})'''| < \infty$ and $q(x) \to \infty$ as $x \to \infty$ imply that condition (ii) is satisfied.

In this note we extend the method of Lazer to derive both upper and lower bounds for $y(x)^2 + z(x)^2$ where y and z are two linearly independent solutions of Eq. (1). From these bounds we will have as a corollary a necessary and sufficient condition that condition (iii) be satisfied. In addition we obtain that condition (ii) is satisfied under a hypothesis weaker than that of Lazer.

Throughout q is supposed to satisfy the conditions

$$q > 0 \quad \text{on} \quad [a, \infty) \quad \text{with} \quad q(x) \to \infty \quad \text{as} \quad x \to \infty, \tag{2}$$

$$q \quad \text{has three continuous derivatives on} \quad [a, \infty) \tag{3}$$

and

$$|q'(x)| = 0(q(x)^{3/2}) \quad \text{as} \quad x \to \infty. \tag{4}$$

We define the function η on $[a, \infty)$ by: $\eta(b)$ is the first zero to the right of b of the solution y of (1) with initial conditions $y(b) = 0$ and $y'(b) = 1$. Equation (2) above clearly implies that all solutions of (1) are oscillatory. We define the function e on $[a, \infty)$ by:

$$e(b) = \text{l.u.b.} \left\{ \frac{|q'(t)|}{q(t)^{3/2}} : b \leqslant t < \infty \right\}.$$

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The hypotheses on q imply that e is a positive-valued, nonincreasing function.

LEMMA 1. *If* $e(b) < (2\pi)^{-1}$ *and* $b \leqslant t \leqslant \eta(b)$, *then*

$$| q(t)^{1/2} [\eta(b) - b] - \pi | < (2\pi^2) e(b). \tag{5}$$

PROOF. By the Sturm comparision theorem,

$$\pi[q_{\max}]^{-1/2} \leqslant \eta(b) - b \leqslant \pi[q_{\min}]^{-1/2},$$

where the minimum and maximum of q are calculated over the interval $[b, \eta(b)]$. By the intermediate value theorem there is a number z, $b \leqslant z \leqslant \eta(b)$, such that $\eta(b) - b = \pi q(z)^{-1/2}$. Define f on $[b, \eta(b)]$ by $f(t) = \pi[q(t)/q(z)]^{1/2}$. Hence $f(z) = \pi$ and

$$|f'(t)| = \frac{\pi}{2} [q(t) q(z)]^{-1/2} | q'(t) | \leqslant \frac{1}{2\pi} e(b) q(z)^{1/2} f(t)^2.$$

If inequality (5) does not hold, then the continuity of q implies there is a $t^* \in [b, \eta(b)]$ such that

$$|f(t^*) - \pi| = (2\pi^2) e(b),$$

and for t between z and t^*,

$$|f(t) - \pi| \leqslant (2\pi^2) e(b) < \pi.$$

For such t we then have $f(t) < 2\pi$. Thus

$$\begin{aligned}(2\pi^2) e(b) = |f(t^*) - \pi| &= \left| \int_z^{t^*} f'(t)\, dt \right| \\ &\leqslant \frac{1}{2\pi} e(b) q(z)^{1/2} \left| \int_z^{t^*} f(t)^2\, dt \right| \\ &< \frac{1}{2\pi} e(b) q(z)^{1/2} (2\pi)^2 [\eta(b) - b] = (2\pi^2) e(b).\end{aligned}$$

This contradiction proves the lemma.

If $| q' | = o(q^{3/2})$ as $x \to \infty$, then $e(b) \to 0$ as $b \to \infty$. In such case there is then by Lemma 1 a number b_0 such that if $b_0 \leqslant b$ and t and $s \in [b, \eta(b)]$, then

$$2^{-1} \leqslant \left[\frac{q(t)}{q(s)}\right]^{1/2} \leqslant 2. \tag{6}$$

LEMMA 2. *If* $|(q^{-1/2})''| = 0(1)$ *as* $x \to \infty$, *then* $|q'| = o(q^{3/2})$ *as* $x \to \infty$.

Proof. Since $q^{-1/2} \to 0$ as $x \to \infty$, we have by Problem 15 of [8, p. 101] that $(q^{-1/2})' \to 0$ as $x \to \infty$. Since $(q^{-1/2})' = -q'/2q^{3/2}$, the lemma is proved.

Following [7] we note that if y is a solution of (1), then

$$\{(y')^2 q^{-1/2} - (q^{-1/2})' yy' + [q^{1/2} + 2^{-1}(q^{-1/2})''] y^2\}' = 2^{-1}(q^{-1/2})''' y^2. \tag{7}$$

If y and z are linearly independent solutions of (1) with $yz' - zy' \equiv 1$ and $r = (y^2 + z^2)^{1/2}$, then it is wellknown that $r'' + qr = r^{-3}$. This differential equation in r implies the relation

$$\{(r')^2 q^{-1/2} - (q^{-1/2})' rr' + [r^2 q^{1/2}]^{-1} + [q^{1/2} + 2^{-1}(q^{-1/2})''] r^2\}' = 2^{-1}(q^{-1/2})''' r^2, \tag{8}$$

which may be verified by differentiation.

THEOREM 1. *If* $|(q^{-1/2})''| = 0(1)$ as $x \to \infty$, $\int_a^\infty q^{-1/2} |(q^{-1/2})'''| < \infty$ *and* y *is a solution of* (1), *then*

$$\limsup y(x)^2 q(x)^{1/2} < \infty \qquad \text{as} \qquad x \to \infty. \tag{9}$$

PROOF. By Lemma 2, $e(b) \to 0$ as $b \to \infty$. Let b_0 be as in the remark following Lemma 1. Let M be a bound for $|(q^{-1/2})''|$ on $[a, \infty)$. By the above hypothesis, there is a number $b \geqslant b_0$ such that for all $x \geqslant b$,

$$\int_b^x q^{-1/2} |(q^{-1/2})'''| < \frac{1}{12} \tag{10}$$

and

$$q(x)^{-1/2} M < \tfrac{1}{4}. \tag{11}$$

If Eq. (9) does not hold then there is an increasing sequence $\{c_n\}_1^\infty$ such that each $c_n > b$, $c_n \to \infty$ as $n \to \infty$,

$$y(c_n)^2 q(c_n)^{1/2} = \max\{y(x)^2 q(x)^{1/2} : b \leqslant x \leqslant c_n\}$$

and

$$y(c_n)^2 q(c_n)^{1/2} \to \infty \qquad \text{as} \qquad n \to \infty. \tag{12}$$

Clearly $y(c_n) \neq 0$. Let a_n and b_n be the greatest zero of y less than c_n and least zero of y greater than c_n, respectively. There is no lost in generality in supposing $b < a_1$. Since $q > 0$, y' has only one zero between a_n and b_n. Denote this zero by c_n^* and let

$$F = (y')^2 q^{-1/2} - (q^{-1/2})' yy' + [q^{1/2} + 2^{-1}(q^{-1/2})''] y^2.$$

By Eqs. (6), (7), and (10) and the inequality $y(c_n)^2 \leqslant y(c_n^*)^2$, we have

$$F(c_n^*) = [q(c_n^*)^{1/2} + 2^{-1}(q^{-1/2})''(c_n^*)]\, y(c_n^*)^2$$

$$= F(b) + 2^{-1} \int_b^{c_n^*} [q^{-1/2}(q^{-1/2})'''] [q^{1/2} y^2]$$

$$\leqslant F(b) + 2^{-1} \int_b^{c_n} [q^{-1/2} \mid (q^{-1/2})''' \mid] [q(c_n)^{1/2} y(c_n^*)^2]$$

$$+ 2^{-1} \left| \int_{c_n}^{c_n^*} [q^{-1/2} \mid (q^{-1/2})''' \mid] [q^{1/2} q(c_n)^{-1/2}] [q(c_n)^{1/2} y(c_n^*)^2] \right|$$

$$\leqslant F(b) + 2^{-1} q(c_n)^{1/2} y(c_n^*)^2 \left[\int_b^{c_n} q^{-1/2} \mid (q^{-1/2})''' \mid + 2 \int_{a_n}^{b_n} q^{-1/2} \mid (q^{-1/2})''' \mid \right]$$

$$\leqslant F(b) + \frac{1}{8}\, q(c_n)^{1/2} y(c_n^*)^2.$$

Solving this inequality for $q(c_n)^{1/2} y(c_n^*)^2$ and applying (6) and (11) yields

$$q(c_n)^{1/2} y(c_n^*)^2 \leqslant F(b) \left[\left(\frac{q(c_n^*)}{q(c_n)} \right)^{1/2} + 2^{-1} q(c_n)^{-1/2} (q^{-1/2})''(c_n^*) - \frac{1}{8} \right]^{-1}$$

$$\leqslant F(b) \left[\frac{1}{2} - \frac{1}{8} - \frac{1}{8} \right]^{-1} = 4F(b).$$

Thus we now have

$$q(c_n)^{1/2} y(c_n)^2 \leqslant q(c_n)^{1/2} y(c_n^*)^2 \leqslant 4F(b),$$

which is a contradiction to condition (12). This contradiction proves the theorem.

We note that the hypothesis of Theorem 1 is weaker than that of Theorem 1 of [7] since the condition $\int_a^\infty \mid (q^{-1/2})''' \mid < \infty$ implies that limit as $x \to \infty$ of $(q^{-1/2})''(x)$ exists.

LEMMA 3. *Under the hypothesis of Theorem 1,*

$$\limsup y'(x)^2 q(x)^{-1/2} < \infty \qquad as \qquad x \to \infty \tag{13}$$

and

$$\limsup \mid y(x) y'(x) (q^{-1/2})'(x) \mid < \infty \qquad as \qquad x \to \infty. \tag{14}$$

PROOF. Theorem 1 implies that there is a number $L > 0$ such that on $[a, \infty)$,

$$\mid y'(x)^2 q(x)^{-1/2} - (q^{-1/2})'(x)\, y(x)\, y'(x) \mid \leqslant L. \tag{15}$$

Denote the zeros of y on $[a, \infty)$ by the sequence $\{d_n\}_1^\infty$. Since $q > 0$, the maximum value of $| y'(x) |$ on $[d_n, d_{n+1}]$ is $\max\{| y'(d_n) |, | y'(d_{n+1}) |\}$. For $x = d_n$, Eq. (15) is $y'(d_n)^2 q(d_n)^{-1/2} \leqslant L$. By the remark following Lemma 1 we have for all sufficiently large n and x and $t \in [d_n, d_{n+1}]$ that $[q(t)/q(x)]^{1/2} \leqslant 2$. Hence for all sufficiently large n and $d_n \leqslant x \leqslant d_{n+1}$, $y'(x)^2 q(x)^{-1/2} \leqslant 2L$, thus proving inequality (13).

Inequality (14) follows immediately from (13) and (15).

THEOREM 2. *If r is as in Eq. (8), then under the hypothesis of Theorem 1,*

$$\liminf r(x)^2 q(x)^{1/2} > 0 \qquad as \qquad x \to \infty. \tag{16}$$

PROOF. Since $rr' = yy' + zz'$, we have by Lemma 3 that

$$\limsup | (q^{-1/2})' rr' | < \infty \qquad \text{as} \qquad x \to \infty. \tag{17}$$

Applying Eq. (8) we have

$$\begin{aligned}[(r')^2 q^{-1/2} + [r^2 q^{1/2}]^{-1}] (x) &= [(q^{-1/2})' rr' - [q^{1/2} + 2^{-1}(q^{-1/2})''] r^2] (x) \\ &+ \{(r')^2 q^{-1/2} - (q^{-1/2}) rr' + [r^2 q^{1/2}]^{-1} + [q^{1/2} + 2^{-1}(q^{-1/2})''] r^2\} (a) \\ &+ 2^{-1} \int_a^x [q^{-1/2}(q^{-1/2})'''] (q^{1/2} r^2).\end{aligned} \tag{18}$$

By (17) and Theorem 1, the right-hand side of (18) is bounded for $x \in [a, \infty)$. Hence there is a number $Q > 0$ such that for $x \in [a, \infty)$,

$$[(r')^2 q^{-1/2} + [r^2 q^{1/2}]^{-1}] (x) \leqslant Q. \tag{19}$$

However, each term of the left-hand side of (19) is non-negative so that $[r(x)^2 q(x)^{1/2}]^{-1} \leqslant Q$ or

$$0 < Q^{-1} \leqslant r(x)^2 q(x)^{1/2}, \tag{20}$$

thus proving Theorem 2.

COROLLARY 2.1. *Under the hypothesis of Theorem 1 we have that as $x \to \infty$,*

$$0 < \limsup y(x)^2 q(x)^{1/2} < \infty \tag{21}$$

and

$$0 < \limsup y'(x)^2 q(x)^{-1/2} < \infty. \tag{22}$$

PROOF. The right-hand parts of inequalities (21) and (22) are contained in Theorem 1 and Lemma 3. To prove the left-hand side of (21), let z be a solution of Eq. (1) such that $yz' - zy' \equiv 1$. If b is a zero of z and Q is as in Eq. (20), then

$$Q^{-1} \leqslant r(b)^2 q(b)^{1/2} = y(b)^2 q(b)^{1/2}.$$

Since z has arbitrary large zeros, the left-hand part of (21) now follows.

If c is a zero of y, then

$$1 = [yz' - zy'](c) = -z(c)\,y'(c)$$

so that

$$y'(c)^2\,q(c)^{-1/2} = [z(c)^2\,q(c)^{1/2}]^{-1}. \tag{23}$$

By (21), lim sup $z(x)^2\,q(x)^{1/2} < \infty$ as $x \to \infty$. This inequality and Eq. (23) imply the left-hand part of (22).

Combining Theorems 1 and 2 gives immediately the following corollary concerning p-integrable solutions of Eq. (1).

COROLLARY 2.2. *If $p \geqslant 1$ and the hypothesis of Theorem 1 is satisfied, then $\int_a^\infty q(x)^{-p/4}\,dx < \infty$ is a necessary and sufficient condition that for all solutions y of* (1) *we have $\int_a^\infty |\,y(x)\,|^p\,dx < \infty$.*

We now give some examples which illustrate these results.

EXAMPLE 1. For $q(x) = x^n$ with $n > 0$, there are positive numbers M_1 and M_2 such that for r as in Eq. (8),

$$M_1 x^{-n/2} \leqslant r(x)^2 \leqslant M_2 x^{-n/2}.$$

The condition $\int_1^\infty q(x)^{-p/4}\,dx < \infty$ in Corollary 2.2 is equivalent to $n > 4/p$.

EXAMPLE 2. Let $q(x) = x^{2/3} + \sin x$ on $[1, \infty)$. Since

$$(q^{-1/2})'' = (\tfrac{3}{4})\,q^{-5/2}(q')^2 - (\tfrac{1}{2})\,q^{-3/2}\,q'' \tag{24}$$

and

$$(q^{-1/2})''' = (-\tfrac{15}{8})\,q^{-7/2}(q')^3 + (\tfrac{9}{4})\,q^{-5/2}q'q'' - (\tfrac{1}{2})\,q^{-3/2}q''', \tag{25}$$

and each of q', q'' and q''' is of $0(1)$ as $x \to \infty$, the hypothesis of Theorem 1 is satisfied. Hence if y is a solution of equation (1), then $y(x) \to 0$ as $x \to \infty$. The condition $\int_1^\infty |\,(q^{-1/2})'''\,| < \infty$ is not satisfied.

EXAMPLE 3. Let $q(x) = x^3(1 + 2^{-1}\sin x)$ on $[1, \infty)$. Then each of q', q'', and q''' is $0(x^3)$ as $x \to \infty$. Since $q(x) \geqslant (\tfrac{1}{2})\,x^3$, we have by Eq. (24) that $(q^{-1/2})'' \to 0$ as $x \to \infty$, and by Eq. (25) $\int_1^\infty |\,(q^{-1/2})'''\,| < \infty$. Hence there are positive numbers N_1 and N_2 such that if r is as in Eq. (8), then

$$N_1 x^{-3/2} \leqslant r(x)^2 \leqslant N_2 x^{-3/2}.$$

This inequality implies $\int_1^\infty r^2 < \infty$, and hence Eq. (1) is of the limit circle type. The limit circle criteria which require monotonicity of q and q' are not appliciable to this example since q' and q'' are oscillatory.