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A Degenerate Diffusion Problem Not in Divergence Form*

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0. INTRODUCTION

The purpose of this article is to investigate nonnegative solutions of the degenerate nonlinear diffusion problem

$$u_t = u\Delta u + g(u) \quad \text{in} \quad (0, \infty) \times \Omega \tag{0.1}$$

$$u = 0 \quad \text{on} \quad (0, \infty) \times \partial\Omega \tag{0.2}$$

$$u = u_0 \quad \text{on} \quad \{0\} \times \Omega. \tag{0.3}$$

A biological model of this form has been proposed in [1]. However, the main interest of this equation is the behaviour different from equations in divergence form

$$u_t = \Delta\phi(u) + g(u). \tag{0.4}$$

Whereas for equations of type (0.4) existence of weak solutions and also uniqueness in most cases have been established by various techniques (see [2, 3, 4]), these results lack for equations like (0.1).

And indeed, what we will show is that uniqueness fails dramatically since there is, for every $\delta > 0$, a weak solution with extinction time δ. We only get a unique maximal solution, and for nice initial data this is characterized by the property that its support remains constant in time.

The paper is organized as follows: Section 1 is devoted to existence, the properties of the solutions are studied in Section 2, and uniqueness is discussed in Section 3.

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We would like to mention that independently M. Ughi [5] has investigated (0.1) in one space dimension.

In her article there is an explicit counterexample to uniqueness along the lines of Remark 4 at the end of this article. Nonexpansion of the support of u is proved there, also and a uniqueness theorem is obtained for a divergence form reformulation of (0.1). The last result requires logarithmic integrability of u_0 on its support.

1. Existence

The following assumptions on the data will be made throughout:

(H.1) $\Omega \subset \mathbb{R}^n$ is an open bounded domain, with Lipschitz boundary $\partial\Omega$.

(H.2) g is locally Lipschitz continuous on $[0, \infty)$, $g(0) = 0$ and $\limsup_{u \to +\infty} (g(u)/u^2) = K < \lambda_1$, where λ_1 denotes the first eigenvalue of Laplace's equation in Ω with Dirichlet boundary values.

(H.3) $u_0 \in C^0_+ := \{u \in C^0 \mid u \geqslant 0 \text{ a.e. in } \Omega\}$.

Let $Q_T := (0, T) \times \Omega$, $V := L^\infty(Q_T) \cap L^2(0, T, H_0^{1,2}(\Omega))$, then we define.

DEFINITION. $u \in V$ is a weak solution of (0.1)–(0.3) on $[0, T]$ if

$$\partial_t u \in V^* \tag{1.1}$$

holds with initial value u_0, that is,

$$\int_0^T \langle \partial_t u, \varphi \rangle + \int_0^T \int_\Omega (u - u_0)\, \partial_t \varphi = 0$$

for every test function $\varphi \in V \cap H^{1,1}(Q_T)$, $\varphi(T) = 0$, and

$$\int_0^T \langle \partial_t u, \varphi \rangle + \int_0^T \int_\Omega \nabla u \cdot \nabla(u\varphi) = \int_0^T \int_\Omega g(u)\, \varphi \tag{1.2}$$

for every $\varphi \in V$.

The following regularity property gives us the main tool for proving the existence of a weak solution.

LEMMA 1. *If u is a weak solution then for all $\alpha \in (0, 1)$,*

$$\int_0^T \int_\Omega \frac{|\nabla u|^2}{u^\alpha} < c_\alpha \tag{1.3}$$

holds.

Proof. Since u is nonnegative, we may take as a test function

$$\varphi := \frac{1}{(u+\varepsilon)^\alpha}, \qquad \varepsilon > 0, \alpha \in (0, 1).$$

Then

$$\int_0^T \left\langle u_t, \frac{1}{(u+\varepsilon)^\alpha} \right\rangle + \int_0^T \int_\Omega \nabla u \nabla \left(\frac{u}{(u+\varepsilon)^\alpha} \right) = \int_0^T \int_\Omega \frac{g(u)}{(u+\varepsilon)^\alpha}$$

from which, observing that $(u+\varepsilon)^{1-\alpha}$ is concave function of u, we have

$$\frac{1}{1-\alpha} \int_\Omega (u+\varepsilon)^{1-\alpha}(T) + (1-\alpha) \int_0^T \int_\Omega \frac{|\nabla u|^2}{(u+\varepsilon)^\alpha}$$
$$\leqslant \int_0^T \int_\Omega \frac{g(u)}{(u+\varepsilon)^\alpha} + \frac{1}{1-\alpha} \int_\Omega (u+\varepsilon)^{1-\alpha}(0).$$

The integral on the right is bounded independently of ε. This proves (1.3).

THEOREM 1 (Existence). *Under assumptions* (H.1)–(H.3), problem (0.1)–(0.3) admits a weak solution.

Proof. Let us consider the sequence of nondegenerate problems

$$\begin{aligned} u_{\varepsilon t} &= u_\varepsilon \Delta u_\varepsilon + g_\varepsilon(u_\varepsilon) && \text{in} \quad (0, T) \times \Omega \\ u_\varepsilon &= \varepsilon && \text{on} \quad (0, T) \times \partial\Omega \end{aligned} \tag{1.4}$$

where $\varepsilon > 0$ and the functions g_ε are defined by

$$g_\varepsilon := g + a_\varepsilon \qquad \text{in} \quad [0, +\infty)$$

$$a_\varepsilon := \max_{0 \leqslant u \leqslant \varepsilon} \max\{-g(u), 0\}.$$

Observe that the sequence $(g_\varepsilon)_{\varepsilon>0}$ is monotone increasing.

Equation (1.4) admits a unique classical solution. And by the known comparison theorem (or Theorem 6) we have

$$\varepsilon \leqslant u_\varepsilon \leqslant c(|u_0|_\infty, \kappa), \qquad u_{\varepsilon_1} \leqslant u_{\varepsilon_2} \qquad \text{if} \quad \varepsilon_1 < \varepsilon_2, \tag{1.5}$$

where $c(|u_0|_\infty, \kappa)$ denotes a constant depending on the terms with brackets, and κ is the constant in assumption (H.2), which guarantees that the solution exists for all time $T > 0$.

With the same argument of Lemma 1, it results that

$$\int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon)^\alpha} \leqslant c_\alpha, \qquad \alpha \in (0, 1). \tag{1.6}$$

From (1.5), (1.6) it follows that the sequence $u_{\varepsilon t}$ is bounded in V^*, and we can assume without loss of generality that

$$\begin{aligned} u_\varepsilon &\to u && \text{weakly in } L^2(0, T; H^{1,2}(\Omega)) \\ u_\varepsilon &\to u && \text{strongly in } L^2(Q_T) \\ u_{\varepsilon t} &\to u_t && \text{weakly in } V^* \end{aligned}$$ [1]

and that u_t satisfies (1.1).

Now we have to show that, indeed u satisfies (1.2), and then we need "strong" convergence of $(\nabla u_\varepsilon)_{\varepsilon > 0}$ in order to go to the limit in

$$\int_0^T \langle u_{\varepsilon t}, \varphi \rangle + \int_0^T \int_\Omega (u_\varepsilon \nabla u_\varepsilon \cdot \nabla \varphi + |\nabla u_\varepsilon|^2 \varphi)$$
$$= \int_0^T \int_\Omega g_\varepsilon(u_\varepsilon)\, \varphi \qquad (\varphi \in V). \tag{1.7}$$

For this purpose, we take as a test function in (1.7), $\varphi := u_\varepsilon^2 - (\varepsilon^2 + u^2)$. This yields

$$\int_0^T \langle u_{\varepsilon t}, u_\varepsilon^2 - (\varepsilon^2 + u^2) \rangle + \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 (u_\varepsilon^2 - (\varepsilon^2 + u^2)) + \tfrac{1}{2} \int_0^T \int_\Omega |\nabla (u_\varepsilon^2 - u^2)|^2$$
$$= \int_0^T \int_\Omega (\tfrac{1}{2} \nabla (u^2) \cdot \nabla (u_\varepsilon^2 - u^2) + g_\varepsilon(u_\varepsilon)(u_\varepsilon^2 - (\varepsilon^2 + u^2))).$$

Due to the estimations previously obtained the first term on the left (for a.a.T) and the expression on the right converge to zero as $\varepsilon \to 0$, then

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega |\nabla (u_\varepsilon^2 - u^2)|^2 = 0.$$

The strong convergence of $\nabla(u_\varepsilon^2)$ to $\nabla(u^2)$ in $L^2((0, T) \times \Omega)$ furthermore implies

$$\nabla u_\varepsilon \to \nabla u \qquad \text{in} \quad L^2(K) \tag{1.8}$$

[1] An additional consideration is necessary, since the time derivatives have a L^1 part, which might converge to a negative measure. Estimate (1.3) shows that nothing can happen at the lateral boundary. Standard methods then give local L^2 bounds for the time derivatives which extend up to the time $t = 0$ if the initial values are in $H^{1,2}$. Approximation from below for continuous initial data shows that there is no jump at $t = 0$. The continuity requirement might be relaxed to: For almost all x such that $u_0(x) > 0$, there exists an $H^{1,2}$ function v between 0 and u_0 such that $v(x) > 0$.

strongly, where K denotes any set of the form

$$\{(t, x) \in Q_T \mid \delta \in R, \delta > 0, u(x, t) \geqslant \delta\}.$$

Combining (1.6) with (1.7), we obtain $|\nabla u_\varepsilon|^2 \to |\nabla u|^2$ strongly in $L^1((0, T) \times \Omega)$. In fact, we have

$$\int_0^T \int_\Omega \left| |\nabla u_\varepsilon|^2 - |\nabla u|^2 \right| \leqslant \delta^\alpha \int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^\alpha} + \int_0^T \int_\Omega |\chi_{\{u_\varepsilon \geqslant \delta\}} |\nabla u_\varepsilon|^2 - \chi_{\{u \geqslant \delta\}} |\nabla u|^2|$$
$$+ \int_0^T \int_\Omega (1 - \chi_{\{u \geqslant \delta\}}) |\nabla u|^2$$

when $\delta > 0$ is such that $\{(t, x) \in Q_T \mid u(x, t) \geqslant \delta\}$ is not empty, (otherwise it is trivial).

Concluding, we can go to the limit in (1.7) as $\varepsilon \to 0$ obtaining that u satisfies (1.2), which proves the claim.

We will refer to u as regular solution of the problem.

Remark 1. We have considered assumption (H.2) for the sake of simplicity. This assumption indeed can be relaxed, that is, Theorem 1 remains valid also when $g \in C[0, \infty)$ and locally Lipschitz on $(0, +\infty)$, requiring that

$$\lim_{u \to 0} \sup \left| \frac{g(u)}{u^\alpha} \right| < \text{const.}$$

for some $\alpha \in (\frac{1}{2}, 1)$, in order to obtain an estimate like (1.6).

2. Structure of the Solutions

In this section we give some characterizations of the solution u of (0.1)–(0.3) in term of extendibility and localization phenomena.

Theorem 2. *Let $\tilde{\Omega} \subset \mathbb{R}^n$ open and connected, $\tilde{\Omega} \supset \Omega$; $\tilde{Q}_T := (0, T) \times \tilde{\Omega}$. If u is a weak solution of* (0.1)–(0.3) *on Q_T, the function*

$$\tilde{u} := \begin{cases} u & \text{in } Q_T \\ 0 & \text{in } \tilde{Q}_T \setminus Q_T \end{cases}$$

is a weak solution of (0.1)–(0.3) *on $\tilde{Q}_T$, with initial data*

$$\tilde{u}_0 := \begin{cases} u_0 & \text{in } \{0\} \times \Omega \\ 0 & \text{in } \{0\} \times (\tilde{\Omega} \setminus \Omega) \end{cases}$$

Proof. We have only to verify (1.2). For small δ, let

$\psi_\delta(w) := \inf(w/\delta, 1)$. Supp $\psi_\delta(\tilde{u}) \subset Q_T$, then we can write for every $\varphi \in \tilde{V} := L^\infty(\tilde{Q}_T) \cap L^2(0, T, H_0^{1,2}(\tilde{\Omega}))$,

$$\int_0^T \langle \partial_t \tilde{u}, \varphi \rangle + \int_0^T \int_{\tilde{\Omega}} \nabla\tilde{u}\nabla(\tilde{u}\varphi) - \int_0^T \int_{\tilde{\Omega}} g(\tilde{u})\, \varphi$$

$$= \int_0^T \langle \partial_t \tilde{u}, (1-\psi_\delta(\tilde{u}))\, \varphi \rangle + \int_0^T \int_{\tilde{\Omega}} \nabla\tilde{u}\nabla(\tilde{u}(1-\psi_\delta(\tilde{u}))\, \varphi)$$

$$- \int_0^T \int_{\tilde{\Omega}} g(\tilde{u})(1-\psi_\delta(\tilde{u}))\, \varphi$$

$$= \int_0^T \langle \partial_t \tilde{u}, (1-\psi_\delta(\tilde{u}))\, \varphi \rangle + \int_0^T \int_{\Omega} \chi_{\{0<u<\delta\}}(1-\psi_\delta(u))$$

$$\times [(|\nabla u|^2 - g(u))\, \varphi + u\nabla u\nabla\varphi] - \int_0^T \int_{\Omega} \chi_{\{0<u<\delta\}} \frac{u}{\delta} |\nabla u|^2 \varphi,$$

which converges to zero as $\delta \to 0$ in virtue of L^2-boundedness of ∇u.

More generally, with the same argument, we can obtain a weak solution for the problem (0.1)–(0.2) in Q_T, with opportune initial data, "gluing" together weak solutions for (0.1)–(0.3) on subcylinders.

Before stating the theorem on the localization property, we specify that for a function $w \colon \Omega \to \mathbb{R}^+ \cup \{0\}$, we denote by $G = \{x \in \Omega \mid w(x) > 0\}$ and define

$$\operatorname{supp} w := \overline{\left\{ x \in G \,\middle|\, \lim_{\rho \to 0} \frac{\mu(G \cap B_\rho(x))}{\mu(B_\rho(x))} > 0 \right\}}$$

(when B_ρ denotes $\{y \mid |x-y| \leqslant \rho\}$).

THEOREM 3. *Let u be a weak solution of* (0.1)–(0.3) *with* $\operatorname{supp} u_0 \subsetneq \Omega$. *Then* $\operatorname{supp} u(t) \subset \operatorname{supp} u_0$ *a.e. in* $(0, T)$.

Proof. Let $\psi \colon \bar{\Omega} \to R$, which satisfies the conditions.

(i) $\operatorname{supp} \psi \subset \overline{\bar{\Omega} \backslash \operatorname{supp} u_0}$,

(ii) $\psi \in C^0 \cap H^{1,2}(\Omega)$.

(For example: $\psi(x) := \inf((\operatorname{dist}(x, \operatorname{supp} u_0) \cup \partial\Omega))/\sigma, 1)$, $\sigma \leqslant 1$.) Observe that condition (i) implies $\psi \cdot u_0 \equiv 0$ on Ω. We take as a test function in (1, 2) $\varphi := \psi/(u+\varepsilon)$ obtaining

$$\int_0^t \left\langle \partial_t u, \frac{\psi}{u+\varepsilon} \right\rangle + \int_0^t \int_\Omega \nabla u \nabla \left(\frac{u\psi}{u+\varepsilon} \right) = \int_0^t \int_\Omega g(u)\, \psi.$$

Then

$$\int_\Omega \chi_{\{\operatorname{supp}\psi\}}(\log(u(t)+\varepsilon)-\log\varepsilon)\,\psi+\varepsilon\int_0^t\int_\Omega\frac{|\nabla u|^2}{(u+\varepsilon)^2}\,\psi$$

$$\leqslant \int_0^t\int_\Omega (g(u)\,\psi+|\nabla u\cdot\nabla\psi|\leqslant C,$$

from which, for every δ sufficiently small, we also have

$$\int_\Omega \chi_{(\{u(t,x)>\delta\}\cap\{\psi=1\})}(\log(u(t)+\varepsilon)-\log\varepsilon)\leqslant c,$$

where c is independent on ε. We can conclude

$$\mu(\{(t,x)\in\{t\}\times\{\psi=1\}\mid u(t,x)>\delta\})=0 \qquad \text{for all } t\in(0,T),$$

which implies the claim.

3. Uniqueness

In the first part of this section we show that the problem (0.1)–(0.3) does not admit a unique solution, giving a counterexample; in the second part we state a comparison and a uniqueness theorem under some auxiliary assumptions.

Definition. We call $u\in V$ a weak subsolution (supersolution) if $u\leqslant(\geqslant)\,0$ on $(0,T)\times\partial\Omega$, and if (1.1)–(1.2) are fulfilled with $=$ replaced by $\leqslant$ $(\geqslant)$ for all test function φ with $\varphi(0)\leqslant 0$ in (1.1) and $\varphi\geqslant 0$ in (1.2).

3.1. *A Counterexample*

For this purpose, we consider the problem

$$v_t=v\Delta v+\bar c\cdot\nabla v+g(v) \qquad \text{in} \quad (0,\infty)\times\Omega \tag{3.1}$$

$$v=0 \qquad \text{on} \quad (0,\infty)\times\partial\Omega \tag{3.2}$$

$$v=u_0 \qquad \text{on} \quad \{0\}\times\Omega, \tag{3.3}$$

when $\bar c\in R^n$, with assumptions (H.1)–(H.3). We define solution (sub, super) of this problem, as usual, in a weak sense, in the same manner as for the problem (0.1)–(0.3).

Theorem 4. *For all* $\bar c\in R^n$, *the problem* (3.1)–(3.3) *admits a weak solution.*

Proof. Exactly as our Theorem 1, considering the approximating problems

$$\begin{aligned} v_{\varepsilon t} &= v_\varepsilon \Delta v_\varepsilon + \bar{c}\cdot\nabla v_\varepsilon + g_\varepsilon(v_\varepsilon) && \text{in} \quad (0,\infty)\times\Omega \\ v_\varepsilon &= \varepsilon && \text{on} \quad (0,\infty)\times\partial\Omega \\ v_{\varepsilon 0} &= u_0+\varepsilon && \text{on} \quad \{0\}\times\Omega. \end{aligned} \tag{3.4}$$

It is immediate to verify that the estimate (1.3) and the counterpart for this problem of Theorem 2 continue to be valid.

We observe that to solve (3.1)–(3.3) in a fixed cylinder is equivalent to solving the original problem with a prescribed direction of propagation. This yields

THEOREM 5. *For every $\bar{c}\in R^n$ there exists a weak solution of* (0, 1)–(0, 3), $u_{\bar{c}}$, *such that*

$$\operatorname{supp} u_{\bar{c}} \subset (R^+\times\operatorname{supp} u_0) \\ \cap\{(t,x)\in R^+\times R^n \mid x_i-y_i-\bar{c}_i t=0,\ i=1,-,n,\ y\in\bar{\Omega}\}.$$

Proof. Let $\bar{c}\in R^n$ and v a solution of (3.1)–(3.3). By the counterpart of Theorem 2 we can extend v by zero obtaining a solution on $R^+\times R^n$, from which $u(x,t):=v(x-\bar{c}t,t)$ is a solution of the original problem in $R^+\times R^n$, whose support is continued in $\{(t,x)\in R^+\times R^n \mid x_i-y_i-c_i t=0,\ i=1,\ldots,n,\ y\in\bar{\Omega}\}$. However, by virtue of Theorem 3, $\operatorname{supp} u(t)\subset\operatorname{supp} u_0$, $t>0$, which implies the claim.

In other words, we can say that for every $T>0$ there exists a solution with extinction time T. This suggests a uniqueness criteria, that is, "positivity" of the solution for all times in a sense that we shall specify later.

3.2. *Comparison and Uniqueness Results*

We can formulate a comparison theorem under a stronger assumption on the data, namely,

(H.4) $f(u):=g(u)/u$ is uniformly Lipschitz continuous from above on $[0,\infty)$, i.e., there exists a $\kappa\in R$ such that

$$f(u)-f(w)\leqslant\kappa(u-w), \qquad 0\leqslant w\leqslant u.$$

THEOREM 6. *Under condition* (H.1)–(H.4), *let u^+ be a supersolution such that $u^+\geqslant\varepsilon>0$ in Q_T, and u^- a subsolution. If u^+ or u^- belongs to $H^{1,2}((0,T)\times\Omega)$, and $u^-(0,\cdot)\leqslant u^+(0,\cdot)$, then $u^-\leqslant u^+$.*

To prove the theorem we use

LEMMA 2. *Let* $u \in V, u_t \in V^*$, $v \in H^{1,2}((0,T)\times\Omega)$, ϕ *be a monotone smooth, convex scalar valued function,* $\operatorname{sign}_\delta(z) := \operatorname{sign}(z)\inf(|z|/\delta, 1)$ *then*

$$\int_0^t \langle (u-v)_t, \operatorname{sign}_\delta(\phi(u)-\phi(v))_+\rangle$$

$$= \int_\Omega \left(\int_0^u \operatorname{sign}_\delta(\phi(z)-\phi(v))_+ \, dz - \int_0^v \operatorname{sign}_\delta(\phi(0)-\phi(z))_+ \, dz \right)\Bigg|_0^t$$

$$- \int_0^t \int_\Omega \partial_t v \left(\int_0^u \operatorname{sign}'_\delta(\phi(z)-\phi(v))(\phi'(z)-\phi'(v))_+ \, dz \right).$$

Proof. For $t=0$ the term on the left is equal to the term on the right, then we have only to verify that they have the same derivative respect to the time. For this purpose we have to check the first term on the right, that we denote by $F(u,v)$. It results $F(u,v)$ is C^2 convex-function with respect to u for any fixed v, then we can apply, by known result, the chain rule; the same is true respect to the variable v due to the regularity assumption on it.

Proof of Theorem 6. Suppose that $u^+ \in H^{1,2}((0,T)\times\Omega)$ and take as test functions in (1.2)

$$\varphi^+ := \frac{\operatorname{sign}_\delta((u^- - u^+)_+)}{u^+} \qquad \text{and} \qquad \varphi^- := \frac{\operatorname{sign}_\delta((u^- - u^+)_+)}{u^-};$$

then we get

$$\int_0^T \langle (\log(u^-) - \log(u^+))_t, \operatorname{sign}_\delta((u^- - u^+)_+)\rangle$$

$$\leqslant \int_0^T \int_\Omega (f(u^-) - f(u^+)) \operatorname{sign}_\delta((u^- - u^+)_+)$$

$$- \int_0^T \int_\Omega |\nabla(u^- - u^+)|^2 \operatorname{sign}'_\delta((u^- - u^+)_+)$$

and by (H.4),

$$\leqslant c(\kappa, |u^-|_{L^2(0,T,L^\infty)}, |u^+|_{L^2(0,T,L^\infty)}) \int_0^T \int_\Omega [\log(u^-) - \log(u^+)]_+$$

(where $c(\cdot, \cdot, \cdot)$ denotes a constant). Putting $u = \log u^-$, $v = \log u^+$ by Lemma 2, and the inequality in (1.1) for u^- and u^+, we obtain

$$\int_\Omega \left(\int_0^{u(T)} \operatorname{sign}[e^z - e^v]_+ \, dz - \int_0^{v(T)} \operatorname{sign}[1 - e^z]_+ \, dz \right)$$

$$\leqslant \int_0^T \int_\Omega (v)_t \left(\int_0^u (e^z - e^v)_+ \operatorname{sign}'_\delta[e^z - e^v] \, dz \right) + c \int_0^T \int_\Omega |u - v|_+ .$$

The term on the left converges to $\int_\Omega |u - v|_+(T)$ and the first term on right to zero as $\delta \to 0$, obtaining

$$\int_\Omega |u - v|_+(T) \leqslant c \int_0^T \int_\Omega |u - v|_+,$$

from which, by Gronwall's Lemma, it results

$$\log u_- = \log u_+ \qquad \text{on the set} \quad \{u_- > u_+\},$$

where the strict monotonicity of the function log implies a contradiction, therefore $u_- \leqslant u_+$.

Remark 2. From Theorem 6, it follows that the set of the limits of sequences of "regular" strictly positive solutions of (0.1)–(0.3) contains only one element.

Remark 3. The solutions of the kind $u_{\bar{c}}(x, t) := v(x - \bar{c}t, t)$, $\bar{c} \in \mathbb{R}^n$, when v is the (regular) solution of (3.1)–(3.3), have the property

$$\operatorname{supp} u_{\bar{c}} = (\mathbb{R}^+ \times \operatorname{supp} u_0) \cap \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \mid x - y + \bar{c}t = 0, y \in \bar{\Omega}\},$$

i.e., equality holds for the supports in Theorem 5, so all the $u_{\bar{c}}$ are distinct. Indeed, we compare v_ε with the functions constructed in the following way: for $y \in \operatorname{supp} u_0$ and $\delta > 0$ such that then exists $\rho > 0$ with $B_\rho(y) = \{x \mid |x - y| < \rho\} \subset \operatorname{supp} u_0$ and $\inf_{B\rho(y)} u_0 \geqslant \delta$, let $\phi_{\delta y}$ the first eigenfunction of Laplaces's equation on $B\rho(y)$ with $|\phi_{\delta y}|_\infty = \delta$, defined

$$w_{\bar{c}} := \begin{cases} \phi_{\delta y}(x + \bar{c}t)\, e^{-\gamma t} & \text{if} \quad x \in B\rho(y),\ t \in (0, \bar{t}), \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{t} = \sup\{t \mid B_\rho(y + ct) \subset \Omega\}$. We get that $w_{\bar{c}}$ is subsolution of (3.1)–(3.3) for γ large enough, independently on ε. That proves the claim.

To state a uniqueness theorem we assume:

(H5) (i) $u_0 \in H^{1,2}(\Omega)$: for every $\delta > 0$, there exists $\alpha_\delta > 0$ such that $\inf_{\Omega_{0\delta}} u_0 \geqslant \alpha_\delta$, where $\Omega_{0\delta} := \{x \in \operatorname{supp} u_0 \mid \operatorname{dist}(x, \partial \operatorname{supp} u_0) \geqslant \delta\}$. Also suppose $\{u \in H_0^{1,2}(\Omega) \mid u = 0$ in $C(\operatorname{supp} u_0)\} = H_0^{1,2}(\operatorname{supp} u_0)$.

(ii) u is a solution of (0.1)–(0.3) and there exists $\alpha_\delta(t) > 0$, $t \in (0, T)$ such that $\inf_{\Omega_{0\delta}} u(t) \geqslant \alpha_\delta(t)$.

LEMMA 3. *Assume* (H.1)–(H.5)(i). *Then* $u = \lim_{\varepsilon \to 0} u_\varepsilon$ (u_ε, $\varepsilon > 0$ *solutions of* (1.4)) *belongs to* $H^{1,2}((0, T) \times \Omega)$ *and satisfies* (*H*.5) (ii).

Proof. Let us consider the problems (1.4), $\varepsilon > 0$. Multiplying by $u_{\varepsilon t}/u_\varepsilon$ in (1.4) and integrating by parts we obtain

$$\int_0^T \int_\Omega \frac{(u_{\varepsilon t})^2}{u_\varepsilon} + \int_\Omega |\nabla u_\varepsilon|^2(t) = \int_\Omega \int_0^{u_\varepsilon(t)} \frac{g_\varepsilon(s)}{s}\, ds - \int_\Omega \int_0^{u_\varepsilon(0)} \frac{g_\varepsilon(s)}{s}\, ds + \int_\Omega |\nabla u_\varepsilon|^2(0),$$

where, using (H.2) and (H.5)(i), the term on the right side is bounded; from which $|u_{\varepsilon t}|_{L^2} \leqslant c$, c independent on ε. Then we can conclude $u \in H^{1,2}((0, T) \times \Omega))$.

To prove the second part of the claim, let $\delta > 0$ and

$$\mathbf{u}_\delta := \begin{cases} \alpha_{\delta/2} \phi_{0\delta/2}\, e^{-(\lambda_{0\delta/2} + K)t} & \text{in } (0, T) \times \Omega_{\delta/2} \\ 0 & \text{otherwise,} \end{cases}$$

$\alpha_{\delta/2}$, $\Omega_{0\delta/2}$ defined in (H.5)(i) and $\Delta \phi_{0\delta} + \lambda_{0\delta} \phi_{0\delta} = 0$ in $\Omega_{0\delta/2}$, $\phi_{0\delta} \geqslant 0$ in $\Omega_{\delta/2}$, $\phi_{0\delta/2} = 0$ on $\partial \Omega_{\delta/2}$ and K Lipschitz constant of $g(s)/s$ on $[0, \alpha_{\delta/2}]$. It is easy to verify that $\mathbf{u}_\delta$ is subsolution of (0.1)–(0.3) for every $\varepsilon > 0$ and then by comparison theorem

$$u_\varepsilon \geqslant u_\delta \quad \text{on} \quad (0, T) \times \Omega, \varepsilon > 0,$$

and in particular

$$u_\varepsilon \geqslant \alpha_{\delta/2} \inf_{\Omega_{0\delta}} \phi_{0\delta|2}\, e^{-(t_{0\delta/2} + K)t} = \alpha_\delta(t), \varepsilon > 0,$$

from which going to the limit as $\varepsilon \to 0$ $u \geqslant \alpha_\delta(t)$ on $(0, T) \times \Omega_{0\delta}$.

THEOREM 7. *Under assumptions* (H.1)–(H.5)(i) *the problem* (0.1) – (0.3) *admits a unique solution satisfying* (H.5)(ii).

Proof. For the sake of simplicity, we assume that supp u_0 be connected.

Let u, v be solutions of (0.1)–(0.3), where u is defined as in Lemma 3. By Theorem 6, $v \leqslant u$.

In order to prove $u = v$, let ϕ be the first eigenfunction of Δ on $\operatorname{supp} u_0$ and, for $\delta > 0$, sufficiently small,

$$\Phi_\delta \in C_0^\infty(\operatorname{supp} u_0), \qquad |\Phi - \Phi_\delta|_{H^{1,2}} \leqslant \delta.$$

We take as test functions in (1.2). $\varphi_u = \phi_\delta/u$ and $\varphi_v = \phi_\delta/v$, obtaining, after obvious manipulations,

$$\int_\Omega |\log u - \log v|(t)\, \phi_\delta + \int_0^t \int_\Omega (\nabla u - \nabla v)\, \nabla \phi_\delta$$
$$= \int_0^t \int_\Omega [f(u) - f(v)]\, \phi_\delta,$$

and taking the limit as $\delta \to 0$ and integrating by parts

$$\int_\Omega |\log u - \log v|(t)\, \phi + \lambda \int_0^t \int_\Omega |u - v|\, \phi = \int_0^t \int_\Omega (f(u) - f(v))\, \phi.$$

This yields

$$\int_\Omega |\log u - \log v|(t)\, \phi \leqslant c(\kappa, |u|_{L^\infty}, |v|_{L^\infty}) \int_0^t \int_\Omega |\log u - \log v|\, \phi$$

(c denotes a constant) and then, by Gronwall's lemma, it results $u = v$.

In general, if the $\operatorname{supp} u_0$ is not connected, i.e., $\operatorname{supp} u_0 = \bigcup \Omega_i$, Ω_i connected and $\Omega_i \cap \Omega_j = \varnothing$ we take in place of ϕ, the function $\psi := \sup_{i \in I}(\phi^i)$ where ϕ^i is the first eigenfunction in Ω_i, with eigenvalue λ_i, extended by zero on the whole domain.

Remark 4. *Assumption* (H.5) *cannot be weakened* as it is shown in following counterexample.

For the sake of simplicity, let us consider in (0.1)–(0.3) $g \equiv 0$, $n = 1$ and $0 \in \Omega = (-a, b)$, $a, b > 0$. Assume $u_0 \in W^{1,\infty}(\Omega)$, with $\inf_\Omega \Delta u_0 \geqslant \bar{c}$, $u_0(0) = 0$ and there exist $\rho > 0$ such that $u_0(x) \geqslant \alpha x^2$ in $B_\rho(0)$. We know that in this case a solution of (0.1)–(0.3) can be obtained solving the problem in $(0, T) \times (-a, 0)$ and in $(0, T) \times (0, b)$; but it is possible to prove that the regular solution u becomes positive in 0. For this purpose we construct a comparison solution $\tilde{u}$ with positivity property in 0. Observe that under the previous assumptions, we have $u(\rho, t) \geqslant \alpha \rho^2 e^{\bar{c}t}$, $t > 0$; in fact $v_\varepsilon := u_{\varepsilon t}/u_\varepsilon = \Delta u_\varepsilon$, satisfy the equation

$$v_{\varepsilon t} = u_\varepsilon\, \Delta v_\varepsilon + 2 \nabla u_\varepsilon\, \nabla v_\varepsilon + v_\varepsilon^2,$$

for which a minimum principle holds. Thus, let $\tilde{u}$ the regular solution in $(0, \tilde{T}) \times B_\rho(0)$, with initial data

$$\tilde{u}_0 := \begin{cases} \alpha x^2 & \text{if } \ |x| < \bar{x} \\ 2\alpha\bar{x}\,|x - (x/2)| & \text{if } \ \rho > |x| \geqslant \bar{x}, \end{cases}$$

$0 < \bar{x} < \rho$; and boundary data $b := 2\alpha\bar{x}(\rho - (\bar{x}/2)) > 0$. We choose $\tilde{T} = (1/|\bar{c}|)\log(\alpha\rho^2/b)$, that yields $u|_{(0,\tilde{T}) \times \partial B\rho(0)} \geqslant b$ and by the comparison theorem $\tilde{u} \leqslant u$ in $(0, \tilde{T}) \times B_\rho(0)$.

Since $\tilde{u}_0$ is a convex function we can affirm that $\tilde{u}$ is not decreasing in time or equivalently $\tilde{u}$ is convex with respect to x for all $t > 0$. For regular solutions multiplying (0.1) with 1, u, u_t/u, respectively, we get the three equations,

(i) $\partial_t \int_\Omega \log u = \int_{\partial\Omega} \partial_\nu u,$

(ii) $\int_\Omega \partial_t u + \int_\Omega |\nabla u|^2 = b \int_{\partial\Omega} \partial_\nu u,$

(iii) $\int_\Omega ((u_t)^2/u) + \partial_t \int_\Omega (|\nabla u|^2/2) = 0$

$(b = u|_{(0,T) \times \partial\Omega})$.

Equations (i) and (ii) lead to

$$\int_{B_\rho(0)} \log \tilde{u}(t) - \int_{B_\rho(0)} \log \tilde{u}(0) \geqslant \frac{t}{b} \int_{B_\rho(0)} |\nabla \tilde{u}|^2. \tag{3.5}$$

Moreover, from (iii), we can deduce that $\int_{B_\rho(0)} |\nabla \tilde{u}|^2$ is decreasing in time. In fact, if we assume $\int_{B_\rho(0)} |\nabla \tilde{u}|^2$ constant, that is, $\tilde{u}_t = 0$ a.e. in $(0, T) \times B_\rho(0)$, by (3.5), we obtain a contradiction. Assume $\tilde{u}(0, t) = 0$, $t > 0$; then

$$\int_{B_\rho(0)} |\nabla \tilde{u}|^2 > \frac{2b^2}{\rho}$$

which implies

$$\int_{B_\rho(0)} \log \tilde{u}(t) - \int_{B_\rho(0)} \log \tilde{u}(0) > \frac{2tb}{\rho}, \qquad t > 0. \tag{3.6}$$

By virtue of the convexity property, $\tilde{u} \leqslant (b/\rho)\,|x|$, $t > 0$, then (3.6) has to be verified when we consider $\omega := (b/\rho)\,|x|$ in place of $\tilde{u}(t)$. In order to prove $\tilde{u}(\cdot, 0) \not\equiv 0$, we show that

$$\int_{B_\rho(0)} \log \frac{b}{\rho}\,|x| - \int_{B_\rho(0)} \log \tilde{u}(0) \leqslant \frac{2b}{\rho}\,\tilde{T}, \tag{3.7}$$

which leads to a contradiction.

By obvious calculation in (3.6), we obtain

$$\bar{x}+\frac{\bar{x}}{2}\log\frac{b}{\alpha\bar{x}^2}\leqslant\frac{b}{\rho\,|\bar{c}|}\log\frac{\alpha\rho^2}{b}$$

or equivalently

$$\frac{\alpha}{|\bar{c}|}\geqslant\left(\rho\log\frac{e^2b}{\alpha\bar{x}^2}\right)\Big/\left(2(2\rho-\bar{x})\log\frac{\alpha\rho^2}{b}\right) \tag{3.8}$$

when, choosing $\rho/\bar{x}$ big enough, (3.7) is satisfied for $\alpha\geqslant|\bar{c}|$, which implies the claim.

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