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ABSTRACT

We call a subspace Y of a Banach space X a DBR subspace if its unit ball B_Y admits farthest points from a dense set of points of X. In this paper, we study DBR subspaces of $C(K)$. In the process, we study boundaries, in particular, the Choquet boundary of any general subspace of $C(K)$. An infinite compact Hausdorff space K has no isolated point if and only if any finite co-dimensional subspace, in particular, any hyperplane is DBR in $C(K)$. As a consequence, we show that a Banach space X is reflexive if and only if X is a DBR subspace of any superspace. As applications, we prove that any M-ideal or any closed $*$-subalgebra of $C(K)$ is a DBR subspace of $C(K)$. It follows that $C(K)$ is ball remotal in $C(K)^{**}$.

1. INTRODUCTION

For a closed and bounded set A in a Banach space X, the farthest distance map ϕ_A is defined as $\phi_A(x) = \sup\{\|z - x\|\colon z \in A\}$, $x \in X$. For $x \in X$, we define the farthest point map as $F_A(x) = \{z \in A\colon \|z - x\| = \phi_A(x)\}$, i.e., the set of points of A farthest from x. Note that this set may be empty. Let $R(A) = \{x \in X\colon F_A(x) \neq \emptyset\}$. Call a closed and bounded set A remotal if $R(A) = X$ and densely remotal if $R(A)$ is norm dense in X.

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Sets that are remotal or densely remotal have been studied in [4,8,13]. In particular, any compact set is clearly remotal while any weakly compact set is densely remotal [8].

The following definition is motivated by a recent paper [2].

Definition 1.1. Let us call a subspace Y of a Banach space X:

(a) ball remotal (BR) if its closed unit ball B_Y is remotal in X;
(b) densely ball remotal (DBR) if B_Y is densely remotal in X.

Let $C(K)$ denote the Banach space of all scalar-valued continuous functions on a compact Hausdorff space K with the supremum norm. In this paper we study DBR/BR subspaces of $C(K)$. Most of our results hold for both real and complex scalars and most often we prove them for complex scalars with more or less obvious modifications in the real case.

In Section 2, we study a special class of subspaces Y, called $(*)$-subspaces, such that $\phi_{B_Y}(x) = \phi_{B_X}(x)$ for all $x \in X$. We completely characterize:

(a) subspaces of $C(K)$ that are $(*)$-subspaces, and
(b) subspaces of $C(K)$ that are both $(*)$- and DBR/BR subspaces

in terms of the density of certain subsets of K. In the process, we prove that any Banach space embeds isometrically as a $(*)$- and DBR subspace of some $C(K)$ space.

In Section 3, we study boundaries of a general subspace Y of $C(K)$. In particular, we relate the Choquet boundary of Y with other boundaries, in the process recapturing some classical results. We also show that if Y is a subspace of co-dimension n in $C(K)$, then any closed boundary for Y can miss at most n points of K. In particular, if K has no isolated points, then any finite co-dimensional subspace cannot have any proper closed boundary.

Applying these results to the question of DBR subspaces, in Section 4, we show that an infinite compact Hausdorff space K has no isolated point if and only if any finite co-dimensional subspace, in particular, any hyperplane is DBR in $C(K)$. We characterize $(*)$- and DBR hyperplanes in $C(K)$ in terms of the defining measure. We also show that a Banach space X is reflexive if and only if X is a DBR subspace of any superspace in which it embeds isometrically as a hyperplane.

In Section 5, we obtain some partial results in the remaining cases. As applications, we prove that any M-ideal or any closed $*$-subalgebra of $C(K)$ is a DBR subspace of $C(K)$. It follows that $C(K)$ is BR in $C(K)^{**}$.

The closed unit ball and the unit sphere of a Banach space X are denoted by B_X and S_X respectively. We denote by $NA_1(X)$ the set of all $x^* \in S_{X^*}$ which attain their norm on B_X. For $t \in K$, let δ_t be Dirac measure at t, and for a subspace Y of $C(K)$, let $e_t = \delta_t|_Y$. $\mathbb{T} \subseteq \mathbb{C}$ denotes the unit circle.

Any unexplained terminology can be found in [7].

Let Y be a subspace of a Banach space X. Notice that $\phi_{B_Y}(x) \leqslant \phi_{B_X}(x) = \|x\| + 1$ for all $x \in X$.

Definition 2.1. Let us call a subspace Y a $(*)$-subspace of X if

$$\phi_{B_Y}(x) = \|x\| + 1 \quad \text{for all } x \in X.$$

Remark 2.2. This notion was introduced in [2] in a slightly different form.

Several examples of $(*)$-subspaces are discussed in [2]. For example, any Banach space is a $(*)$-subspace of its bidual.

Definition 2.3. We say that $A \subseteq B_{X^*}$ is a norming set for X if $\|x\| = \|x|_A\| := \sup\{|x^*(x)|\colon x^* \in A\}$ for all $x \in X$.

In [2, Proposition 2.4] it is noted that if

$$A_Y = \left\{x^* \in S_{X^*}\colon \left\|x^*|_Y\right\| = 1\right\}$$

is a norming set for X, then Y is a $(*)$-subspace of X. We do not know if the converse is true in general.[1] However, for subspaces of $C(K)$, we have the following result.

We will encounter the following subsets of K repeatedly throughout the paper. So let us fix our notations here.

Definition 2.4. Let Y be a subspace of $C(K)$. We define:

$$K' = \{t \in K\colon \delta_t \in A_Y\} = \{t \in K\colon \|e_t\| = 1\},$$
$$K_0 = \{t \in K\colon |g(t)| = 1 \text{ for some } g \in S_Y\} = \{t \in K\colon e_t \in NA_1(Y)\}.$$

Clearly, $K_0 \subseteq K'$. Also let $K_1 = \overline{K}_0$.

Theorem 2.5. *Let Y be a subspace of $C(K)$. The following are equivalent:*

(a) *Y is a $(*)$-subspace of $C(K)$.*
(b) *A_Y is a norming set for $C(K)$.*
(c) *K' is dense in K.*
(d) *K' is residual, i.e., contains a dense G_δ set in K.*

Proof. (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) is clear.

(a) $\Rightarrow$ (d). Let $F(t) = \|e_t\|$. Then F is clearly lower semicontinuous.

CLAIM: *$K' = \{t \in K\colon F$ is continuous at $t\}$, and hence, is residual* [5].

Since F is lower semicontinuous and $F \leqslant 1$, it is easy to see that F is continuous at points of $K' = \{t \in K\colon F(t) = 1\}$.

[1] We have recently shown that the converse is always true.

Now, suppose F is continuous at some $t_0 \notin K'$. Then $F(t_0) < 1$. Let $0 < \varepsilon < (1 - F(t_0))/2$. By continuity, there is an open neighborhood U of t_0 such that $|F(t) - F(t_0)| < \varepsilon$ for all $t \in U$.

Let $f \in C(K)$ be such that $f(K) \subseteq [0, 1]$, $f(t_0) = 1$ and $f|_{K \setminus U} \equiv 0$.

By (a), $\phi_{B_Y}(f) = \|f\| + 1 = 2$. Therefore, there exists $g \in B_Y$ such that $\|f - g\|_\infty > 2 - \varepsilon$. It follows that $f - g$ must attain its norm at some $t_1 \in U$. But

$$|g(t_1)| = |e_{t_1}(g)| \leqslant \|e_{t_1}\| = F(t_1) < F(t_0) + \varepsilon.$$

Thus,

$$\|f - g\|_\infty = |f(t_1) - g(t_1)| \leqslant |f(t_1)| + |g(t_1)| < 1 + F(t_0) + \varepsilon < 2 - \varepsilon,$$

a contradiction that completes the proof. □

Remark 2.6. If the scalars are real, the above claim can also be proved by using the characterization of sets in $C(K)$ that are intersection of closed balls given in [9, Proposition 4.1] and observing that Y is a $(*)$-subspace of X if and only if the intersection of closed balls containing B_Y equals B_X.

Coming to ball remotality, it is easy to see that:

Proposition 2.7. *If Y is a subspace of $C(K)$ such that K_0 is finite, then Y is BR.*

Proof. Clearly, Y embeds isometrically into $C(\overline{K}_0)$. If K_0 is finite, it follows that Y is finite-dimensional, and hence, BR. □

Proposition 2.8. *Let Y be a subspace of $C(K)$, K_0 as defined above and let $A = \{f \in C(K)\colon\ f(t) = \|f\|_\infty$ for some $t \in K_0\}$. Then $A \subseteq R(B_Y)$.*

If Y is a $(*)$-subspace, then $A = R(B_Y)$.

Proof. Let $f \in A$ and $t \in K_0$ be such that $|f(t)| = \|f\|_\infty$. By definition of K_0, there exists $g \in S_Y$ such that $|g(t)| = 1$. Then for a suitable scalar $\alpha \in \mathbb{T}$, $\|f - \alpha g\|_\infty = |f(t) - \alpha g(t)| = \|f\|_\infty + 1$. Thus, $f \in R(B_Y)$.

Conversely, if Y is a $(*)$-subspace and $f \in R(B_Y)$, then $\|f - g\|_\infty = \|f\|_\infty + 1$ for some $g \in B_Y$. Now, if $f - g$ attains its norm at $t_0 \in K$, then $\|f\|_\infty = |f(t_0)|$ and $|g(t_0)| = 1$. Thus $t_0 \in K_0$ and hence, $f \in A$. □

Proposition 2.9. *For a compact Hausdorff space K, the following are equivalent:*

(a) *Each singleton in K is a G_δ.*

(b) *For any $t_0 \in K$, there exists $f \in C(K)$ such that $f(K) \subseteq [0, 1]$, $f(t_0) = 1$ and $f(t) < 1$ for all $t \neq t_0$.*

Proof. (a) $\Rightarrow$ (b). Let $t_0 \in K$. By (a), there exists open sets $\{U_n\}$ such that $\{t_0\} = \bigcap_n U_n$. Get $\{f_n\} \subseteq C(K)$ such that $f_n : K \to [0, 1]$, $f_n(t_0) = 1$ and $f_n(U_n^c) = 0$.

Define $f(t) = \sum_n 2^{-n} f_n(t)$. Clearly, $f \in C(K)$ and $f(t_0) = 1$. If $t \in K$ and $t \neq t_0$, there exists U_m such that $t \notin U_m$. So $f_m(t) = 0$, and hence, $f(t) < 1$.

(b) $\Rightarrow$ (a). If such an $f \in C(K)$ exists, then $\{t_0\} = f^{-1}(\{1\})$ is a G_δ. □

Theorem 2.10. *Let Y be a subspace of $C(K)$ and K_0 as above.*

If $K_0 = K$, then Y is $(*)$- and BR in $C(K)$.

And if each singleton in K is a G_δ, in particular if K is metrizable, then the converse is also true.

Proof. If $K_0 = K$, it follows from Theorem 2.5 that Y is a $(*)$-subspace. Moreover, in Proposition 2.8, $A = C(K)$ and therefore, Y is BR.

Conversely, if Y is $(*)$ and BR, let $t_0 \in K$. By Proposition 2.9, there exists $f \in C(K)$ such that $f(K) \subseteq [0, 1]$, $f(t_0) = 1$ and $f(t) < 1$ for all $t \neq t_0$. Since $f \in R(B_Y)$, by Proposition 2.8, $t_0 \in K_0$, that is, $K_0 = K$. □

Corollary 2.11. *Let Y be a subspace of $C(K)$.*

(a) *If Y contains a unimodular function, in particular, if Y contains constants, then Y is BR in $C(K)$.*

(b) *$Y = \{g \in C[0, 1]\colon \int_0^1 g(t)\,dt = 0\}$ is BR in $C[0, 1]$.*

For a Banach space X, let $C(K, X)$ denote the space of all continuous functions from K to X. We will need the following result only when X is the scalars, but the general result follows at no extra effort.

Lemma 2.12. *Let $L \subseteq K$ be such that $\overline{L} = K$. For $f \in C(K, X)$ and $\varepsilon > 0$, there exists $g \in C(K, X)$ such that g attains its norm on L and $\|f - g\|_\infty < \varepsilon$.*

Proof. Let $\|f\|_\infty = M$. There exists $x_0 \in f(K)$ such that $\|x_0\| = M$.

Since L is dense in K, there exists $u \in L$ such that $\|f(u) - x_0\| < \varepsilon$.

Let $x_1 = f(u)$. Choose r_0 such that $\|x_1 - x_0\| < r_0 < \varepsilon$ and split X into three disjoint regions:

$$X_1 = \{x \in X\colon \|x - x_0\| > \varepsilon\},$$
$$X_2 = \{x \in X\colon \|x - x_0\| \leqslant r_0\}$$

and

$$X_3 = \{x \in X\colon r_0 < \|x - x_0\| \leqslant \varepsilon\}.$$

Define $\phi : X_1 \cup X_2 \to X$ as follows:

$$\phi(x) = x \quad \text{if } x \in X_1, \qquad \phi(x) = x_0 \quad \text{if } x \in X_2.$$

To define ϕ on X_3, notice that any point in X_3 is of the form $x_0 + ry$ for some $r \in (r_0, \varepsilon]$ and $y \in S_X$. Define $h : [r_0, \varepsilon] \to [0, \varepsilon]$ by $h(r) = \frac{r - r_0}{\varepsilon - r_0}\varepsilon$ and define $\phi : X_3 \to X$ by $\phi(x_0 + ry) = x_0 + h(r)y$.

CLAIM: $\phi : X \to X$ *is continuous.*

It clearly suffices to check the continuity of ϕ on $\overline{X}_3$.

Let $(z_n), z_0 \in \overline{X}_3$ such that $z_n \to z_0$. Then $z_n = x_0 + r_n y_n$ and $z_0 = x_0 + ry$, for some $r_n, r \in [r_0, \varepsilon]$ and $y_n, y \in S_X$.

Clearly, $r_n \to r$, and since $r \geqslant r_0 > 0$, $\frac{r_n}{r} y_n \to y$. Therefore, $\|y_n - y\| \leqslant \|y_n - \frac{r_n}{r} y_n\| + \|\frac{r_n}{r} y_n - y\| \to 0$ as $n \to \infty$. Now

$$\begin{aligned} \|\phi(z_n) - \phi(z_0)\| &= \|h(r_n)y_n - h(r)y\| \leqslant |h(r_n) - h(r)| + |h(r)|\|y_n - y\| \\ &\leqslant \varepsilon\left(\left|\frac{r_n - r}{\varepsilon - r_0}\right| + \|y_n - y\|\right) \to 0 \end{aligned}$$

as $n \to \infty$. This proves the claim.

Define $g : K \to X$ by $g = \phi \circ f$.

Note that $g(K) \subseteq \phi(MB_X) \subseteq MB_X$ as the last set is convex and ϕ maps a point z of X_3 to a point on the straight line $[z, x_0]$. It follows that $\|g\|_\infty \leqslant M = \|x_0\| = \|\phi(x_1)\| = \|g(u)\|$. Thus, g attains its norm on L.

Moreover,

$$\begin{aligned} \|f - g\|_\infty &\leqslant \sup\big\{\|x - \phi(x)\|\colon\ x \in X\big\} = \sup\{\|x - \phi(x)\|\colon\ x \in X_2 \cup X_3\} \\ &= \max\big\{\sup\{\|x - \phi(x)\|\colon\ x \in X_2\}, \sup\{\|x - \phi(x)\|\colon\ x \in X_3\}\big\} \\ &= \max\big\{r_0, \sup\{|r - h(r)|\colon\ r \in (r_0, \varepsilon]\}\big\} \\ &= \max\left\{r_0, \sup\left\{\frac{r_0(\varepsilon - r)}{\varepsilon - r_0}\colon\ r \in (r_0, \varepsilon]\right\}\right\} \leqslant r_0 < \varepsilon. \end{aligned}$$

This completes the proof. $\square$

Theorem 2.13. *Let Y be a subspace of $C(K)$ and K_0 as above. Then Y is a $(*)$-and DBR subspace of $C(K)$ if and only if $\overline{K}_0 = K$.*

Proof. With the notation of Definition 2.4, since $K_0 \subseteq K'$, if $\overline{K}_0 = K$, then by Theorem 2.5, Y is a $(*)$-subspace of $C(K)$.

Putting $L = K_0$ in Lemma 2.12, we get that the set A in Proposition 2.8 is dense in $C(K)$. And hence, Y is DBR.

Conversely, if Y is a $(*)$-subspace of $C(K)$, by Proposition 2.8, $A = R(B_Y)$. It follows that for any $f \in R(B_Y)$, $\|f\|_\infty = \|f|_{K_0}\|_\infty$. Since $\{f \in C(K)\colon\ \|f\|_\infty = \|f|_{K_0}\|_\infty\}$ is a closed set containing $R(B_Y)$, it follows that if Y is also DBR, then $\|f\|_\infty = \|f|_{K_0}\|_\infty$ for all $f \in C(K)$. Hence $\overline{K}_0 = K$. $\square$

Corollary 2.14. *Any Banach space X is a $(*)$-subspace as well as DBR subspace of $C(K)$, where $K = (B_{X^*}, w^*)$ if X is infinite-dimensional, and $K = (S_{X^*}, norm)$, if X is finite-dimensional.*

Proof. If X is finite-dimensional, it is BR in any superspace.

If X is infinite-dimensional, let $K = (B_{X^*}, w^*)$. Then X embeds isometrically as a subspace of $C(K)$. Now notice that $K_0 = NA_1(X)$, which is norm dense in S_{X^*} by Bishop–Phelps Theorem, and hence w^*-dense in B_{X^*}. $\square$

Some of the results in this section may be folklore, but we have not found them recorded anywhere, hence we include proofs.

Definition 3.1. Let Y be a subspace of $C(K)$. A set $B \subseteq K$ is said to be a boundary for Y if for every $g \in Y$, there exists $t \in B$ such that $|g(t)| = \|g\|_\infty$.

For a subspace Y of $C(K)$, recall (Definition 2.4) that $K_0 = \{t \in K\colon |g(t)| = 1 \text{ for some } g \in S_Y\}$ and $K_1 = \overline{K}_0$. Clearly, K_0 is a boundary for Y and K_1 is a closed boundary.

Lemma 3.2. *Let Y be a subspace of $C(K)$. Let $B \subseteq K$ be a closed boundary for Y. For any $t \in K$, there exists a regular complex Borel measure μ_t on B such that $\|\mu_t\| = \|e_t\|$ and $g(t) = \int_B g\, d\mu_t$ for all $g \in Y$. Call μ_t a representing measure for t on B.*

If Y separates points of K, the map $t \to \mu_t$ is one-one.

Proof. Since B is a closed boundary for Y, the map $g \to g|_B$ is an isometry between Y and $Y|_B \subseteq C(B)$. Therefore, e_t induces a functional $\Lambda \in (Y|_B)^*$ with $\|\Lambda\| = \|e_t\|$. Any norm preserving extension of Λ on $C(B)$ corresponds to a regular Borel measure μ_t on B such that $\|\Lambda\| = \|\mu_t\|$.

The last statement in the lemma is obvious. □

Theorem 3.3. *If Y is a subspace of co-dimension n in $C(K)$ and $B \subseteq K$ is a closed boundary for Y, then $K \setminus B$ contains at most n distinct points.*

In particular, if K has no isolated points, then $B = K$.

Proof. Suppose there are $(n+1)$ distinct points $t_1, t_2, \ldots, t_{n+1}$ in $K \setminus B$. Let μ_i be a representing measure for t_i on B. If $e_{t_i} = 0$ for some i, then $\mu_i = 0$ and $\delta_{t_i} \in Y^\perp$. Since each μ_i has no point mass outside of B, it is clear that the measures $\mu_i - \delta_{t_i}$ are linearly independent. Since each $\mu_i - \delta_{t_i} \in Y^\perp$, this contradicts the fact that Y has co-dimension n.

Now, if $K \setminus B$ is nonempty, it contains at most n points and necessarily these points are isolated. Thus, if K has no isolated points, then $B = K$. □

Definition 3.4. Let Y be a subspace of $C(K)$. The Choquet boundary of Y is defined as

$$\partial Y = \{t \in K\colon e_t \in \operatorname{ext} B_{Y^*}\},$$

where $\operatorname{ext} B_{Y^*}$ denotes the set of extreme points of B_{Y^*}.

This definition (given in [10, p. 29], when $1 \in Y$) coincides with the classical definition of the Choquet boundary when Y separates points of K and contains the constants [10, Proposition 6.2, p. 29]. We use the same definition even when $1 \notin Y$. It is clear that ∂Y is a boundary and therefore we obtain the following corollary.

Corollary 3.5. *If Y is a subspace of co-dimension n in $C(K)$, then $K \setminus \overline{\partial Y}$ contains at most n distinct points. And if K has no isolated points, then $\overline{\partial Y} = K$.*

Remark 3.6. The stronger result that the set $K \setminus \partial Y$ itself contains at most n points, was proved, under the additional assumption that Y separates points of K, in [6, Lemma 5.6, Theorem 7.3] and in full generality in [1, Proposition 3.1]. Our argument is significantly simpler.

It is well known that when Y separates points of K and contains the constants, the Choquet boundary is contained in any closed boundary [10, Proposition 6.4, p. 30]. In our next result, we relate the Choquet boundary with other closed boundaries for a general subspace Y of $C(K)$.

Let $B \subseteq K$ be a closed boundary for Y. The map $e : K \to (B_{Y^*}, w^*)$ defined by $e(t) = e_t$ is clearly continuous and hence, $\mathbb{T}e(B)$ is a w^*-compact subset of B_{Y^*}.

Theorem 3.7. *Let Y be a subspace of $C(K)$ and $B \subseteq K$ a closed boundary for Y. Then:*

(a) $e(\partial Y) \subseteq \mathbb{T}e(B)$.
(b) *If Y contains the constants and separates points of K, then $\partial Y \subseteq B$.*
(c) *If $\{|f| \colon f \in Y\}$ separates points of K then also $\partial Y \subseteq B$.*
(d) *If K_0, as in Definition 2.4, is closed, then $\partial Y \subseteq K_0$.*

Proof. (a). Since $e(B)$ is a norming set for Y, by separation arguments,

$$B_{Y^*} = \overline{\text{conv}}^{w^*}\big(\mathbb{T}e(B)\big).$$

By Milman's theorem [10, Proposition 1.5, p. 6], we have

$$e(\partial Y) \subseteq \text{ext}\, B_{Y^*} \subseteq \mathbb{T}e(B).$$

(b). If $t \in \partial Y$, then by (a), there are $\gamma \in \mathbb{T}$ and $b \in B$ such that $g(t) = \gamma g(b)$ for all $g \in Y$. (b) follows.

(c). By (a), if $t \in \partial Y$, then there is $b \in B$ such that

$$|g(t)| = |g(b)| \quad \text{for all } g \in Y. \tag{1}$$

Now the hypothesis implies $t = b$, i.e., $\partial Y \subseteq B$.

(d) follows from (1) and the definition of K_0. □

Remark 3.8. (a) If $Y = \{f \in C(K) \colon f|_D \equiv 0\}$, where $D \subseteq K$ is a closed set, then $K' = \{t \in K \colon \|\delta_t|_Y\| = 1\} = K \setminus D$ and points of K' are separated by nonnegative functions in Y. Therefore, (c) holds.

(b) Though $\partial Y \subseteq K_0$ when K_0 is closed, the two sets need not be equal. For example, if $Y = \{f \in C[0,1]\colon\ f(0) = \int_0^1 f(t)\,dt\}$, then $K_0 = [0,1]$ is closed but $0 \notin \partial Y$ as it has a representing measure other than δ_0, namely, the Lebesgue measure on $[0,1]$.

4. FINITE CO-DIMENSIONAL SUBSPACES OF $C(K)$

Coming back to DBR subspaces, let Y be a subspace of finite co-dimension in $C(K)$. Let K_0 and K_1 be as in Definition 2.4. Now, Theorem 3.3 yields the following corollary.

Corollary 4.1. *If Y has co-dimension n, then there can be at most n distinct points in $K \setminus K_1$. And if K has no isolated points, then $K_1 = K$.*

And therefore:

Theorem 4.2. *If K has no isolated points, then any finite co-dimensional subspace of $C(K)$ is a $(*)$- and DBR subspace.*

Remark 4.3. If K is infinite and has no isolated points, e.g., $K = [0,1]$, then $C(K)$ clearly has hyperplanes that are not proximinal. Thus, DBR subspaces need not be proximinal.

It also follows that:

Corollary 4.4. *If $\mu_1, \mu_2, \ldots, \mu_n$ are non-atomic measures, then $Y = \bigcap_{i=1}^n \ker(\mu_i)$ is a $(*)$- and DBR subspace of $C(K)$.*

Proof. If $K \setminus K_1$ is nonempty, let $t_1, t_2, \ldots, t_m \in K \setminus K_1$ for some $m \leqslant n$.

Let μ_i be a representing measure for t_i on K_1. Then $\mu_i - \delta_{t_i} \in Y^\perp$. It follows that at least some elements of $Y^\perp$ must put nonzero mass on the points $t_1, t_2, \ldots, t_m$. Hence the result follows. $\square$

Theorem 4.5. *If Y is of co-dimension n and $K \setminus K_1$ contains exactly n points, then K_0 is closed. Moreover, $\partial Y = \overline{\partial Y} = K_0 = K_1$.*

Proof. For simplicity, we give the proof for $n = 2$ as no new ideas are required for other values of n.

Let $t_1, t_2 \in K \setminus K_1$ with representing measures μ_1, μ_2 respectively. Let $Y_i = \ker(\mu_i - \delta_{t_i})$, $i = 1, 2$. Then $Y = Y_1 \cap Y_2$.

Find $f_1, f_2 \in C(K)$ such that $f_i(t_j) = \delta_{ij}$ and $f_i|_{K_1} = 0$ for $i, j = 1, 2$. Then $(\mu_i - \delta_{t_i})(f_j) = -\delta_{ij}$. It follows that for any $f \in C(K)$, $g = f + (\mu_1 - \delta_{t_1})(f) \cdot f_1 + (\mu_2 - \delta_{t_2})(f) \cdot f_2 \in Y$.

Now define $f \in C(K)$ by $f|_{K_1} = 1$, $f(t_1) = f(t_2) = 0$ and consider $g \in Y$ as above. Then $g|_{K_1} = 1$ and $g(t_i) = \mu_i(K_1)$, $i = 1, 2$. Thus, $\|g\|_\infty = \max\{1, |\mu_1(K_1)|, |\mu_2(K_1)|\}$.

Since g attains its norm only on $K_0 \subseteq K_1$, we must have $|\mu_1(K_1)| < 1$, $|\mu_2(K_1)| < 1$ and $K_1 \subseteq K_0$, and so, K_0 is closed.

Now by Theorem 3.7(d), $\partial Y \subseteq \overline{\partial Y} \subseteq K_0$. If $K_0 \setminus \partial Y$ were nonempty, there would be more than n points outside ∂Y contradicting [1, Proposition 3.1]. □

Remark 4.6. What happens if Y is of co-dimension n but $K \setminus K_1$ has fewer than n points? We don't know the answer but a look at some examples seems to suggest that if $t \notin K_1$, then $e_t \notin \operatorname{ext} B_{Y^*}$, and so $\overline{\partial Y} \subseteq K_1$.

Theorem 4.7. *Let K be an infinite compact Hausdorff space. The following are equivalent:*

(a) *K has no isolated point;*
(b) *any finite co-dimensional subspace of $C(K)$ is DBR;*
(c) *any hyperplane in $C(K)$ is DBR.*

Proof. (a) ⇒ (b) ⇒ (c) is clear from Theorem 4.2.

(c) ⇒ (a). Suppose $t_0 \in K$ is an isolated point. Then $K = T \cup \{t_0\}$, where T is closed. Since $C(T)$ is nonreflexive, there exists $\mu \in S_{C(T)^*}$ such that μ is not norm attaining on $C(T)$. Now let $Y = \ker(\delta_{t_0} - \mu)$, i.e.,

$$Y = \{f \in C(K)\colon\ f(t_0) = \mu(f|_T)\}.$$

It follows that given any $h \in C(T)$, if we define $f : K \to \mathbb{C}$ as

$$f(t) = \begin{cases} h(t) & \text{if } t \in T, \\ \mu(h) & \text{if } t = t_0, \end{cases}$$

then $f \in Y$ and $\|f\|_\infty = \|h\|_\infty$. Thus, $T \subseteq K_0$.

CLAIM: $\|e_{t_0}\| = 1$, *but* $t_0 \notin K_0$.

Since $\|\mu\| = 1$, there exists $(h_n) \subseteq S_{C(T)}$ such that $\mu(h_n) \to 1$. If we define the corresponding $f_n \in S_Y$ as above, then $f_n(t_0) \to 1$. Thus, $\|e_{t_0}\| = 1$.

On the other hand, since μ is not norm attaining on $C(T)$, $t_0 \notin K_0$.

It follows that $K_0 = T$ and $K' = K$. Therefore, Y is a $(*)$-subspace, and hence, cannot be a DBR subspace of $C(K)$. □

Remark 4.8. (a) If K is finite, $C(K)$ is finite-dimensional and hence, any subspace is DBR, but any point of K is also isolated.

(b) Since $K = T \cup \{t_0\}$, $C(K) = C(T) \bigoplus_\infty \mathbb{C}$. Therefore by [2, Lemma 3.1], $C(T)$ is BR in $C(K)$. Clearly, Y is isometric to $C(T)$, but Y is not even DBR in $C(K)$. This emphasizes the fact that this property not only depends on the norm, but also on how Y 'sits' in X.

A simple example of the above phenomenon is given by the following example.

Example 4.9. Let $K = [0,1] \cup \{2\}$ and μ be the measure on $[0,1]$ defined by $\mu = \lambda|_{[0,1/2]} - \lambda|_{[1/2,1]}$, where λ denotes the Lebesgue measure on $[0,1]$.

Now define

$$Y = \left\{ F \in C(K)\colon\ F(2) = \int_0^{1/2} F(x)\,dx - \int_{1/2}^{1} F(x)\,dx \right\}.$$

It is easy to see that μ is not norm attaining on $C[0,1]$. It follows that $K_0 = [0,1]$ and $K' = K$. Therefore, Y is a $(*)$-subspace of $C(K)$. But Y cannot be a DBR subspace of $C(K)$.

Similar examples can be constructed for any finite co-dimension. For example for $n = 2$, let $K = [0,1] \cup \{-1\} \cup \{2\}$, and let $Y = \ker(\mu_1) \cap \ker(\mu_2)$ where

$$\mu_1 = \frac{1}{2} \cdot \delta_2 + \lambda|_{[\frac{1}{2},\frac{3}{4}]} - \lambda|_{[\frac{3}{4},1]},$$
$$\mu_2 = \frac{1}{2} \cdot \delta_{-1} + \lambda|_{[\frac{1}{4},\frac{1}{2}]} - \lambda|_{[0,\frac{1}{4}]}.$$

One can check as before that Y is a $(*)$-subspace, but not DBR in $C(K)$.

Now we can characterize $(*)$- and DBR hyperplanes in $C(K)$.

Theorem 4.10. *Let $\mu \in C(K)^*$, $\|\mu\| = 1$. Then $Y = \ker(\mu)$ is not $(*)$ and DBR in $C(K)$ if and only if the following conditions hold:*

(a) *There is an isolated point $t_0 \in K$ such that $|\mu(\{t_0\})| \geqslant 1/2$.*
(b) *If we write $\mu = \alpha\delta_{t_0} + \nu$ and $|\alpha| = 1/2$, then ν is not norm attaining on $C(K \setminus \{t_0\})$.*

Proof. First assume Y is not $(*)$ and DBR in $C(K)$.

Then $K_1 \neq K$ and hence, by Theorem 4.5, K_0 is closed and there exists exactly one isolated point $t_0 \in K$ such that $t_0 \notin K_0$ and $K = K_0 \cup \{t_0\}$.

Now we can write $\mu = \alpha\delta_{t_0} + \nu$, where ν is supported on K_0. Moreover, $1 = \|\mu\| = |\alpha| + \|\nu\|$.

If $|\alpha| < 1/2$, then $\|\nu\| = 1 - |\alpha| > 1/2 > |\alpha|$. So there exists $g \in B_{C(K_0)}$ such that $\nu(g) = |\alpha|$. Define $f \in B_{C(K)}$ by

$$f(t) = \begin{cases} g(t) & \text{if } t \in K_0, \\ -\operatorname{sgn}(\alpha) & \text{if } t = t_0. \end{cases}$$

Clearly $f \in S_Y$ and $|f(t_0)| = 1$, which implies $t_0 \in K_0$. This contradiction ensures that $|\alpha| \geqslant 1/2$.

Now suppose $|\alpha| = 1/2$, then $\|\nu\| = 1/2$. If ν is norm attaining on $C(K_0)$, we can get $g \in B_{C(K_0)}$ such that $\nu(g) = \|\nu\| = 1/2 = |\alpha|$ and hence, $f \in B_{C(K)}$ defined as above satisfies $f \in S_Y$ and $|f(t_0)| = 1$. This again implies $t_0 \in K_0$. A contradiction!

Conversely assume that (a) and (b) hold. It is enough to prove that $t_0 \notin K_0$.

If $t_0 \in K_0$, then there exists $f \in B_Y$ with $|f(t_0)| = 1$. It follows that

$$|\alpha| = |\alpha f(t_0)| = |\nu(f)| \leqslant \|\nu\| = 1 - |\alpha|$$

which implies $|\alpha| \leqslant 1/2$. This together with (a) implies $|\alpha| = 1/2$, and hence, $\|\nu\| = 1/2$. It follows that $|\nu(f)| = |\alpha| = 1/2$. Thus ν is norm attaining, contradicting (b). Hence $t_0 \notin K_0$. $\square$

Remark 4.11. In the above, if $|\alpha| = \|\nu\| = 1/2$ and ν is norm attaining on $C(K \setminus \{t_0\})$, then $Y = \ker(\mu)$ is actually $(*)$ and ball remotal. Indeed, from the above proof, it follows that $t_0 \in K_0$. Now define g on $B_{C(K)}$ by

$$g(t) = \begin{cases} 1 & \text{if } t \neq t_0, \\ -\nu(1)/\alpha & \text{if } t = t_0. \end{cases}$$

Since $|\nu(1)| \leqslant \|\nu\| = |\alpha| = 1/2$, $g \in S_Y$ and therefore, $K = K_0$.

We now obtain a characterization of reflexivity.

Let Y be a subspace of a Banach space X. It is not difficult to see that $R(B_Y) \supseteq \{x \in X\colon$ there are $x^* \in S_{X^*}$, $y \in S_Y$ such that $x^*(x) = \|x\|$ and $x^*(y) = 1\}$ and [2, Proposition 2.9] observed that if Y is a $(*)$-subspace of X, then the two sets coincide.

Proposition 4.12. *Let $N_Y = \{x^* \in S_{X^*}\colon x^*(y) = 1$ for some $y \in S_Y\}$. If Y is a $(*)$-and DBR subspace of X, then N_Y is norming for X.*

Proof. If Y is a $(*)$-subspace of X, it follows from the result quoted above that $\|x\| = \|x|_{N_Y}\|$ for any $x \in R(B_Y)$. If Y is a DBR subspace, as in Theorem 2.13, it follows that N_Y is a norming set for X. $\square$

In the following result, we use real scalars just for notational convenience.

Theorem 4.13. *Let X be a nonreflexive Banach space. Then there exists a Banach space Z and a hyperplane Y in Z such that X is isometric to Y and Y is not a DBR subspace of Z.*

Proof. Define $Z = X \bigoplus_\infty \mathbb{R}$. Since X is nonreflexive, there exists $x_0^* \in S_{X^*}$ which is not norm attaining.

Let $Y = \{(x, x_0^*(x))\colon x \in X\}$. Clearly, Y is a hyperplane in Z. Since $\|x_0^*\| = 1$, Y is isometric to X.

CLAIM 1: *Y is a $(*)$-subspace of Z.*

Clearly, $\{(x^*, 0)\colon x^* \in S_{X^*}\} \subseteq A_Y$. And since $\|x_0^*\| = 1$, it also follows that $(0, 1) \in A_Y$. Thus A_Y is norming for Z.

CLAIM 2: $N_Y = \{(x^*, 0)\colon x^* \in NA_1(X)\}$.

Let $z^* = (x^*, \alpha) \in N_Y$. Then $|(x^*, \alpha)(x, x_0^*(x))| = 1$ for some $x \in S_X$. But then, $1 = |x^*(x) + \alpha x_0^*(x)| \leqslant |x^*(x)| + |\alpha| \cdot |x_0^*(x)| \leqslant \|x^*\| + |\alpha| = 1$. Since x_0^* is not norm attaining, $\alpha = 0$ and $|x^*(x)| = \|x^*\|$. Hence the claim.

But clearly, N_Y cannot be norming for Z and hence, by Proposition 4.12, Y cannot be a DBR subspace of Z. $\square$

Corollary 4.14. *For a Banach space X, the following are equivalent:*

(a) *X is reflexive.*
(b) *X is a DBR subspace of any superspace.*
(c) *X is a DBR subspace of any superspace in which it embeds isometrically as a hyperplane.*

5. OTHER DBR SUBSPACES OF $C(K)$

Let Y be a subspace of $C(K)$. Recall that (Definition 2.4) $K_0 = \{t \in K\colon |g(t)| = 1 \text{ for some } g \in S_Y\}$ and $K_1 = \overline{K_0}$. We may assume $K_1 \neq K$. Note that $g \mapsto g|_{K_1}$ is an isometric embedding of Y into $C(K_1)$, and $B_Y|_{K_1} = B_{(Y|_{K_1})}$ is densely remotal in $C(K_1)$.

Theorem 5.1. *Let $K_2 = \overline{K \setminus K_1}$. Suppose $B_Y|_{K_2}$ is remotal in $C(K_2)$, then Y is a DBR subspace of $C(K)$.*

Proof. Let $h \in C(K)$ and $\varepsilon > 0$. Let $h_1 = h|_{K_1}$. Since $B_Y|_{K_1}$ is densely remotal in $C(K_1)$, there is some $f_1 \in C(K_1)$ such that $\|f_1 - h_1\|_\infty < \varepsilon$ and $f_1 \in R(B_Y|_{K_1})$. By Tietze's extension theorem, there is $f \in C(K)$ such that $\|f - h\|_\infty < \varepsilon$ and $f|_{K_1} = f_1$. Let $g_1 \in B_Y$ be such that for all $g \in B_Y$, $\|(f+g)|_{K_1}\|_\infty \leqslant \|(f+g_1)|_{K_1}\|_\infty$. Let $f_2 = f|_{K_2}$. Since $B_Y|_{K_2}$ is remotal in $C(K_2)$, there exist $g_2 \in B_Y$ such that $\|(f+g)|_{K_2}\|_\infty \leqslant \|(f+g_2)|_{K_2}\|_\infty$ for all $g \in B_Y$. Then for all $g \in B_Y$, $\|f+g\|_\infty = \max\{\|(f+g)|_{K_1}\|_\infty, \|(f+g)|_{K_2}\|_\infty\} \leqslant \max\{\|(f+g_1)|_{K_1}\|_\infty, \|(f+g_2)|_{K_2}\|_\infty\}$. Now, depending on which of the two terms on the RHS is bigger, either $-g_1$ or $-g_2$ is farthest from f in B_Y. Hence Y is DBR in $C(K)$. $\square$

Remark 5.2. If Y is finite co-dimensional, K_2 is finite and hence, as soon as $B_Y|_{K_2}$ is closed in $C(K_2)$, it is remotal and Theorem 5.1 applies. However, as Theorem 4.7 shows, $B_Y|_{K_2}$ need not be closed in $C(K_2)$.

Interchanging the roles of K_1 and K_2 in the above argument, we also obtain the following theorem.

Theorem 5.3. *Suppose K_0 is closed and $B_Y|_{K_2}$ is densely remotal in $C(K_2)$, then Y is a DBR subspace of $C(K)$.*

Corollary 5.4. *Suppose for all $g \in Y$, $g|_{K_2} \equiv 0$, then Y is a DBR subspace of $C(K)$.*

Theorem 5.5. *Any M-ideal in $C(K)$ is a DBR subspace of $C(K)$.*

Proof. Recall that any M-ideal in $C(K)$ is of the form $Y = \{f \in C(K)\colon f|_D \equiv 0\}$ for some closed set $D \subseteq K$ (see [7, Example 1.4(a)]).

It is easy to see that in this case, $K_0 = K \setminus D$ and therefore, $K_2 = D$. Thus, the result follows from Corollary 5.4. $\square$

Remark 5.6. It is observed in [2] that in general an M-ideal may fail to be a DBR subspace.

Recently, we have shown [3] that in many function spaces and function algebras, an M-ideal is a DBR subspace, including an alternative proof of Theorem 5.5.

Theorem 5.7. *Let $\{\mu_n\}$ be countable family of regular Borel measures on K. Let $S(\mu_n)$ denote the support of μ_n. Suppose*

(a) *for each $n \geqslant 1$, $K \setminus S(\mu_n)$ is dense in K, and*
(b) *$\bigcup_n S(\mu_n)$ is a closed subset of K.*

Then $Y = \bigcap_n \ker(\mu_n)$ is a DBR subspace of $C(K)$.

Proof. Let $D = \bigcup_n S(\mu_n)$. Let $Z = \{f \in C(K)\colon\ f|_D \equiv 0\}$. By Baire Category Theorem, $K \setminus D$ is dense in K. Therefore, by Theorem 2.13 and Theorem 5.5, Z is a DBR $(*)$-subspace of $C(K)$. Since $Z \subseteq Y \subseteq C(K)$, Y is also a DBR $(*)$-subspace of $C(K)$. □

Proposition 5.8. *Let K and S be compact Hausdorff spaces, $\sigma : K \to S$ a continuous onto map, and $s_0 \in S$. Then*

$$Y = \{h \circ \sigma\colon\ h \in C(S) \text{ and } h(s_0) = 0\}$$

is a DBR subspace of $C(K)$.

Proof. Since σ is onto, $\|h \circ \sigma\|_K = \|h\|_S$ for $h \in C(S)$.

Let $D = \sigma^{-1}(\{s_0\})$. If $t \notin D$, then there is $h \in C(S)$ such that $h(S) \subseteq [0,1]$, $h(s_0) = 0$ and $h(\sigma(t)) = 1$. It follows that $g = h \circ \sigma \in Y$ and $\|g\|_K = 1$.

Thus $K_0 = K \setminus D$ and therefore, $K_2 = D$. Clearly, $Y|_{K_2} \equiv 0$ and the result again follows from Corollary 5.4. □

Theorem 5.9. *Any closed $*$-subalgebra $\mathcal{A}$ of $C(K)$ is a DBR subspace of $C(K)$.*

Proof. Let $\mathcal{A}$ be a closed $*$-subalgebra of $C(K)$.

If $\mathcal{A}$ contains the unit, i.e., the constant function 1, then by Corollary 2.11, $\mathcal{A}$ is a BR subspace of $C(K)$.

If $\mathcal{A}$ does not contain the unit, then it follows from [11,12] that there is a compact Hausdorff space S, $s_0 \in S$ and a continuous onto map $\sigma : K \to S$ such that $\mathcal{A} = \{h \circ \sigma\colon\ h \in C(S) \text{ and } h(s_0) = 0\}$. Now by Proposition 5.8, $\mathcal{A}$ is a DBR subspace of $C(K)$. □

Corollary 5.10. *$C(K)$ is BR in $C(K)^{**}$.*

Proof. It is known that $C(K)^{**}$ is a $C(T)$ space for some compact Hausdorff space T, and $C(K)$ is a closed $*$-subalgebra of $C(T)$ containing the unit. Thus the result follows from the proof of Theorem 5.9. □

Remark 5.11. In [2], it is shown that c_0 is a DBR subspace of its bidual, while the space $X = C(\mathbb{T})/A$, where A is the disc algebra, is not.

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