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Realization Theory in Hilbert Space for a Class of Transfer Functions*

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1. INTRODUCTION

In this paper we continue our investigation into the realization problem started in [6, 7] and simultaneously in [1, 11]. Our main results deal with characterization of exact controllability and exact observability of a class of restricted shift systems and the characterization of all transfer functions admitting realizations by exactly controllable and exactly observable discrete time linear systems. For general motivation and background as well as the related problem for continuous time systems we refer to our previous paper [7].

Consider a bounded $p \times r$ matrix valued analytic function A in the open unit disc D having a Taylor expansion $A(z) = \sum_{k=0}^{\infty} A_k z^k$. Let X be a Hilbert space $F \in B(X, X)$, $G \in B(\mathbb{C}^p, X)$ and $H \in B(X, \mathbb{C}^r)$ we say the system $\{F, G, H\}$ realizes A if $A_i = HF^iG$ for all $i \geqslant 0$. This is equivalent to the system having internal description given by

$$\begin{aligned} x_{n+1} &= Fx_n + Gu_n , \\ y_n &= Hx_n , \end{aligned} \tag{1.1}$$

having the input/output relation given by A.

The discrete time system given by (1.1) will be called controllable (observable) if $\bigcap_i \ker G^*F^{*i} = \{0\}$ $(\bigcap_i \ker HF^i = \{0\})$.

To introduce the concept of exact controllability consider the space $l^2(0, \infty; \mathbb{C}^p)$ and in it the dense subset Δ of all finite nonzero sequences.

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We define the controllability operator $\mathscr{C}$ of the system $\{F, G, H\}$ as the map $\mathscr{C}: \Delta \to X$ defined by

$$\mathscr{C}(\{\xi_k\}) = \sum_{k=0}^{\infty} F^k G \xi_k . \tag{1.2}$$

Since the sequence $\{\xi_k\}$ has only a finite number of nonzero elements $\mathscr{C}$ is well defined. We will say that the system $\{F, G, H\}$ is exactly controllable if $\mathscr{C}$ can be extended to a continuous map of $l^2(0, \infty; \mathbb{C}^p)$ onto X.

Similarly we define the observability operator $\mathscr{O}: \Delta_1 \to X$, where Δ_1 is the dense subset of $l^2(0, \infty; \mathbb{C}^r)$ of all finite nonzero sequences, by

$$\mathscr{O}(\{\eta_k\}) = \sum_{k=0}^{\infty} F^{*k} H^* \eta_k . \tag{1.3}$$

We define exact observability analogously.

2. Some Operator Theoretic Results

Let M, N be two separable Hilbert spaces. In the next section they will be identified with the complex euclidean spaces $\mathbb{C}^p$, $\mathbb{C}^r$, etc. By $l^2(0, \infty; N)$ we denote the space of all sequences $\{\alpha_k \mid k \geqslant 0\}$ whose values lie in the Hilbert space N and for which $\Sigma \| \alpha_k \|_N^2 < \infty$. The inner product in $l^2(0, \infty; N)$ is defined in the natural way. In similar fashion we define $l^2(-\infty, \infty; N)$. $L^2(\mathbb{T}; N)$ denotes the Hilbert space of all (equivalence classes) weakly measurable functions from the unit circle to N for which the norm defined by $\| f \|^2 = \int \| f(e^{it}) \|_N^2 \, dt$ is finite. The Fourier transform $\mathscr{F}$ is the map from $L^2(-\infty, \infty; N)$ onto $L^2(\mathbb{T}; N)$ defined by $\mathscr{F}(\{\alpha_n\}) = \Sigma \alpha_n e^{\text{int}} = \alpha(e^{it})$. $\mathscr{F}$ is a unitary map which maps $l^2(0, \infty; N)$ onto $H^2(N)$. Here $H^2(N)$ is the Hardy space of all N-vector valued functions that is the subspace of $L^2(\mathbb{T}; N)$ of all functions whose negative indexed Fourier coefficients vanish. $H^2(N)$ functions have analytic extensions into the unit disc from which they can be recaptured as radial limits almost everywhere. We will use the same letters to denote a function in $H^2(N)$ and its analytic extension into the open disc. We let χ denote the identiy function in the closed unit disc, i.e., $\chi(z) = z$. We define the right shift S in $l^2(\infty, \infty; N)$ by $S\{(\alpha_n\}) = \{\beta_n\}$ and $\beta_n = \alpha_{n-1}$. The subspace $l^2(0, \infty; N)$ is invariant under the right shift. We will use S also to denote the right shift restricted to $l^2(0, \infty; N)$. In an abuse of notation we will use S also for the image of the right shift under the Fourier transform. Thus

$Sf = \chi f$ for all f in $H^2(N)$. We note that $(S^*f)(z) = (f(z) - f(0))/z$. We let $B(N, M)$ be the space of all bounded operators from N to M. $H^\infty(B(N, M))$ is the space of all $B(N, M)$ valued analytic functions A in the open unit disc for which $\| A \|_\infty = \sup\{\| A(z)\| \mid | z | < 1\} < \infty$. If $N = M$ the space $H^\infty(B(N, N))$ is a Banach algebra. We have a natural map of $H^\infty(B(N, M))$ onto $H^\infty(B(M, N))$ given by $A \to \tilde{A}$ where

$$\tilde{A}(z) = A(\bar{z})^*. \tag{2.1}$$

We recall that an operator W in a Hilbert space H is called a partial isometry if there exists a subspace M such that $\| Wx \| = \| x \|$ for $x \in M$ and $W \mid M = 0$. The subspace M is called the initial subspace of W. An element of $H^\infty(B(N, N))$ is called a rigid function if $\| Q \|_\infty \leqslant 1$ and almost everywhere on $\mathbb{T}$ the operators $Q(e^{it})$ are partial isometries with a fixed initial space. If almost everywhere $Q(e^{it})$ is unitary in N the function will be called inner [10]. Rigid functions are important because of their connection with invariant subspace structure.

THEOREM 2.1 [2, 12, 9]. *Every right invariant subspace of $H^2(N)$ is of the form $QH^2(N)$ for some rigid function Q.*

The rigid function in this representation can be taken to be inner if and only if the invariant subspace has full range [10].

Given a rigid function Q in $H^\infty(B(N, W))$ we denote by $H(Q)$ the left invariant subspace of $H^2(N)$ defined by $H(Q) = \{QH^2(N)\}^\perp$. We will use $P_{H(Q)}$ for the orthogonal projection of $H^2(N)$ onto $H(Q)$. We define an operator $S(Q)$ in $H(Q)$ by

$$S(Q)f = P_{H(Q)}\chi f \qquad \text{for all} \quad f \in H(Q). \tag{2.2}$$

We refer to $S(Q)$ as the restricted right shift, or the compression of the shift. We clearly have $S(Q)^* = S^* \mid H(Q)$ that is $S(Q)^*$ is the restriction of the left shift in $H^2(N)$ to the left invariant subspace $H(Q)$. The spectral analysis of $S(Q)$ is based on the study of the corresponding rigid function Q. This has been carried out by Moeller, Helson, etc. [13, 10, 14].

THEOREM 2.2. *Let Q be a rigid function in $H(B(N, W))$ and let $S(Q)$ be defined by* (2.2).

(a) *If $| \lambda | < 1$ then $\bar{\lambda} \in \sigma_p(S(Q)^*)$ if and only if* $\operatorname{Ker} Q(\lambda)^* \neq \{0\}$. *The normalized eigenfunctions of $S(Q^*)$ are of the form $(1 - | \lambda |^2)^{1/2}(1 - \bar{\lambda}\chi)^{-1}\xi$ where ξ is a unit vector in* $\operatorname{Ker} Q(\lambda)^*$.

(b) *If* $|\lambda| < 1$ *then* $\lambda \in \sigma_p(S(Q))$ *if and only if* $\operatorname{Ker} Q(\lambda) \cap L \neq \{0\}$ *where L is almost everywhere on the unit circle the initial space of* $Q(e^{it})$. *The normalized eigenfunctions of* $S(Q)$ *are of the form* $(1 - |\lambda|^2)^{1/2}(\chi - \lambda)^{-1}Q\xi$ *where* ξ *is a unit vector in* $\operatorname{Ker} Q(\lambda) \cap L$. *For an inner function* Q *the condition reduces to* $\operatorname{Ker} Q(\lambda) \neq \{0\}$.

(c) *For an inner function* Q, $\sigma(S(Q))$ *is the union of the set of all points* λ, $|\lambda| < 1$, *for which* $Q(\lambda)$ *is not boundedly invertible and the set of all points* λ, $|\lambda| = 1$, *where* Q *has no analytic continuation outside the unit disc.*

We note that if N is finite dimensional and Q is a noninner rigid function then $\sigma(S(Q))$ is the closed unit disc. In case Q is inner $\sigma(S(Q))$ contains at most a countable number of points in the open unit disc which have to coincide with the zero set of a Blaschke product.

For an inner function Q we define the map $\tau_Q: L^2(\mathbb{T}; N) \to L^2(\mathbb{T}; N)$ by

$$(\tau_Q f)(e^{it}) = e^{-it}\tilde{Q}(e^{it}) f(e^{-it}). \tag{2.3}$$

τ_Q is a unitary map in $L^2(\mathbb{T}; N)$, maps $H(Q)$ onto $H(\tilde{Q})$ making the following diagram commutative [4].

$$\begin{array}{ccc} H(Q) & \xrightarrow{\tau_Q} & H(Q) \\ {\scriptstyle S(Q)^*}\downarrow & & \downarrow{\scriptstyle S(\tilde{Q})} \\ H(Q) & \xrightarrow{\tau_Q} & H(\tilde{Q}) \end{array} \tag{2.4}$$

An inner function P in $H^\infty(B(M, M))$ is a left inner factor of a function A in $H^\infty(B(N, M))$ if $A = PA'$ for some A' in $H^\infty(B(N,M))$. Two functions A and A_1 in $H^\infty(B(N, M))$ and $H^\infty(B(N_1, M))$, respectively, are left prime if A and A_1 have no common nontrivial left inner factor. We will use the notation $(A, A_1)_L = I_M$ to denote the left primeness of A and A_1. We will say that A and A_1 are strongly left prime if there exists a $\delta > 0$ such that for all z in the open unit disc

$$\inf\{\| A(z)^*\xi \| + \| A_1(z)^*\xi \| \mid \xi \in M, \| \xi \| = 1\} \geq \delta. \tag{2.5}$$

We will use $[A, A_1]_L = I_M$ to denote the strong left primeness of A and A_1. Similarly given A in $H^\infty(B(N, M))$ and A_1 in $H^\infty(B(N, M_1))$ we define right and strong right primeness analogously. We clearly have $(A, A_1)_R = I_N$ if and only if $(A, A_1)_L = I_N$ and $[A, A_1]_R = I_N$

if and only if $[A, A_1]_L = I_N$. Thus $[A, A_1]_R = I_N$ is equivalent to the existence of a $\delta > 0$ such that for all z in the open unit disc

$$\inf\{\| A(z)\eta \| + \| A_1(z)\eta \| \mid \eta \in N, \| \eta \| = 1\} \geqslant \delta. \tag{2.6}$$

Let Q_1 and Q_2 be two inner functions in $H^\infty(B(N, N))$ and $H^\infty(B(M, M))$, respectively. For any function A in $H^\infty(B(N, M))$ we define a bounded operator from $H(Q_1)$ to $H(Q_2)$ by

$$\mathfrak{A}f = P_{H(Q_2)}Af \qquad \text{for every} \quad f \in H(Q_1). \tag{2.7}$$

If we make the assumption that

$$AQ_1H^2(N) \subset Q_2H^2(M), \tag{2.8}$$

then these operators admit natural composition rules. The inclusion relation (2.8) is equivalent to the existence of a function A_1 in $H^\infty(B(N, M))$ for which

$$AQ_1 = Q_2A_1. \tag{2.9}$$

We refer to [5] for more details on this operational calculus. The importance of this class of operators is that every bounded operator $\mathfrak{A}$ which intertwines $S(Q_1)$ and $S(Q_2)$, i.e., for which $\mathfrak{A}S(Q_1) = S(Q_2)\mathfrak{A}$, has a representation in the form (2.7) for some A in $H^\infty(B(N, M))$. This is the content of the Sz.-Nagy and Foias lifting theorem [14]. A matrix version of the Carleson corona theorem proved in [4] yields the following theorem.

THEOREM 2.3. *Let Q_1 and Q_2 be inner functions in $H^\infty(B(N, N))$ and $H^\infty(B(M, M))$, respectively, where N and M are finite-dimensional Hilbert spaces. Let A and A_1 be functions in $H^\infty(B(N, M))$ for which* (2.9) *holds and let $\mathfrak{A}$ be the operator from $H(Q_1)$ into $H(Q_2)$ defined by* (2.7) *then $\mathfrak{A}$ has a bounded left inverse if and only if $[A, Q_2]_L = I_M$ and it has a bounded right inverse if and only if $[A_1, Q_1]_R = I_N$.*

Let $A \in H^\infty(B(N, M))$ then A has a Taylor expansion around the origin given by $A(z) = \Sigma A_n z^n$. We define the Hankel operator induced by A, H_A, as the map of $l^2(0, \infty; N)$ into $l^2(0, \infty; M)$ given by $H_A(\{\alpha_n\}) = \{\beta_n\}$ with $\beta_n = \Sigma A_{n+j}\alpha_j$. H_A is a bounded operator for which $S^*H_A = H_AS$. Here we used the same letter to denote the shifts in $l^2(0, \infty; M)$ and $l^2(0, \infty; N)$. From the above relation it is clear that Ker H_A is a right invariant subspace of $l^2(0, \infty; N)$ whereas $\overline{\text{Range } H_A}$ is a left invariant subspace of $l^2(0, \infty; M)$. Applying the Fourier transform it is easy to get a functional representation for H_A.

In fact if $J: L^2(\mathbb{T}; N) \to L^2(\mathbb{T}; N)$ is the unitary map defined by $(Jf)(e^{it}) = f(e^{-it})$ then we have $H_A: H^2(N) \to H^2(M)$ is defined by

$$H_A f = P_{H^2(M)} A(Jf), \tag{2.10}$$

for all f in $H_2(N)$. Clearly $\overline{\text{Range } H_A}$ is the smallest left invariant subspace of $H^2(M)$ containing all functions of the form $A\xi$ with ξ in N.

3. Restricted Shift Systems

Definition 3.1. A (multiple input/multiple output) restricted shift system is a triple $\{S(Q), G, H\}$ where $S(Q)$ is the operator defined by (2.2) in the left invariant subspace $H(Q)$ of $H^2(\mathbb{C}^n)$. $G: \mathbb{C}^p \to H(Q)$ and $H: H(Q) \to \mathbb{C}^r$ are bounded operators.

Remark 3.2. For each $\xi \in \mathbb{C}^p$ the function $G\xi$ is an element of $H(Q)$ and hence a $\mathbb{C}^n$-valued analytic function. It follows that there exists a $n \times p$ matrix valued analytic function with H^2 elements, which we denote by $C(z)$, and which satisfies $(G\xi)(z) = C(z)\xi$ for all $\xi \in \mathbb{C}^p$. Similarly there exists an $n \times r$ matrix valued analytic function $D(z)$ for which $(H^*\eta)(z) = D(z)\eta$ for all $\eta \in \mathbb{C}^r$.

In terms of restricted shift systems the realization problem is easily solved [6, 11]. Let $A \in H^\infty(B(\mathbb{C}^p, \mathbb{C}^n))$ and let H_A be the Hankel operator induced by A, defined by (2.10). If $\{\text{Range } H_A\}^\perp = QH^2(\mathbb{C}^n)$ and G and H are defined by

$$G\xi = A\xi \qquad \text{for} \quad \xi \in \mathbb{C}^p, \tag{3.1}$$

and

$$Hf = f(0) \qquad \text{for} \quad f \in H(Q), \tag{3.2}$$

then the system $\{S(Q)^*, G, H\}$ is a realization of A.

It is clear that the controllability operator of the above system is H_A. Therefore the question of characterizing exact controllability is equivalent to characterizing the range closure of Hankel operators.

Remark 3.3. In the rest of the paper we will study only restricted shifts in subspaces $H(Q)$ associated with inner functions. We will refer to functions A for which $\{\text{Range } H_A\}^\perp$ is an invariant subspace of full range as strictly noncyclic functions [8]. This seems to be the right generalization to the vector valued case of the notion of noncyclic vectors for the left shift introduced and studied in depth in [3]. For an intrinsic characterization of strictly noncyclic functions we refer to [8]. The relation between the class of strictly noncyclic functions with spectral minimality of realizations [1] will be the subject of another publication.

THEOREM 3.4. (a) *The restricted shift system* $\{S(Q), G, H\}$ *is controllable if and only if*

$$(C, Q)_L = I_{\mathbb{C}^n}. \tag{3.3}$$

(b) *The restricted shift system* $\{S(Q), G, H\}$ *is exactly controllable if and only if*

$$G\xi = P_{H(Q)}\Gamma\xi \qquad \textit{for all} \quad \xi \in \mathbb{C}^p, \tag{3.4}$$

where $\Gamma \in H^\infty(B(\mathbb{C}^p, \mathbb{C}^n))$ *and*

$$[\Gamma, Q]_L = I_{\mathbb{C}^n}. \tag{3.5}$$

If we assume that $C \in H^\infty(B(\mathbb{C}^p, \mathbb{C}^n))$ *then condition* (3.5) *can be replaced by*

$$[C, Q]_L = I_{\mathbb{C}^n}. \tag{3.6}$$

Proof. (a) The system $\{S(Q), G, H\}$ is controllable if and only if the set of vectors of the form $S(Q)^n G$ with $n \geqslant 0$ and $\xi \in \mathbb{C}^p$ is dense in $H(Q)$, i.e., if and only if the map Φ_C: $H^2(\mathbb{C}^p) \to H(Q)$ defined by $\Phi_C f = P_{H(Q)}Cf$ has dense range. Of course if C is not bounded Φ_C is defined only on a dense subset of $H^2(\mathbb{C}^p)$ containing all vector polynomials. The range of Φ_C is not dense in $H(Q)$ if and only if for some nonzero f in $H(Q)$ all $n \geqslant 0$ and all vectors $\xi \in \mathbb{C}^p$ we have $(f, P_{H(Q)}\chi^n C) = (f, \chi^n C) = 0$. Since $(f, Q\chi^n\eta) = 0$ for all $n \geqslant 0$ and all $\eta \in \mathbb{C}^n$ the function f is orthogonal to the span of the ranges of Q and Φ_C which is given by $Q_1 H^2(\mathbb{C}^n)$ where Q_1 is the greatest common left inner factor of Q and C. Hence the result follows.

(b) Let S be the right shift in $l^2(0, \infty; \mathbb{C}^p)$. If is obvious from the definition of the controllability operator that the following relation

$$\mathscr{C}S = S(Q)\mathscr{C} \tag{3.7}$$

holds.

This implies that Ker $\mathscr{C}$ is a right invariant subspace of $l^2(0, \infty; \mathbb{C}^p)$. Using the Fourier transform $\mathscr{F}$ we may as well assume $\mathscr{C}$ to be defined on $H^2(\mathbb{C}^p)$ in a natural way. By the Beurling–Lax theorem Ker $\mathscr{C} = Q_{co}H^2(\mathbb{C}^p)$ for some rigid function Q_{co}. If $H(Q_{co}) = \{Q_{co}H^2(\mathbb{C}^p)\}^\perp$ then $\mathscr{C} \mid H(Q_{co})$ is a one-to-one, and assuming exact controllability, onto map of $H(Q_{co})$ onto $H(Q)$. We keep the notation $\mathscr{C}$ for this boundedly invertible map. It follows from (3.7) that

$$\mathscr{C}S(Q_{co}) = S(Q)\mathscr{C}, \tag{3.8}$$

that is, $S(Q_{co})$ and $S(Q)$ are similar. Since Q is inner it follows from spectral considerations, based on the Moeller–Helson theorem, that Q_{co} is also inner. Now $\mathscr{C}$ is an intertwining operator for $S(Q_{co})$ and $S(Q)$ and hence the Sz.-Nagy–Foias lifting theorem [14] we have the following representation for $\mathscr{C}$:

$$\mathscr{C}f = P_{H(Q)}\Gamma f \qquad \text{for all} \quad f \in H(Q_{co}), \tag{3.9}$$

where $\Gamma \in H^\infty(B(\mathbb{C}^p, \mathbb{C}^n))$ is such that

$$\Gamma Q_{co} H^2(\mathbb{C}^p) \subset QH^2(\mathbb{C}^n). \tag{3.10}$$

Condition (3.10) is, by Theorem 2.3 in [5], equivalent to the existence of a function Γ_1 in $H^\infty(B(\mathbb{C}^p, \mathbb{C}^n))$ for which

$$\Gamma Q_{co} = Q\Gamma_1 . \tag{3.11}$$

We point out that if $C \in H^\infty(B(\mathbb{C}^p, \mathbb{C}^n))$ then Γ can be identified with C. From these considerations it is clear that the exact controllability of the system $\{S(Q), G, H\}$ is equivalent to the bounded right invertibility of the map $\mathscr{C}$ in (3.9). This, by Theorem 2.3, is equivalent to $[\Gamma, Q]_L = I_{\mathbb{C}^n}$.

Consider the special case $n = 1$. Then $G\colon \mathbb{C}^p \to H(q)$ is given by $G(\alpha_1, \ldots, \alpha_p) = \Sigma \alpha_i g_i$ with $g_i \in H^2$. Here q is a scalar inner function.

COROLLARY 3.5. (a) *The system* $\{S(q), G, H\}$ *is controllable if and only if* $(g_1, \ldots, g_p, q) = 1$, *i.e., if and only if* $g_1, \ldots, g_p, q$ *are relatively prime.*

(b) *The system* $\{S(q), G, H\}$ *is exactly controllable if and only if* $[g_1, \ldots, g_p, q] = 1$, *i.e., if and only if* $g_1, \ldots, g_p, q$ *are strongly relatively prime.*

The study of the observability of the restricted shift system is done by reduction to questions of controllability through the use of duality considerations. The system $\{S(Q), G, H\}$ is observable or exactly observable if and only if the adjoint system $\{S(Q)^*, H^*, G^*\}$ is controllable as exactly controllable, respectively. Since controllability and exact controllability are invariant under unitary transformations it suffices to study the controllability properties of the system

$\{S(\tilde{Q}), \tilde{H}, \tilde{G}\}$ which is the image of $\{S(Q)^*, H^*, G^*\}$ under the unitary map τ_Q. Therefore $\tilde{H} = \tau_Q H^*$ and $G^* = \tilde{G}\tau_Q$. Here we used also relation (2.4).

In order to derive the observability criteria we need an explicit form for H. By our assumption $(H^*\eta)(z) = D(z)\eta$ for all $\eta \in \mathbb{C}^r$. Therefore

$$\tilde{H}\eta = \tau_Q H^*\eta = \bar{\chi}\tilde{Q}J(D\eta) = \bar{\chi}\tilde{Q}\tilde{D}^*\eta. \tag{3.12}$$

Let us define the function E by

$$E = \bar{\chi}D^*Q, \tag{3.13}$$

then

$$D = \bar{\chi}QE^*, \tag{3.14}$$

and the controllability of the system $\{S(Q), H, G\}$ is equivalent, by Theorem 3.4, to $(\tilde{E}, \tilde{Q})_L = I_{\mathbb{C}^n}$ or $(E, Q)_R = I_{\mathbb{C}^n}$. The system $\{S(\tilde{Q}), \tilde{H}, \tilde{G}\}$ is exactly controllable if and only if

$$\tilde{H}\eta = P_{H(\tilde{Q})}\tilde{\Delta}\eta \qquad \text{for all} \quad \eta \in \mathbb{C}^r, \tag{3.15}$$

where $\Delta \in H^\infty(B(\mathbb{C}^n, \mathbb{C}^r))$ and satisfies $[\tilde{\Delta}, \tilde{Q}]_L = I_{\mathbb{C}^n}$. If

$$D \in H^\infty(B(\mathbb{C}^r, \mathbb{C}^n))$$

then Δ can be identified with E defined by (3.13). These results can be easily translated back in terms of the original system $\{S(Q), G, H\}$ and we get the following theorem.

THEOREM 3.6. (a) *The restricted shift system $\{S(Q), G, H\}$ is observable if and only if*

$$(E, Q)_R = I_{\mathbb{C}^n}, \tag{3.16}$$

where E is defined by (3.13).

(b) *The restricted shift system $\{S(Q), G, H\}$ is exactly observable if and only if*

$$H^*\eta = P_{H^2(\mathbb{C}^n)}\bar{\chi}Q\,\Delta^*\eta \qquad \textit{for all} \quad \eta \in \mathbb{C}^r, \tag{3.17}$$

where

$$\Delta \in H^\infty(B(\mathbb{C}^r, \mathbb{C}^n)) \quad \textit{and,}$$

$$[\Delta, Q]_R = I_{\mathbb{C}^n} \tag{3.18}$$

is satisfied.

We note that from (3.17) it follows that the observability operator of the system $\{S(Q), G, H\}$ is the Hankel operator induced $\bar{\chi}Q\Delta^*$. This yields a characterization of Hankel operator with closed range.

THEOREM 3.7. *Let $A \in H^\infty(B(\mathbb{C}^p, \mathbb{C}^n))$ then a necessary and sufficient condition for Range H_A to be closed is that on the unit circle A is factorable in the form*

$$A = \bar{\chi}QE^*, \tag{3.19}$$

where Q is inner, $E \in H^\infty(B(\mathbb{C}^n, \mathbb{C}^p))$ and the strong relative primeness condition

$$[Q, E]_R = I_{\mathbb{C}^n} \tag{3.20}$$

is satisfied.

Proof. The sufficiency part follows as a corollary the proof of Theorem 3.6. Suppose conversely that Range H_A is closed. Since from the definition of the Hankel operator we have

$$S^*H_A = H_AS, \tag{3.21}$$

where S denotes the right shift in $H^2(\mathbb{C}^n)$ as well as in $H^2(\mathbb{C}^p)$. This in turn implies that Ker H_A is a right invariant subspace of $H^2(\mathbb{C}^p)$ and Range H_A is left invariant in $H^2(\mathbb{C}^n)$ and closed by assumption. Let Q and R be rigid functions for which Ker $H_A = RH^2(\mathbb{C}^p)$ and Range $H_A = H(Q) = \{QH^2(\mathbb{C}^n)\}^\perp$. From (3.21) we get the following relation

$$S(Q)^*H_A = H_AS(R). \tag{3.22}$$

Thus $S(Q)^*$ and $S(R)$ are similar. By spectral considerations, based on Theorem 2.2, it follows that necessarily R and Q are both inner. We apply now the transformation τ_Q defined by (2.3) to both sides of (3.22). Using the commutativity of diagram (2.4), we see that

$$S(\tilde{Q})(\tau_Q H_A) = (\tau_Q H_A)\, S(R). \tag{3.23}$$

The operator $\tau_Q H_A$ is therefore an invertible operator that intertwines two restricted right shifts. By the Sz.-Nagy–Foias lifting theorem $\tau_Q H_A \xi = F\xi$ for some $F \in H^\infty(B(\mathbb{C}^p, \mathbb{C}^n))$. This, by a simple calculation, implies that $A = \bar{\chi}Q\tilde{F}^*$ and (3.19) follows with $E = \tilde{F}$. The strong primeness condition now follows from the preceeding theorem.

THEOREM 3.8. *A function A in* $H^\infty(B(\mathbb{C}^p, \mathbb{C}^r))$ *has a realization by an exactly controllable and exactly observable system if and only if it is factorable on the unit circle in the forms*

$$A = \bar{\chi} Q E^*, \tag{3.24}$$

and

$$A = \bar{\chi} E_1^* Q_1, \tag{3.25}$$

where Q *and* Q_1 *are inner function in* $H^\infty(B(\mathbb{C}^r, \mathbb{C}^r))$ *and* $H^\infty(B(\mathbb{C}^p, \mathbb{C}^p))$, *respectively*, E, $E_1 \in H^\infty(B(\mathbb{C}^r, \mathbb{C}^p))$ *and the strong relative primeness conditions*

$$[Q_1, E_1]_L = I_{\mathbb{C}^p}, \tag{3.24}$$

and

$$[Q, E]_R = I_{\mathbb{C}^r} \tag{3.27}$$

are satisfied.

Proof. Assume $A \in H^\infty(B(\mathbb{C}^p, \mathbb{C}^r))$ has a realization by an exactly controllable and exactly observable system. The shift realization of A given in $H(Q)$ by the system $\{S(Q)^*, G, H\}$ with G and H defined by (3.1) and (3.2) is controllable and exactly observable. Hence, by the state space isomorphism of Helton [11], the two realizing systems are similar. Thus it suffices to study the shift realization of A. Since the controllability operator of the shift realization is H_A then everything reduces to the study of the range closure of H_A. By Theorem 3.7 Range H_A is closed if and only if A has a factorization (3.24) satisfying (3.26). Since $H_A{}^* = H_{\tilde{A}}$, with $\tilde{A}$ defined by (2.1), and as bounded operators have closed range if and only if their adjoints have closed range, it follows that Range $H_{\tilde{A}}$ is closed. Thus $\tilde{A}$ is also factorable on the unit circle with the factors satisfying a strong relative primeness condition. Going back to A this implies the factorization (3.25) satisfying (3.27).

Conversely if A has a factorization (3.24) satisfying (3.26) then the shift realization of A is exactly controllable and exactly observable. This completes the proof.

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