



Irreducible Lie–Yamaguti algebras

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ABSTRACT

Lie–Yamaguti algebras (or generalized Lie triple systems) are binary–ternary algebras intimately related to reductive homogeneous spaces. The Lie–Yamaguti algebras which are irreducible as modules over their Lie inner derivation algebra are the algebraic counterpart of the isotropy irreducible homogeneous spaces.

These systems will be shown to split into three disjoint types: adjoint type, non-simple type and generic type. The systems of the first two types will be classified and most of them will be shown to be related to a Generalized Tits Construction of Lie algebras.

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1. Introduction

Let G be a connected Lie group with Lie algebra \mathfrak{g} , H a closed subgroup of G , and let \mathfrak{h} be the associated subalgebra of \mathfrak{g} . The corresponding homogeneous space $M = G/H$ is said to be *reductive* ([23, Section 7]) in case there is a subspace \mathfrak{m} of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\text{Ad}(H)(\mathfrak{m}) \subseteq \mathfrak{m}$.

In this situation, Nomizu proved [23, Theorem 8.1] that there is a one-to-one correspondence between the set of all G -invariant affine connections on M and the set of bilinear multiplications $\alpha : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ such that the restriction of $\text{Ad}(H)$ to \mathfrak{m} is a subgroup of the automorphism group of the nonassociative algebra (\mathfrak{m}, α) .

There exist natural binary and ternary products defined in \mathfrak{m} , given by

$$\begin{aligned} x \cdot y &= \pi_{\mathfrak{m}}([x, y]), \\ [x, y, z] &= [\pi_{\mathfrak{h}}([x, y]), z], \end{aligned} \quad (1.1)$$

for any $x, y, z \in \mathfrak{m}$, where $\pi_{\mathfrak{h}}$ and $\pi_{\mathfrak{m}}$ denote the projections on \mathfrak{h} and \mathfrak{m} respectively, relative to the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Note that the condition $\text{Ad}(H)(\mathfrak{m}) \subseteq \mathfrak{m}$ implies the condition $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, the converse being valid if H is connected.

There are two distinguished invariant affine connections: the natural connection (or canonical connection of the first kind), which corresponds to the bilinear multiplication given by $\alpha(x, y) = \frac{1}{2}x \cdot y$ for any $x, y \in \mathfrak{m}$, which has trivial torsion, and the canonical connection corresponding to the trivial multiplication: $\alpha(x, y) = 0$ for any $x, y \in \mathfrak{m}$. In case the reductive homogeneous space is symmetric, so $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$, these two connections coincide. For the canonical connection, the torsion and curvature tensors are given on the tangent space to the point $eH \in M$ (e denotes the identity element of G), which can be naturally identified with \mathfrak{m} , by

$$T(x, y) = -x \cdot y, \quad R(x, y)z = -[x, y, z],$$

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for any $x, y, z \in \mathfrak{m}$ (see [23, Theorem 10.3]).

Moreover, Nomizu showed too that the affine connections on manifolds with parallel torsion and curvature are locally equivalent to canonical connections on reductive homogeneous spaces.

Yamaguti [31] considered the properties of the torsion and curvature of these canonical connections (or alternatively, of the binary and ternary multiplications in (1.1)), and thus defined what he called the *general Lie triple systems*, later renamed as *Lie triple algebras* in [15]. We will follow here the notation in [18, Definition 5.1], and will call these systems *Lie–Yamaguti algebras*:

Definition 1.1. A *Lie–Yamaguti algebra* $(\mathfrak{m}, x \cdot y, [x, y, z])$ (*LY-algebra* for short) is a vector space \mathfrak{m} equipped with a bilinear operation $\cdot : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ and a trilinear operation $[, ,] : \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ such that, for all $x, y, z, u, v, w \in \mathfrak{m}$:

$$(LY1) \quad x \cdot x = 0,$$

$$(LY2) \quad [x, x, y] = 0,$$

$$(LY3) \quad \sum_{(x,y,z)} ([x, y, z] + (x \cdot y) \cdot z) = 0,$$

$$(LY4) \quad \sum_{(x,y,z)} [x \cdot y, z, t] = 0,$$

$$(LY5) \quad [x, y, u \cdot v] = [x, y, u] \cdot v + u \cdot [x, y, v],$$

$$(LY6) \quad [x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]].$$

Here $\sum_{(x,y,z)}$ means the cyclic sum on x, y, z .

The LY-algebras with $x \cdot y = 0$ for any x, y are exactly the Lie triple systems, closely related with symmetric spaces, while the LY-algebras with $[x, y, z] = 0$ are the Lie algebras. Less known examples can be found in [2] where a detailed analysis on the algebraic structure of LY-algebras arising from homogeneous spaces which are quotients of the compact Lie group G_2 is given.

These nonassociative binary–ternary algebras have been treated by several authors in connection with geometric problems on homogeneous spaces [16,17,24–26], but no much information on their algebraic structure is available yet.

Given a Lie–Yamaguti algebra $(\mathfrak{m}, x \cdot y, [x, y, z])$ and any two elements $x, y \in \mathfrak{m}$, the linear map $D(x, y) : \mathfrak{m} \rightarrow \mathfrak{m}$, $z \mapsto D(x, y)(z) = [x, y, z]$ is, due to (LY5) and (LY6), a derivation of both the binary and ternary products. These derivations will be called *inner derivations*. Moreover, let $D(\mathfrak{m}, \mathfrak{m})$ denote the linear span of the inner derivations. Then $D(\mathfrak{m}, \mathfrak{m})$ is closed under commutation thanks to (LY6). Consider the vector space $\mathfrak{g}(\mathfrak{m}) = D(\mathfrak{m}, \mathfrak{m}) \oplus \mathfrak{m}$, and endow it with the anticommutative multiplication given, for any $x, y, z, t \in \mathfrak{m}$, by:

$$[D(x, y), D(z, t)] = D([x, y, z], t) + D(z, [x, y, t]),$$

$$[D(x, y), z] = D(x, y)(z) = [x, y, z],$$

$$[z, t] = D(z, t) + z \cdot t.$$

(1.2)

Note that the Lie algebra $D(\mathfrak{m}, \mathfrak{m})$ becomes a subalgebra of $\mathfrak{g}(\mathfrak{m})$.

Then it is straightforward [31] to check that $\mathfrak{g}(\mathfrak{m})$ is a Lie algebra, called the *standard enveloping Lie algebra* of the Lie–Yamaguti algebra \mathfrak{m} . The binary and ternary products in \mathfrak{m} coincide with those given by (1.1), where $\mathfrak{h} = D(\mathfrak{m}, \mathfrak{m})$.

As was mentioned above, the Lie triple systems are precisely those LY-algebras with trivial binary product. These correspond to the symmetric homogeneous spaces. Following [23, Section 16], a symmetric homogeneous space G/H is said to be *irreducible* if the action of $\text{ad } \mathfrak{h}$ on \mathfrak{m} is irreducible, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the canonical decomposition of the Lie algebra \mathfrak{g} of G .

This suggests the following definition:

Definition 1.2. A Lie–Yamaguti algebra $(\mathfrak{m}, x \cdot y, [x, y, z])$ is said to be *irreducible* if \mathfrak{m} is an irreducible module for its Lie algebra of inner derivations $D(\mathfrak{m}, \mathfrak{m})$.

Geometrically, the irreducible LY-algebras correspond to the isotropy irreducible homogeneous spaces studied by Wolf in [30] “as a first step toward understanding the geometry of the Riemannian homogeneous spaces”. Likewise, the classification of the irreducible LY-algebras constitutes a first step in our understanding of this variety of algebras. Concerning the isotropy irreducible homogeneous spaces, Wolf remarks that “the results are surprising, for there are a large number of nonsymmetric isotropy irreducible coset spaces G/K , and only a few examples had been known before. One of the most interesting class is $\mathbf{SO}(\dim K)/\text{ad } K$ for an arbitrary compact simple Lie group K ”. These spaces $\mathbf{SO}(\dim K)/\text{ad } K$ show a clear pattern, but there appear many more examples in the classification, where no such clear pattern appears.

Here it will be shown that most of the irreducible LY-algebras follow clear patterns if several kinds of nonassociative algebraic systems are used, not just Lie algebras. In fact, most of the irreducible LY-algebras will be shown, here and in the forthcoming paper [3], to appear inside simple Lie algebras as orthogonal complements of subalgebras of derivations of Lie and Jordan algebras, Freudenthal triple systems and Jordan pairs.

Let us fix some notation to be used throughout this paper. All the algebraic systems will be assumed to be finite dimensional over an algebraically closed ground field k of characteristic 0. Unadorned tensor products will be considered over this ground field k . Given a Lie algebra \mathfrak{g} and a subalgebra \mathfrak{h} , the pair $(\mathfrak{g}, \mathfrak{h})$ will be said to be a *reductive pair* (see [25]) if there is a complementary subspace \mathfrak{m} of \mathfrak{h} with $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. The decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ will then be called a *reductive decomposition* of the Lie algebra \mathfrak{g} .

In particular, given an LY-algebra $(\mathfrak{m}, x \cdot y, [x, y, z])$, the pair $(\mathfrak{g}(\mathfrak{m}), D(\mathfrak{m}, \mathfrak{m}))$ is a reductive pair.

The following result is instrumental:

Proposition 1.3. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition of a simple Lie algebra \mathfrak{g} , with $\mathfrak{m} \neq 0$. Then \mathfrak{g} and \mathfrak{h} are isomorphic, respectively, to the standard enveloping Lie algebra and the inner derivation algebra of the Lie–Yamaguti algebra $(\mathfrak{m}, x \cdot y, [x, y, z])$ given by (1.1). Moreover, in case \mathfrak{h} is semisimple and \mathfrak{m} is irreducible as a module for \mathfrak{h} , either \mathfrak{h} and \mathfrak{m} are isomorphic as $\text{ad } \mathfrak{h}$ -modules or $\mathfrak{m} = \mathfrak{h}^\perp$, the orthogonal complement of \mathfrak{h} relative to the Killing form of \mathfrak{g} .*

Proof. For the first assertion it is enough to note that $\pi_{\mathfrak{h}}([\mathfrak{m}, \mathfrak{m}]) \oplus \mathfrak{m} (= [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m})$ and $\{x \in \mathfrak{h} : [x, \mathfrak{m}] = 0\}$ are ideals of \mathfrak{g} . Hence, if \mathfrak{g} is simple, $\pi_{\mathfrak{h}}([\mathfrak{m}, \mathfrak{m}]) = \mathfrak{h}$ holds, and \mathfrak{h} embeds naturally in $D(\mathfrak{m}, \mathfrak{m}) \subseteq \text{End}_k(\mathfrak{m})$. From here it follows that the map $\mathfrak{g} \rightarrow \mathfrak{g}(\mathfrak{m})$ given by $h \in \mathfrak{h} \mapsto \text{ad } h|_{\mathfrak{m}}$ and $x \in \mathfrak{m} \mapsto x$ is an isomorphism from \mathfrak{g} to $\mathfrak{g}(\mathfrak{m})$ which sends \mathfrak{h} onto $D(\mathfrak{m}, \mathfrak{m})$. Moreover, in case \mathfrak{h} is semisimple, \mathfrak{h} is anisotropic with respect to the Killing form of \mathfrak{g} (by Cartan’s criterion, as \mathfrak{g} is a faithful $\text{ad}_{\mathfrak{g}}$ \mathfrak{h} -module), so $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ and the orthogonal projection, $\pi_{\mathfrak{h}}(\mathfrak{m})$ from \mathfrak{m} onto \mathfrak{h} is an ideal of \mathfrak{h} . By irreducibility of \mathfrak{m} , either $\pi_{\mathfrak{h}}(\mathfrak{m}) = 0$ and therefore $\mathfrak{m} = \mathfrak{h}^\perp$, or \mathfrak{m} is isomorphic to $\pi_{\mathfrak{h}}(\mathfrak{m})$. In the latter case, since the action of \mathfrak{h} on \mathfrak{m} is faithful, it follows that $\mathfrak{h} = \pi_{\mathfrak{h}}(\mathfrak{m})$, as required. \square

The paper is organized as follows. Section 2 will be devoted to establish the main structural features on Lie inner derivations and standard enveloping Lie algebras of the irreducible LY-algebras. These will be split into three non-overlapping types: adjoint, non-simple and generic. The final result in this section shows that LY-algebras of adjoint type are essentially simple Lie algebras. The classification of the LY-algebras of non-simple type is the goal of the rest of the paper, while the generic type will be treated in a forthcoming paper. Section 3 will give examples of irreducible LY-algebras, many of them appearing inside Lie algebras obtained by means of the Tits construction of Lie algebras in [29] in terms of composition algebras and suitable Jordan algebras. Then in Section 4 these examples will be shown to exhaust the irreducible LY-algebras of non-simple type.

2. Irreducible Lie–Yamaguti algebras. Initial classification

For irreducible LY-algebras \mathfrak{m} , the irreducibility as a module for $D(\mathfrak{m}, \mathfrak{m})$, together with Schur’s Lemma, quickly leads to the following result:

Theorem 2.1. *Let $(\mathfrak{m}, x \cdot y, [x, y, z])$ be an irreducible LY-algebra. Then $D(\mathfrak{m}, \mathfrak{m})$ is a semisimple and maximal subalgebra of the standard enveloping Lie algebra $\mathfrak{g}(\mathfrak{m})$. Moreover, $\mathfrak{g}(\mathfrak{m})$ is simple in case \mathfrak{m} and $D(\mathfrak{m}, \mathfrak{m})$ are not isomorphic as $D(\mathfrak{m}, \mathfrak{m})$ -modules.*

Proof. Any subalgebra M of $\mathfrak{g}(\mathfrak{m})$ containing $D(\mathfrak{m}, \mathfrak{m})$ decomposes as $M = D(\mathfrak{m}, \mathfrak{m}) \oplus (M \cap \mathfrak{m})$, thus $M = D(\mathfrak{m}, \mathfrak{m})$ or $\mathfrak{g}(\mathfrak{m})$ by the irreducibility of \mathfrak{m} . Hence $D(\mathfrak{m}, \mathfrak{m})$ is a maximal subalgebra. The irreducibility of \mathfrak{m} also implies that $D(\mathfrak{m}, \mathfrak{m})$ is a reductive algebra with $\dim Z(D(\mathfrak{m}, \mathfrak{m})) \leq 1$ (see [11, Proposition 19.1]). If $Z(D(\mathfrak{m}, \mathfrak{m})) = Fz$, Schur’s Lemma shows that there is a scalar $\alpha \in k$ such that $\text{ad}_{\mathfrak{g}(\mathfrak{m})} z|_{\mathfrak{m}} = \alpha \text{Id}$ holds. In this case, for any $x, y \in \mathfrak{m}$ we have

$$\text{ad}_{\mathfrak{g}(\mathfrak{m})} z([x, y]) = 2\alpha[x, y]. \tag{2.1}$$

If $\alpha \neq 0$, since 2α is not an eigenvalue of $\text{ad}_{\mathfrak{g}(\mathfrak{m})} z$, from (2.1) it follows that $[\mathfrak{m}, \mathfrak{m}] = 0$, so $D(\mathfrak{m}, \mathfrak{m}) = 0$, a contradiction. Hence $\alpha = 0$ which implies $z = 0$ because \mathfrak{m} is a faithful module for $D(\mathfrak{m}, \mathfrak{m})$, and therefore $D(\mathfrak{m}, \mathfrak{m})$ is semisimple.

Finally, if \mathfrak{m} is not the adjoint module for $D(\mathfrak{m}, \mathfrak{m})$, given a proper ideal I of $\mathfrak{g}(\mathfrak{m})$, we have $I \cap \mathfrak{m} = 0$: otherwise, $I \cap \mathfrak{m} = \mathfrak{m}$ and then $\mathfrak{g}(\mathfrak{m}) = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] \subseteq I$, a contradiction. Hence $[I \cap D(\mathfrak{m}, \mathfrak{m}), \mathfrak{m}] = 0$ and therefore $I \cap D(\mathfrak{m}, \mathfrak{m}) = 0$. By maximality of $D(\mathfrak{m}, \mathfrak{m})$, $\mathfrak{g}(\mathfrak{m})$ can be decomposed as

$$\mathfrak{g}(\mathfrak{m}) = D(\mathfrak{m}, \mathfrak{m}) \oplus I = D(\mathfrak{m}, \mathfrak{m}) \oplus \mathfrak{m} \tag{2.2}$$

thus I is isomorphic to \mathfrak{m} as $D(\mathfrak{m}, \mathfrak{m})$ -modules. From (2.2), $\mathfrak{m} \oplus I$ is a $D(\mathfrak{m}, \mathfrak{m})$ -module isomorphic to $\mathfrak{m} \oplus \mathfrak{m}$ and it is easily checked that $P = (\mathfrak{m} \oplus I) \cap D(\mathfrak{m}, \mathfrak{m})$ is a nonzero ideal of $D(\mathfrak{m}, \mathfrak{m})$ isomorphic to \mathfrak{m} . So that $D(\mathfrak{m}, \mathfrak{m}) = P \oplus Q$ (direct sum of ideals). Now, as $[P, Q] = 0$ and P is isomorphic to \mathfrak{m} as $D(\mathfrak{m}, \mathfrak{m})$ -modules, $[Q, \mathfrak{m}] = 0$ follows, and therefore, since \mathfrak{m} is a faithful module, $Q = 0$ and this contradicts the fact that \mathfrak{m} is not the adjoint module for $D(\mathfrak{m}, \mathfrak{m})$. \square

The previous theorem points out two different situations depending on the LY-algebra module behavior. This observation, together with Proposition 1.3, leads to the following definition and structure result:

Definition 2.2. An LY-algebra \mathfrak{m} is said to be of *adjoint type* if \mathfrak{m} is the adjoint module for the inner derivation algebra $D(\mathfrak{m}, \mathfrak{m})$.

Corollary 2.3. *The irreducible LY-algebras which are not of adjoint type are the orthogonal subspaces of their inner derivation algebras relative to the Killing form of their standard enveloping Lie algebras. In particular, these irreducible LY-algebras are contragredient modules for $D(\mathfrak{m}, \mathfrak{m})$.* \square

Note that Theorem 2.1 guarantees the simplicity of standard enveloping Lie algebras of the non-adjoint irreducible LY-algebras. In the adjoint type, according to Theorem 2.4, the standard enveloping Lie algebras are never simple. So these

results split the classification of irreducible LY-algebras into the following non-overlapping types:

- ADJOINT TYPE: \mathfrak{m} is the adjoint module for $D(\mathfrak{m}, \mathfrak{m})$
 - NON-SIMPLE TYPE: $D(\mathfrak{m}, \mathfrak{m})$ is not simple
 - GENERIC TYPE: Both $\mathfrak{g}(\mathfrak{m})$ and $D(\mathfrak{m}, \mathfrak{m})$ are simple
- (2.3)

Moreover, the complete classification of the first type is easily obtained as we shall show in what follows. The non-simple type will be studied in Section 4, while the generic type will be the object of a forthcoming paper [3].

Given any irreducible LY-algebra of adjoint type $(\mathfrak{m}, x \cdot y, [x, y, z])$, the inner derivation Lie algebra $D(\mathfrak{m}, \mathfrak{m})$ is simple. Thus from [4] the subspace

$$\text{Hom}_{D(\mathfrak{m}, \mathfrak{m})}(\Lambda^2 \mathfrak{m}, \mathfrak{m}) \tag{2.4}$$

is one dimensional and spanned by the Lie bracket in $D(\mathfrak{m}, \mathfrak{m})$. So, given a $D(\mathfrak{m}, \mathfrak{m})$ -module isomorphism $\varphi : D(\mathfrak{m}, \mathfrak{m}) \rightarrow \mathfrak{m}$, the maps

$$\cdot : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}, \quad (x, y) \mapsto x \cdot y \tag{2.5}$$

and

$$\tilde{D} : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}, \quad (x, y) \mapsto \varphi(D(x, y)) = \varphi([x, y, -]) \tag{2.6}$$

belong to the vector space in (2.4), and hence there exist scalars $\alpha, \beta \in k, \beta \neq 0$, such that

$$\varphi(x) \cdot \varphi(y) = \alpha \varphi([x, y]) \tag{2.7}$$

$$\tilde{D}(\varphi(x), \varphi(y)) = \beta \varphi([x, y]) \tag{2.8}$$

for any $x, y \in D(\mathfrak{m}, \mathfrak{m})$. Moreover, there is then an isomorphism of Lie algebras:

$$\mathfrak{g}(\mathfrak{m}) = D(\mathfrak{m}, \mathfrak{m}) \oplus \varphi(D(\mathfrak{m}, \mathfrak{m})) \cong K \otimes D(\mathfrak{m}, \mathfrak{m}), \tag{2.9}$$

where K is the quotient $k[t]/(t^2 - \alpha t - \beta)$ of the polynomial ring on the variable t , that maps $x + \varphi(y)$ to $1 \otimes x + \bar{t} \otimes y$, for any $x, y \in D(\mathfrak{m}, \mathfrak{m})$, where \bar{t} denotes the class of the variable t modulo the ideal $(t^2 - \alpha t - \beta)$. Now, depending on α , two different situations appear:

- If $\alpha = 0$, it can be assumed that $\beta = 1$ (by taking $\frac{1}{\sqrt{\beta}}\varphi$ instead of φ). In this case, \mathfrak{m} is an LY-algebra with trivial binary product, so a Lie triple system, isomorphic to the triple system given by the Lie algebra $D(\mathfrak{m}, \mathfrak{m})$ with trivial binary product and ternary product given by $[x, y, z] = [[x, y], z]$. In this case, $\mathfrak{g}(\mathfrak{m})$ is the direct sum of two copies of $D(\mathfrak{m}, \mathfrak{m})$.
- If $\alpha \neq 0$, it can be assumed that $\alpha = 1$ (by taking $\frac{1}{\alpha}\varphi$ instead of φ). Then \mathfrak{m} is isomorphic to the LY-algebra $D(\mathfrak{m}, \mathfrak{m})$ with binary and ternary products given by $x \cdot y = [x, y]$ and $[x, y, z] := \beta [[x, y], z]$. Moreover, if $\beta \neq -1/4$ (equivalently, $K \cong k \times k$), $\mathfrak{g}(\mathfrak{m})$ is the direct sum of two copies of $D(\mathfrak{m}, \mathfrak{m})$. In case $\beta = -1/4$, the enveloping Lie algebra $\mathfrak{g}(\mathfrak{m})$ is isomorphic to the Lie algebra $k[t]/(t^2) \otimes D(\mathfrak{m}, \mathfrak{m})$, whose solvable (actually abelian) radical is $(t)/(t^2) \otimes D(\mathfrak{m}, \mathfrak{m})$.

Now, from our previous discussion we obtain:

Theorem 2.4. *Up to isomorphism, the LY-algebras of adjoint type are the simple Lie algebras L with binary and ternary products of one of the following types:*

- (i) $x \cdot y = 0$ and $[x, y, z] = [[x, y], z]$
- (ii) $x \cdot y = [x, y]$ and $[x, y, z] = \beta [[x, y], z], \beta \neq 0$

where $[x, y]$ is the Lie bracket in L . Moreover, the standard enveloping Lie algebra is a direct sum of two copies of the simple Lie algebra L in case (i) or case (ii) with $\beta \neq -1/4$. In case (ii) with $\beta = -1/4$, the standard enveloping Lie algebra is isomorphic to $k[t]/(t^2) \otimes L$. \square

Remark 2.5. This theorem, together with Theorem 2.1, shows that the adjoint type in (2.3) does not overlap with the other two types, as the standard enveloping Lie algebra is never simple for the adjoint type, while it is always simple in the non-simple and generic types. \square

3. Examples of non-simple type irreducible LY-algebras

Several examples of irreducible LY-algebras and of its enveloping Lie algebras will be shown in this section. In the next section, these examples will be proved to exhaust all the possibilities for non-simple type irreducible LY-algebras.

3.1. Classical examples

Given a vector space V and a nondegenerate ϵ -symmetric bilinear form φ on V (that is, φ is symmetric if $\epsilon = 1$ and skew-symmetric if $\epsilon = -1$), consider the Lie algebra $\mathfrak{skew}(V, \varphi) = \{f \in \mathfrak{gl}(V) : \varphi(f(v), w) = -\varphi(v, f(w)) \forall v, w \in V\}$ of skew-symmetric linear maps relative to φ . Thus, $\mathfrak{skew}(V, \varphi) = \mathfrak{so}(V, \varphi)$ (respectively $\mathfrak{sp}(V, \varphi)$) if φ is symmetric (respectively

skew-symmetric). This Lie algebra $\mathfrak{skew}(V, \varphi)$ is spanned by the linear maps $\varphi_{v,w} = \varphi(v, \cdot)w - \epsilon\varphi(w, \cdot)v$, for $v, w \in V$. The bracket of two such linear maps is given by:

$$\begin{aligned} [\varphi_{a,b}, \varphi_{x,y}] &= \varphi_{\varphi_{a,b}(x),y} + \varphi_{x,\varphi_{a,b}(y)} \\ &= \varphi(a, x)\varphi_{b,y} - \varphi(x, b)\varphi_{a,y} - \varphi(y, a)\varphi_{b,x} + \varphi(b, y)\varphi_{a,x}, \end{aligned} \tag{3.1}$$

for any $a, b, x, y \in V$.

Moreover, the subspace $\mathfrak{sym}(V, \varphi) = \{f \in \text{End}_k(V) : \varphi(f(v), w) = \varphi(v, f(w)) \forall v, w \in V\}$ of the symmetric linear maps relative to φ is closed under the symmetrized product:

$$f \bullet g = \frac{1}{2}(fg + gf).$$

($\mathfrak{sym}(V, \varphi)$ is a special Jordan algebra) Use will be made of the subspace of trace zero symmetric linear maps, which will be denoted by $\mathfrak{sym}_0(V, \varphi)$. It is clear that $\mathfrak{sym}(V, \varphi) = k1_V \oplus \mathfrak{sym}_0(V, \varphi)$, where 1_V denotes the identity map on V .

Example 3.1. Let $(V_i, \varphi_i), i = 1, 2$, be two vector spaces endowed with nondegenerate ϵ -symmetric bilinear forms ($\epsilon = \pm 1$), with $1 \leq \dim V_1 \leq \dim V_2$. Consider the direct sum $V_1 \oplus V_2$ with the nondegenerate ϵ -symmetric bilinear form given by the orthogonal sum $\varphi = \varphi_1 \perp \varphi_2$. Then, under the natural identifications,

$$\begin{aligned} \mathfrak{skew}(V_1 \oplus V_2, \varphi) &= (\varphi_{V_1, V_1} \oplus \varphi_{V_2, V_2}) \oplus \varphi_{V_1, V_2} \\ &= (\mathfrak{skew}(V_1, \varphi_1) \oplus \mathfrak{skew}(V_2, \varphi_2)) \oplus \varphi_{V_1, V_2}. \end{aligned}$$

This gives a \mathbb{Z}_2 -grading of $\mathfrak{skew}(V_1 \oplus V_2, \varphi)$. As a module for the even part $\mathfrak{skew}(V_1, \varphi_1) \oplus \mathfrak{skew}(V_2, \varphi_2)$, the odd part φ_{V_1, V_2} is isomorphic to $V_1 \otimes V_2$, and it is irreducible unless $\epsilon = 1$ and either $\dim V_1 = 1$ and $1 \leq \dim V_2 \leq 2$, or $\dim V_1 = 2$. The Lie bracket of two basic elements in φ_{V_1, V_2} is, due to (3.1) and since V_1 and V_2 are orthogonal, given by:

$$[\varphi_{x_1, x_2}, \varphi_{y_1, y_2}] = \varphi_2(x_2, y_2)(\varphi_1)_{x_1, y_1} + \varphi_1(x_1, y_1)(\varphi_2)_{x_2, y_2},$$

for any $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$.

Therefore, unless $\epsilon = 1$ and either $\dim V_1 = 1$ and $1 \leq \dim V_2 \leq 2$, or $\dim V_1 = 2, m = V_1 \otimes V_2$ is an irreducible LY-algebra (actually an irreducible Lie triple system) with trivial binary product, and ternary product given by (see (1.1)):

$$[x_1 \otimes x_2, y_1 \otimes y_2, z_1 \otimes z_2] = \varphi_2(x_2, y_2) ((\varphi_1)_{x_1, y_1}(z_1) \otimes z_2) + \varphi_1(x_1, y_1) (z_1 \otimes (\varphi_2)_{x_2, y_2}(z_2)). \tag{3.2}$$

Example 3.2. Let (V_i, φ_i) be a vector space endowed with a nondegenerate ϵ_i -symmetric bilinear form ($i = 1, 2$), with $2 \leq \dim V_1 \leq \dim V_2$. Then $V_1 \otimes V_2$ is endowed with the nondegenerate $\epsilon_1\epsilon_2$ -symmetric bilinear form $\varphi = \varphi_1 \otimes \varphi_2$. For $i = 1, 2$, we have:

$$\mathfrak{gl}(V_i) = \mathfrak{skew}(V_i, \varphi_i) \oplus \mathfrak{sym}(V_i, \varphi_i) = \mathfrak{skew}(V_i, \varphi_i) \oplus \mathfrak{sym}_0(V_i, \varphi_i) \oplus k1_{V_i},$$

and

$$\begin{aligned} \mathfrak{skew}(V_1 \otimes V_2, \varphi) &= (\mathfrak{skew}(V_1, \varphi_1) \otimes k1_{V_2} \oplus k1_{V_1} \otimes \mathfrak{skew}(V_2, \varphi_2)) \\ &\quad \oplus ((\mathfrak{skew}(V_1, \varphi_1) \otimes \mathfrak{sym}_0(V_2, \varphi_2)) \oplus (\mathfrak{sym}_0(V_1, \varphi_1) \otimes \mathfrak{skew}(V_2, \varphi_2))). \end{aligned}$$

This provides a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of $\mathfrak{g} = \mathfrak{skew}(V_1 \otimes V_2, \varphi)$, where $\mathfrak{h} \simeq \mathfrak{skew}(V_1, \varphi_1) \oplus \mathfrak{skew}(V_2, \varphi_2)$ and $\mathfrak{m} = (\mathfrak{skew}(V_1, \varphi_1) \otimes \mathfrak{sym}_0(V_2, \varphi_2)) \oplus (\mathfrak{sym}_0(V_1, \varphi_1) \otimes \mathfrak{skew}(V_2, \varphi_2))$.

In this situation, if \mathfrak{m} is an irreducible module for \mathfrak{h} , then $\dim V_1 = 2$ and $\epsilon_1 = -1$ (which forces $\mathfrak{sym}_0(V_1, \varphi_1)$ to be trivial).

Assuming $\dim V_1 = 2, \epsilon_1 = -1$, and $\dim V_2 = n \geq 2$, then $\mathfrak{m} = \mathfrak{sp}(V_1, \varphi_1) \otimes \mathfrak{sym}_0(V_2, \varphi_2)$ is an irreducible module for \mathfrak{h} if and only if either $\epsilon_2 = -1$ and $\dim V_2 = 2m \geq 4$, or $\epsilon_2 = 1$ and $\dim V_2 \geq 3$.

With these assumptions, for $a, b \in \mathfrak{sp}(V_1, \varphi_1)$ and $f, g \in \mathfrak{sym}_0(V_2, \varphi_2)$, $ab + ba = \text{tr}(ab)1_{V_1}$ (as $\mathfrak{sp}(V_1, \varphi_1)$ is isomorphic to the Lie algebra $\mathfrak{sl}_2(k)$), and hence $ab = \frac{1}{2}([a, b] + \text{tr}(ab)1_{V_1})$ and $ba = \frac{1}{2}(-[a, b] + \text{tr}(ab)1_{V_1})$ hold. Moreover, if the dimension of V_2 is n , then for any $f, g \in \mathfrak{sym}_0(V_2, \varphi_2)$, the element $fg + gf - \frac{2}{n} \text{tr}(fg)1_{V_2}$ also belongs to $\mathfrak{sym}_0(V_2, \varphi_2)$.

Now, for any $a, b \in \mathfrak{sp}(V_1, \varphi_1)$ and $f, g \in \mathfrak{sym}_0(V_2, \varphi_2)$:

$$\begin{aligned} [a \otimes f, b \otimes g] &= ab \otimes fg - ba \otimes gf \\ &= \frac{1}{2}[a, b] \otimes (fg + gf) + \frac{1}{2} \text{tr}(ab)1_{V_1} \otimes [f, g] \\ &= \left([a, b] \otimes \frac{1}{n} \text{tr}(fg)1_{V_2} + \frac{1}{2} \text{tr}(ab)1_{V_1} \otimes [f, g] \right) + \frac{1}{2}[a, b] \otimes \left(fg + gf - \frac{2}{n} \text{tr}(fg)1_{V_2} \right). \end{aligned} \tag{3.3}$$

Therefore, the binary and ternary products in the irreducible LY-algebra $\mathfrak{m} = \mathfrak{sl}(V_1, \varphi_1) \otimes \mathfrak{sl}(V_2, \varphi_2)$ are given by:

$$(a \otimes f) \cdot (b \otimes g) = \frac{1}{2}[a, b] \otimes \left(fg + gf - \frac{2}{n} \operatorname{tr}(fg)1_{V_2} \right),$$

$$[a \otimes f, b \otimes g, c \otimes h] = \frac{1}{n} \operatorname{tr}(fg)[[a, b], c] \otimes h + \frac{1}{2} \operatorname{tr}(ab)c \otimes [[f, g], h],$$
(3.4)

for any $a, b, c \in \mathfrak{sl}(V_1, \varphi_1) = \mathfrak{sl}(V_1)$ and $f, g, h \in \mathfrak{sl}(V_2, \varphi_2)$.

Note that for $\epsilon_2 = -1$ and $\dim V_2 = 4$, it is easily checked that $[[a, b], c] = 2 \operatorname{tr}(bc)a - 2 \operatorname{tr}(ac)b$ for any $a, b, c \in \mathfrak{sl}(V_1)$, while $fg + gf - \frac{1}{2} \operatorname{tr}(fg)1_{V_2} = 0$ and $[[f, g], h] = \operatorname{tr}(gh)f - \operatorname{tr}(fh)g$ for any $f, g, h \in \mathfrak{sl}(V_2, \varphi_2)$. Hence (3.4) becomes in this case

$$(a \otimes f) \cdot (b \otimes g) = 0,$$

$$[a \otimes f, b \otimes g, c \otimes h] = \frac{1}{2} \operatorname{tr}(fg) (\operatorname{tr}(bc)a - \operatorname{tr}(ac)b) \otimes h + \frac{1}{2} \operatorname{tr}(ab)c \otimes (\operatorname{tr}(gh)f - \operatorname{tr}(fh)g),$$

for any $a, b, c \in \mathfrak{sl}(V_1, \varphi_1) = \mathfrak{sl}(V_1)$ and $f, g, h \in \mathfrak{sl}(V_2, \varphi_2)$, and thus the triple product coincides with the expression in (3.2) for $\varphi_1(a, b) = \operatorname{tr}(ab)$ and $\varphi_2(f, g) = -\frac{1}{2} \operatorname{tr}(fg)$. Therefore, the irreducible Lie–Yamaguti algebras obtained here for $\dim V_1 = 2, \dim V_2 = 4$ and $\epsilon_1 = -1 = \epsilon_2$ coincides with the one obtained in Example 3.1 for two vector spaces of dimension 3 and 5. \square

Example 3.3. Let now V_1 and V_2 be two vector spaces with $2 \leq \dim V_1 \leq \dim V_2$. The algebra of endomorphisms of the tensor product $V_1 \otimes V_2$ can be identified with the tensor product of the algebras of endomorphisms of V_1 and V_2 . Moreover, the general Lie algebra $\mathfrak{gl}(V_i)$ decomposes as $\mathfrak{gl}(V_i) = k1_{V_i} \oplus \mathfrak{sl}(V_i)$. Then

$$\begin{aligned} \mathfrak{sl}(V_1 \otimes V_2) &= (\mathfrak{sl}(V_1) \otimes k1_{V_2}) \oplus (k1_{V_1} \otimes \mathfrak{sl}(V_2)) \oplus (\mathfrak{sl}(V_1) \otimes \mathfrak{sl}(V_2)) \\ &\simeq (\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)) \oplus (\mathfrak{sl}(V_1) \otimes \mathfrak{sl}(V_2)) \end{aligned}$$

gives a reductive decomposition, and this shows that $\mathfrak{m} = \mathfrak{sl}(V_1) \otimes \mathfrak{sl}(V_2)$ is an irreducible LY-algebra. For $a, b \in \mathfrak{sl}(V_1)$, both $[a, b] = ab - ba$ and $ab + ba - \frac{2}{n_1} \operatorname{tr}(ab)1_{V_1}$ belong to $\mathfrak{sl}(V_1)$, where n_i denotes the dimension of $V_i, i = 1, 2$. Therefore, for any $a, b \in \mathfrak{sl}(V_1)$ and $f, g \in \mathfrak{sl}(V_2)$:

$$\begin{aligned} [a \otimes f, b \otimes g] &= ab \otimes fg - ba \otimes gf \\ &= \left([a, b] \otimes \frac{1}{n_2} \operatorname{tr}(fg)1_{V_2} + \frac{1}{n_1} \operatorname{tr}(ab)1_{V_1} \otimes [f, g] \right) \\ &\quad + \left(\frac{1}{2}[a, b] \otimes \left(fg + gf - \frac{2}{n_2} \operatorname{tr}(fg)1_{V_2} \right) + \left(ab + ba - \frac{2}{n_1} \operatorname{tr}(ab)1_{V_1} \right) \otimes \frac{1}{2}[f, g] \right). \end{aligned}$$
(3.5)

Hence, the binary and the ternary products in the irreducible LY-algebra $\mathfrak{m} = \mathfrak{sl}(V_1) \otimes \mathfrak{sl}(V_2)$ are given by:

$$(a \otimes f) \cdot (b \otimes g) = \frac{1}{2}[a, b] \otimes \left(fg + gf - \frac{2}{n_2} \operatorname{tr}(fg)1_{V_2} \right) + \left(ab + ba - \frac{2}{n_1} \operatorname{tr}(ab)1_{V_1} \right) \otimes \frac{1}{2}[f, g],$$

$$[a \otimes f, b \otimes g, c \otimes h] = [[a, b], c] \otimes \frac{1}{n_2} \operatorname{tr}(fg)h + \frac{1}{n_1} \operatorname{tr}(ab)c \otimes [[f, g], h],$$
(3.6)

for any $a, b, c \in \mathfrak{sl}(V_1)$ and $f, g, h \in \mathfrak{sl}(V_2)$.

Note that, as noted in Example 3.2, if $\dim V_1 = 2$, then for any $a, b, c \in \mathfrak{sl}(V_1), ab + ba - \operatorname{tr}(ab)1_{V_1} = 0$, while $[[a, b], c] = 2 \operatorname{tr}(bc)a - 2 \operatorname{tr}(ac)b$. Hence, if $\dim V_1 = \dim V_2 = 2$, (3.6) becomes:

$$(a \otimes f) \cdot (b \otimes g) = 0,$$

$$[a \otimes f, b \otimes g, c \otimes h] = \operatorname{tr}(fg) (\operatorname{tr}(bc)a - \operatorname{tr}(ac)b) \otimes h + \operatorname{tr}(ab)c \otimes (\operatorname{tr}(gh)f - \operatorname{tr}(fh)g),$$

for any $a, b, c \in \mathfrak{sl}(V_1)$ and $f, g, h \in \mathfrak{sl}(V_2)$, and thus the triple product coincides with the expression in (3.2) for $\varphi_1(a, b) = \operatorname{tr}(ab)$ and $\varphi_2(f, g) = -\operatorname{tr}(fg)$. Therefore, the irreducible Lie–Yamaguti algebras obtained here for $\dim V_1 = 2 = \dim V_2$ coincides with the one obtained in Example 3.1 for two vector spaces of dimension 3. \square

3.2. Generalized Tits Construction

Examples 3.1 and 3.2 can be seen as instances of a Generalized Tits Construction, due to Benkart and Zelmanov [5], which will now be reviewed in a way suitable for our purposes.

Let X be a unital k -algebra endowed with a *normalized* trace $t : X \rightarrow k$. This means that t is a linear map with $t(1) = 1, t(xy) = t(yx)$ and $t((xy)z) = t(x(yz))$ for any $x, y, z \in X$. Then $X = k1 \oplus X_0$, where $X_0 = \{x \in X : t(x) = 0\}$ is the set of trace zero elements in X . For $x, y \in X_0$, the element $x*y = xy - t(xy)1$ lies in X_0 too, and this defines a bilinear multiplication on X_0 . Assume that there is a skew-symmetric bilinear transformation $D : X_0 \times X_0 \rightarrow \operatorname{Der}(X)$, where $\operatorname{Der}(X)$ denotes the Lie

algebra of derivations of X , such that $D_{x,y}$ leaves invariant X_0 and $[E, D_{x,y}] = D_{E(x),y} + D_{x,E(y)}$, for any $x, y \in X_0$ and $E \in D_{X_0, X_0}$. Here D_{X_0, X_0} denotes the Lie subalgebra of $\text{Der}(X)$ spanned by the image of the map D .

An easy example of this situation is given by the Jordan algebras of symmetric bilinear forms: let V be a vector space endowed with a symmetric bilinear form φ , then $\mathcal{J}(V, \varphi) = k1 \oplus V$, with commutative multiplication given by

$$(\alpha 1 + v)(\beta 1 + w) = (\alpha\beta + \varphi(v, w)) 1 + (\alpha w + \beta v),$$

for any $\alpha, \beta \in k$ and $v, w \in V$. Here the normalized trace is given by $t(1) = 1$ and $t(v) = 0$ for any $v \in V$, while the skew-symmetric map D is given by $D(v, w) = \varphi_{v,w}$ for any $v, w \in V$.

Let $Y = k1 \oplus Y_0$ be another such algebra, with normalized trace also denoted by t , multiplication on Y_0 denoted by \star and analogous skew-symmetric bilinear map $d : Y_0 \times Y_0 \rightarrow \text{Der}(Y)$. Then the vector space

$$\mathcal{T}(X, Y) = D_{X_0, X_0} \oplus (X_0 \otimes Y_0) \oplus d_{Y_0, Y_0} \tag{3.7}$$

is an anticommutative algebra with multiplication defined by

$$\begin{aligned} &D_{X_0, X_0} \text{ and } d_{Y_0, Y_0} \text{ are subalgebras of } \mathcal{T}(X, Y), \\ &[D_{X_0, X_0}, d_{Y_0, Y_0}] = 0, \\ &[D, x \otimes y] = D(x) \otimes y, \\ &[d, x \otimes y] = x \otimes d(y), \\ &[x \otimes y, x' \otimes y'] = t(yy')D_{x,x'} + (x \star x') \otimes (y \star y') + t(xx')d_{y,y'}, \end{aligned} \tag{3.8}$$

for any $x, x' \in X_0, y, y' \in Y_0, D \in D_{X_0, X_0}$ and $d \in d_{Y_0, Y_0}$.

Proposition 3.4 ([5, Proposition 3.9]). *The algebra $\mathcal{T}(X, Y)$ above is a Lie algebra provided the following relations hold*

$$\begin{aligned} \text{(i)} \quad &\sum_{\circlearrowleft} t((x_1 \star x_2)x_3) d_{y_1 \star y_2, y_3} = 0, \\ \text{(ii)} \quad &\sum_{\circlearrowleft} t((y_1 \star y_2)y_3) D_{x_1 \star x_2, x_3} = 0, \\ \text{(iii)} \quad &\sum_{\circlearrowleft} (D_{x_1, x_2}(x_3) \otimes t(y_1 y_2) y_3 + (x_1 \star x_2) \star x_3 \otimes (y_1 \star y_2) \star y_3 + t(x_1 x_2)x_3 \otimes d_{y_1, y_2}(y_3)) = 0 \end{aligned}$$

for any $x_1, x_2, x_3 \in X_0$ and any $y_1, y_2, y_3 \in Y_0$. The notation “ \sum_{\circlearrowleft} ” indicates summation over the cyclic permutation of the indices.

Note that, in case $\mathcal{T}(X, Y)$ is a Lie algebra, then $X_0 \otimes Y_0$ becomes an LY-algebra with binary and ternary products given by

$$\begin{aligned} (x_1 \otimes y_1) \cdot (x_2 \otimes y_2) &= (x_1 \star x_2) \otimes (y_1 \star y_2), \\ [x_1 \otimes y_1, x_2 \otimes y_2, x_3 \otimes y_3] &= D_{x_1, x_2}(x_3) \otimes t(y_1 y_2) y_3 + t(x_1 x_2)x_3 \otimes d_{y_1, y_2}(y_3), \end{aligned} \tag{3.9}$$

for any $x_1, x_2, x_3 \in X_0$ and $y_1, y_2, y_3 \in Y_0$. This will be called the *Lie–Yamaguti algebra inside $\mathcal{T}(X, Y)$* .

Remark 3.5. An important example where $\mathcal{T}(X, Y)$ is a Lie algebra arises when Jordan algebras of symmetric bilinear forms are used as the ingredients [5, 3.28]. If (V_1, φ_1) and (V_2, φ_2) are two vector spaces endowed with nondegenerate symmetric bilinear forms and $\mathcal{J}_1 = \mathcal{J}(V_1, \varphi_1)$ and $\mathcal{J}_2 = \mathcal{J}(V_2, \varphi_2)$ are the corresponding Jordan algebras, then $D_{(\mathcal{J}_i)_0, (\mathcal{J}_i)_0} = \mathfrak{so}(V_i, \varphi_i) = \mathfrak{skew}(V_i, \varphi_i)$, $i = 1, 2$, and the reductive decomposition

$$\begin{aligned} \mathcal{T}(\mathcal{J}_1, \mathcal{J}_2) &= (D_{(\mathcal{J}_1)_0, (\mathcal{J}_1)_0} \oplus D_{(\mathcal{J}_2)_0, (\mathcal{J}_2)_0}) \oplus ((\mathcal{J}_1)_0 \otimes (\mathcal{J}_2)_0) \\ &\simeq (\mathfrak{so}(V_1, \varphi_1) \oplus \mathfrak{so}(V_2, \varphi_2)) \oplus (V_1 \otimes V_2) \end{aligned}$$

coincides, with the natural identifications, with the reductive decomposition in Example 3.1 with $\epsilon = 1$. Therefore, the LY-algebras in Example 3.1 with $\epsilon = 1$, are the LY-algebras obtained inside the Generalized Tits Construction $\mathcal{T}(\mathcal{J}_1, \mathcal{J}_2)$, where \mathcal{J}_1 and \mathcal{J}_2 are Jordan algebras of nondegenerate symmetric bilinear forms.

Moreover, the Generalized Tits Construction $\mathcal{T}(X, Y)$ can be assumed to be associated with algebras (X_0, \star) and (Y_0, \star) having skew-symmetric bilinear forms, and with symmetric maps D and d (see [5, 3.33]). In particular, it works when $\mathcal{J}_i = k1 \oplus V_i$ is the Jordan superalgebra of a nondegenerate skew-symmetric bilinear form φ_i , $i = 1, 2$. Here the even part of the superalgebra \mathcal{J}_i is just $k1$, while the odd part is V_i . With exactly the same arguments as above, it is checked that the LY-algebras in Example 3.1 with $\epsilon = -1$, are exactly the LY-algebras obtained inside the Generalized Tits Construction $\mathcal{T}(\mathcal{J}_1, \mathcal{J}_2)$, where \mathcal{J}_1 and \mathcal{J}_2 are Jordan superalgebras of nondegenerate skew-symmetric bilinear forms. \square

But the Generalized Tits Construction has its origin in the Classical Tits Construction in [29], which is the source of further examples of LY-algebras.

Table 1
Freudenthal's magic square

$\mathcal{T}(\mathcal{C}, \mathcal{J})$	$H_3(k)$	$H_3(\mathcal{K})$	$H_3(\mathcal{Q})$	$H_3(\mathcal{O})$
k	A_1	A_2	C_3	F_4
\mathcal{K}	A_2	$A_2 \oplus A_2$	A_5	E_6
\mathcal{Q}	C_3	A_5	D_6	E_7
\mathcal{O}	F_4	E_6	E_7	E_8

Example 3.6 (Classical Tits Construction). Let \mathcal{C} be a unital composition algebra with norm n (see [12]). Thus, \mathcal{C} is a finite-dimensional unital k -algebra, with the nondegenerate quadratic form $n : \mathcal{C} \rightarrow k$ such that $n(ab) = n(a)n(b)$ for any $a, b \in \mathcal{C}$. Then, each element satisfies the degree 2 equation

$$a^2 - \text{tr}(a)a + n(a)1 = 0, \tag{3.10}$$

where $\text{tr}(a) = n(a, 1)$ ($= n(a + 1) - n(a) - n(1)$) is called the *trace*. The subspace of trace zero elements will be denoted by \mathcal{C}_0 . The algebra \mathcal{C} is endowed of a canonical involution, given by $\bar{x} = \text{tr}(x)1 - x$.

Moreover, for any $a, b \in \mathcal{C}$, the linear map $D_{a,b} : \mathcal{C} \rightarrow \mathcal{C}$ given by

$$D_{a,b}(c) = \frac{1}{4} ([a, b], c) + 3(a, c, b) \tag{3.11}$$

where $[a, b] = ab - ba$ is the commutator, and $(a, c, b) = (ac)b - a(cb)$ the associator, is a derivation: the *inner derivation* determined by the elements a, b (see [27, Chapter III, Section 8]). These derivations span the whole Lie algebra of derivations $\text{Der}(\mathcal{C})$. Moreover, they satisfy

$$D_{a,b} = -D_{b,a}, \quad D_{ab,c} + D_{bc,a} + D_{ca,b} = 0, \tag{3.12}$$

for any $a, b, c \in \mathcal{C}$. The normalized trace here is $t = \frac{1}{2} \text{tr}$, and the multiplication $*$ on \mathcal{C}_0 is just $a * b = ab - t(ab)1 = \frac{1}{2}[a, b]$, since $ab + ba = \text{tr}(ab)1$, for any $a, b \in \mathcal{C}_0$.

The only unital composition algebras (recall that the ground field is being assumed to be algebraically closed) are, up to isomorphism, the ground field k , the cartesian product of two copies of the ground field $\mathcal{K} = k \times k$, the split quaternion algebra, which is the algebra of two by two matrices $\mathcal{Q} = \text{Mat}_2(k)$, and the split octonion algebra \mathcal{O} (see, for instance, [33, Chapter 2]).

On the other hand, given a finite-dimensional unital Jordan algebra \mathcal{J} of degree n (see [13]), we denote by $T(x)$ its *generic trace* ($T(1) = n$), by $N(x)$ its *generic norm* and by \mathcal{J}_0 the subspace of trace zero elements. Then $t = \frac{1}{n}T$ is a normalized trace. If R_x is the right multiplication by x , the map $d_{x,y} : \mathcal{J} \rightarrow \mathcal{J}$ given by

$$d_{x,y}(z) = [R_x, R_y] \tag{3.13}$$

is a derivation.

Now, given a unital composition algebra \mathcal{C} , one may consider the subspace $H_n(\mathcal{C})$ of $n \times n$ hermitian matrices over \mathcal{C} with respect to the standard involution $(x_{ij})^* = (\bar{x}_{ji})$. This is a Jordan algebra with the symmetrized product $x \bullet y = \frac{1}{2}(xy + yx)$ if either \mathcal{C} is associative or $n \leq 3$. For $\mathcal{C} = k$, this is just the algebra of symmetric $n \times n$ matrices, for $\mathcal{C} = \mathcal{K}$ this is isomorphic to the algebra $\text{Mat}_n(k)$ with the symmetrized product, while for $\mathcal{C} = \mathcal{Q}$ this is the algebra of symmetric matrices for the symplectic involution in $\text{Mat}_n(\text{Mat}_2(k)) \simeq \text{Mat}_{2n}(k)$.

Up to isomorphisms, the simple Jordan algebras are the following:

degree 1: The ground field k .

degree 2: The Jordan algebras of nondegenerate symmetric bilinear forms $\mathcal{J}(V, \varphi)$.

degree $n \geq 3$: The Jordan algebras $H_n(k)$, $H_n(\mathcal{K})$ and $H_n(\mathcal{Q})$, plus the degree three Jordan algebra $H_3(\mathcal{O})$.

For the simple Jordan algebras, the derivations $d_{x,y}$'s span the whole Lie algebra of derivations $\text{Der}(\mathcal{J})$.

It turns out that the conditions in Proposition 3.4 are satisfied if $X = \mathcal{C}$ is a unital composition algebra and $Y = \mathcal{J}$ is a degree three Jordan algebra (see [29] and [5, Proposition 3.24]). This is the Classical Tits Construction, which gives rise to Freudenthal's Magic Square (Table 1), if the simple Jordan algebras of degree three are taken as the second ingredient.

In the third and fourth rows of this Magic Square (that is, if the composition algebras \mathcal{Q} and \mathcal{O} are considered), there appears the reductive decomposition:

$$\mathcal{T}(\mathcal{C}, \mathcal{J}) = (\text{Der}(\mathcal{C}) \oplus \text{Der}(\mathcal{J})) \oplus (\mathcal{C}_0 \otimes \mathcal{J}_0),$$

and this shows that, with $\dim \mathcal{C}$ being either 4 or 8 and \mathcal{J} being a simple degree three Jordan algebra, $\mathcal{C}_0 \otimes \mathcal{J}_0$ is an irreducible LY-algebra with binary and ternary products given by

$$(a \otimes x) \cdot (b \otimes y) = \frac{1}{2} [a, b] \otimes (x \bullet y - t(x \bullet y)1), \tag{3.14}$$

$$[a_1 \otimes x_1, a_2 \otimes x_2, a_3 \otimes x_3] = D_{a_1, a_2}(a_3) \otimes t(x_1 \bullet x_2)x_3 + t(a_1 a_2)a_3 \otimes d_{x_1, x_2}(x_3)$$

for any $a, b, a_1, a_2, a_3 \in \mathcal{C}$ and $x, y, x_1, x_2, x_3 \in \mathcal{J}$. \square

Consider the third row of the Classical Tits Construction, with an arbitrary unital Jordan algebra of degree n . Since \mathcal{Q} is associative, the inner derivation $D_{a,b}$ in (3.11) is just $\frac{1}{4} \text{ad}_{[a,b]}$, thus $\text{Der}(\mathcal{Q})$ can be identified to \mathcal{Q}_0 . The linear map $(\mathcal{Q}_0 \otimes \mathcal{J}) \oplus \text{Der}(\mathcal{J}) \rightarrow \mathcal{T}(\mathcal{Q}, \mathcal{J})$, which is the identity on $\text{Der}(\mathcal{J})$ and takes $a \otimes 1$ to $\text{ad}_a \in \text{Der}(\mathcal{Q})$ and $a \otimes x$ to $2(a \otimes x)$, for any $a \in \mathcal{Q}_0$ and $x \in \mathcal{J}_0$, is then a bijection. Under this bijection, the anticommutative product on $\mathcal{T}(\mathcal{Q}, \mathcal{J})$ is transferred to the following product on $\mathfrak{g} = (\mathcal{Q}_0 \otimes \mathcal{J}) \oplus \text{Der}(\mathcal{J})$:

$$\begin{aligned} &\text{Der}(\mathcal{J}) \text{ is a subalgebra of } \mathfrak{g}, \\ &[d, a \otimes x] = a \otimes d(x), \\ &[a \otimes x, b \otimes y] = ([a, b] \otimes x \bullet y) + 2 \text{tr}(ab)d_{x,y} \end{aligned} \tag{3.15}$$

for any $a, b \in \mathcal{Q}_0, x, y \in \mathcal{J}$ and $d \in \text{Der}(\mathcal{J})$.

For any Jordan algebra \mathcal{J} , Tits showed in [28] that this bracket gives a Lie algebra \mathfrak{g} . This is the well-known Tits–Kantor–Koecher Lie algebra attached to \mathcal{J} (see [28,14,19]). Therefore, the third row of the Classical Tits Construction is valid for any unital Jordan algebra, not just for degree three Jordan algebras.

Remark 3.7. Take, for instance, the Jordan algebra $\mathcal{J} = H_n(\mathcal{K})$, which can be identified with the algebra of $n \times n$ matrices $\text{Mat}_n(k)$, but with the Jordan product $x \bullet y = \frac{1}{2}(xy + yx) = \frac{1}{2}(l_x + r_x)(y)$, where l_x and r_x denote, respectively, the left and right multiplication in the associative algebra $\text{Mat}_n(k)$. Then for any $x, y \in \mathcal{J}$, the inner derivation $d_{x,y}$ equals $\frac{1}{4}[l_x + r_x, l_y + r_y] = \frac{1}{4} \text{ad}_{[x,y]}$. Since $\mathcal{Q} = \text{Mat}_2(k)$, for any $a, b \in \mathcal{Q}_0$ and $x, y \in \mathcal{J}_0$, the Lie bracket in (3.15) gives, for any $a, b \in \mathcal{Q}_0 = \mathfrak{sl}_2(k)$ and $x, y \in \mathcal{J}_0 = \mathfrak{sl}_n(k)$:

$$[a \otimes x, b \otimes y] = \frac{1}{n} \text{tr}(xy)[a, b] + \frac{1}{2}[a, b] \otimes \left(xy + yx - \frac{2}{n} \text{tr}(xy)1 \right) + \frac{1}{2} \text{tr}(ab)[x, y].$$

This is exactly the multiplication in (3.5) with $n_1 = 2$ and $n_2 = n$.

Actually, we can think of the construction in Example 3.3 as a sort of Generalized Tits Construction $\mathcal{T}(H_{n_1}(\mathcal{K}), H_{n_2}(\mathcal{K}))$.

On the other hand, let (V_2, φ_2) be a vector space endowed with a nondegenerate ϵ -symmetric bilinear form. Then $\mathcal{J} = \text{sym}(V_2, \varphi_2)$ is a Jordan algebra with the symmetrized product $f \bullet g = \frac{1}{2}(fg + gf)$. If $\epsilon = 1$ and $\dim W = n$, then \mathcal{J} is isomorphic to $H_n(k)$, while if $\epsilon = -1$ and $\dim W = 2n$, then \mathcal{J} is isomorphic to $H_n(\mathcal{Q})$. As in the previous remark, and since $\mathcal{Q}_0 = \mathfrak{sl}_2(k) \simeq \text{sp}(V_1, \varphi_1)$, where V_1 is a two-dimensional vector space endowed with a nonzero skew-symmetric bilinear form φ_1 , the Lie bracket in (3.15) is exactly the multiplication in (3.3). This means that the irreducible LY-algebra in Example 3.2 is the LY-algebra obtained inside $\mathcal{T}(\mathcal{Q}, \text{sym}(V_2, \varphi_2))$.

Finally, if again (V_2, φ_2) is a vector space endowed with a nondegenerate symmetric bilinear form and $\mathcal{J}_2 = \mathcal{J}(V_2, \varphi_2)$ is the associated Jordan algebra, since $\text{ad}_{\mathcal{Q}_0}$ is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(\mathcal{Q}_0, n|_{\mathcal{Q}_0})$ (recall that n denotes the norm of the composition algebra \mathcal{Q} , which in this case coincides with the determinant of 2×2 matrices), it follows easily that $\mathcal{T}(\mathcal{Q}, \mathcal{J}_2)$ is isomorphic to $\mathcal{T}(\mathcal{J}_1, \mathcal{J}_2)$ (see Remark 3.5), where \mathcal{J}_1 is the Jordan algebra of the nondegenerate symmetric bilinear form $n|_{\mathcal{Q}_0}$.

Therefore, concerning the LY-algebras inside the Classical Tits Construction, only the cases $\mathcal{T}(\mathcal{Q}, H_3(\mathcal{O}))$ and $\mathcal{T}(\mathcal{O}, H_3(\mathcal{C}))$ for $\mathcal{C} = k, \mathcal{K}, \mathcal{Q}$, or \mathcal{O} are not covered by the previous examples. \square

3.3. Symplectic triple systems

There is another type of examples of irreducible LY-algebras (actually, of irreducible Lie triple systems) with exceptional enveloping Lie algebra, which appears in terms of the so-called *symplectic triple systems* or, equivalently, of Freudenthal triple systems.

Symplectic triple systems were introduced first in [32]. They are basic ingredients in the construction of some 5-graded Lie algebras (and hence \mathbb{Z}_2 -graded algebras). They consist of a vector space \mathcal{T} endowed with a trilinear product $\{xyz\}$ and a nonzero skew-symmetric bilinear form (x, y) satisfying some conditions (see Definition 2.1 in [7] for a complete description). Following [7], from any symplectic triple system \mathcal{T} , a Lie algebra can be defined on the vector space

$$\mathfrak{g}(\mathcal{T}) = \text{sp}(V) \oplus (V \otimes \mathcal{T}) \oplus \text{Inder}(\mathcal{T}) \tag{3.16}$$

where V is a two-dimensional space endowed with a nonzero skew-symmetric bilinear form φ and $\text{Inder} \mathcal{T} = \text{span}\{d_{x,y} = \{xy \cdot\} : x, y \in \mathcal{T}\}$ is the Lie algebra of inner derivations of \mathcal{T} , by considering the anticommutative product given by:

- $\text{sp}(V)$ and $\text{Inder}(\mathcal{T})$ are Lie subalgebras of $\mathfrak{g}(\mathcal{T})$,
- $[\text{sp}(V), \text{Inder}(\mathcal{T})] = 0$,
- $[f + d, v \otimes x] = f(v) \otimes x + v \otimes d(x)$,
- with $\varphi_{u,v} = \varphi(u, \cdot)v + \varphi(v, \cdot)u$ (as usual),

$$[u \otimes x, v \otimes y] = (x, y)\varphi_{u,v} + \varphi(u, v)d_{x,y} \tag{3.17}$$

Table 2
 $\mathfrak{g}(\mathcal{T}_{\mathcal{J}})$ -algebras

$\mathcal{T}_{\mathcal{J}}$	\mathcal{T}_k	$\mathcal{T}_{H_3(k)}$	$\mathcal{T}_{H_3(\mathcal{C})}$	$\mathcal{T}_{H_3(\mathcal{Q})}$	$\mathcal{T}_{H_3(\mathcal{O})}$
$\text{Inder}\mathcal{T}_{\mathcal{J}}$	A_1	C_3	A_5	D_6	E_7
$\mathfrak{g}(\mathcal{T}_{\mathcal{J}})$	G_2	F_4	E_6	E_7	E_8

for all $f \in \mathfrak{sp}(V)$, $d \in \text{Inder}(\mathcal{T})$, $u, v \in V$ and $x, y \in \mathcal{T}$. The decomposition $\mathfrak{g}_0 = \mathfrak{sp}(V) \oplus \text{Inder}(\mathcal{T})$ and $\mathfrak{g}_1 = V \otimes \mathcal{T}$ provides a \mathbb{Z}_2 -graduation on $\mathfrak{g}(\mathcal{T})$, so the odd part $\mathfrak{g}_1 = V \otimes \mathcal{T}$ is a LY-algebra with trivial binary product (Lie triple system). The simplicity of $\mathfrak{g}(\mathcal{T})$ is equivalent to that of \mathcal{T} , which is characterized by the nondegeneracy of the associated bilinear form (x, y) . Note that viewing $\mathfrak{sp}(V)$ as $\mathfrak{sl}(V)$, and V as its natural module, a 5-grading is obtained by looking at the eigenspaces of the adjoint action of a Cartan subalgebra in $\mathfrak{sl}(V)$. This feature relates symplectic triples with structurable algebras with a one-dimensional space of skew-hermitian elements (see [1]).

Symplectic triple systems are also related to Freudenthal triple systems (see [22]) and to Faulkner ternary algebras introduced in [8,9]. In fact, in the simple case all these systems are essentially equivalent (see [7]).

Among the simple symplectic triple systems (see [7]) use will be made of the following ones:

$$\mathcal{T}_{\mathcal{J}} = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} : \alpha, \beta \in k, a, b \in \mathcal{J} \right\} \tag{3.18}$$

where $\mathcal{J} = \mathcal{J}ordan(n, c)$ is the Jordan algebra of a nondegenerate cubic form n with basepoint (see [21, II.4.3] for a definition) of one of the following types: $\mathcal{J} = k$, $n(\alpha) = \alpha^3$ and $t(\alpha, \beta) = 3\alpha\beta$ or $\mathcal{J} = H_3(\mathcal{C})$ for a unital composition algebra \mathcal{C} . Theorem 2.21 in [7] displays carefully the product and bilinear form for the triple systems $\mathcal{T}_{\mathcal{J}}$ and Theorem 2.30 describes the structure of $\mathfrak{g}(\mathcal{T}_{\mathcal{J}})$. The information on the Lie algebras involved is given in Table 2.

From these symplectic triple systems, five new constructions of exceptional Lie algebras, exactly one for each simple Jordan algebra \mathcal{J} above, and hence a new family of LY-algebras appears:

Example 3.8. Let $\mathcal{T}_{\mathcal{J}}$ be the symplectic triple system defined in (3.18) where either \mathcal{J} is k with norm $n(\alpha) = \alpha^3$, or it is $H_3(\mathcal{C})$ with its generic norm for a unital composition algebra \mathcal{C} . The Lie algebra $\mathfrak{g}(\mathcal{T}_{\mathcal{J}})$ given in (3.16) is simple and presents the reductive decomposition $\mathfrak{g}(\mathcal{T}_{\mathcal{J}}) = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} = \mathfrak{sp}(V) \oplus \text{Inder}\mathcal{T}_{\mathcal{J}}$ and $\mathfrak{m} = V \otimes \mathcal{T}_{\mathcal{J}}$. In these cases, \mathfrak{h} is isomorphic to the semisimple Lie algebra of type $A_1 \oplus L$, with $L = A_1, C_3, A_5, D_6$ or E_7 as in Table 2. Moreover, \mathfrak{h} acts irreducible on \mathfrak{m} and therefore $V \otimes \mathcal{T}_{\mathcal{J}}$ becomes an irreducible LY-algebra with trivial binary product (that is, it is an irreducible Lie triple system) and ternary product given by:

$$[u \otimes x, v \otimes y, w \otimes z] = (x, y)\varphi_{u,v}(w) \otimes z + \varphi(u, v)w \otimes \{xyz\} \tag{3.19}$$

where (x, y) and $\{xyz\}$ are the alternating form and the triple product of $\mathcal{T}_{\mathcal{J}}$. Its standard enveloping Lie algebra is, because of Proposition 1.3, the Lie algebra $\mathfrak{g}(\mathcal{T}_{\mathcal{J}})$, whose type is given in Table 2 too. \square

4. Classification

As shown in Section 2, the irreducible Lie–Yamaguti algebras of non-simple type are those for which the inner derivation algebra is semisimple but not simple. According to Theorem 2.1, the standard enveloping Lie algebras of such LY-algebras are simple Lie algebras, so following Proposition 1.3 the classification of such LY-algebras can be reduced to determine the reductive decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satisfying

- (a) \mathfrak{g} is a simple Lie algebra
 - (b) \mathfrak{h} is a semisimple but not simple subalgebra of \mathfrak{g}
 - (c) \mathfrak{m} is an irreducible ad \mathfrak{h} -module.
- $$\tag{4.1}$$

In this section we classify the irreducible LY-algebras of non-simple type and, first of all, the irreducible LY-algebras whose standard enveloping is classical, that is, isomorphic to either $\mathfrak{sl}_n(k)$ (special), $n \geq 2$, $\mathfrak{so}_n(k)$ (orthogonal), $n \geq 3$, or $\mathfrak{sp}_{2n}(k)$ (symplectic), $n \geq 1$.

Theorem 4.1. *Let $(\mathfrak{m}, x \cdot y, [x, y, z])$ be an irreducible LY-algebra of non-simple type whose standard enveloping Lie algebra is simple and classical. Then, up to isomorphism, either:*

- (i) $\mathfrak{m} = \mathfrak{sl}(V_1) \otimes \mathfrak{sl}(V_2)$ for some vector spaces V_1 and V_2 with $2 \leq \dim V_1 \leq \dim V_2$ and $(\dim V_1, \dim V_2) \neq (2, 2)$, as in Example 3.3, with binary and ternary products given in (3.6).
 In this case the standard enveloping Lie algebra is isomorphic to the special linear algebra $\mathfrak{sl}(V_1 \otimes V_2)$ and the inner derivation algebra to $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$.
- (ii) $\mathfrak{m} = V_1 \otimes V_2$ for some vector spaces V_1 and V_2 endowed with nondegenerate symmetric bilinear forms with $3 \leq \dim V_1 \leq \dim V_2$ as in Example 3.1. This is an irreducible Lie triple system, whose triple product is given in (3.2). Alternatively, this is the LY-algebra inside the Tits construction $\mathcal{T}(\mathcal{J}(V_1), \mathcal{J}(V_2))$ for two Jordan algebras of symmetric bilinear forms in Remark 3.5.
 In this case the standard enveloping Lie algebra is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(V_1 \oplus V_2)$ and the inner derivation algebra to $\mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2)$.

(iii) $\mathfrak{m} = V_1 \otimes V_2$ for some vector spaces V_1 and V_2 endowed with nondegenerate skew-symmetric bilinear forms with $2 \leq \dim V_1 \leq \dim V_2$ as in Example 3.1. This is an irreducible Lie triple system, whose triple product is given in (3.2). Alternatively, this is the LY-algebra inside the Tits construction $\mathcal{T}(\mathcal{J}(V_1), \mathcal{J}(V_2))$ for two Jordan superalgebras of skew-symmetric bilinear forms in Remark 3.5.

In this case the standard enveloping Lie algebra is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(V_1 \oplus V_2)$ and the inner derivation algebra to $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$.

(iv) $\mathfrak{m} = \mathfrak{sp}(V_1) \otimes \mathcal{J}_0$, where V_1 is a two-dimensional vector space endowed with a nonzero skew-symmetric bilinear form and \mathcal{J} is the Jordan algebra $H_n(k)$ for $n \geq 3$ (that is, isomorphic to $\mathfrak{sym}(V_2, \varphi_2)$, for a vector space V_2 of dimension n endowed with a nondegenerate symmetric bilinear form φ_2). The binary and ternary products are given in (3.4). Alternatively, this is the LY-algebra inside the Tits construction $\mathcal{T}(\mathcal{Q}, H_n(k))$ (see Remark 3.7).

In this case the standard enveloping Lie algebra is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(V_1 \otimes V_2) \simeq \mathfrak{sp}_{2n}(k)$, and the inner derivation algebra to $\mathfrak{sp}(V_1) \oplus \mathfrak{so}(V_2)$.

(v) $\mathfrak{m} = \mathfrak{sp}(V_1) \otimes \mathcal{J}_0$, where V_1 is a two-dimensional vector space endowed with a nonzero skew-symmetric bilinear form and \mathcal{J} is the Jordan algebra $H_n(\mathcal{Q})$ for $n \geq 3$ (that is, isomorphic to $\mathfrak{sym}(V_2, \varphi_2)$, for a vector space V_2 of dimension $2n$ endowed with a nondegenerate skew-symmetric bilinear form φ_2). The binary and ternary products are given in (3.4). Alternatively, this is the LY-algebra inside the Tits construction $\mathcal{T}(\mathcal{Q}, H_n(\mathcal{Q}))$ (see Remark 3.7).

In this case the standard enveloping Lie algebra is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(V_1 \otimes V_2) \simeq \mathfrak{so}_{4n}(k)$, and the inner derivation algebra to $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$.

Proof. The irreducible LY-algebras of non-simple type with classical enveloping Lie algebras are those obtained from reductive decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satisfying (4.1), where \mathfrak{g} is a classical simple Lie algebra and $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, $0 \neq \mathfrak{h}_i$ semisimple. In this case, \mathfrak{h} is a maximal subalgebra of \mathfrak{g} and Proposition 1.3 asserts that \mathfrak{m} is exactly the orthogonal complement of \mathfrak{h} with respect to the Killing form of \mathfrak{g} .

Suppose first that \mathfrak{g} is (isomorphic to) the special linear Lie algebra $\mathfrak{sl}(V)$ for some vector space V of dimension ≥ 2 . If V were not irreducible as a module for \mathfrak{h} , then by Weyl’s Theorem, there would exist \mathfrak{h} -invariant subspaces V_1 and V_2 with $V = V_1 \oplus V_2$, but then \mathfrak{h} would be contained in the subalgebra $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$ which is not maximal. Therefore, V is irreducible too as a module for \mathfrak{h} . Hence, up to isomorphism, the \mathfrak{h} -module V decomposes as a tensor product $V = V_1 \otimes V_2$ for some irreducible module V_1 for \mathfrak{h}_1 and some irreducible module V_2 for \mathfrak{h}_2 . It can be assumed that $2 \leq \dim V_1 \leq \dim V_2$. Then \mathfrak{h} is contained in the subalgebra $\mathfrak{sl}(V_1) \otimes k1_{V_2} \oplus k1_{V_1} \otimes \mathfrak{sl}(V_2)$ of $\mathfrak{sl}(V_1 \otimes V_2)$ and, by maximality, \mathfrak{h} is exactly this subalgebra. Hence, we are in the situation of Example 3.3 and Proposition 1.3 shows that the only complementary subspace to \mathfrak{h} in \mathfrak{g} which is \mathfrak{h} -invariant is its orthogonal complement relative to the Killing form. This uniqueness shows that we are dealing with the irreducible LY-algebra in Example 3.3, thus obtaining case (i).

Suppose now that \mathfrak{g} is isomorphic to the Lie algebra of skew-symmetric linear maps of a vector space V endowed with a nondegenerate symmetric or skew-symmetric bilinear map φ .

If V is not irreducible as a module for \mathfrak{h} , and W is an irreducible \mathfrak{h} -submodule of V with $\varphi(W, W) \neq 0$, then by irreducibility the restriction of φ to W is nondegenerate, so V is the orthogonal sum $V = W \oplus W^\perp$. By maximality of \mathfrak{h} , \mathfrak{h} is precisely the subalgebra $\mathfrak{skew}(W) \oplus \mathfrak{skew}(W^\perp)$, and the situation of Example 3.1 appears. Because of the uniqueness in Proposition 1.3, items (ii) (for symmetric φ) or (iii) (for skew-symmetric φ) are obtained.

On the other hand, if V is not irreducible as a module for \mathfrak{h} , and the restriction of φ to any irreducible \mathfrak{h} -submodule of V is trivial then, by Weyl’s theorem on complete reducibility, given an irreducible submodule W_1 , there is another irreducible submodule W_2 with $\varphi(W_1, W_2) \neq 0$. Since $\varphi(W_1, W_1) = 0 = \varphi(W_2, W_2)$, W_1 and W_2 are contragredient modules and $V = (W_1 \oplus W_2) \oplus (W_1 \oplus W_2)^\perp$. Proceeding in the same way with $(W_1 \oplus W_2)^\perp$, it is obtained that $V = V_1 \oplus V_2$ for some \mathfrak{h} -invariant subspaces V_1 and V_2 such that the restrictions of φ to V_1 and V_2 are trivial. Then \mathfrak{h} is contained in $\{f \in \mathfrak{skew}(V, \varphi) : f(V_i) \subseteq V_i, i = 1, 2\}$, which is φ_{V_1, V_2} . But this contradicts the maximality of \mathfrak{h} , since φ_{V_1, V_2} is contained in the subalgebra $\varphi_{V_1, V_2} \oplus \varphi_{V_1, V_1}$.

Finally, if V remains irreducible as a module for \mathfrak{h} then, as above, there is a decomposition $V = V_1 \otimes V_2$ for an irreducible module V_i for \mathfrak{h}_i , $i = 1, 2$, endowed with a nondegenerate symmetric or skew-symmetric bilinear form φ_i such that $\varphi = \varphi_1 \otimes \varphi_2$. By maximality of \mathfrak{h} and Proposition 1.3, we are in the situation of Example 3.2, thus obtaining cases (iv) and (v) depending on φ being either skew-symmetric or symmetric respectively. \square

Now it is time to deal with the irreducible LY-algebras with exceptional standard enveloping Lie algebras. These algebras appear inside reductive decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satisfying (4.1) with \mathfrak{g} a simple exceptional Lie algebra, and hence of type G_2, F_4, E_6, E_7 or E_8 . Over the complex field, a thorough description of the maximal semisimple subalgebras of the simple exceptional Lie algebras is given in [6]. The following result shows that the reductive decomposition we are looking for can be transferred to the complex field, so the results in [6] can be used over our ground field to get the classification of the exceptional irreducible LY-algebras of non-simple type.

Lemma 4.2. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition over our ground field k . Then there is an algebraically closed subfield k' of k , an embedding $\iota : k' \rightarrow \mathbb{C}$ and a Lie algebra \mathfrak{g}' over k' with a reductive decomposition $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$ such that $\mathfrak{g} = \mathfrak{g}' \otimes_{k'} k$, $\mathfrak{h} = \mathfrak{h}' \otimes_{k'} k$ and $\mathfrak{m} = \mathfrak{m}' \otimes_{k'} k$.*

Proof. Let $\{x_i : i = 1, \dots, n\}$ be a basis of \mathfrak{g} over k such that $\{x_i : i = 1, \dots, m\}$ is a basis of \mathfrak{h} and $\{x_{m+1}, \dots, x_n\}$ is a basis of \mathfrak{m} ($1 < m < n$). For any $1 \leq i \leq j \leq n$, $[x_i, x_j] = \sum_{k=1}^n \alpha_{ij}^k x_k$, for some $\alpha_{ij}^k \in k$ (the structure constants). Note that the decomposition being reductive means that $\alpha_{ij}^k = 0$ for $1 \leq i \leq j \leq m$ and $m + 1 \leq k \leq n$ (\mathfrak{h} is a subalgebra), and

for $1 \leq i, k \leq m$ and $m + 1 \leq j \leq n$. Let k'' be the subfield of k generated (over the rational numbers) by the structure constants. Since the transcendence degree of the extension \mathbb{C}/\mathbb{Q} is infinite, there is an embedding $\iota'' : k'' \rightarrow \mathbb{C}$. Finally, let k' be the algebraic closure of k'' on k . By uniqueness of the algebraic closure, ι'' extends to an embedding $\iota : k' \rightarrow \mathbb{C}$. Now, it is enough to take $\mathfrak{h}' = \sum_{i=1}^m k'x_i$, $\mathfrak{m}' = \sum_{i=m+1}^n k'x_i$ and $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$. \square

Therefore, if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition of a simple exceptional Lie algebra over our ground field k , with \mathfrak{h} semisimple but not simple, and with \mathfrak{m} an irreducible module for \mathfrak{h} , take \mathfrak{g}' , \mathfrak{h}' and \mathfrak{m}' as in the previous Lemma 4.2. Then there exists the reductive decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{m}}$ over \mathbb{C} , where $\tilde{\mathfrak{g}} = \mathfrak{g}' \otimes_{k'} \mathbb{C}$ (via ι) and also $\tilde{\mathfrak{h}} = \mathfrak{h}' \otimes_{k'} \mathbb{C}$ and $\tilde{\mathfrak{m}} = \mathfrak{m}' \otimes_{k'} \mathbb{C}$. Since \mathfrak{g} is simple and \mathfrak{g}' is a form of \mathfrak{g} , \mathfrak{g}' is simple too and of the same type as \mathfrak{g} , and hence so is $\tilde{\mathfrak{g}}$. In the same vein, \mathfrak{h} , \mathfrak{h}' and $\tilde{\mathfrak{h}}$ are semisimple Lie algebras of the same type, and the highest weights of \mathfrak{m} and $\tilde{\mathfrak{m}}$ “coincide”, as both are obtained from the highest weight of \mathfrak{m}' relative to a Cartan subalgebra and an ordering of the roots for \mathfrak{h}' .

The displayed list of maximal subalgebras of complex semisimple Lie algebras given in [6] distinguishes the regular maximal subalgebras and the so-called S -subalgebras. Following [6], a subalgebra \mathfrak{r} of a semisimple Lie algebra \mathfrak{g} is said to be *regular* in case \mathfrak{r} has a basis formed by some elements of a Cartan subalgebra of \mathfrak{g} and some elements of its root spaces. On the other hand, an S -subalgebra is a subalgebra \mathfrak{s} not contained in any regular subalgebra. We observe that maximal subalgebras are either regular or S -subalgebras and regular maximal subalgebras have maximal rank, that is, the rank of the semisimple algebras they are living in. Hence, the inner derivation Lie algebras of the irreducible LY-algebras belong to one of these classes of subalgebras and, in the case of nonzero binary product, they are necessarily S -subalgebras:

Lemma 4.3. *Let \mathfrak{m} be an irreducible LY-algebra which is not of adjoint type. If the binary product in \mathfrak{m} is not trivial, then the inner derivation Lie algebra $D(\mathfrak{m}, \mathfrak{m})$ is a maximal semisimple S -subalgebra of the simple standard enveloping Lie algebra of \mathfrak{m} .*

Proof. Following Theorem 2.1 and Corollary 2.3, $D(\mathfrak{m}, \mathfrak{m})$ is a maximal semisimple subalgebra of the simple enveloping Lie algebra $\mathfrak{g}(\mathfrak{m})$ and \mathfrak{m} is a self-dual $D(\mathfrak{m}, \mathfrak{m})$ -module. Let λ be the highest weight of \mathfrak{m} as a module for $D(\mathfrak{m}, \mathfrak{m})$ with respect to a Cartan subalgebra H of $D(\mathfrak{m}, \mathfrak{m})$ and an ordering of the roots, so $\mathfrak{m} = V(\lambda)$ as a module. Then $-\lambda$ is its lowest weight (\mathfrak{m} is self-dual). Since the binary product on \mathfrak{m} is nonzero, so is the vector space $\text{Hom}_{D(\mathfrak{m}, \mathfrak{m})}(V(\lambda) \otimes V(\lambda), V(\lambda))$. Moreover, any map φ in this space is determined by $\varphi(v_\lambda \otimes v_{-\lambda}) \in V(\lambda)_0$ with v_λ and $v_{-\lambda}$ weight vectors of weights λ and $-\lambda$, and $V(\lambda)_0$ the zero weight space in $V(\lambda)$. Then $V(\lambda)_0$ must be nontrivial and, as $V(\lambda)_0$ is contained in the centralizer of H in $\mathfrak{g}(\mathfrak{m})$, the subalgebra H is not a Cartan subalgebra of $\mathfrak{g}(\mathfrak{m})$. Therefore, $D(\mathfrak{m}, \mathfrak{m})$ is not a maximal rank subalgebra of $\mathfrak{g}(\mathfrak{m})$ and hence it is an S -subalgebra. \square

The irreducible LY-algebras of non-simple type whose standard enveloping Lie algebra is exceptional are classified in the next result.

Theorem 4.4. *Let $(\mathfrak{m}, \chi \cdot y, [x, y, z])$ be an irreducible LY-algebra of non-simple type whose standard enveloping Lie algebra is a simple exceptional Lie algebra. Then, up to isomorphism, either:*

- (i) $\mathfrak{m} = V \otimes \mathcal{T}_{\mathfrak{J}}$, where V is a two-dimensional vector space endowed with a nonzero skew-symmetric bilinear form and $\mathcal{T}_{\mathfrak{J}}$ is the symplectic triple system associated to a Jordan algebra \mathfrak{J} isomorphic either to $k, H_3(k), H_3(\mathcal{K}), H_3(\mathcal{Q})$ or $H_3(\mathcal{O})$, as in Example 3.8. This is an irreducible Lie triple system whose ternary product is given in (3.19).
 In this case, the standard enveloping Lie algebra is the exceptional simple Lie algebra of type G_2 for $\mathfrak{J} = k, F_4$ for $\mathfrak{J} = H_3(k), E_6$ for $\mathfrak{J} = H_3(\mathcal{K}), E_7$ for $\mathfrak{J} = H_3(\mathcal{Q})$ and E_8 for $\mathfrak{J} = H_3(\mathcal{O})$, while its inner derivation Lie algebra is isomorphic respectively to $\mathfrak{sl}_2(k) \oplus \mathfrak{sl}_2(k), \mathfrak{sl}_2(k) \oplus \mathfrak{sp}_6(k), \mathfrak{sl}_2(k) \oplus \mathfrak{sl}_6(k), \mathfrak{sl}_2(k) \oplus \mathfrak{so}_{12}(k)$ and $\mathfrak{sl}_2(k) \oplus E_7$.
- (ii) $\mathfrak{m} = \mathcal{O}_0 \otimes \mathfrak{J}_0$, where \mathfrak{J} is one of the Jordan algebras $H_3(k), H_3(\mathcal{K}), H_3(\mathcal{Q})$ or $H_3(\mathcal{O})$. This is the irreducible LY-algebra inside the Classical Tits Construction $\mathcal{T}(\mathcal{O}, \mathfrak{J})$ in Example 3.6. The binary and ternary products are given in (3.14).
 In this case, the standard enveloping Lie algebra is the exceptional simple Lie algebra of type F_4 for $\mathfrak{J} = H_3(k), E_6$ for $\mathfrak{J} = H_3(\mathcal{K}), E_7$ for $\mathfrak{J} = H_3(\mathcal{Q})$ and E_8 for $\mathfrak{J} = H_3(\mathcal{O})$, while its inner derivation Lie algebra is isomorphic respectively to $G_2 \oplus \mathfrak{sl}_2(k), G_2 \oplus \mathfrak{sl}_3(k), G_2 \oplus \mathfrak{sp}_6(k)$ and $G_2 \oplus F_4$.
- (iii) $\mathfrak{m} = \mathcal{Q}_0 \otimes H_3(\mathcal{O})_0$ is the irreducible LY-algebra inside the Classical Tits Construction $\mathcal{T}(\mathcal{O}, H_3(\mathcal{O}))$ in Example 3.6. The binary and ternary products are given in (3.14).
 In this case, the standard enveloping Lie algebra is the exceptional simple Lie algebra of type E_7 , while its inner derivation Lie algebra is isomorphic $\mathfrak{sl}_2(k) \oplus F_4$.

Proof. Following (4.1), we must find reductive decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with \mathfrak{g} exceptional simple, \mathfrak{h} semisimple but not simple and \mathfrak{m} irreducible. In case the binary product is trivial, \mathfrak{m} is an irreducible Lie triple system. Up to isomorphism, these triple systems fit into one of the following $(\mathfrak{g}(\mathfrak{m}), D(\mathfrak{m}, \mathfrak{m}), \mathfrak{m})$ possibilities (see [10]): $(G_2, A_1 \times A_1, V(\lambda_1) \otimes V(3\mu_1))$, $(F_4, A_1 \times C_3, V(\lambda_1) \otimes V(\mu_3))$, $(E_6, A_1 \times A_5, V(\lambda_1) \otimes V(\mu_3))$, $(E_7, A_1 \times D_6, V(\lambda_1) \otimes V(\mu_6))$, $(E_8, A_1 \times E_7, V(\lambda_1) \otimes V(\mu_7))$. In the above list, $V(\lambda) \otimes V(\mu)$ indicates the irreducible module structure of \mathfrak{m} , described by means of the fundamental weights λ_i and μ_i relative to fixed Cartan subalgebras in each component of $\mathfrak{h} = L_1 \times L_2$. The notation follows [11]. In all these cases, \mathfrak{g} is a \mathbb{Z}_2 -graded simple Lie algebra in which the odd part contains a three-dimensional simple ideal of type A_1 for which the even part is a sum of copies of a two-dimensional irreducible module. Identifying A_1 and $V(\lambda_1)$ with $\mathfrak{sp}(V)$ and V respectively, for a two-dimensional vector space V endowed with a nonzero skew-symmetric bilinear form, the following general description for these reductive decompositions follows:

$$\mathfrak{g} = \mathfrak{sp}(V) \oplus \mathfrak{s} \oplus (V \otimes \mathcal{T}) \tag{4.2}$$

where \mathfrak{s} is a simple Lie algebra. Then, Theorem 2.9 in [7] shows that \mathcal{T} is endowed with a structure of a simple symplectic triple system obtained from the Lie bracket of \mathfrak{g} for which $\mathfrak{s} = \text{Inder}(\mathcal{T})$. It follows that \mathfrak{g} is the Lie algebra $\mathfrak{g}(\mathcal{T})$ in (3.16). An inspection of the classification of the simple symplectic triple systems displayed in [7, Theorem 2.30] shows that the only possibilities for \mathcal{T} are those given in Example 3.8. Thus item (i) is obtained.

Now let us assume that the binary product is not trivial. From Lemma 4.3, it follows that \mathfrak{h} is a maximal semisimple S -subalgebra of \mathfrak{g} . Because of [6, Theorem 14.1], there exist only eight possible pairs $(\mathfrak{g}, \mathfrak{h})$ with \mathfrak{h} not simple and \mathfrak{g} exceptional: $(F_4, G_2 \oplus A_1)$, $(E_6, G_2 \oplus A_2)$, $(E_7, G_2 \oplus C_3)$, $(E_7, F_4 \oplus A_1)$, $(E_7, G_2 \oplus A_1)$, $(E_7, A_1 \oplus A_1)$, $(E_8, G_2 \oplus F_4)$, $(E_8, A_2 \oplus A_1)$. Now, the irreducible and nontrivial action of \mathfrak{h} on \mathfrak{m} implies that this is a tensor product $\mathfrak{m} = V(\lambda) \otimes V(\mu)$ with $V(\lambda), V(\mu)$ irreducible modules of nonzero dominant weights λ and μ for each one of the simple components in \mathfrak{h} . Computing dimensions and possible irreducible modules of the involved algebras, the following descriptions of \mathfrak{m} , as a module for \mathfrak{h} are obtained:

- $(F_4, G_2 \oplus A_1)$: Here $\dim \mathfrak{m} = 52 - (14 + 3) = 35 = 7 \times 5$. The only possibility for \mathfrak{m} is to be the tensor product of the seven-dimensional irreducible module for G_2 and the five-dimensional irreducible module for A_1 : $\mathfrak{m} = V(\lambda_1) \otimes V(4\mu_1)$.
- $(E_6, G_2 \oplus A_2)$: Here $\dim \mathfrak{m} = 78 - (14 + 8) = 56$. The only possibility for \mathfrak{m} is to be the tensor product of the seven-dimensional irreducible module for G_2 and the adjoint module for A_2 : $\mathfrak{m} = V(\lambda_1) \otimes V(\mu_1 + \mu_2)$.
- $(E_7, G_2 \oplus C_3)$: Here $\dim \mathfrak{m} = 133 - (14 + 21) = 98$. The only possibility for \mathfrak{m} is to be the tensor product of the seven-dimensional irreducible module for G_2 and a fourteen-dimensional module for C_3 : $\mathfrak{m} = V(\lambda_1) \otimes V(\mu_2)$. (The weight μ_3 for C_3 cannot occur as this module is not self-dual.)
- $(E_7, F_4 \oplus A_1)$: Here $\dim \mathfrak{m} = 133 - (52 + 3) = 78$. The only possibility for \mathfrak{m} is to be the tensor product of the twenty-six-dimensional irreducible module for F_4 and the adjoint module for A_1 : $\mathfrak{m} = V(\lambda_4) \otimes V(2\mu_1)$.
- $(E_8, G_2 \oplus F_4)$: Here $\dim \mathfrak{m} = 248 - (14 + 52) = 182$. The only possibility for \mathfrak{m} is to be the tensor product of the seven-dimensional irreducible module for G_2 and the twenty-six-dimensional module for F_4 : $\mathfrak{m} = V(\lambda_1) \otimes V(\mu_4)$.
- $(E_7, G_2 \oplus A_1)$: Here $\dim \mathfrak{m} = 133 - (14 + 3) = 116 = 2^2 \times 29$. As G_2 has no irreducible modules of dimension 2, 4, 29 or 58, this case is not possible.
- $(E_7, A_1 \oplus A_1)$: Here $\dim \mathfrak{m} = 133 - (3 + 3) = 127$. Since 127 is prime, there is no possible factorization.
- $(E_8, A_2 \oplus A_1)$: Here $\dim \mathfrak{m} = 248 - (8 + 3) = 237 = 3 \times 79$. As A_2 has no irreducible module of dimension 79 and its modules of dimension 3 are not self-dual, this case is impossible too.

Note that the possible reductive decompositions above fit exactly into the Classical Tits Construction of exceptional Lie algebras given in Example 3.6. By identifying G_2 with $\text{Der}(\mathcal{O})$ and $V(\lambda_1)$ with \mathcal{O}_0 , and F_4 with $\text{Der}(H_3(\mathcal{O}))$ and $V(\lambda_4)$ with $H_3(\mathcal{O})_0$, the case $(E_8, G_2 \oplus F_4)$ corresponds to $\mathcal{T}(\mathcal{O}, H_3(\mathcal{O}))$. Also, with the identifications $A_1 \simeq \text{Der } H_3(k)$ and $V(4\mu_1) \simeq H_3(k)_0$, $A_2 \simeq \text{Der } H_3(\mathcal{K})$ and $V(\mu_1 + \mu_2) \simeq H_3(\mathcal{K})_0$ (recall $\mathcal{K} = k \times k$), $C_3 \simeq \text{Der } H_3(\mathcal{Q})$ and $V(\mu_2) \simeq H_3(\mathcal{Q})_0$, the cases $(F_4, G_2 \oplus A_1)$, $(E_6, G_2 \oplus A_2)$ and $(E_7, G_2 \oplus C_3)$ are given by $\mathcal{T}(\mathcal{O}, \mathcal{J})$ with $\mathcal{J} = H_3(k), H_3(\mathcal{K})$ or $H_3(\mathcal{Q})$. Finally, the case $(E_7, F_4 \oplus A_1)$ corresponds to $\mathcal{T}(\mathcal{Q}, H_3(\mathcal{O}))$ under the identifications $F_4 \simeq \text{Der } H_3(\mathcal{O})$ and $V(\lambda_4) \simeq H_3(\mathcal{O})_0, A_1 \simeq \text{Der } \mathcal{Q}$ and $V(2\mu_1) \simeq \mathcal{O}_0$.

On the other hand, if \mathcal{A} denotes either the algebra of quaternions or octonions, the subspaces $\text{Hom}_{\text{Der } \mathcal{A}}(\mathcal{A}_0 \otimes \mathcal{A}_0, \text{Der } \mathcal{A})$, $\text{Hom}_{\text{Der } \mathcal{A}}(\mathcal{A}_0 \otimes \mathcal{A}_0, k)$ and $\text{Hom}_{\text{Der } \mathcal{A}}(\mathcal{A}_0 \otimes \mathcal{A}_0, \mathcal{A}_0)$ are spanned by $a \otimes b \mapsto D_{a,b}, a \otimes b \mapsto \text{tr}(ab)$ and $a \otimes b \mapsto [a, b]$ respectively, where $D_{a,b}$ is defined in (3.11) and $\text{tr}(a)$ is the trace form, while if \mathcal{J} denotes one of the Jordan algebras $H_3(k), H_3(\mathcal{Q})$, or $H_3(\mathcal{O})$, the subspaces $\text{Hom}_{\text{Der } \mathcal{J}}(\mathcal{J}_0 \otimes \mathcal{J}_0, \text{Der } \mathcal{J})$, $\text{Hom}_{\text{Der } \mathcal{J}}(\mathcal{J}_0 \otimes \mathcal{J}_0, k)$ and $\text{Hom}_{\text{Der } \mathcal{J}}(\mathcal{J}_0 \otimes \mathcal{J}_0, \mathcal{J}_0)$ are spanned by $x \otimes y \mapsto d_{x,y}, x \otimes y \mapsto T(xy)$ and $x \otimes y \mapsto x \star y = x \bullet y - \frac{1}{3}T(xy)1$, with $d_{x,y}$ as in (3.13) and $T(x)$ the generic trace. Then, by imposing the Jacobi identity, it is easily checked that, up to scalars, there exists only one way to introduce a Lie product in the vector space $(\text{Der } \mathcal{A} \oplus \text{Der } \mathcal{J}) \oplus (\mathcal{A}_0 \otimes \mathcal{J})$, for $\mathcal{A} = \mathcal{Q}$ or $\mathcal{A} = \mathcal{O}$, with the natural actions of the derivation algebras on \mathcal{A} and \mathcal{J} . This product is given by

$$[a \otimes x, b \otimes y] = \frac{\alpha^2}{3}T(xy)D_{a,b} + 2\alpha^2 \text{tr}(ab)d_{x,y} + \alpha[a, b] \otimes x \star y \tag{4.3}$$

where $\alpha \in k$. The resulting algebras for the same ingredients and different nonzero scalars α are all isomorphic and hence isomorphic to the Classical Tits Construction $\mathcal{T}(\mathcal{O}, \mathcal{J})$ with $\mathcal{J} \neq H_3(\mathcal{K})$, or $\mathcal{T}(\mathcal{Q}, H_3(\mathcal{O}))$.

For $\mathcal{J} = H_3(\mathcal{K})$ (which is isomorphic to the algebra $\text{Mat}_3(k)$ with the symmetrized product), \mathcal{J}_0 is isomorphic to the adjoint module $\text{Der } \mathcal{J}$, and hence the subspaces $\text{Hom}_{\text{Der } \mathcal{J}}(\mathcal{J}_0 \otimes \mathcal{J}_0, \mathcal{J}_0)$ and $\text{Hom}_{\text{Der } \mathcal{J}}(\mathcal{J}_0 \otimes \mathcal{J}_0, \text{Der } \mathcal{J})$ have dimension 2, being spanned by the symmetric product $x \star y$ and the skew product $d_{x,y}$. Since the products in $\text{Hom}_{\text{Der } \mathcal{O}}(\mathcal{O}_0 \otimes \mathcal{O}_0, \mathcal{O}_0)$ are skew and symmetric in $\text{Hom}_{\text{Der } \mathcal{O}}(\mathcal{O}_0 \otimes \mathcal{O}_0, k)$, the anticommutativity imposed in the construction of a Lie algebra on the vector space $(\text{Der } \mathcal{O} \oplus \text{Der } \mathcal{J}) \oplus (\mathcal{O}_0 \otimes \mathcal{J})$ with the natural actions of the derivation algebras on \mathcal{O} and \mathcal{J} , can only be guaranteed if a symmetric product in $\text{Hom}_{\text{Der } \mathcal{J}}(\mathcal{J}_0 \otimes \mathcal{J}_0, \mathcal{J}_0)$ and a skew-symmetric one in $\text{Hom}_{\text{Der } \mathcal{J}}(\mathcal{J}_0 \otimes \mathcal{J}_0, \text{Der } \mathcal{J})$ are used. This yields again the Lie product in (4.3) and, up to isomorphism, the corresponding Classical Tits Construction $\mathcal{T}(\mathcal{O}, H_3(\mathcal{K}))$ given in Example 3.6. This provides cases (ii) and (iii) in the theorem. \square

5. Concluding remarks

As mentioned in the introduction, concerning the isotropy irreducible homogeneous spaces, Wolf remarked in [30] that only the irreducible homogeneous spaces $\mathbf{SO}(\dim K)/\text{ad } K$ for an arbitrary compact simple Lie group follow a clear pattern.

These are related to the reductive pairs $(\mathfrak{so}(L), \text{ad } L)$ for a simple Lie algebra L , so $\text{ad } L = \text{Der}(L) = [\text{Der}(L), \text{Der}(L)]$, and hence the reductive pair can be written as $(\mathfrak{so}(L), \text{Der}(L))$.

The examples in Section 3 follow clear patterns too. Moreover, a closer look at the classification of the non-simple type irreducible LY-algebras shows that, apart from the irreducible Lie triple systems and the exceptional cases that appear related to the Classical Tits Construction in Theorem 4.4, there are two more classes, that correspond to Examples 3.2 and 3.3.

Concerning the irreducible LY-algebras in Example 3.2, let (V_1, φ_1) be a two-dimensional vector space endowed with a nonzero skew-symmetric bilinear form, and let (V_2, φ_2) be another vector space of dimension ≥ 3 endowed with a nondegenerate ϵ -symmetric bilinear form. Then $T = V_1 \otimes V_2$ is an irreducible Lie triple system, as in Example 3.1, whose Lie algebra of derivations is $\text{Der}(T) = \mathfrak{sp}(V_1, \varphi_1) \oplus \mathfrak{stew}(V_2, \varphi_2)$. Hence, the reductive pair $(\mathfrak{g}, \mathfrak{h})$ in Example 3.2 (or in Theorem 4.1, items (iv) and (v)), is nothing else but $(\mathfrak{stew}(T, \varphi_1 \otimes \varphi_2), \text{Der}(T))$.

Also, in Example 3.3 (or the first item in Theorem 4.1) two vector spaces V_1 and V_2 of dimension n_1 and n_2 are considered. The tensor product $V_1 \otimes V_2$ can be identified to $k^{n_1} \otimes k^{n_2}$ or to the space of rectangular matrices $V = \text{Mat}_{n_1 \times n_2}(k)$. The pair $\mathcal{V} = (V, V)$ is a Jordan pair (see [20]) under the product given by $\{xyz\} = xy^t z + zy^t x$ for any $x, y, z \in V$. The Lie algebra of derivations is $\text{Der}(\mathcal{V}) = \mathfrak{sl}_{n_1}(k) \oplus \mathfrak{sl}_{n_2}(k) \oplus k$, which acts naturally on V , and then its derived algebra is $\text{Der}_0(\mathcal{V}) = [\text{Der}(\mathcal{V}), \text{Der}(\mathcal{V})] = \mathfrak{sl}_{n_1}(k) \oplus \mathfrak{sl}_{n_2}(k)$. Hence the reductive pair associated to the irreducible LY-algebra in Example 3.3 is the pair $(\mathfrak{sl}(V), \text{Der}_0(\mathcal{V}))$.

This sort of patterns will explain most of the situations that arise in the generic case [3].

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