Asymptotic Distributions of the Likelihood Ratio Test Statistics for Covariance Structures of the Complex Multivariate Normal Distributions*

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In this paper, the authors derived asymptotic expressions for the null distributions of the likelihood ratio test statistics for multiple independence and multiple homogeneity of the covariance matrices when the underlying distributions are complex multivariate normal. Also, asymptotic expressions are obtained in the non-null cases for the likelihood ratio test statistics for independence of two sets of variables and the equality of two covariance matrices. The expressions obtained in this paper are in terms of beta series. In the null cases, the accuracy of the first terms alone is sufficient for many practical purposes.

1. Introduction

The problems of testing the hypotheses on the structures of the covariance matrices of the real multivariate normal populations received considerable attention in recent years. But, not much work was done on the covariance structures of the complex multivariate normal populations. Investigations on covariance structures of the complex multivariate normal populations have important applications in the area of inference on multiple time series since certain estimates of the spectral density matrix of the multivariate stationary
Gaussian time series are approximately distributed as the complex Wishart matrix. The object of this paper is to investigate the null and non-null asymptotic distributions of the likelihood ratio statistics for testing the hypotheses on the covariance structures of the complex multivariate normal populations. For some discussions on the applications of the complex multivariate distributions, the reader is referred to Brillinger [1], Brillinger and Krishnaiah [2], Hannan [3] and Krishnaiah [6].

In Section 2 of this paper, we derived the asymptotic null distribution of the likelihood ratio statistic for multiple independence, whereas Section 3 is devoted to the corresponding non-null distribution for the case of two sets of variables. An expression is derived in Section 4 for the asymptotic null distribution of the likelihood ratio statistic for multiple homogeneity of the covariance matrices of the complex multivariate normal populations. In Section 5, we derived an expression for the asymptotic non-null distribution of the likelihood ratio test statistic for homogeneity of the covariance matrices of two complex multivariate normal populations under certain alternatives. The hypotheses considered in Sections 2–5 arise in studying certain linear structures of the covariance matrices. For a discussion of these problems in the real case, the reader is referred to Krishnaiah and Lee [7]. The expressions obtained in this paper are in terms of the beta series. In the null cases, it is found that the accuracy of the approximations based on the first terms of the asymptotic series are sufficient for many practical purposes. Krishnaiah, Lee and Chang [8] approximated the null distributions of certain powers of the likelihood ratio test statistics for multiple independence and multiple homogeneity of the covariance matrices of the complex multivariate normal populations with Pearson's Type I distributions. But these approximations are based upon empirical investigations, whereas the investigations in the present paper are analytic in nature. In the real case, Rao [10] gave a useful approximation, in terms of beta series, for the null distribution of certain power of the multivariate beta matrix.

2. ASYMPTOTIC DISTRIBUTION OF THE LIKELIHOOD RATIO STATISTIC FOR TESTING MULTIPLE INDEPENDENCE

Let \( Z' = (Z_1', \ldots, Z_d') \) be distributed as a complex multivariate normal with mean vector \( \mu' = (\mu_1', \ldots, \mu_d') \) and covariance matrix \( \Sigma \). Also, let

\[
E((Z_i - \mu_i)(\bar{Z}_j - \bar{\mu}_j)') = \Sigma_{ij},
\]

where \( \bar{Z}_j \) is the complex conjugate of \( Z_j \). We will assume that \( Z_i \) is of order \( p_i \times 1 \) and \( b = p_1 + \cdots + p_d \). Consider the hypothesis \( H_0 \) where

\[
H_0: \Sigma_{ij} = 0 \quad (i \neq j = 1, \ldots, d).
\]
Let \((Z_{ij}, \ldots, Z_{ij})\), \(j = 1, 2, \ldots, N\), be \(N\) independent observations on \(Z'\). Also, let \(\mathcal{S}' = (\mathcal{S}'_{lm})\), where

\[
\mathcal{S}'_{lm} = \frac{1}{N} \sum_{j=1}^{N} (Z_{ij} - \bar{Z}_l)(\bar{Z}_{mj} - \bar{Z}_m), \quad \bar{Z}_l = \frac{1}{N} \sum_{j=1}^{N} Z_{ij}
\]

\((l, m = 1, \ldots, d)\). (2.2)

The likelihood ratio statistic for testing \(H_0\) is

\[
\lambda = |\mathcal{S}'| \left\{ \prod_{i=1}^{d} |\mathcal{S}'_{ii}| \right\}^{-1}.
\]

Make the transformation \(u = \lambda^{1/s}\), where \(s\) is a constant to be chosen to govern the rate of convergence. The \(h\)th moment of \(u\) is

\[
E(u^h) = \left\{ \prod_{j=1}^{b} \frac{\Gamma(n + h/s - j + 1)}{\Gamma(n - j + 1)} \right\} \left\{ \prod_{i=1}^{d} \prod_{a=1}^{p_i} \frac{\Gamma(n - \alpha + 1)}{\Gamma(n + h/s - \alpha + 1)} \right\}. (2.4)
\]

By using the Mellin's inverse transform, the density of \(u\) becomes

\[
f(u) = \frac{K(b, d, p_i, n)}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^{-h-1} \left\{ \prod_{j=1}^{b} \frac{\Gamma(n + h/s - j + 1)}{\Gamma(n - j + 1)} \right\} \left\{ \prod_{i=1}^{d} \prod_{a=1}^{p_i} \frac{\Gamma(n - \alpha + 1)}{\Gamma(n + h/s - \alpha + 1)} \right\} dh, (2.5)
\]

where

\[
K(b, d, p_i, n) = \prod_{i=1}^{d} \prod_{a=1}^{p_i} \Gamma(n - \alpha + 1) \prod_{j=1}^{b} \Gamma(n - j + 1).
\]

Now, let \(n = m + \delta, e = c + ms\) where \(n = N - 1\) and \(m + h/s = t/s\) where \(\delta\) is also a converging factor to be chosen. Then we have

\[
f(u) = \frac{K(b, d, p_i, n)}{2\pi i} u^{s m - 1} \int_{e^{-i\infty}}^{e^{+i\infty}} u^{-i\phi(t)} dt \quad (2.6)
\]

and

\[
\phi(t) = \left( \prod_{j=1}^{b} \frac{\Gamma(t/s + \delta - j + 1)}{\Gamma(t/s + \delta - \alpha + 1)} \right) \left( \prod_{i=1}^{d} \prod_{a=1}^{p_i} \frac{\Gamma(t/s + \delta - j + 1)}{\Gamma(t/s + \delta - \alpha + 1)} \right). (2.7)
\]

Using the asymptotic expansion for the logarithm of a gamma function, we have

\[
\log \phi(t) = \log s^v + \log t^{-v} + \sum_{r=1}^{\infty} \frac{A_r}{t^r}. (2.8)
\]
where
\[
\nu = \frac{b(b + 1)}{2} - \sum_{i=1}^{d} \frac{p_i(p_i + 1)}{2},
\]
(2.9)

\[
A_r = \frac{(-1)^r s^r}{2} \left[ \frac{1}{2} - \sum_{i=1}^{d} \sum_{\alpha=1}^{p_i} B_{r-1,1}(\delta - \alpha + 1) - \sum_{i=1}^{b} B_{r-1,1}(\delta - i + 1) \right],
\]
(2.10)

and \(B_r(\cdot)\) is the Bernoulli polynomial of degree \(r\) and order one. Proceeding as in Nagarsenker [9], the c.d.f. of \(u\) is

\[
\text{Prob}(u \leq x) = K(b, d, p_i, n) s^v \times \sum_{j=0}^{\infty} R_j I_x(sm + a, v + j) \frac{\Gamma(sm + a)}{\Gamma(sm + a + v + j)},
\]
(2.11)

where \(I_x(\cdot, \cdot)\) is the incomplete beta function, the coefficients \(R_i\) and \(a\) are defined by the relations

\[
t^{-v} \left\{ 1 + \sum_{r=1}^{\infty} \frac{Q_r}{t^r} \right\} = \sum_{j=0}^{\infty} R_j \frac{\Gamma(t + a)}{\Gamma(t + a + v + j)},
\]
(2.12)

where

\[
Q_r = \frac{1}{r} \sum_{i=1}^{r} l A_i Q_{r-i}; \quad Q_0 = 1.
\]
(2.13)

Further use of the asymptotic expansion for the logarithm of gamma functions in \(K(b, d, p_i, n), \Gamma(sm + a)/\Gamma(sm + a + v + j)\) and \(\Gamma(t + a)/\Gamma(t + a + v + j)\) reduces Eq. (2.11) to

\[
\text{Prob}(u \leq x) = I_x(sm + a, v) + \sum_{i=1}^{\infty} \frac{G_i}{m^i},
\]
(2.14)

where

\[
G_i = \sum_{j=0}^{i} R_{i-j} I_x(sm + a, v + i - j) \sum_{l=0}^{j} \frac{Q_{l}^* C_{i-j, j-l}}{s^l}. \quad (2.15)
\]

The coefficient \(Q_{r}^*, C_{ij}\) are calculated as follows:

\[
Q_{r}^* = \frac{1}{r} \sum_{i=1}^{r} l A_i^* Q_{r-i}^*; \quad Q_0^* = 1.
\]
(2.16)
where

\[ A_i^* = -\frac{A_i}{s^i}, \]  

(2.17)

\[ C_{ij} = \frac{1}{j} \sum_{i=1}^{j} IA_{il}C_{i,j-l}, \quad C_{i0} = 1 \]  

(2.18)

and

\[ A_{il} = \frac{(-1)^l}{l(l+1)} [B_{l+1}(v + a + i) - B_{l+1}(a)]. \]  

(2.19)

Also, after expansion of gamma functions \( \Gamma(t + a)/\Gamma(t + a + v + j) \), \( R_i \) can be computed from Eq. (2.12) as

\[ \sum_{j=0}^{i} R_{i-j}C_{i-j,j} = Q_i; \quad R_0 = 1. \]  

(2.20)

The exact c.d.f. can be calculated using Eq. (2.14) when the sample size \( n \) is small. However, if \( n \) is moderately large, we may choose

\[ \delta = \left[ b(b + 1)(2b + 1) - \sum_{i=1}^{d} p_i(p_i + 1)(2p_i + 1) - 6v \right]/12v, \]

\[ a = (1 - v)/2, \quad s^2 = (1 - v^2)v/(24A_x^*). \]  

(2.21)

We have then from Eq. (2.14) the following asymptotic expansion of the distribution of \( u \):

\[ \text{Prob}(u \leq x) = I_x(sm + a, v) + \frac{1}{m^3} \frac{R_3}{s^3} [I_x(sm + a, v + 3) \]

\[ - I_x(sm + a, v)] + O(m^{-4}). \]  

(2.22)

For \( d = 2, p_1 = 1 \), we have \( u = \lambda \), and \( v = b - 1 \), \( a = (2 - b)/2 \), \( \delta = b/2 \), \( s = 1 \), and

\[ \text{Prob}(\lambda \leq x) = I_x(N - b, b - 1). \]  

(2.23)

Table I gives a comparison of the accuracy of the approximation by taking the first term in (2.22) with the approximation obtained by taking the first two terms in (2.22). In the table, the constant \( \tilde{v} \) is defined by \( c = \exp(-\tilde{v}/2) \) where \( P(\tilde{\lambda} \leq c) = \alpha \) and the values of \( \tilde{v} \) are taken from the tables of
TABLE I
Significance Level Associated with the Asymptotic Expression
for the Likelihood Ratio Test for Independence

<table>
<thead>
<tr>
<th>n</th>
<th>d = 3, p = 1, α = 0.05</th>
<th>d = 3, p = 2, α = 0.05</th>
<th>d = 5, p = 2, α = 0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>w̃</td>
<td>a₁</td>
<td>a₂</td>
</tr>
<tr>
<td>10</td>
<td>1.459</td>
<td>0.0499</td>
<td>0.0499</td>
</tr>
<tr>
<td>15</td>
<td>0.923</td>
<td>0.0499</td>
<td>0.0499</td>
</tr>
<tr>
<td>20</td>
<td>0.675</td>
<td>0.0500</td>
<td>0.0500</td>
</tr>
<tr>
<td>30</td>
<td>0.439</td>
<td>0.0502</td>
<td>0.0502</td>
</tr>
</tbody>
</table>

Krishnaiah, Lee and Chang [8]. The value of $P[λ ≤ c]$ is denoted by $a_1$ or $a_2$ according as one term or two terms in (2.22) are used. Also, $p_i - p$ for $i = 1, 2, ..., d$.

3. NON-NULL ASYMPTOTIC DISTRIBUTION OF THE LIKELIHOOD RATIO TEST STATISTIC FOR INDEPENDENCE OF TWO SETS OF VARIABLES

Consider the case $d = 2$, $p_1 = p$ and $p_2 = q$ in Section 2. $(Z_1, Z_2)$ is distributed as complex multivariate normal with mean vector $μ'$ and covariance matrix $Σ$ where

$$Σ = \begin{pmatrix} Σ_{11} & Σ_{12} \\ Σ_{21} & Σ_{22} \end{pmatrix}. \tag{3.1}$$

Also, $A = (A_{ij})$ is as defined in Eq. (2.2) and $p ≤ q$. The likelihood ratio statistic for testing $Σ_{12} = 0$ against the alternative $Σ_{12} ≠ 0$ is given by

$$λ = |A| |A_{11}| |A_{22}| = |I - A_{11} A_{11}^{-1} A_{12} A_{22}^{-1}|$$

$$= |I - B^2|, \tag{3.2}$$
where $\mathcal{S}^2 = \text{diag}(\tau_1^2, ..., \tau_p^2)$, $\tau_1^2 \geq \cdots \geq \tau_p^2$ and $\tau_i$'s are the sample canonical correlations between $Z_1$ and $Z_2$.

Let $n = N - 1 = m + \delta$, where $\delta$ is defined as in Eq. (2.21) and assume

$$P^2 = \frac{W}{m}, \quad (3.3)$$

where $W$ is fixed as $m \to \infty$. Also, $P^2 = \text{diag}(\rho_1^2, ..., \rho_p^2)$ and $\rho_1^2 \geq \cdots \geq \rho_p^2$ are the roots of the characteristic equation

$$|\Sigma_{12} \Sigma_{21}^{-1} \Sigma_{11} - \rho^2 \Sigma_{11}| = 0. \quad (3.4)$$

Using the density of $\tau_1^2, ..., \tau_p^2$ given in James [5] the $h$th moment of $u = \lambda^{1/s}$, $s$ being defined in Eq. (2.21), is given by

$$E(u^h) = |I - P^2|^n \sum_k \sum_\kappa \frac{C_k(p^2)}{k!} \prod_{i=1}^p \frac{[\Gamma(n + k_i - i + 1)]^2}{\Gamma(n - i + 1) \Gamma(n - q - i + 1)}$$

$$\times \frac{\Gamma(n - q + h/s - i + 1)}{\Gamma(n + h/s + k_i - i + 1)}, \quad (3.5)$$

where $C_k(M)$ is the zonal polynomial of Hermitian matrix $M$, $\kappa = (k_1, ..., k_p)$ is a partition of $k$, $k_1 \geq k_2 \geq \cdots \geq k_p \geq 0$ and $k = k_1 + k_2 + \cdots + k_p$. We need the following lemma, which is easily proved using Lemma 1 of Hayakawa [4].

**Lemma 3.1.** Let

$$\tilde{a}_1(\kappa) = \sum_{j=1}^p k_j(k_j - 2j)$$

and

$$\tilde{a}_2(\kappa) = 2 \sum_{j=1}^p k_j(k_j^2 - 3jk_j + 3j^2).$$

Then for any Hermitian positive definite $V$, we have

1. $\sum_\kappa \tilde{a}_1(\kappa) C_k(V) = k(k - 1) \text{tr } V^2 (\text{tr } V)^{k-2} - k (\text{tr } V)^k,$

2. $\sum_\kappa \tilde{a}_1^2(\kappa) C_k(V) = (3k^2 - 2k)(\text{tr } V)^k - 2k^2(k - 1) \text{tr } V^2 (\text{tr } V)^{k-2}$

$$+ 4k(k - 1)(k - 2) \text{tr } V^3 (\text{tr } V)^{k-3}$$

$$+ k(k - 1)(k - 2)(k - 3)(\text{tr } V^2)^2 (\text{tr } V)^{k-4},$$
and

\[ (3) \sum_{k} \tilde{a}_2(\kappa) \tilde{C}_\kappa(V) = (3k^2 - k)(\text{tr } V)^k - 3k(k - 1) \text{tr } V^3(\text{tr } V)^{k-2} \]
\[ + 2k(k - 1)(k - 2) \text{tr } V^3(\text{tr } V)^{k-3}. \]

Using the inverse Mellin transform to (3.5) and Lemma 3.1, we have, proceeding as in Section 2, the following asymptotic expansion of the non-null distribution of \( u \) up to the order \( m^{-2} \) and under the sequence of local alternatives \( P^2 = W/m \) where \( W \) is a fixed matrix.

\[
\text{Prob}(u \leq x) = \beta_\delta(sm + a, v) + \frac{1}{m} \left( \beta_\delta(sm + a, v) \left( -\delta \text{ tr } W - \frac{1}{2} \text{ tr } W^2 \right) \right. \\
\left. + \beta_\delta(sm + a, v + 1) \left( 2\delta - \frac{a + v}{s} \right) \text{ tr } W \right. \\
\left. + \beta_\delta(sm + a, v + 2) \left[ \left( -\delta + \frac{a + v}{s} \right) \text{ tr } W + \text{ tr } W^2 - \frac{1}{2s} (\text{tr } W)^2 \right] \right. \\
\left. + \beta_\delta(sm + a, v + 3) \left[ \frac{1}{2s} (\text{tr } W)^2 - \frac{1}{2} \text{ tr } W^2 \right] \right. \\
\left. + \frac{1}{m} \left( \beta_\delta(sm + a, v) \left[ \begin{array}{c} \delta \text{ tr } W^2 - \frac{\delta}{2} \text{ tr } W^2 - \frac{1}{3} \text{ tr } W^3 \\
+ \frac{\delta}{2} \text{ tr } W \text{ tr } W^2 + \frac{1}{8} (\text{tr } W^2)^2 \end{array} \right) \right. \right. \\
\left. + \beta_\delta(sm + a, v + 1) \left[ \left( \delta - \frac{a + v}{s} \right)^2 \text{ tr } W \right. \\
\left. - \left( 2\delta^2 - \frac{\delta(a + v)}{s} \right) (\text{tr } W)^2 - \left( \delta - \frac{a + v}{2s} \right) \text{ tr } W \text{ tr } W^2 \right] \right. \\
\left. + \beta_\delta(sm + a, v + 2) \left[ \left( -2\delta^2 + \frac{\delta}{s} + \frac{4\delta(a + v)}{s} - \frac{(a + v)}{s^2} \right. \right. \\
\left. \frac{2(a + v)^2}{s^2} \text{ tr } W + \left( 3\delta - \frac{2(a + v)}{s} - \frac{1}{s} \right) \text{ tr } W^2 \right. \\
\left. + \left( \frac{1}{2s^2} ((v + a + 1)^2 + (v + a)) - \frac{3\delta(a + v)}{s} \right. \right. \\
\left. \frac{2\delta}{s} + 3\delta^2 + \frac{1}{2} \right) (\text{tr } W)^2 + \frac{\delta}{2s} (\text{tr } W)^3 \right) \]
\[- \left( \frac{\delta}{2} + \frac{a + v}{2s} \right) \text{tr } W \text{tr } W^2 - \frac{1}{2} \left( \text{tr } W^2 \right)^2 + \frac{1}{4s} \text{tr } W^2 (\text{tr } W)^2 \right]

+ \beta_6 (sm + a, v + 3) \left[ \left( \frac{\delta^2 + (a + v)^2 + (a + v)}{s^2} - \frac{2\delta(a + v)}{s} - \frac{\delta}{s} \right) \text{tr } W \right.

+ \left( \frac{7(a + v)}{2s} + \frac{5}{2s} - 4\delta \right) \text{tr } W^2 + \frac{5}{3} \text{tr } W^3

- \left( \frac{2(a + v)^2 + 3(a + v) + 1}{2s^2} - \frac{3\delta(a + v)}{s} - 4\delta \right) \left( \frac{2\delta}{2s} + 2 \delta^2 + 1 \right) (\text{tr } W)^2

+ \left( \frac{a + v}{2s^2} + \frac{2}{3s^2} - \frac{2\delta}{2s} \right) (\text{tr } W)^3 + \left( \frac{5\delta}{2} - \frac{a + v + 2}{s} \right) \text{tr } W \text{tr } W^2

- \frac{1}{4s} \text{tr } W^2 (\text{tr } W^2) + \frac{1}{4} \left( \text{tr } W^2 \right)^2 \right]

+ \beta_6 (sm + a, v + 4) \left[ \left( \frac{3}{2} \delta - \frac{3(a + v + 1)}{2s} \right) \text{tr } W^2 - 2 \text{tr } W^3 \right.

+ \left( \frac{(a + v)^2 + 4(a + v) + 2}{2s^2} - \frac{\delta(a + v + 2)}{s} + \frac{\delta^2 + 1}{2} \right) (\text{tr } W)^2

- \left( \frac{a + v}{s^2} + \frac{3}{2s} - \frac{3\delta}{2s} \right) (\text{tr } W)^3 + \left( \frac{3(a + v) + 7}{2s} - 2\delta \right) \text{tr } W \text{tr } W^2

- \frac{1}{2s} \text{tr } W^2 (\text{tr } W^2) + \frac{1}{2} \left( \text{tr } W^2 \right)^2 + \frac{1}{8s^2} (\text{tr } W)^4 \right]

+ \beta_6 (sm + a, v + 5) \left[ \frac{2}{3} \text{tr } W^3 + \left( \frac{5}{6s^2} + \frac{a + v}{2s^2} - \frac{\delta}{2s} \right) (\text{tr } W)^3 \right.

+ \left( \frac{\delta}{2} - \frac{a + v + 3}{2s} \right) \text{tr } W \text{tr } W^2 + \frac{3}{4s} \text{tr } W^2 (\text{tr } W^2)

- \frac{1}{2} \left( \text{tr } W^2 \right)^2 - \frac{1}{4s^2} (\text{tr } W)^4 \right]

+ \beta_6 (sm + a, v + 6) \left[ - \frac{1}{4s} \text{tr } W^2 (\text{tr } W)^2 \right.

+ \frac{1}{8} \left( \text{tr } W^2 \right)^2 \left\{ O(m^{-3}) \right\} \right\}$
where
\[ \beta_\theta(sm + a, v + j) = \exp(-\theta) \sum_k \frac{\theta^k}{k!} I_x(sm + a, v + j + k) \quad (3.7) \]
and \( \theta = \text{tr} W, v \) and \( a \) are defined in Eq. (2.9) and Eq. (2.21), respectively.

When the alternative hypotheses are close to the null hypothesis, the accuracy obtained by using the first three terms in the asymptotic expressions is sufficient for practical purposes. When the alternative hypotheses deviate from the null hypothesis significantly, we require higher-order terms to obtain very good accuracy.

4. Null Asymptotic Distribution of the Test Statistic for Multiple Homogeneity of Covariance Matrices

Let \( Z_1, \ldots, Z_q \) be independent complex \( p \)-variate normal variables with mean vectors \( \mu_1, \ldots, \mu_q \) and covariance matrices \( \Sigma_1, \ldots, \Sigma_q \), respectively. Also, let \( Z_{ij} \) \((j = 1, \ldots, N_i)\) be \( j \)th independent observations on \( Z_i \). Let \( H_0 = \bigcap_{j=1}^d H_{0j} \) where \( H_{0j} \) \((j = 1, \ldots, d)\) is given by
\[ H_{0j} : \Sigma_{q_{j-1}+1} = \cdots = \Sigma_{q_j} \quad (4.1) \]
and
\[ q_j^* = q_1 + \cdots + q_j, \quad q_0^* = 0, \quad q_1^* = q_1 \quad \text{and} \quad q_d^* = q. \quad (4.2) \]
The modified likelihood ratio statistic for testing \( H_0 \) is given by
\[ \lambda = \prod_{i=1}^q \left| \mathcal{A}_i/n_i \right|^{n_i} \left( \prod_{j=1}^d \sum_{i=q_{j-1}+1}^{q_j} \mathcal{A}_i/n_j^* \right)^{n_j^*}^{-1}, \quad (4.3) \]
where
\[ n_i = N_i - 1, \quad n_j^* = \sum_{i=q_{j-1}+1}^{q_j} n_i, \quad (4.4) \]
\[ \mathcal{A}_i = \sum_{j=1}^{N_i} (Z_{ij} - Z_i)(\overline{Z}_{ij} - \overline{Z}_i)' \quad \text{and} \quad Z_i = \frac{1}{N_i} \sum_{j=1}^{N_i} Z_{ij} \quad (4.5) \]
and \( \overline{Z} \) denotes the complex conjugate.

Let \( w = \lambda^{1/n} \) where \( n = \sum_{i=1}^q n_i \) and define \( \gamma_i = n_i/n, n = m + \delta \) where \( \delta \) is determined as in Section 2. Thus \( n_i = (m + \delta) \gamma_i \), and \( \gamma_j^* = \sum_{i=q_{j-1}+1}^{q_j} \gamma_i \), \( u = w^{1/s} \), where \( s \) is also a convergent factor. The \( h \)th moment of \( u \) is
The modified likelihood ratio test statistic $\lambda$ and its moments are given in Krishnaiah, Lee and Chang [8]. Using the Mellin inverse transform on $E(u^h)$, and proceeding as in Section 2, we have the following asymptotic expansion for the c.d.f. of $u$ up to the order $m^{-3}$.

$$\text{Prob}(u \leq x) = I_x(sm + a, v) + \frac{1}{m^3} R_3 \left[ I_x(sm + a, v + 3) - I_x(sm + a, v) \right] + O(m^{-4}),$$  

(4.7)

where

$$v = p^2(q - d)/2, \quad \delta = \frac{(2p^2 - 1)}{6p(q - d)} \left[ \sum_{a=1}^{q_a} \frac{1}{\gamma_a} - \frac{1}{\gamma_a^*} \right], \quad \left(4.8\right)$$

and

$$A_r = \frac{p}{r} \sum_{a=1}^{q_a} \left[ B_{r+1}(\gamma_a^* \delta - i + 1) \gamma_a^r \right] - \frac{1}{s^2} \sum_{s=q_{a-1}+1}^{q_a} \frac{B_{r+1}(\gamma_s^* \delta - i + 1)}{\gamma_s^r}, \quad \left(4.10\right)$$

$$A_2^* = -\frac{A_2}{s^2},$$

$$R_3 = Q_3 - C_{03},$$

where

$$Q_3 = \frac{1}{3} \sum_{i=1}^{3} IA_i Q_{3-i}; \quad Q_0 = 1$$

and $C_{03}$ is as defined in Eqs. (2.18) and (2.19).
Table II gives a comparison of the accuracy of the approximation by taking the first term in (4.7) with the approximation obtained by taking the first two terms in (4.7). In the table, the constant $\tilde{\nu}$ is defined by $c = \{\exp(-\tilde{\nu}/2)\}^{1/n}$ where $P[w \leq c] = \alpha$ and the values of $\tilde{\nu}$ are taken from the tables of Krishnaiah, Lee and Chang [8]. The value of $P[w \leq c]$ is denoted by $\alpha_1$ or $\alpha_2$ according as one term or two terms in (4.7) are used. Also, $q_1 = \cdots = q_d = q/d$ and $n_i = n_0$ for $i = 1, 2, \ldots, q, n = qn_0$.

The likelihood ratio test statistic $\lambda$ given by Eq. (4.3) can be expressed as

$$\lambda = \lambda_1 \cdots \lambda_d,$$

where

$$\lambda_j = \frac{\prod_{l=q_{j-1}+1}^{q_j} \frac{1}{P_i/n_i^{n_l}}}{\sum_{l=q_{j-1}+1}^{q_j} \frac{1}{P_i/n_i^{n_l}}}$$

and $\lambda_j$ is the likelihood ratio statistic for testing $H_{0j}$. An alternative procedure to test $H_0$ is given below. We accept or reject $H_{0j}$, according as

$$\lambda_j \geq c_\alpha,$$

where

$$P[\lambda_j \geq c_\alpha; j = 1, \ldots, d | H_0] = \prod_{j=1}^{d} P[\lambda_j \geq c_\alpha | H_{0j}] = (1 - \alpha). \quad (4.11)$$

**TABLE II**

Significance Level Associated with the Asymptotic Expression for the Likelihood Ratio Test for the Multiple Homogeneity of the Covariance Matrices

<table>
<thead>
<tr>
<th>$n_0$</th>
<th>$q$</th>
<th>$d$</th>
<th>$p = 3, \alpha = 0.05$</th>
<th>$p = 4, \alpha = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\tilde{\nu}$</td>
<td>$\alpha_1$</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>1</td>
<td>37.24</td>
<td>0.0499</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>1</td>
<td>64.10</td>
<td>0.0497</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>1</td>
<td>33.21</td>
<td>0.0500</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>1</td>
<td>56.85</td>
<td>0.0499</td>
</tr>
<tr>
<td>20</td>
<td>3</td>
<td>1</td>
<td>30.85</td>
<td>0.0499</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
<td>1</td>
<td>54.13</td>
<td>0.0500</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>3</td>
<td>46.97</td>
<td>0.0501</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>2</td>
<td>58.66</td>
<td>0.0499</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>3</td>
<td>43.22</td>
<td>0.0500</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>2</td>
<td>54.49</td>
<td>0.0500</td>
</tr>
<tr>
<td>30</td>
<td>6</td>
<td>3</td>
<td>42.12</td>
<td>0.0500</td>
</tr>
<tr>
<td>30</td>
<td>6</td>
<td>2</td>
<td>53.26</td>
<td>0.0500</td>
</tr>
</tbody>
</table>
The c.d.f. of $\lambda_1^{1/(s_1/n_1^*)}$ can be obtained by replacing $d$, $q$, and $n$ with 1, $q_1$, and $n_1^*$ respectively in Eqs. (4.7) and (4.8); the value of $s_1$ is obtained by taking the square root of the right side of Eq. (4.9) after replacing $d$ and $q$ with 1 and $q_1$, respectively. We can similarly determine the values of $s_2, \ldots, s_d$ and obtain the c.d.f. of $\lambda_j^{1/(s_j/n_j^*)}$ ($j = 2, \ldots, d$). So, the value of $c_\alpha$ in (4.11) can be determined approximately.

The test procedure discussed in this section is useful in testing the hypothesis that the diagonal blocks in the covariance matrix of a complex multivariate normal are equal when the off-diagonal blocks are null matrices.

5. Non Null Distribution of the Test Statistic for Equality of Two Covariance Matrices

Here we shall consider the case for $d = 1$ and $q = 2$ in the framework of Section 4. The likelihood ratio criterion for testing the hypothesis $H_0: \Sigma_1 = \Sigma_2$ against the alternative $H_1: \Sigma_1 \neq \Sigma_2$ is based on the statistic

$$\lambda = \frac{n^{pn}}{n_1^{p_1n_2^{p_2n_2}}} \frac{|\mathcal{A}_1|^{n_1} |\mathcal{A}_2|^{n_2}}{|\mathcal{A}|^n},$$

(5.1)

where $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, $\mathcal{A}_i$ is as defined in Eq. (4.5), and $n = n_1 + n_2$. Let

$$n = m + \delta, \quad n_1/n = \gamma_1, \quad n_2/n = \gamma_2, \quad w = \lambda^{1/n},$$

(5.2)

where $\delta$ is defined in Eq. (4.8) and assume that

(i) $(I - \Sigma_1^{-1}\Sigma_2) = \frac{W}{\gamma_1 m}; W$ is fixed as $m \to \infty$

(ii) $0 < \lim \gamma_i < 1; i = 1, 2.$

(5.3)

The non-null $h$th moment of $u = w^{1/s}$ is

$$E(u^h) = \left(\frac{n}{n_1^{p_1}n_2^{p_2}}\right)^{ph/s} \frac{\tilde{F}_p(n_3(n/h + 1))}{\tilde{F}_p(n_1(n_4))} |\Omega|^{-n_1} \times \sum_k \sum_\kappa \tilde{C}_k(I - \Omega^{-1}) \frac{\tilde{F}_p(n, k)}{k!} \frac{\tilde{F}_p(n_1(n/h + 1), \kappa)}{\tilde{F}_p(n(h/n + 1), \kappa)},$$

(5.4)

where

$$\tilde{F}_p(g) = \pi^{(p-1)/2} \prod_{i=1}^{\rho} \Gamma(g - i + 1),$$
\[ F_p(g, k) = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(g - i + 1 + k_i). \]

is defined in Eq. (4.9) and \( \Omega = \Sigma_2^{-1}\Sigma_1. \)

Now proceeding as in Section 3, we have under the sequence of alternatives \( \Omega^{-1} = I - W/m\gamma_1, \) where \( W \) is a fixed matrix, the following asymptotic expansion of the non-null distribution of \( u \) up to the order \( m^{-2} \):

\[
\text{Prob}(u \leq x) = I_x(sm + a, v) + \frac{1}{m!} d_1[I_x(sm + a, v) - I_x(sm + a, v + 1)]
\]
\[ + \frac{1}{m^2} \sum_{i=1}^{3} \alpha_i I_x(sm + a, v + i - 1) + O(m^{-3}), \quad (5.8) \]

where

\[
d_1 = \frac{1}{2} \left( 1 - \frac{1}{\gamma_1} \right) \text{tr} W^2,
\]

\[
\alpha_1 = \delta d_1 + \frac{1}{2} d_1^2 + \frac{1}{3} \left( 1 - \frac{1}{\gamma_1^2} \right) \text{tr} W^3,
\]

\[
\alpha_2 = -\frac{1}{2} \left( 4\delta - \frac{v + 1}{s} \right) d_1 - d_1^2 \left( 1 - \frac{1}{\gamma_1} \right) \left[ \text{tr} W^3 + \frac{1}{2} (\text{tr} W)^2 \right],
\]

\[
\alpha_3 = \left( \delta - \frac{v + 1}{2s} + 1 \right) d_1 + \frac{1}{2} d_1^2 + \left( \frac{2}{3} - \frac{1}{\gamma_1} + \frac{1}{3\gamma_1^2} \right) \text{tr} W^3,
\]

and \( \delta, v \) are defined in Eq. (4.8).

REFERENCES


