Hadamard Products and Multivariate Statistical Analysis

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ABSTRACT

The *Hadamard product* of two matrices multiplied together elementwise is a rather neglected concept in matrix theory and has found only brief and scattered application in statistical analysis. We survey the known results on Hadamard products in a historical perspective and obtain various extensions. New applications to multivariate analysis are developed with complicated expressions appearing in closed form. These lead to new results concerning Hadamard products of positive definite matrices. The paper ends with an exhaustive bibliography of books and articles related to Hadamard products.

1. INTRODUCTION

If $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B} = \{b_{ij}\}$ are each $m \times n$ matrices, then their *Hadamard product* is the $m \times n$ matrix of elementwise products

$$\mathbf{A} * \mathbf{B} = \{a_{ij}b_{ij}\}. \tag{1.1}$$

[Matrices (which will all have real elements) are denoted by capital letters, vectors by lower case letters, and both appear in bold face print. Transposition will be indicated by a prime, with row vectors always appearing primed.]

Halmos [**13**, p. **144**] appears to be the first to give the name Hadamard product to Eq. (1.1). It is not clear why this product was so named. The French mathematician Jacques Hadamard (**1865**–**1963**) wrote about **400** scientific papers (cf. Hadamard [**11***], Cartwright [**3***], Mandelbrojt and Schwartz [**18***] as well as several books. The two references to Hadamard most frequently cited by later writers in this area date to 1893 and 1903. In the first, Hadamard obtained an upper bound for an arbitrary determinant, the special case of which, for a positive semidefinite matrix, we give as Lemma 3.3. This result is used in establishing lower bounds for the determinant of $\mathbf{A} * \mathbf{B}$ (Corollary 3.6 and Theorem 3.7). In the 1903 book, Hadamard [12] considers quadratic forms of the type $\mathbf{x}'(\mathbf{A} * \mathbf{B})\mathbf{x}$, but as far as we can determine only for the special case $\mathbf{x} = \mathbf{e}$, the column vector with each element unity.

Apparently unaware of any previous work concerning the product (1.1), the German mathematician Issai Schur (1875–1941) proved (Theorem 3.1) that whenever $\mathbf{A}$ and $\mathbf{B}$ are positive semidefinite, then so is $\mathbf{A} * \mathbf{B}$. Schur [30] also proved an interesting inequality (Theorem 3.4) concerning the characteristic roots of $\mathbf{A} * \mathbf{B}$ which appears to have been overlooked by subsequent writers.

Fan [7] introduced the product $\mathbf{C} = \mathbf{A} \circ \mathbf{B}$ where $c_{ii} = a_{ii}b_{ii}$ and $c_{ij} = -a_{ij}b_{ij}$, $i \neq j$, and related this concept to the Hadamard product. This was motivated by the property that when $\mathbf{A}$ and $\mathbf{B}$ are each an M-matrix, then so is $\mathbf{A} \circ \mathbf{B}$; an M-matrix is a square matrix of the form $\rho\mathbf{I} - \mathbf{N}$, where $\mathbf{N}$ has nonnegative elements and the nonnegative real scalar ρ exceeds in absolute value every characteristic root of $\mathbf{N}$. Lynn [16] generalized some of the theorems in Sect. 3 (in particular, Theorem 3.7) from positive semidefinite matrices to M-matrices.

The product (1.1) merits the name *Schur product*, with Bellman [2, p. 30], Davis [4], Majindar [17], Lynn [16], and Srivastava [32] having used this term. Following Halmos [13, 14], later writers including Olkin and Pratt [25], Marcus and Khan [20], Fiedler [10], Marcus and Thompson [22], Marcus and Minc [21, p. 120], Djoković [6], Ballantine [1], and Davis [5] call Eq. (1.1) the Hadamard product. Other writers using the product fail to name it.

The notation used in Eq. (1.1) follows that of Marcus and Minc [21, p. 120] as well as Srivastava [32], Ballantine [1], and McDonald [23]. All the other literature on this topic that we have found uses a different notation. Fiedler [9, 10], Marcus and Khan [20], Davis [4], Marcus and Thompson [22], Lynn [16], and Davis [5] use $\mathbf{A} \circ \mathbf{B}$, while Mirsky [24, p. 421], Olkin and Pratt [25], and Olkin and Siotani [26] use $\mathbf{A} \times \mathbf{B}$. Djoković [6] uses $\mathbf{A} \odot \mathbf{B}$ and Rao [29] $\mathbf{A} \boxdot \mathbf{B}$.

We have found only brief and scattered use of the Hadamard product in statistical analysis. Olkin and Pratt [25] use the Hadamard product of a matrix with itself in the context of multivariate Tchebycheff inequalities, while Srivastava [32] and Rao [29] use the Hadamard product of two different positive definite matrices in the study of general linear models. McDonald [23] represents a generalized factor analysis model using Hadamard products. Olkin and Siotani [26], in an unpublished technical report, use the Hadamard product in the maximum likelihood equations (cf. Sec. 4) for the variances in a multivariate normal population. No other published use of the Hadamard product in statistical analysis has been found.

We examine the algebra of matrices multiplied together elementwise and obtain expressions for the diagonal matrix and trace of a matrix which are useful in applications. In Sec. 3 we present the known results on Hadamard products. Most of these involve positive semidefinite matrices (which we define to be symmetric and to have nonnegative characteristic roots). New inequalities for the characteristic roots of the Hadamard product of two symmetric matrices are given as Theorems 3.11 and 3.12. Applications in multivariate analysis are developed in Sec. 4 with the study of maximum likelihood estimation in a multivariate normal population with known correlation matrix $\boldsymbol{R}$. We show that $\boldsymbol{R} * \boldsymbol{R} - 2(\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I})^{-1}$ is positive semidefinite using a probabilistic argument (Theorem 4.1); a matrix-theoretic proof has eluded us. Various extensions of this result are explored. Section 5 concludes the paper with an exhaustive bibliography of books and papers related to Hadamard products.

We will use $|\mathbf{A}|$ to denote determinant, $\mathrm{tr}(\mathbf{A})$, trace, and $\mathrm{ch}_j(\mathbf{A})$, the jth largest characteristic root of a square matrix $\mathbf{A}$. When $\mathbf{A}$ is symmetric, the roots are real and we mean largest numerically. When the roots are possibly complex we mean largest in absolute value. Proofs terminate with (Q.E.D.).

2. PRELIMINARIES

The Hadamard product differs from the usual product in many ways. To begin with, conformability of the orders of the component matrices is quite different. When $\mathbf{A}$ and $\mathbf{B}$ are two matrices of orders $m \times n$ and $p \times q$, respectively, then we can define $\mathbf{A} * \mathbf{B}$ only when $m = p$ and $n = q$, while $\mathbf{AB}$ is defined only if $n = p$, with no restrictions on m and q.

For matrices of unit rank, however, the two kinds of product enjoy an interesting transitive property. Let $\mathbf{A} = \mathbf{uv}'$ and $\mathbf{B} = \mathbf{wx}'$, where $\mathbf{u}$ and $\mathbf{w}$ are $m \times 1$ and $\mathbf{v}'$ and $\mathbf{x}'$ are $1 \times n$. Then

$$\mathbf{A} * \mathbf{B} = (\mathbf{uv}') * (\mathbf{wx}') = (\mathbf{u} * \mathbf{w})(\mathbf{v} * \mathbf{x})', \tag{2.1}$$

so that the Hadamard product of two matrices of unit rank has rank at most one.

The role of identity matrix in Hadamard products is taken by $\mathbf{ee}'$, the matrix with each component unity

$$\mathbf{A} * (\mathbf{ee}') = \mathbf{A} = (\mathbf{ee}') * \mathbf{A}, \tag{2.2}$$

while the null matrix retains its role

$$\mathbf{A} * \mathbf{0} = \mathbf{0}. \tag{2.3}$$

Hadamard multiplication is commutative unlike regular matrix multiplication

$$\mathbf{A} * \mathbf{B} = \mathbf{B} * \mathbf{A} = \{a_{ij}b_{ij}\}, \tag{2.4}$$

but the distributive property is retained

$$(\mathbf{A} + \mathbf{B}) * \mathbf{C} = \mathbf{A} * \mathbf{C} + \mathbf{B} * \mathbf{C} = \{a_{ij}c_{ij} + b_{ij}c_{ij}\}, \tag{2.5}$$

where $\mathbf{C}$ has the same order as $\mathbf{A}$ and $\mathbf{B}$.

Diagonal matrices are conveniently handled in Hadamard products. The diagonal matrix formed from the square matrix $\mathbf{A}$ may be written

$$\mathbf{A}_{dg} = \mathbf{A} * \mathbf{I}. \tag{2.6}$$

When $\mathbf{A}$ and $\mathbf{B}$ are both square, the row sums of $\mathbf{A} * \mathbf{B}$ are the diagonal elements of $\mathbf{AB}'$ or $\mathbf{BA}'$. Hence we may write

$$(\mathbf{A} * \mathbf{B})\mathbf{e} = (\mathbf{AB}')_{dg}\mathbf{e} = [(\mathbf{AB}') * \mathbf{I}]\mathbf{e} = (\mathbf{BA}')_{dg}\mathbf{e} = [(\mathbf{BA}') * \mathbf{I}]\mathbf{e} \tag{2.7}$$

which becomes $(\mathbf{AB})_{dg}\mathbf{e} = [(\mathbf{AB}) * \mathbf{I}]\mathbf{e}$, when $\mathbf{B}$ is symmetric, and $(\mathbf{BA})_{dg}\mathbf{e} = [(\mathbf{BA}) * \mathbf{I}]\mathbf{e}$, when $\mathbf{A}$ is symmetric.

The trace of $\mathbf{AB}$ is the sum of all the elements of $\mathbf{A} * \mathbf{B}'$, or $\mathbf{A} * \mathbf{B}$ when $\mathbf{B}$ is square and symmetric. Thus

$$\operatorname{tr}(\mathbf{AB}) = \mathbf{e}'(\mathbf{A} * \mathbf{B}')\mathbf{e}, \tag{2.8}$$

which also follows from Eq. (2.7).

Multiplication of a Hadamard product by diagonal matrices enjoys a useful associative property. With $\mathbf{D}$ a diagonal matrix, we have

$$\mathbf{D}(\mathbf{A} * \mathbf{B}) = (\mathbf{DA}) * \mathbf{B} = \mathbf{A} * (\mathbf{DB}); \quad (\mathbf{A} * \mathbf{B})\mathbf{D} = (\mathbf{AD}) * \mathbf{B} = \mathbf{A} * (\mathbf{BD}); \tag{2.9}$$

$$\mathbf{D}(\mathbf{A} * \mathbf{B})\mathbf{D} = (\mathbf{DAD}) * \mathbf{B} = \mathbf{A} * (\mathbf{DBD}) = (\mathbf{DA}) * (\mathbf{BD}) = (\mathbf{AD}) * (\mathbf{DB}). \quad (2.10)$$

If $\mathbf{A}$ has unit rank we may write $\mathbf{A} = \mathbf{uv}'$. Let $\mathbf{D}_u$ and $\mathbf{D}_v$ be diagonal matrices formed from $\mathbf{u}$ and $\mathbf{v}$, respectively. Then

$$\mathbf{A} * \mathbf{B} = (\mathbf{uv}') * \mathbf{B} = (\mathbf{D}_u\mathbf{ee}'\mathbf{D}_v) * \mathbf{B} = \mathbf{D}_u[(\mathbf{ee}') * \mathbf{B}]\mathbf{D}_v = \mathbf{D}_u\mathbf{B}\mathbf{D}_v, \quad (2.11)$$

as noted by Ballantine [1] for $\mathbf{u} = \mathbf{v}$.

3. RESULTS

The most widely used and possibly most important result concerning Hadamard products was proved, probably for the first time, by Issai Schur [30] in 1911. We will assume throughout this section, unless stated to the contrary, that $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B} = \{b_{ij}\}$ are square symmetric matrices of order p.

THEOREM 3.1 (Schur [30]). *When* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive semidefinite, then so is their Hadamard product* $\mathbf{A} * \mathbf{B}$. *When both* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive definite then so also is* $\mathbf{A} * \mathbf{B}$.

Proof. Suppose $\mathbf{A}$ and $\mathbf{B}$ are positive semidefinite, and consider the quadratic form

$$\mathbf{x}'(\mathbf{A} * \mathbf{B})\mathbf{x}, \quad (3.1)$$

where $\mathbf{x}$ is $p \times 1$, $\mathbf{x} \neq \mathbf{0}$. There exists a matrix $\mathbf{T}$, $p \times p$, such that $\mathbf{B} = \mathbf{TT}'$. Substituting in Eq. (3.1) gives

$$\sum_{i,j=1}^{p} x_i a_{ij} \left(\sum_{k=1}^{p} t_{ik} t_{jk} \right) x_j = \sum_{k=1}^{p} (\mathbf{x} * \mathbf{t}_k)' \mathbf{A} (\mathbf{x} * \mathbf{t}_k) \geqslant 0, \quad (3.2)$$

where $\mathbf{t}_k$ is the kth column of $\mathbf{T}$. When $\mathbf{B}$ is nonsingular, so is $\mathbf{T}$, and if in addition $\mathbf{A}$ is nonsingular, Eq. (3.2) is positive. (Q.E.D.)

The above proof shortens the original version given by Schur [30], which is also given by Fejér [8], Pólya and Szegö [28, p. 307], Oppenheim [27], Halmos [13, pp. 143–144], and [14, pp. 173–174], Mirsky [24, p. 421], Bellman [2, p. 94 (1960), and p. 95 (1970)].

When $\mathbf{A}$ is positive definite but $\mathbf{B}$ is positive semidefinite and singular then $\mathbf{A} * \mathbf{B}$ may or may not be positive definite. If $\mathbf{B} = \mathbf{ee}'$ then, as in Eq. (2.2),

$\mathbf{A} * \mathbf{B} = \mathbf{A} * (\mathbf{ee}') = \mathbf{A}$ and $\mathbf{A} * \mathbf{B}$ is positive definite. But if $\mathbf{B} = \mathbf{0}$ then, as in Eq. (2.3), $\mathbf{A} * \mathbf{B} = \mathbf{A} * \mathbf{0} = \mathbf{0}$ and so $\mathbf{A} * \mathbf{B}$ is not positive definite. Necessary and sufficient conditions on $\mathbf{B}$ for $\mathbf{A} * \mathbf{B}$ to be positive definite were sought by Djoković [6] and found by Ballantine [1]; these conditions are derived from Theorem 3.6 and so will be presented later (Theorem 3.8).

A general result obtained by Ballantine [1] and Styan [33, 34] is

THEOREM 3.2 (Ballantine [1]). *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices of order* $m \times n$. *Then*

$$\operatorname{rank}(\mathbf{A} * \mathbf{B}) \leqslant \operatorname{rank}(\mathbf{A}) \cdot \operatorname{rank}(\mathbf{B}). \tag{3.3}$$

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ have rank a and b respectively. Then there exist matrices $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_a)$, $m \times a$, $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_a)$, $n \times a$, $\mathbf{W} = (\mathbf{w}_1, \ldots, \mathbf{w}_b)$, $m \times b$, and $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_b)$, $n \times b$, such that

$$\mathbf{A} = \mathbf{U}\mathbf{V}' = \sum_{i=1}^{a} \mathbf{u}_i \mathbf{v}_i'; \qquad \mathbf{B} = \mathbf{W}\mathbf{X}' = \sum_{j=1}^{b} \mathbf{w}_j \mathbf{x}_j'. \tag{3.4}$$

Hence

$$\mathbf{A} * \mathbf{B} = \left(\sum_{i=1}^{a} \mathbf{u}_i \mathbf{v}_i'\right) * \left(\sum_{j=1}^{b} \mathbf{w}_j \mathbf{x}_j'\right) = \sum_{i,j=1}^{a,b} (\mathbf{u}_i * \mathbf{w}_j)(\mathbf{v}_i * \mathbf{x}_j)', \tag{3.5}$$

using Eq. (2.1). Since there are ab terms in Eq. (3.5), Eq. (3.3) follows directly. (Q.E.D.)

We note that equality in Eq. (3.3) is possible when the ranks exceed one [cf. Eq. (2.1)]. If $m = n = p$, say, we must, however, have $p \geqslant 4$. When $p = 4$,

$$\mathbf{A} = \begin{bmatrix} \mathbf{ee}' & \mathbf{0} \\ \mathbf{0} & \mathbf{ee}' \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{bmatrix}, \tag{3.6}$$

where the matrices within the partitioning are all 2×2, we obtain $\mathbf{A} * \mathbf{B} = \mathbf{I}$, which has rank 4. Since $\mathbf{A}$ and $\mathbf{B}$ each have rank 2, equality is attained in Eq. (3.3).

An interesting alternate proof of Theorem 3.1 follows from the following lemma given by Marcus and Khan [20], Marcus and Minc [21, pp. 120–121], Srivastava [32], and Davis [5]. The *Kronecker product* $\mathbf{A} \otimes \mathbf{B}$, with $\mathbf{A}$ $m \times n$

and $\mathbf{B}$ $p \times q$, is the $mp \times nq$ matrix $\{a_{ij}\mathbf{B}\}$, with every element of $\mathbf{A}$ multiplied by the matrix $\mathbf{B}$.

LEMMA 3.1 (Marcus and Khan [20]). *The Hadamard product is a principal submatrix of the Kronecker product.*

Theorem 3.1 was extended in 1963 by Majindar [17], who showed that any positive definite matrix may be expressed as a Hadamard product of two positive definite matrices, though never uniquely. When the matrices are positive semidefinite and singular the result is immediate using $\mathbf{ee}'$ as a factor.

THEOREM 3.3 (Schur [30], Majindar [17], Djoković [6]). *A symmetric matrix is positive definite if and only if it can be written as the Hadamard product of two positive definite matrices.*

A further result proved by Issai Schur [30] in 1911 appears to have been overlooked by later writers:

THEOREM 3.4 (Schur [30]). *When* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive semidefinite,*

$$\mathrm{ch}_p(\mathbf{A}) \cdot b_{\min} \leqslant \mathrm{ch}_j(\mathbf{A} * \mathbf{B}) \leqslant \mathrm{ch}_1(\mathbf{A}) \cdot b_{\max}, \qquad j = 1, \ldots, p, \tag{3.7}$$

where $\mathrm{ch}_j(\cdot)$ *denotes* jth *largest characteristic root, and* $b_{\min}$ *and* $b_{\max}$ *are the smallest and largest diagonal elements of* $\mathbf{B}$.

Proof. We write $\mathbf{t}_k$ as the kth column of $\mathbf{T}$, with $\mathbf{B} = \mathbf{TT}'$. Using Eq. (3.2) gives

$$\mathbf{x}'(\mathbf{A} * \mathbf{B})\mathbf{x} = \sum_{k=1}^{p} (\mathbf{x} * \mathbf{t}_k)'\mathbf{A}(\mathbf{x} * \mathbf{t}_k) \leqslant \mathrm{ch}_1(\mathbf{A}) \sum_{k=1}^{p} (\mathbf{x} * \mathbf{t}_k)'(\mathbf{x} * \mathbf{t}_k)$$

$$= \mathrm{ch}_1(\mathbf{A})\mathbf{x}'(\mathbf{B} * \mathbf{I})\mathbf{x} \leqslant \mathrm{ch}_1(\mathbf{A}) b_{\max} \mathbf{x}'\mathbf{x}. \tag{3.8}$$

This proves the right side of Eq. (3.7). The left side follows similarly. (Q.E.D.)

A positive semidefinite matrix with each diagonal entry 1 is called a *correlation matrix.*

COROLLARY 3.1. *When* $\mathbf{R}$ *is a correlation matrix and* $\mathbf{A}$ *is positive semidefinite,*

$$\mathrm{ch}_p(\mathbf{A}) \leqslant \mathrm{ch}_j(\mathbf{A} * \mathbf{R}) \leqslant \mathrm{ch}_1(\mathbf{A}), \qquad j = 1,\ldots,p. \tag{3.9}$$

Since $\mathrm{ch}_p(\mathbf{B}) \leqslant b_{jj} \leqslant \mathrm{ch}_1(\mathbf{B})$, $j = 1,\ldots,p$, whenever $\mathbf{B}$ is symmetric, Theorem 3.4 also implies

COROLLARY 3.2. *When* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive semidefinite,*

$$\mathrm{ch}_p(\mathbf{A})\,\mathrm{ch}_p(\mathbf{B}) \leqslant \mathrm{ch}_j(\mathbf{A} * \mathbf{B}) \leqslant \mathrm{ch}_1(\mathbf{A})\,\mathrm{ch}_1(\mathbf{B}), \qquad j = 1,\ldots,p. \tag{3.10}$$

COROLLARY 3.3. *When* $\mathbf{A}$ *is positive semidefinite,*

$$\mathrm{ch}_p^2(\mathbf{A}) \leqslant a_{\min}\,\mathrm{ch}_p(\mathbf{A}) \leqslant \mathrm{ch}_j(\mathbf{A} * \mathbf{A}) \leqslant a_{\max}\,\mathrm{ch}_1(\mathbf{A}) \leqslant \mathrm{ch}_1^2(\mathbf{A}),$$
$$j = 1,\ldots,p, \tag{3.11}$$

where $a_{\min}$ *and* $a_{\max}$ *are the smallest and largest diagonal elements of* $\mathbf{A}$.

The assumption of $\mathbf{A}$ positive semidefinite in Theorem 3.4 and Corollaries 3.1 and 3.2 may be relaxed to $\mathbf{A}$ symmetric but not negative definite, for in Eq. (3.8) we need only $\mathrm{ch}_1(\mathbf{A}) \geqslant 0$. When just symmetry is assumed for both $\mathbf{A}$ and $\mathbf{B}$, Davis [4] obtained an upper bound for the absolute value of $\mathrm{ch}(\mathbf{A} * \mathbf{B})$; this bound reduces to that in Eq. (3.7), i.e., $\mathrm{ch}_1(\mathbf{A} * \mathbf{B}) \leqslant \mathrm{ch}_1(\mathbf{A}) \cdot b_{\max}$, when $\mathbf{A}$ and $\mathbf{B}$ are positive semidefinite. Further details, and some extensions, are developed in Theorems 3.11 and 3.12 at the end of this section.

Using Lemma 3.1 we may obtain bounds for $\mathrm{ch}(\mathbf{A} * \mathbf{B})$ in terms of $\mathrm{ch}(\mathbf{A} \otimes \mathbf{B})$ with $\mathbf{A}$ and $\mathbf{B}$ positive semidefinite. If $\mathbf{A}_{p-r}$ denotes an $r \times r$ principal submatrix of the symmetric matrix $\mathbf{A}$, then $\mathrm{ch}_{s+t}(\mathbf{A}) \leqslant \mathrm{ch}_s(\mathbf{A}_t) \leqslant \mathrm{ch}_s(\mathbf{A})$; $s = 1,\ldots,p-t$, $t = 1,\ldots,p-1$.

THEOREM 3.5 (Marcus and Khan [20]). *When* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive semidefinite,*

$$\mathrm{ch}_p(\mathbf{A})\,\mathrm{ch}_p(\mathbf{B}) \leqslant \mathrm{ch}_{j+p^2-p}(\mathbf{A} \otimes \mathbf{B}) \leqslant \mathrm{ch}_j(\mathbf{A} * \mathbf{B}) \leqslant \mathrm{ch}_j(\mathbf{A} \otimes \mathbf{B})$$
$$\leqslant \mathrm{ch}_1(\mathbf{A})\,\mathrm{ch}_1(\mathbf{B}), \qquad j = 1,\ldots,p. \tag{3.12}$$

If $\alpha_1,\ldots,\alpha_p$ and $\beta_1,\ldots,\beta_p$ are the characteristic roots of $\mathbf{A}$ and $\mathbf{B}$ respectively, then the characteristic roots of $\mathbf{A} \otimes \mathbf{B}$ are the p^2 quantities

$\alpha_s\beta_t$; $s, t = 1,\ldots,p$ (Marcus [19, p. 5]). The jth largest characteristic root of $\mathbf{A} * \mathbf{B}$ thus lies between the jth and $(j + p^2 - p)$th largest of the pairs $\alpha_s\beta_t$; $s, t = 1,\ldots,p$. Davis [5], apparently unaware of many of the above results, used $\mathrm{ch}_p(\mathbf{A})\,\mathrm{ch}_p(\mathbf{B}) \leqslant \mathrm{ch}_j(\mathbf{A} \otimes \mathbf{B}) \leqslant \mathrm{ch}_1(\mathbf{A})\,\mathrm{ch}_1(\mathbf{B})$ and Lemma 3.1 to prove Theorem 3.1 and Corollary 3.2.

Extending Theorem 3.5 we obtain

COROLLARY 3.4. *When* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive semidefinite,*

$$\prod_{t=0}^{p-1} \mathrm{ch}_{p^2-t}(\mathbf{A} \otimes \mathbf{B}) \leqslant |\mathbf{A} * \mathbf{B}| \leqslant \prod_{s=1}^{p} \mathrm{ch}_s(\mathbf{A} \otimes \mathbf{B}). \tag{3.13}$$

We note that the determinant $|\mathbf{A} * \mathbf{B}|$ lies between the products of the p largest and p smallest characteristic roots of $\mathbf{A} \otimes \mathbf{B}$. Different *lower* bounds are obtained below.

Now let $\mathbf{A}_{p-r}$ denote the lower $r \times r$ principal submatrix of $\mathbf{A}$, with $\mathbf{A}_0 = \mathbf{A}$. Then

LEMMA 3.2 (Mirsky [24, p. 416]). *When* $\mathbf{A}$ *is positive semidefinite,*

$$\mathbf{A}^0 = \mathbf{A} - \alpha\mathbf{e}_1\mathbf{e}_1', \tag{3.14}$$

is positive semidefinite, where $\alpha = |\mathbf{A}|/|\mathbf{A}_1|$ *when* $|\mathbf{A}| \neq 0$ *and zero otherwise, and where* $\mathbf{e}_1 = (1, 0,\ldots, 0)'$.

Proof. When $\mathbf{A}$ is singular, Eq. (3.14) is $\mathbf{A}$ and so positive semidefinite by definition. When $\mathbf{A}$ is nonsingular, $\alpha = 1/\mathbf{e}_1'\mathbf{A}^{-1}\mathbf{e}_1$ and

$$\mathbf{A}^0\mathbf{A}^{-1}\mathbf{A}^0 = (\mathbf{A} - \alpha\mathbf{e}_1\mathbf{e}_1')(\mathbf{I} - \alpha\mathbf{A}^{-1}\mathbf{e}_1\mathbf{e}_1') = \mathbf{A} - \alpha\mathbf{e}_1\mathbf{e}_1' = \mathbf{A}^0, \tag{3.15}$$

and so $\mathbf{A}^{-1}$ is a generalized inverse of $\mathbf{A}^0$ (cf., e.g., Searle [31*, p. 1]). Since $\mathbf{A}^0$ is symmetric it is positive semidefinite. (Q.E.D.)

From this lemma we obtain immediately, with $\mathbf{A}^{-1} = \{a^{ij}\}$,

$$a_{11}a^{11} \geqslant 1, \tag{3.16}$$

and so $a_{ii}a^{ii} \geqslant 1$, $i = 1,\ldots, p$ (cf. Fiedler [10]). Also Eq. (3.16) may be written $|\mathbf{A}| \leqslant a_{11}|\mathbf{A}_1|$. Similarly $|\mathbf{A}_1| \leqslant a_{22}|\mathbf{A}_2|$ and so $|\mathbf{A}| \leqslant a_{11}a_{22}|\mathbf{A}_2|$. Proceeding inductively we obtain the classic result

LEMMA 3.3 (Hadamard [11*]). *When* $\mathbf{A}$ *is positive semidefinite,*

$$|\mathbf{A}| \leqslant a_{11}a_{22}\cdots a_{pp}. \tag{3.17}$$

Marcus [19, p. 14] calls Lemma 3.3 the *Hadamard determinant theorem.* An alternative proof of Eq. (3.17) is due to Hardy, Littlewood, and Pólya [15*, pp. 34, 35] writing $\mathbf{A}$ in terms of a correlation matrix.

COROLLARY 3.5. *When* $\mathbf{R}$ *is a correlation matrix, the diagonal elements of* $\mathbf{R}^{-1}$,

$$r^{ii} \geqslant 1, \qquad i = 1,\ldots, p, \tag{3.18}$$

and the determinant

$$|\mathbf{R}| \leqslant 1. \tag{3.19}$$

Proof. Eq. (3.18) follows directly from Eq. (3.16). To show Eq. (3.19) we use the arithmetic mean/geometric mean inequality

$$|\mathbf{R}| = \prod_{s=1}^{p} \mathrm{ch}_s(\mathbf{R}) \leqslant \left[\frac{\sum_{s=1}^{p} \mathrm{ch}_s(\mathbf{R})}{p}\right]^p = [\mathrm{tr}(\mathbf{R})/p]^p = 1, \tag{3.20}$$

and Eq. (3.19) is proved. (Q.E.D.)

If $\mathbf{A}$ has a diagonal element equal to 0, Eq. (3.17) is identically 0. Otherwise there exists a nonsingular diagonal matrix $\mathbf{D} = (\mathbf{A} * \mathbf{I})^{1/2}$ such that $\mathbf{A} = \mathbf{DRD}$, where $\mathbf{R}$ is a correlation matrix; in such cases Eq. (3.19) is equivalent to Eq. (3.17).

We now establish a lower bound for $|\mathbf{A} * \mathbf{B}|$, first proved in 1930 by the British mathematician (later *Sir*) Alexander Oppenheim (1903–).

THEOREM 3.6 (Oppenheim [27]). *When* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive semidefinite*

$$|\mathbf{A} * \mathbf{B}| \geqslant |\mathbf{A}| b_{11}\cdots b_{pp}. \tag{3.21}$$

Proof. When $\mathbf{A}$ is singular or $\mathbf{B}$ has a zero diagonal element, Eq. (3.21) is trivially satisfied. When $\mathbf{A}$ is nonsingular and $\mathbf{B}$ has no zero diagonal elements we may write $\mathbf{B} = \mathbf{DRD}$, where $\mathbf{D} = (\mathbf{B} * \mathbf{I})^{1/2}$, and Eq. (3.21) is equivalent to

$$|\mathbf{A} * \mathbf{R}| \geqslant |\mathbf{A}|. \tag{3.22}$$

Using Theorem 3.1 and Lemma 3.2, we have

$$0 \leqslant |\mathbf{A}^0 * \mathbf{R}| = |(\mathbf{A} - \mathbf{e}_1\mathbf{e}_1'/a^{11}) * \mathbf{R}|$$

$$= |\mathbf{A} * \mathbf{R} - \mathbf{e}_1\mathbf{e}_1'/a^{11}|$$

$$= |\mathbf{A} * \mathbf{R}| - |\mathbf{A}_1 * \mathbf{R}_1|/a^{11}. \tag{3.23}$$

Thus $|\mathbf{A} * \mathbf{R}| \geqslant |\mathbf{A}_1 * \mathbf{R}_1| \cdot |\mathbf{A}|/|\mathbf{A}_1|$. Similarly $|\mathbf{A}_1 * \mathbf{R}_1| \geqslant |\mathbf{A}_2 * \mathbf{R}_2| \cdot |\mathbf{A}_1|/|\mathbf{A}_2|$, so that $|\mathbf{A} * \mathbf{R}| \geqslant |\mathbf{A}_2 * \mathbf{R}_2| \cdot |\mathbf{A}|/|\mathbf{A}_2|$. Proceeding inductively we obtain Eq. (3.22) since $|\mathbf{A}_{p-1} * \mathbf{R}_{p-1}|/|\mathbf{A}_{p-1}| = a_{pp}/a_{pp} = 1$. (Q.E.D.)

Applying Lemma 3.3 to Theorem 3.6 yields the following additional lower bound for $|\mathbf{A} * \mathbf{B}|$:

COROLLARY 3.6 (Oppenheim [27]). *When* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive semi-definite,*

$$|\mathbf{A} * \mathbf{B}| \geqslant |\mathbf{A}| \cdot |\mathbf{B}|. \tag{3.24}$$

We use Theorem 3.6 to obtain a tighter lower bound than that in Eq. (3.21). The *only* proof we have found in the literature (cf. Sec. 5) is in the same 1930 paper of Oppenheim [27], who credits it to Schur [30, p. 14], who, however, presents only Theorems 3.1 and 3.4. Mirsky [24, p. 421] mentions the sharpening of Eq. (3.21) but gives no proof. Mirsky credits Schur, but clearly is following Oppenheim [27]. Marcus [19, p. 14] calls Eq. (3.25) the *Schur inequality*. Lynn [16] establishes Eq. (3.25) for *M*-matrices.

THEOREM 3.7 (Oppenheim [27]). *When* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive semi-definite,*

$$|\mathbf{A} * \mathbf{B}| + |\mathbf{A}| \cdot |\mathbf{B}| \geqslant |\mathbf{A}| b_{11} \cdots b_{pp} + a_{11} \cdots a_{pp} |\mathbf{B}|. \tag{3.25}$$

Proof. If either $\mathbf{A}$ or $\mathbf{B}$ is singular, Eq. (3.25) reduces to Eq. (3.21). Thus let $\mathbf{A}$ and $\mathbf{B}$ be positive definite. Then we may write $\mathbf{A}$ and $\mathbf{B}$ in terms of correlation matrices $\mathbf{Q}$ and $\mathbf{R}$, so that using Eq. (2.9), we may write Eq. (3.25) as

$$|\mathbf{Q} * \mathbf{R}| + |\mathbf{Q}| \cdot |\mathbf{R}| \geqslant |\mathbf{Q}| + |\mathbf{R}|. \tag{3.26}$$

From Lemma 3.2, $\mathbf{R}^0 = \mathbf{R} - \mathbf{e}_1\mathbf{e}_1'/r^{11}$, where $r^{11} = \mathbf{e}_1'\mathbf{R}^{-1}\mathbf{e}_1$, is positive semidefinite. Hence by Theorem 3.1,$\mathbf{Q} * \mathbf{R}^0$ is positive definite. Thus by Eq. (3.21),

$$|\mathbf{Q}|(1 - 1/r^{11}) \leqslant |\mathbf{Q} * \mathbf{R}^0| = |\mathbf{Q} * \mathbf{R} - \mathbf{e}_1\mathbf{e}_1'/r^{11}| = |\mathbf{Q} * \mathbf{R}|\left(1 - \frac{|\mathbf{Q}_1 * \mathbf{R}_1|}{|\mathbf{Q} * \mathbf{R}|r^{11}}\right). \tag{3.27}$$

That is,

$$|\mathbf{Q} * \mathbf{R}| - |\mathbf{Q}_1 * \mathbf{R}_1|/r^{11} \geqslant |\mathbf{Q}| - |\mathbf{Q}|/r^{11}. \tag{3.28}$$

Let

$$l_{i+1} = |\mathbf{Q}_i * \mathbf{R}_i| + |\mathbf{Q}_i| \cdot |\mathbf{R}_i| - |\mathbf{Q}_i| - |\mathbf{R}_i|, \qquad i = 0, 1, \ldots, p - 1.$$

Then $l_1 \geqslant 0$ is equivalent to Eq. (3.26). We may write Eq. (3.28), after some rearrangement, as

$$l_1 - l_2/r^{11} \geqslant (1/r^{11} - |\mathbf{R}|)(|\mathbf{Q}_1| - |\mathbf{Q}|). \tag{3.29}$$

The first factor is $(1 - |\mathbf{R}_1|)/r^{11}$, which is nonnegative by Eq. (3.19). The second factor is nonnegative from Eq. (3.16), and hence so is each side of Eq. (3.29). Thus $l_1 \geqslant l_2|\mathbf{R}|/|\mathbf{R}_1|$. Similarly $l_2 \geqslant l_3|\mathbf{R}_1|/|\mathbf{R}_2|$, so that $l_1 \geqslant l_2|\mathbf{R}|/|\mathbf{R}_2|$. Proceeding inductively we obtain $l_1 \geqslant 0$ [i.e., Eq. (3.26)], since

$$l_{p-1} = 1 - q^2r^2 + (1 - q^2)(1 - r^2) - (1 - q^2) - (1 - r^2) = 0, \tag{3.30}$$

where $q = q_{p,p-1}$ and $r = r_{p,p-1}$. (Q.E.D.)

The above leads to the following conclusion as to when $\mathbf{A} * \mathbf{B}$ and $\mathbf{AB}$ are equal:

COROLLARY 3.7. *When* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive definite,* $\mathbf{A} * \mathbf{B} = \mathbf{AB}$ *if and only if* $\mathbf{A}$ *and* $\mathbf{B}$ *are both diagonal matrices.*

Proof. Sufficiency is immediate. To show necessity we have from Eqs. (3.17) and (3.21) that $a_{11} \cdots a_{pp}|\mathbf{B}| \geqslant |\mathbf{A}| \cdot |\mathbf{B}| = |\mathbf{A} * \mathbf{B}| \geqslant a_{11} \cdots a_{pp}|\mathbf{B}|$. Hence $|\mathbf{A}| = a_{11} \cdots a_{pp}$. Similarly $|\mathbf{B}| = b_{11} \cdots b_{pp}$ and so the result. (Q.E.D.)

We also use Theorem 3.6 to answer the question posed by Djoković ([6] cf. paragraph before Theorem 3.2).

THEOREM 3.8 (Ballantine [1]). *A positive semidefinite matrix* $\mathbf{C}$ *may be expressed as* $\mathbf{A} * \mathbf{B}$ *for some positive definite* $\mathbf{A}$ *and some positive semidefinite* $\mathbf{B}$ *if and only if the rank of* $\mathbf{C}$ *equals the number of positive diagonal elements of* $\mathbf{C}$.

The proof follows directly from

LEMMA 3.4 (Ballantine [1]). *If* $\mathbf{A}$ *is positive definite and* $\mathbf{B}$ *is positive semidefinite with* r *positive diagonal elements then* $\operatorname{rank}(\mathbf{A} * \mathbf{B}) = r$.

Proof. As $\mathbf{A} * \mathbf{B}$ is positive semidefinite (by Theorem 3.1) and has r positive diagonal elements, $\operatorname{rank}(\mathbf{A} * \mathbf{B}) \leqslant r$. Theorem 3.6 implies that $\mathbf{A} * \mathbf{B}$ has a nonsingular principal submatrix of order r and so $\operatorname{rank}(\mathbf{A} * \mathbf{B}) \geqslant r$, which completes the proof. (Q.E.D.)

When all diagonal elements of $\mathbf{B}$ are positive, $\mathbf{A} * \mathbf{B}$ is positive definite (and nonsingular), thus strengthening Theorem 3.1, as noted by Pólya and Szegö [28, p. 107]. Lemma 3.4 strengthens the result of Djoković [6] who proved $\operatorname{rank}(\mathbf{A} * \mathbf{B}) \geqslant \operatorname{rank}(\mathbf{B})$ when $\mathbf{A}$ is positive definite and $\mathbf{B}$ positive semidefinite.

Now suppose $\mathbf{A}$ and $\mathbf{B}$ are both positive semidefinite and singular. When they both have rank one, $\mathbf{A} * \mathbf{B}$ has rank at most one as we saw from Eq. (2.1). In such a case $\mathbf{A} * \mathbf{B}$ is singular unless the matrices have order one and are scalars. Thus the Hadamard product of two singular positive semidefinite matrices of order two cannot be positive definite. But when the order is at least three $\mathbf{A} * \mathbf{B}$ may or may not be singular. If $\mathbf{A}$ or $\mathbf{B}$ is $\mathbf{0}$ then $\mathbf{A} * \mathbf{B} = \mathbf{0}$ and so $|\mathbf{A} * \mathbf{B}| = 0$. But if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \tag{3.31}$$

which both have rank two, then

$$\mathbf{A} * \mathbf{B} = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \tag{3.32}$$

and $|\mathbf{A} * \mathbf{B}| = 2$, so $\mathbf{A} * \mathbf{B}$ is positive definite. For $p = 4$, cf. Eq. (3.6).

Rao [29] seeks conditions on a symmetric idempotent matrix $\mathbf{M}$ so that $\mathbf{M} * \mathbf{M}$ is nonsingular.

Fiedler [9, 10] studied the characteristic roots of $\mathbf{A} * \mathbf{A}^{-1}$, where $\mathbf{A}$ is positive definite. From Eq. (2.7) it follows that all the row sums are unity, and so $\mathbf{A} * \mathbf{A}^{-1}$ has a characteristic root of unity with $\mathbf{e}$ a corresponding characteristic vector. This result is strengthened when tied in with the reducibility of $\mathbf{A}$. We will say that $\mathbf{A}$ has *reducibility index s*, when by row and column permutations we can write $\mathbf{A}$ as

$$\begin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \mathbf{A}_{s1} & \cdots & \mathbf{A}_{ss} \end{bmatrix}, \tag{3.33}$$

where $\mathbf{A}_{ii}$, $i = 1, \ldots, s$, are square and cannot be reduced further. We may call the $\mathbf{A}_{ii}$ irreducible, or with reducibility index 1. Hence

THEOREM 3.9 (Fiedler [9]). *When $\mathbf{A}$ is positive definite with reducibility index s, then $\mathbf{A} * \mathbf{A}^{-1}$ has minimum characteristic root unity, with multiplicity s, characteristic vector $\mathbf{e}$, and reducibility index s.*

A square, not necessarily symmetric, matrix with nonnegative elements has a dominant real characteristic root which is not less than any other root in absolute value. If $\mathbf{A}$ and $\mathbf{B}$ are such matrices then so is $\mathbf{A} * \mathbf{B}$ and

$$\mathrm{ch}_1(\mathbf{A} * \mathbf{B}) \leqslant \mathrm{ch}_1(\mathbf{A})\, \mathrm{ch}_1(\mathbf{B}), \tag{3.34}$$

as proved by Marcus and Khan [20], and by Lynn [16], who also showed that the inequality is strict if either $\mathbf{A}$ or $\mathbf{B}$ has all its diagonal elements positive.

In 1963, Marcus and Thompson [22] considered the Hadamard product of normal matrices and proved

THEOREM 3.10 (Marcus and Thompson [22]). *Let $\mathbf{A}$ and $\mathbf{B}$ be normal matrices with characteristic roots $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_p$, respectively. Then the characteristic roots of $\mathbf{A} * \mathbf{B}$ lie in a subset of the convex polygon in the plane supported by $\alpha_i\beta_j$ [$\frac{1}{2}(\alpha_i\beta_j + \alpha_j\beta_i)$ when $\mathbf{A}$ and $\mathbf{B}$ commute].*

We now derive a new inequality for the characteristic roots of $\mathbf{A} * \mathbf{B}$, when only symmetry is assumed for $\mathbf{A}$ and $\mathbf{B}$. There exists a unique positive semidefinite matrix $\bar{\mathbf{A}}$ satisfying $\bar{\mathbf{A}}^2 = \mathbf{A}^2$; $\bar{\mathbf{A}}$ and $\mathbf{A}$ have the same characteristic vectors but the characteristic roots of $\bar{\mathbf{A}}$ are the *singular values* of $\mathbf{A}$ (and hence also of $\bar{\mathbf{A}}$). A matrix $\mathbf{F}$, not necessarily square, of rank f has f nonzero singular values

$$\operatorname{sg}_j(\mathbf{F}) = [\operatorname{ch}_j(\mathbf{F}'\mathbf{F})]^{1/2} = [\operatorname{ch}_j(\mathbf{F}\mathbf{F}')]^{1/2}; \qquad j = 1,\ldots,f. \tag{3.35}$$

For symmetric $\mathbf{A}$, therefore, $\operatorname{ch}_j(\bar{\mathbf{A}}) = \operatorname{sg}_j(\mathbf{A}) = |\operatorname{ch}_k(\mathbf{A})|$, $j, k = 1,\ldots,p$, with $|\cdot|$ denoting absolute value. Motivated by Davis [4] we obtain

THEOREM 3.11. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be symmetric matrices and let* $\bar{\mathbf{A}}$ *and* $\bar{\mathbf{B}}$ *denote the positive semidefinite matrices satisfying* $\mathbf{A}^2 = \bar{\mathbf{A}}^2$ *and* $\mathbf{B}^2 = \bar{\mathbf{B}}^2$. *Then*

$$\operatorname{ch}_j(\mathbf{A} * \mathbf{B}) \leqslant \operatorname{ch}_j(\bar{\mathbf{A}} * \bar{\mathbf{B}}), \qquad j = 1,\ldots,p. \tag{3.36}$$

Proof. There exist orthogonal matrices $\mathbf{P} = \{p_{ih}\}$ and $\mathbf{Q} = \{q_{jk}\}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{\Lambda} = \{\lambda_h\}$ and $\mathbf{Q}'\mathbf{B}\mathbf{Q} = \mathbf{\Delta} = \{\delta_k\}$ are diagonal. Thus for any $\mathbf{x} = \{x_i\}$, $p \times 1$,

$$\begin{aligned}\mathbf{x}'(\mathbf{A} * \mathbf{B})\mathbf{x} &= \sum_{i,j=1}^{p} x_i \left[\sum_{h=1}^{p} p_{ih}\lambda_h p_{jh}\right]\left[\sum_{k=1}^{p} q_{ik}\delta_k q_{jk}\right] x_j \\ &= \sum_{h,k=1}^{p} \lambda_h\delta_k \left[\sum_{i=1}^{p} x_i p_{ih} q_{ik}\right]^2 \\ &\leqslant \sum_{h,k=1}^{p} |\lambda_h|\,|\delta_k| \left[\sum_{i=1}^{p} x_i p_{ih} q_{ik}\right]^2 = \mathbf{x}'(\bar{\mathbf{A}} * \bar{\mathbf{B}})\mathbf{x}.\end{aligned} \tag{3.37}$$

Let the $p \times (j-1)$ matrix $\mathbf{G}$ have as its columns the characteristic vectors of $\bar{\mathbf{A}} * \bar{\mathbf{B}}$ corresponding to the largest $j-1$ roots. If $\mathbf{x}'\mathbf{x} = 1$, then

$$\begin{aligned}\operatorname{ch}_j(\mathbf{A} * \mathbf{B}) &\leqslant \max\{\mathbf{x}'(\mathbf{A} * \mathbf{B})\mathbf{x} \colon \mathbf{G}'\mathbf{x} = \mathbf{0}\} \\ &\leqslant \max\{\mathbf{x}'(\bar{\mathbf{A}} * \bar{\mathbf{B}})\mathbf{x} \colon \mathbf{G}'\mathbf{x} = \mathbf{0}\} = \operatorname{ch}_j(\bar{\mathbf{A}} * \bar{\mathbf{B}}),\end{aligned} \tag{3.38}$$

using the Courant-Fischer min-max theorem (cf. Bellman [2, p. 113 (1960) or p. 115 (1970)]). (Q.E.D.)

We enquire if (3.36) can be strengthened to $\mathrm{sg}_j(\mathbf{A} * \mathbf{B}) \leqslant \mathrm{sg}_j(\bar{\mathbf{A}} * \bar{\mathbf{B}})$. Essentially this was proved by Davis [4] for $j = 1$; when $j \geqslant 2$, however, the inequality is *not*, in general, valid.

THEOREM 3.12. *When* $\mathbf{A}$ *and* $\mathbf{B}$ *are symmetric*

$$\mathrm{ch}_1(\mathbf{A} * \mathbf{B}) \leqslant \mathrm{sg}_1(\mathbf{A} * \mathbf{B}) \leqslant \mathrm{sg}_1(\bar{\mathbf{A}} * \bar{\mathbf{B}}) \leqslant \mathrm{sg}_1(\mathbf{A}) \cdot \bar{b}_{\max}, \tag{3.39}$$

where $\bar{b}_{\max}$ *is the largest diagonal element of* $\bar{\mathbf{B}}$. *The inequality* $\mathrm{sg}_j(\mathbf{A} * \mathbf{B}) \leqslant \mathrm{sg}_j(\bar{\mathbf{A}} * \bar{\mathbf{B}})$ *does* not *hold, in general, for* $j \geqslant 2$.

Proof. From (3.37) we have $|\mathbf{x}'(\mathbf{A} * \mathbf{B})\mathbf{x}| \leqslant \mathbf{x}'(\bar{\mathbf{A}} * \bar{\mathbf{B}})\mathbf{x}$ and so $\mathrm{sg}_1(\mathbf{A} * \mathbf{B}) = \max[\mathrm{ch}_1(\mathbf{A} * \mathbf{B}), -\mathrm{ch}_p(\mathbf{A} * \mathbf{B})] = \max[\max\{\mathbf{x}'(\mathbf{A} * \mathbf{B})\mathbf{x}\}, -\min\{\mathbf{x}'(\mathbf{A} * \mathbf{B})\mathbf{x}\}] = \max|\mathbf{x}'(\mathbf{A} * \mathbf{B})\mathbf{x}| \leqslant \max[\mathbf{x}'(\bar{\mathbf{A}} * \bar{\mathbf{B}})\mathbf{x}] = \mathrm{ch}_1(\bar{\mathbf{A}} * \bar{\mathbf{B}}) = \mathrm{sg}_1(\bar{\mathbf{A}} * \bar{\mathbf{B}})$, and this is no greater than $\mathrm{ch}_1(\bar{\mathbf{A}}) \cdot \bar{b}_{\max}$ by Theorem 3.4. Thus (3.39) follows. To see that $\mathrm{sg}_j(\mathbf{A} * \mathbf{B}) \leqslant \mathrm{sg}_j(\bar{\mathbf{A}} * \bar{\mathbf{B}})$ does *not* hold, in general, suppose $j = p = 2$. Consider

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}; \qquad \mathbf{A}_1 * \mathbf{B} = \begin{bmatrix} 0 & 4 \\ 4 & 3 \end{bmatrix} \tag{3.40}$$

has characteristic roots $\frac{1}{2}(3 \pm \sqrt{73}) \simeq 5.77, -2.77$. Moreover

$$\bar{\mathbf{A}}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \bar{\mathbf{B}} = \frac{1}{5}\begin{bmatrix} 8 & 6 \\ 6 & 17 \end{bmatrix}; \qquad \bar{\mathbf{A}}_1 * \bar{\mathbf{B}} = \frac{2}{5}\begin{bmatrix} 8 & 3 \\ 3 & 17 \end{bmatrix} \tag{3.41}$$

has roots $5 \pm 0.6\sqrt{13} \simeq 7.16, 2.84$. Thus $\mathrm{sg}_2(\mathbf{A}_1 * \mathbf{B}) < \mathrm{sg}_2(\bar{\mathbf{A}} * \bar{\mathbf{B}})$. This inequality is reversed, however, when

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}; \qquad \mathbf{A}_2 * \mathbf{B} = \begin{bmatrix} 0 & 8 \\ 8 & 3 \end{bmatrix} \tag{3.42}$$

has roots $\frac{1}{2}(3 \pm \sqrt{265}) \simeq 9.64, -6.64$. But

$$\bar{\mathbf{A}}_2 = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}; \qquad \bar{\mathbf{A}}_2 * \bar{\mathbf{B}} = \frac{2}{5}\begin{bmatrix} 16 & 3 \\ 3 & 34 \end{bmatrix} \tag{3.43}$$

has roots $10 \pm 1.2\sqrt{10} \simeq 13.79, 6.21$, and so $\mathrm{sg}_2(\mathbf{A}_2 * \mathbf{B}) > \mathrm{sg}_2(\bar{\mathbf{A}}_2 * \bar{\mathbf{B}})$. (Q.E.D.)

4. APPLICATIONS OF HADAMARD PRODUCTS TO MULTIVARIATE ANALYSIS

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a random sample from a p-variate normal distribution $N(\mathbf{0}, \mathbf{\Sigma})$. We suppose L denotes the joint likelihood of the n observations and write

$$l = -(2/n) \log L - p \log 2\pi. \tag{4.1}$$

If $\mathrm{S} = (1/n) \sum_{\alpha=1}^{n} \mathbf{x}_\alpha \mathbf{x}_\alpha'$ denotes the sample covariance matrix, then

$$l = \operatorname{tr}(\mathbf{\Sigma}^{-1}\mathrm{S}) + \log|\mathbf{\Sigma}|. \tag{4.2}$$

With $\mathbf{\Sigma}$ and S positive definite, we may write

$$\mathbf{\Sigma} = \mathbf{\Delta} \boldsymbol{R} \mathbf{\Delta}; \qquad \mathrm{S} = \mathbf{DRD}, \tag{4.3}$$

where $\boldsymbol{R}$ and $\mathbf{R}$ are population and sample correlation matrices, while $\mathbf{\Delta}$ and $\mathbf{D}$ are diagonal matrices of population and sample standard deviations. We assume $\boldsymbol{R}$ known.

If

$$\mathbf{\Delta e} = \boldsymbol{\sigma} = \{\sigma_i\}; \qquad \mathbf{\Delta}^{-1}\mathbf{e} = \boldsymbol{\sigma}^{(-1)} = \{1/\sigma_i\}; \qquad \mathbf{\Delta}^2\mathbf{e} = \boldsymbol{\sigma}^{(2)} = \{\sigma_i^2\}, \tag{4.4}$$

Eqs. (2.8) and (2.10) give $\operatorname{tr}(\mathbf{\Sigma}^{-1}\mathrm{S}) = \mathbf{e}'(\mathbf{\Sigma}^{-1} * \mathrm{S})\mathbf{e} = \mathbf{e}'[(\mathbf{\Delta}^{-1}\boldsymbol{R}^{-1}\mathbf{\Delta}^{-1}) * \mathrm{S}]\mathbf{e} = \boldsymbol{\sigma}^{(-1)\prime}(\boldsymbol{R} * \mathrm{S})\boldsymbol{\sigma}^{(-1)}$. Hence

$$\partial l / \partial \boldsymbol{\sigma}^{(-1)} = 2[(\boldsymbol{R}^{-1} * \mathrm{S})\boldsymbol{\sigma}^{(-1)} - \boldsymbol{\sigma}], \tag{4.5}$$

and so the maximum likelihood equations are

$$(\boldsymbol{R}^{-1} * \mathrm{S})\hat{\boldsymbol{\sigma}}^{(-1)} = \hat{\boldsymbol{\sigma}}, \tag{4.6}$$

as observed by Olkin and Siotani [26] and Styan [**33**, **34**].

Theorem **3.1** implies that

$$\frac{\partial^2 l}{\partial \boldsymbol{\sigma}^{(-1)}\, \partial \boldsymbol{\sigma}^{(-1)\prime}} = 2[\boldsymbol{R}^{-1} * \mathrm{S} + \mathbf{\Delta}^2] \tag{4.7}$$

is positive definite and therefore Eq. (4.6) admits a unique solution in the positive orthant.

Iterative solution of Eq. (4.6) by the Newton-Raphson process based on the initial guess $\hat{\boldsymbol{\sigma}}_0 = \mathbf{De}$, the column vector of sample standard deviations, yields the first iterate

$$\hat{\boldsymbol{\sigma}}_1^{(-1)} = 2\mathbf{D}^{-1}(\boldsymbol{R}^{-1} * \mathbf{R} + \mathbf{I})^{-1}\mathbf{e}. \tag{4.8}$$

Styan [33, 34] proved that $\sqrt{n}(\hat{\boldsymbol{\sigma}}_1^{(2)} - \boldsymbol{\sigma}^{(2)})$ and $\sqrt{n}(\hat{\boldsymbol{\sigma}}^{(2)} - \boldsymbol{\sigma}^{(2)})$ have the same limiting normal distribution with covariance matrix

$$4\boldsymbol{\Delta}^2(\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I})^{-1}\boldsymbol{\Delta}^2, \tag{4.9}$$

while $\sqrt{n}(\mathbf{D}^2\mathbf{e} - \boldsymbol{\sigma}^{(2)})$ has a limiting normal distribution with covariance matrix

$$2\boldsymbol{\Delta}^2(\boldsymbol{R} * \boldsymbol{R})\boldsymbol{\Delta}^2. \tag{4.10}$$

This suggests

THEOREM 4.1. *For any positive definite correlation matrix* $\boldsymbol{R}$, *the matrix*

$$\boldsymbol{R} * \boldsymbol{R} - 2(\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I})^{-1} \tag{4.11}$$

is positive semidefinite.

Proof. We evaluate the joint covariance matrix of $\mathbf{s} = (\mathbf{S} * \mathbf{I})\mathbf{e} = \mathbf{D}^2\mathbf{e}$ and $\partial \log L/\partial\boldsymbol{\sigma}^{(2)}$. We prove first that

$$n\mathscr{V}(\mathbf{s}) = 2\boldsymbol{\Sigma} * \boldsymbol{\Sigma} = 2\boldsymbol{\Delta}^2(\boldsymbol{R} * \boldsymbol{R})\boldsymbol{\Delta}^2 \tag{4.12}$$

[cf. Eqs. (4.10) and (2.10)], where $\mathscr{V}(\cdot)$ denotes covariance matrix. The (i, j)th element is

$$\begin{aligned} n \operatorname{cov}(s_i, s_j) &= (1/n) \operatorname{cov}\left[\sum_{\alpha=1}^{n} x_{\alpha i}^2, \sum_{\beta=1}^{n} x_{\beta j}^2\right] = \operatorname{cov}(X_i^2, X_j^2) \\ &= \tfrac{1}{2}[\mathscr{V}(X_i^2 + X_j^2) - \mathscr{V}(X_i^2) - \mathscr{V}(X_j^2)] = \operatorname{tr}(\mathbf{V}^2) - \sigma_{ii}^2 - \sigma_{jj}^2, \end{aligned}$$

where X_i, X_j are bivariate normal with zero means and covariance matrix

$$\mathbf{V} = \begin{bmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ij} & \sigma_{jj} \end{bmatrix},$$

cf., e.g., Searle [31*, p. 57]. As $\operatorname{tr}(\mathbf{V}^2) = \sigma_{ii}^2 + \sigma_{jj}^2 + 2\sigma_{ij}^2$, $n \operatorname{cov}(s_i, s_j) = 2\sigma_{ij}^2$ and Eq. (4.12) follows.

From Eq. (4.5),

$$\mathscr{E}(\partial l/\partial \boldsymbol{\sigma}^{(-1)}) = 2[(\boldsymbol{R}^{-1} * \boldsymbol{\Sigma})\boldsymbol{\sigma}^{(-1)} - \boldsymbol{\sigma}] = 2[\boldsymbol{\Delta}(\boldsymbol{R}^{-1} * \boldsymbol{R})\mathbf{e} - \boldsymbol{\sigma}]$$
$$= 2[\boldsymbol{\Delta}\mathbf{e} - \boldsymbol{\sigma}] = \mathbf{0}, \tag{4.13}$$

using Eqs. (2.7) and (4.4), where $\mathscr{E}(\cdot)$ denotes mathematical expectation. Thus Eq. (4.1) gives $\mathscr{E}(\partial \log L/\partial \boldsymbol{\sigma}^{(-1)}) = \mathbf{0}$ and so

$$\operatorname{cov}(\mathrm{s}, \partial \log L/\partial \boldsymbol{\sigma}^{(2)}) = \mathscr{E}[\mathrm{s}(\partial \log L/\partial \boldsymbol{\sigma}^{(2)})'] = \partial \mathscr{E}(\mathrm{s})/\partial \boldsymbol{\sigma}^{(2)\prime} = \mathbf{I}, \tag{4.14}$$

as $\mathscr{E}(\mathrm{s}) = \boldsymbol{\sigma}^{(2)}$. Moreover $\mathscr{V}(\partial \log L/\partial \boldsymbol{\sigma}^{(2)}) = \frac{1}{4}n^2\mathscr{V}[(\partial \boldsymbol{\sigma}^{(-1)}/\partial \boldsymbol{\sigma}^{(2)\prime})(\partial l/\partial \boldsymbol{\sigma}^{(-1)})]$, and since

$$\partial \boldsymbol{\sigma}^{(-1)}/\partial \boldsymbol{\sigma}^{(2)\prime} = -\tfrac{1}{2}\boldsymbol{\Delta}^{-3}; \quad \mathscr{V}(\partial l/\partial \boldsymbol{\sigma}^{(-1)}) = (2/n)\mathscr{E}\left[\frac{\partial^2 l}{\partial \boldsymbol{\sigma}^{(-1)}\,\partial \boldsymbol{\sigma}^{(-1)\prime}}\right], \tag{4.15}$$

we find, using Eq. (4.7), that

$$\mathscr{V}(\partial \log L/\partial \boldsymbol{\sigma}^{(2)}) = \tfrac{1}{4}n\boldsymbol{\Delta}^{-2}(\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I})\boldsymbol{\Delta}^{-2}, \tag{4.16}$$

as $\mathscr{E}(\mathrm{S}) = \boldsymbol{\Sigma}$. Therefore

$$\mathscr{V}\begin{bmatrix} \mathrm{s} \\ \partial \log L/\partial \boldsymbol{\sigma}^{(2)} \end{bmatrix} = \begin{bmatrix} (2/n)\boldsymbol{\Delta}^2(\boldsymbol{R} * \boldsymbol{R})\boldsymbol{\Delta}^2 & \mathbf{I} \\ \mathbf{I} & \tfrac{1}{4}n\boldsymbol{\Delta}^{-2}(\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I})\boldsymbol{\Delta}^{-2} \end{bmatrix}, \tag{4.17}$$

from whose positive semidefiniteness the theorem follows. (Q.E.D.)

A matrix-theoretic proof of Theorem 4.1 would be of interest.

Conditions for singularity of Eq. (4.11) are examined in

COROLLARY 4.1. *A sufficient but not necessary condition that Eq.* (4.11) *be singular is that* $\boldsymbol{R} - \mathbf{I}$ *has at least one null row.*

Proof. If the ith row of $\boldsymbol{R} - \mathbf{I}$ is null then $\boldsymbol{R} * \boldsymbol{R}$, $\boldsymbol{R}^{-1} * \boldsymbol{R}$ and $2(\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I})^{-1}$ all have the same ith row, and so the ith row of Eq. (4.11) is null.

To show that the converse is false, i.e., that singularity of Eq. (4.11) does not imply that $\boldsymbol{R} - \mathbf{I}$ has a null row, consider the following example with $p = 3$:

$$\boldsymbol{R} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & 1 \end{bmatrix}. \tag{4.18}$$

We obtain

$$\tfrac{1}{2}(\boldsymbol{R} * \boldsymbol{R})(\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I}) - \mathbf{I} = (1/24) \begin{bmatrix} 6 & 3 & 3 \\ 3\frac{3}{4} & 3 & \frac{3}{4} \\ 3\frac{3}{4} & \frac{3}{4} & 3 \end{bmatrix}, \tag{4.19}$$

which is singular, and so, therefore, is (4.11). (Q.E.D.)

Corollary 4.2. *For any positive definite correlation matrix* $\boldsymbol{R}$,

$$\mathrm{ch}_j[(\boldsymbol{R} * \boldsymbol{R})(\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I})] \geqslant 2; \qquad j = 1, \ldots, p, \tag{4.20}$$

$$\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I} - 2(\boldsymbol{R} * \boldsymbol{R})^{-1} \quad \textit{is positive semidefinite,} \tag{4.21}$$

$$[\mathrm{ch}_j(\boldsymbol{R} * \boldsymbol{R})][1 + \mathrm{ch}_k(\boldsymbol{R}^{-1} * \boldsymbol{R})] \geqslant 2, \qquad j + k \leqslant p + 1, \tag{4.22}$$

where $\mathrm{ch}_j(\cdot)$ *denotes the* jth *largest characteristic root.*

Proof. Postmultiplying Eq. (4.11) by $\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I}$ establishes Eq. (4.20), since the product of two matrices, each at least positive semidefinite, has nonnegative characteristic roots. Premultiplying this product by $(\boldsymbol{R} * \boldsymbol{R})^{-1}$ proves Eq. (4.21). Applying the result $\mathrm{ch}_j(\mathbf{A})\,\mathrm{ch}_k(\mathbf{B}) \geqslant \mathrm{ch}_i(\mathbf{AB})$, $j + k \leqslant i + 1$ (cf. Marcus and Minc [21]) to Eq. (4.20) yields Eq. (4.22). (Q.E.D.)

A proof of Eq. (4.22) without using Theorem 4.1 would be of interest; we note $\mathrm{ch}_p(\boldsymbol{R}^{-1} * \boldsymbol{R}) = 1$, and $\mathrm{ch}_p(\boldsymbol{R} * \boldsymbol{R}) \leqslant 1$, as $\mathrm{tr}(\boldsymbol{R} * \boldsymbol{R}) = p$.

Corollary 4.3. *For any positive definite matrix* $\mathbf{A}$, *the matrices*

$$\mathbf{A} * \mathbf{A} - 2(\mathbf{A} * \mathbf{I})(\mathbf{A}^{-1} * \mathbf{A} + \mathbf{I})^{-1}(\mathbf{A} * \mathbf{I}), \tag{4.23}$$

and

$$\mathbf{A}^{-1} * \mathbf{A} + \mathbf{I} - 2(\mathbf{A} * \mathbf{I})(\mathbf{A} * \mathbf{A})^{-1}(\mathbf{A} * \mathbf{I}), \tag{4.24}$$

are positive semidefinite.

Proof. For any diagonal matrix $\mathbf{D}$, $(\mathbf{DAD}) * (\mathbf{DAD}) = \mathbf{D}^2(\mathbf{A} * \mathbf{A})\mathbf{D}^2$ and $(\mathbf{DAD}) * (\mathbf{DAD})^{-1} = \mathbf{A} * \mathbf{A}^{-1}$ [cf. Eqs. (2.9) and (2.10)]. Substituting the correlation matrix $(\mathbf{A} * \mathbf{I})^{-1/2}\mathbf{A}(\mathbf{A} * \mathbf{I})^{-1/2}$ for $\boldsymbol{R}$ in Eq. (4.11) yields Eq. (4.23) after pre- and postmultiplication by $\mathbf{A} * \mathbf{I}$. Similar operations on Eq. (4.21) give Eq. (4.24). (Q.E.D.)

COROLLARY 4.4. *For any positive definite correlation matrix* $\boldsymbol{R}$,

$$\operatorname{tr}(\boldsymbol{R}^{-1} * \boldsymbol{R}) \geqslant 2 \operatorname{tr}(\boldsymbol{R} * \boldsymbol{R})^{-1} - p, \tag{4.25}$$

$$\operatorname{tr}[(\boldsymbol{R} * \boldsymbol{R})(\boldsymbol{R}^{-1} * \boldsymbol{R})] \geqslant p. \tag{4.26}$$

Proof. Taking the trace of Eq. (4.21) yields Eq. (4.25) directly, while Eq. (4.26) follows by summing Eq. (4.20), since $\operatorname{tr}(\boldsymbol{R} * \boldsymbol{R}) = p$. (Q.E.D.)

As the diagonal elements of $\boldsymbol{R}^{-1}$ are at least equal to 1 (Corollary 3.5),

$$\operatorname{tr}(\boldsymbol{R} * \boldsymbol{R})^{-1} \geqslant p; \qquad \operatorname{tr}(\boldsymbol{R}^{-1} * \boldsymbol{R}) \geqslant p. \tag{4.27}$$

Corollary 4.4 does not appear to follow from Eq. (4.27), and we have been unable to prove Eqs. (4.25) or (4.26) without using Theorem 4.1.

COROLLARY 4.5. *For any positive definite matrix* $\mathbf{A}$ *with diagonal elements* a_{ii}, $i = 1, \ldots, p$,

$$|\mathbf{A} * \mathbf{A}| \cdot |\mathbf{A}^{-1} * \mathbf{A} + \mathbf{I}| \geqslant 2^p \prod_{i=1}^{p} a_{ii}^2. \tag{4.28}$$

Proof. We substitute the correlation matrix $(\mathbf{A} * \mathbf{I})^{-1/2}\mathbf{A}(\mathbf{A} * \mathbf{I})^{-1/2}$ for $\boldsymbol{R}$ in Eq. (4.20). (Q.E.D.)

In contrast to Eq. (4.28), the Hadamard determinant theorem (Lemma 3.3) gives

$$|\mathbf{A} * \mathbf{A}| \leqslant \prod_{i=1}^{p} a_{ii}^2, \tag{4.29}$$

while $|\mathbf{A}^{-1} * \mathbf{A} + \mathbf{I}| \geqslant 2^p$ follows from $\operatorname{ch}(\mathbf{A}^{-1} * \mathbf{A}) \geqslant 1$.

Corollary 3.5 implies that $|\boldsymbol{R} * \boldsymbol{R}| \leqslant 1$ and $|\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I}| \leqslant \prod_{i=1}^{p} (1 + \rho^{ii})$, where ρ^{ii} is the ith diagonal element of $\boldsymbol{R}^{-1}$. Hence

$$(|\boldsymbol{R} * \boldsymbol{R}| \cdot \tfrac{1}{2}(\boldsymbol{R}^{-1} * \boldsymbol{R} + \mathbf{I})|)^{-1} \geqslant 2^p \Big/ \prod_{i=1}^{p} (1 + \rho^{ii}). \tag{4.30}$$

It follows from the proof of Theorem 4.1 that the left side of Eq. (4.30) is the relative efficiency of the sample variances when the correlations are known in a multivariate normal population (cf. Styan [33, 34]).

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