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On the Kazhdan-Lusztig conjectures

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§ 1. INTRODUCTION

In this paper we give some intermediatery results obtained while attempting to prove some conjectures, made by D. Kazhdan and G. Lusztig ([4–§ 1.4]); these results seem to be interesting in their own right as well.

The conjectures mentioned above relate the formal character of an irreducible highest weight representation of a complex semisimple Lie algebra with values of certain polynomials (in one variable) with integral coefficients. The point of view taken by this author in tackling this problem is that certain submodules of Verma modules which were introduced by J. Jantzen ([3–§ 5.2]) should be used as they are intimately related with these formal characters as is evident from his results.

Let $\mathfrak{G}$ be a complex semisimple Lie algebra. Let $\mathfrak{H} \subseteq \mathfrak{B}$ be respectively a Cartan subalgebra and a Borel subalgebra. Let W be the Weyl group and ϱ be the half-sum of roots of $(\mathfrak{B}, \mathfrak{H})$. For $\lambda \in \mathfrak{H}^*$, let $M(\lambda)$ be the Verma module "with respect of $\mathfrak{B}$" of highest weight λ. Let $L(\lambda)$ be its unique irreducible quotient. Jantzen (loc. cit.) has introduced certain submodules $\{M(\lambda)^r\}_{r \geq 0}$ of $M(\lambda)$ such that $M(\lambda) = M(\lambda)^0 \supseteq M(\lambda)^1 \supseteq \ldots$ and $M(\lambda)/M(\lambda)^1 = L(\lambda)$. We prove that these submodules are "independent" in certain sense of the parametre λ as long as we move in the "same chamber". More precisely, we have:

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THEOREM 1.1. *Let λ_0, μ_0 be dominant, integral elements of $\mathfrak{H}^*$. Let F be the finite dimensional irreducible $\mathfrak{G}$-module with highest weight μ_0. For $x \in W$ put $\lambda_x = x(\lambda_0 + \varrho) - \varrho$ and $\mu_x = x(\mu_0)$. Then there exists a $\mathfrak{G}$-module isomorphism*

$$\psi_x : (M(\lambda_x) \otimes F)_{\chi_{\lambda_x + \mu_x}} \simeq M(\lambda_x + \mu_x)$$

such that

$$\psi_x((M(\lambda_x)^r \otimes F)_{\chi_{\lambda_x + \mu_x}}) = M(\lambda_x + \mu_x)^r \, \forall r \geq 0.$$

(For a $\mathfrak{G}$-module N and a central character χ, $N_\chi = \{n \in N \mid (z - \chi(z))^r n = 0$ for some $r \geq 0$ $\forall z \in$ centre of enveloping algebra of $\mathfrak{G}\}$.

Using the notation of G. Zuckerman ([6]), the above theorem can be restated as:

$$\varphi^{\lambda_0}_{\lambda_0 + \mu_0}(M(\lambda_x)^r) = M(\lambda_x + \mu_x)^r.$$

It is not difficult to prove (eg [3–§ 2.4]) that

$$\varphi^{\lambda_0}_{\lambda_0 + \mu_0}(L(\lambda_y)) = L(\lambda_y + \mu_y) \;\; \forall y \in W.$$

Combining this fact with the conclusion of the above theorem, we see that the integer $mtp^{(r)}_{\lambda_0}(x,y)$ = multiplicity of $L(\lambda_y)$ in Jordan-Hölder series of $M(\lambda_x)^r$ ($x \leq y \in W$, $\leq$ being the Bruhat ordering on W) depends only on the pair (x,y) and we denote it simply by $mtp^{(r)}(x,y)$.

We now restate the character identity of Jantzen ([3–§ 5.3]) which, as mentioned earlier, has been the motivation behind this approach:

$$(1.2) \qquad \sum_{r \geq 1} mtp^{(r)}(x,y) = \sum_{\substack{x \leq xt \leq y \\ t \text{ a reflection}}} mtp^{(0)}(xt, y).$$

The Kazhdan-Lusztig conjectures now state that

$$(1.3) \qquad mtp^{(0)}(x,y) = P_{x,y}(1),$$

where $P_{x,y}$'s are certain polynomials in one variable defined with the help of the Hecke algebra of W.

One is now led to search for polynomials $P^{(r)}_{x,y}$ which are related to $P_{x,y}$ in a natural way and which are "consistent" with the identity (1.2). We indeed have the following:

THEOREM 1.4. There exists *a unique set* $\{P^{(r)}_{x,y}\}_{r \geq 0}$ of polynomials such that

(i) $\quad P_{x,y} = P^{(0)}_{x,y} \geq P^{(1)}_{x,y} \geq \ldots$

(a polynomial $f = \sum_{i \geq 0} a_i q^i \geq g = \sum_{i \geq 0} b_i q^i$ if $a_i \geq b_i \forall_i$)

(ii) $\quad \deg P^{(r)}_{x,y} \leq \left[\dfrac{l(y) - l(x) - r}{2}\right]$ l being the length function of W.

(By convention, $f=0$ if $\deg f<0$)

(iii) $$\sum_{r\geq 1} P^{(r)}_{x,y}(1) = \sum_{\substack{x\leq xt\leq y \\ t \text{ a reflection}}} P^{(0)}_{xt,y}(1).$$

It is now only logical to formulate:

CONJECTURE 1.5. $mtp^{(r)}(x,y)=P^{(r)}_{x,y}(1)\ \forall r\geq 0,\ \forall x\leq y\in W.$

REMARK 1.6. A natural direction to proceed from this point is to get some more information about these polynomials "$P^{(r)}_{x,y}$". The identity (1.2) provides the natural induction step. The initial investigation in this direction looks to be quite promising.

This paper is arranged as follows: In § 2, we establish some properties of polynomials $P_{x,y}$ and the accompanying polynomials $R_{x,y}$ ([4–§ 2]). § 3 is devoted to the proof of Theorem 1.4 which uses results of § 2. In § 4 we investigate the structure of some modules with "$\mathbb{C}[[t]]$-coefficients". This is an extension of ideas of Jantzen who uses modules with "$\mathbb{C}[t]$-coefficients" to define the filtration $\{M(\lambda)^r\}_{r\geq 0}$ mentioned earlier. We will also compare the two set-ups. In § 5 we give a proof of Theorem 1.1 specializing results of § 4. Even though the conclusion of this theorem has been the starting point for this paper, its proof, apart from being technically involved, is of a different flavour than the discussion in § 2 and 3. That is the reason why this starting point has been arranged at the end.

Thanks are due to J.C. Jantzen for pointing out a simplification in the original proof at Theorem 1.1.

Notes added in proof:

(i) Results similar to ours have been obtained by O. Gabber and A. Joseph, e.g. Proposition 3.2 (see § 3) has been proved by them and also, independently, by G. Lusztig.

(ii) The original conjectures of Kazhdan-Lusztig (Conjecture (1.3) above) have now been proved to be true by Brylinski and Kashiwara and also by Belinson and Bernstein. The generalized conjecture (1.5) (which has been formulated by Gabber and Joseph as well) still remains open.

§ 2. A LEMMA ON THE "SHAPE" OF $R_{x,y}$

We recall the definition of the polynomial $R_{x,y}$ ([4–§ 2]): Let $\tilde{\mathscr{H}}$ be the free $\mathbb{Z}[q]$-module with $\{T_w \mid w\in W\}$ as a basis. Define a multiplication in $\tilde{\mathscr{H}}$ by: For $s\in S$ (the set of simple reflections) and $w\in W$,

$$T_s\cdot T_w = \begin{cases} T_{sw} \text{ if } l(sw)\geq l(w) \\ (q-1)T_w + q\cdot T_{sw} \text{ if } l(sw)\leq l(w). \end{cases}$$

Let $\mathscr{H}=\tilde{\mathscr{H}}\otimes_{\mathbb{Z}[q]}\mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]$. Then the above multiplication extends to $\mathscr{H}$ in which T_y becomes invertible. We write

$$T^{-1}_{y^{-1}} = \frac{1}{q^{l(y)}} \sum_{x\leq y} (-1)^{l(x,y)}\cdot R_{x,y}(q)\cdot T_x,\ (l(x,y)=l(y)-l(x)).$$

It can be proved ([4.-§ 2]) that $R_{x,y}$ satisfies the following (inductive) relations Let $s \in S$ such that $sy \leq y$. Then

(i) $\quad R_{x,y} = R_{sx,sy}$ if $sx \leq x$

(ii) $\quad R_{x,y} = (q-1)R_{x,sy} + qR_{sx,xy}$ if $sx \geq x$.

(By convention, $R_{\sigma,\tau} = 0$ if $\sigma \not\leq \tau$). Let T be the set of all reflections i.e $T = \bigcup_{w \in W} wSw^{-1}$.

We now prove the following lemma:

LEMMA 2.1. *For $x \lneq y \in W$, $R_{x,y}(q)$ is divisible by $(q-1)$. In fact, it is divisible by $(q-1)^2$ as well unless $x^{-1}y \in T$ in which case $(q-1)^2$ divides*

$$\left[R_{x,y}(q) - (q-1)q^{\frac{l(x,y)-1}{2}}\right].$$

PROOF. We proceed by induction on $l(y)$. Since $x \lneq y$, necessarily $l(y) \geq 1$. Choose $s \in S$ such that $sy \leq y$.

Case (i) $sx \leq x$.

Recall the Z-property of Bruhat ordering ([2-Thm. 1.1]) which states that in this case $sx \leq sy$ as well. Also, $R_{x,y} = R_{sx,sy}$ in this case. Thus by induction hypothesis, $(q-1)$ divides $R_{sx,sy}$ and so $(q-1)$ divides $R_{x,y}$. Further, $x^{-1}y \in T$ iff $(sx)^{-1}sy \in T$ and $l(x,y) = l(sx,sy)$. It is now clear that the conclusion of the lemma holds in this case.

Case (ii) $sx \geq x$.

In this case the Z-property gives $sx \leq y$ and $x \leq sy$. Also,

$$R_{x,y}(q) = (q-1)R_{x,sy}(q) + q \cdot R_{sx,sy}(q).$$

$$(R_{sx,sy} = 0 \text{ if } sx \not\leq sy).$$

First note that in case $x = sy$, the conclusion of the lemma holds as $R_{x,y}(q) = (q-1)$ in this case.

So assume $x \neq sy$. Hence induction hypothesis and the above relation show that $(q-1)$ divides $R_{x,y}(q)$. If $x^{-1}y \notin T$ then necessarily $(sx)^{-1}(sy) \notin T$ and so irrespective of whether $sx \leq sy$ or $sx \not\leq sy$, $(q-1)^2$ divides $R_{x,y}(q)$. If $x^{-1}y \in T$ and $x \neq sy$ then necessarily $sx \lneq sy$ and $(sx)^{-1}sy \in T$. Thus by induction hypothesis,

$$R_{sx,sy}(q) = (q-1)q^{\frac{l(sx,sy)-1}{2}} + (q-1)^2 f(q) \text{ for some polynomial } f.$$

Note also that $l(sx,sy) = l(x,y) - 2$ in this case. Thus we get,

$$\begin{aligned} R_{x,y}(q) &= (q-1)R_{x,sy}(q) + q \cdot R_{sx,sy}(q) \\ &= (q-1)R_{x,sy}(q) + q \cdot (q-1)^2 f(q) + (q-1) \cdot q^{\frac{l(x,y)-3}{2}} \cdot q \\ &= [(q-1)R_{x,sy}(q) + q \cdot (q-1)^2 f(q)] + (q-1)q^{\frac{l(x,y)-1}{2}}. \end{aligned}$$

As $x \lneq sy$, $(q-1)$ divides $R_{x,sy}(q)$ and so it is clear that $(q-1)^2$ divides

$$R_{x,y}(q)-(q-1)\cdot q^{\frac{l(x,y)-1}{2}}.$$

The proof of the lemma is now complete.

REMARK 2.2. The above lemma is a special case of the following general fact about $R_{x,y}$'s:

$$R_{x,y}(q)=(q-1)^{l(x,y)}+b_1(q-1)^{l(x,y)-2}\cdot q+\ldots+b_m(q-1)^{l(x,y)-2m}\cdot q^m.$$

where b_i's are positive integers and the integer m is completely determined by the "segment" between x and y. The proof of this general fact is quite subtle and will be given elsewhere. For our purposes here, Lemma 2.1 is quite sufficient. It looks that this integer m will play an important role in understanding the combinatorics of the Bruhat ordering.

§ 3. PROOF OF THEOREM 1.4

Recall ([5-§ 2.1.5]) that the polynomials $P_{x,y}$'s are inductively defined by the equation:

$$(3.1)\qquad \sum_{x<z\leq y} R_{x,z}(q)\cdot P_{z,y}(q)=q^{l(x,y)}P_{x,y}(q^{-1})-P_{x,y}(q)$$

and the condition

$$\deg P_{x,y}\leq\left[\frac{l(x,y)-1}{2}\right]\ (\text{if } x<y).$$

Let $P_{x,y}(q)=a_0+a_1q+\ldots+a_pq^p$, $a_i\in\mathbb{Z}$. It is known that $a_0=1$ ([4-Lemma 2.6]) and that $a_i\geq 0\ \forall i$ ([5-Corollary 4.8]). We now prove:

PROPOSITION 3.2.

$$\sum_{\substack{x<xt\leq y\\ t\in T}} P_{xt,y}(1)=\sum_{0\leq j\leq p} a_j(l(x,y)-2j).$$

PROOF. Split the left-hand-side terms of (3.1) as:

$$\sum_{\substack{x<xt\leq y\\ t\in T}} R_{x,xt}(q)\cdot P_{xt,y}(q)+\sum_{\substack{x<z\leq y\\ x^{-1}z\notin T}} R_{x,z}(q)\cdot P_{z,y}(q).$$

Now by Lemma 2.1, $(q-1)^2$ divides each term in the second summation above. Also,

$$\left(R_{x,xt}(q)-(q-1)\cdot q^{\frac{l(x,xt)-1}{2}}\right)$$

is divisible by $(q-1)^2\ \forall x<xt\leq y$.

Thus, the left-hand-side of (3.1) is equal to

$$\sum_{x<xt\leq y} (q-1)\cdot q^{\frac{l(x,xt)-1}{2}}\cdot P_{xt,y}(q)+(q-1)^2\cdot g(q)$$

for some polynomial g. Dividing the above by $(q-1)$ and then putting $q=1$, we get precisely $\sum_{x<xt\leq y} P_{xt,y}(1)$.

Put $P_{x,y}(q)=a_0+a_1q+\ldots+a_pq^p$ on the right-hand-side of (3.1), divide by $(q-1)$ and then put $q=1$ to get $\sum_{0\leq j\leq p} a_j(l(x,y)-2j)$. The proposition now follows.

3.3. We now come to the proof of Theorem 1.4. Recall the statement:

THEOREM. There exists a unique set $\{P_{x,y}^{(r)}\}_{r\geq 0}$ of polynomials such that

(i) $P_{x,y}=P_{x,y}^{(0)}\geq P_{x,y}^{(1)}\geq\ldots$

(ii) $\deg P_{x,y}^{(r)}\leq\left[\frac{l(x,y)-r}{2}\right]\ \forall r\geq 0$

(iii) $\sum_{r\geq 1} P_{x,y}^{(r)}(1)=\sum_{\substack{x<xt\leq y\\ t\in T}} P_{xt,y}^{(0)}(1)$.

PROOF. Define $P_{x,y}^{(r)}=a_0+a_1q+\ldots+a_iq^i$ where

$$i=\min\left\{p,\ \left[\frac{l(x,y)-r}{2}\right]\right\}.$$

It is clear that the conditions (i) and (ii) are satisfied. Next observe that

$$P_{x,y}^{(r)}=0 \text{ if } r>l(x,y)(=l(y)-l(x))$$

$$P_{x,y}^{(l(x,y))}(q)=P_{x,y}^{(l(x,y)-1)}(q)=a_0$$

$$P_{x,y}^{(l(x,y)-2)}(q)=P_{x,y}^{(l(x,y)-3)}(q)=a_0+a_1q \text{ (if } p\geq 1)$$

and so on. It is then easy to see that

$$\sum_{r\geq 1} P_{x,y}^{(r)}(1)=\sum_{0\leq j\leq p} a_j(l(x,y)-2j).$$

Using Proposition 3.2 it is clear that the condition (iii) is also satisfied. Next we prove the uniqueness of $\{P_{x,y}^{(r)}\}_{r\geq 0}$. Let $\{Q_{x,y}^{(r)}\}_{r\geq 0}$ be any set of polynomials satisfying the conditions (i), (ii) and (iii). Let $Q_{x,y}^{(r)}(q)=\sum_{j\geq 0} b_j^{(r)}q^j$. By (ii),

$$b_j^{(r)}=0 \text{ for } j>\left[\frac{l(x,y)-r}{2}\right].$$

(Note that this implies $Q_{x,y}^{(r)}=0$ for $r>l(x,y)$). Thus

$$\sum_{\substack{j\geq 0\\ r\geq 1}} b_j^{(r)}$$

becomes a finite sum and in fact is equal to

$$\sum_{j\geq 0}(b_j^{(1)}+b_j^{(2)}+\ldots+b_j^{(l(x,y)-2j)})$$

as

$$0=b_j^{(l(x,y)-2j+1)}=b_j^{(l(x,y)-2j+2)}=\ldots.$$

Also, by (i) $a_j\geq b_j^{(r)}\geq b_j^{(r+1)}\,\forall r\geq 1,\ \forall j$. Thus

$$\sum_{\substack{j\geq 0\\ r\geq 1}} b_j^{(r)}=\sum_{j\geq 0}(b_j^{(1)}+\ldots+b_j^{(l(x,y)-2j)})\leq\sum_{j\geq 0}a_j(l(x,y)-2j)-(*).$$

On the other hand, by (iii) and Proposition 3.2,

$$\sum_{\substack{j\geq 0\\ r\geq 1}} b_j^{(r)}=\sum_{r\geq 1}Q_{x,y}^{(r)}(1)=\sum_{\substack{x\leq xt\leq y\\ t\in T}}Q_{xt,y}^{(0)}(1)$$

$$=\sum_{\substack{x\leq xt\leq y\\ t\in T}}P_{xt,y}(1)=\sum_{0\leq j\leq p}a_j(l(x,y)-2j).$$

(Note $Q_{x,y}^{(0)}=P_{x,y}$).

Thus equality must hold in "each term" in (*). Hence

$$a_j=b_j^{(1)}=b_j^{(2)}=\ldots=b_j^{(l(x,y)-2j)}\ (\forall_j\geq 0).$$

It is now easy to see that $Q_{x,y}^{(r)}=P_{x,y}^{(r)}\ \forall r\geq 1$. Already $Q_{x,y}^{(0)}=P_{x,y}=P_{x,y}^{(0)}$. This shows the uniqueness and so completes the proof of the Theorem.

REMARK 3.4. If $x<y$, $P_{x,y}=P_{x,y}^{(0)}=P_{x,y}^{(1)}$ as

$$\deg P_{x,y}\leq\left[\frac{l(x,y)-1}{2}\right]$$

in that case. For $x=y$, $P_{x,y}^{(0)}=1$ and $P_{x,y}^{(r)}=0\ \forall r\geq 1$.

§ 4. MODULES WITH "$\mathbb{C}[[t]]$-COEFFICIENTS"

Let $A=\mathbb{C}[[t]]$ be the ring of formal power series in one variable over $\mathbb{C}$. Let $\mathfrak{G}_A=A\otimes_{\mathbb{C}}\mathfrak{G}$. Then $\mathfrak{G}_A$ becomes an A-Lie algebra under the bracket operation

$$[p\otimes u,p'\otimes u']=pp'\otimes[u,u']\ \forall p,p'\in A,u,u'\in\mathfrak{G}.$$

By modules with "$\mathbb{C}[[t]]$-coefficients" we mean modules for the Lie-algebra $\mathfrak{G}_A$. An example of such a module is $M(\lambda_t)$ which is a "$\mathfrak{G}_A$-Verma-module" associated to a $\lambda_t\in\mathrm{Hom}_{\mathbb{C}}(\mathfrak{H},A)$. (We will come to the definitions of λ_t and $M(\lambda_t)$ later on.) We are interested in these modules and their tensor products (over $\mathbb{C}$) with finite dimensional $\mathfrak{G}$-modules. The discussion is very similar to ([3-§ 5]) where modules with "$\mathbb{C}[t]$-coefficients" are considered. However, the ring A has "many" invertible elements and this, as we will see, allows us a lot of space to manoeuver. (In fact, A "almost behaves" like a field).

In this section, we will develop some general theory and apply it in the next section to a special case to prove Theorem 1.1. Part of the discussion in this section is quite "routine" to such set-ups (cf. [1], [3]); nevertheless we give it here for the sake of completeness.

We first enlarge the notation used earlier and also collect together some of the standard facts about complex semisimple Lie-algebra which will be needed later on.

Let Φ (respectively Φ^+) be the root system of $(\mathfrak{G}, \mathfrak{H})$ (respectively $(\mathfrak{B}, \mathfrak{H})$). Then Φ^+ is a positive root system for Φ. Denote by Δ the set of simple roots in Φ^+. Let $\mathfrak{n}^+$ (respectively $\mathfrak{n}^-$) be the nilpotent Lie algebra corresponding to Φ^+ (respectively $-\Phi^+$). For any subalgebra $\mathfrak{c}$ of $\mathfrak{G}$ let $U(\mathfrak{c})$ denote the Universal enveloping algebra of $\mathfrak{c}$ and identify $U(\mathfrak{c})\hookrightarrow U(\mathfrak{G})$.

Enumerate $\Phi^+ = \{\gamma_1, \ldots, \gamma_k\}$. Choose a Chevalley basis

$$\{x_{\gamma_i}\}_{1 \leq i \leq k} U \{y_{\gamma_i}\}_{1 \leq i \leq k} U \{[x_\alpha, y_\alpha]\}_{\alpha \in \Delta} \text{ of } \mathfrak{G}$$

$(x_{\gamma_i} \in \mathfrak{G}^{\gamma_i},\ y_{\gamma_i} \in \mathfrak{G}^{-\gamma_i}\ \forall i)$.

Let Π be the set of all maps from the set $\{1, \ldots, k\}$ into the set of non-negative integers. For $\pi \in \Pi$, let $y_\pi \in U(\mathfrak{n}^-)$ be defined by $y_\pi = y_{\gamma_1}^{\pi(1)} \ldots y_{\gamma_k}^{\pi(k)}$. Define $\| \pi \| = \sum_{1 \leq i \leq k} \pi(i)\gamma_i$.

Let $Z(\mathfrak{G})$ be the centre of $U(\mathfrak{G})$. Then it is known that

$$Z(\mathfrak{G}) \simeq \mathbb{C}[z_1, \ldots, z_l] \ (z_i \in Z(\mathfrak{G}),\ l = \text{rank } \mathfrak{G}).$$

Let β: $U(\mathfrak{G}) \to U(\mathfrak{H})$ be the projection given by the decomposition

$$U(\mathfrak{G}) \simeq U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{H}) \otimes_{\mathbb{C}} U(\mathfrak{n}^+).$$

For $\lambda \in \mathfrak{H}^*$, let $\chi_\lambda \in Z(\hat{\mathfrak{G}})$ $(= \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(Z(\mathfrak{G}), \mathbb{C}))$ be given by: $\chi_\lambda(z) = \lambda(\beta(z))$ $\forall z \in Z(\mathfrak{G})$ (we note that χ_λ is the central character of the Verma module $M(\lambda)$). One also has: every $\chi \in Z(\hat{\mathfrak{G}})$ is of the form χ_λ for some $\lambda \in \mathfrak{H}^*$; moreover, $\chi_\lambda = \chi_\mu$ iff $\exists \tau \in W$ such that $\lambda + \varrho = \tau(\mu + \varrho)$.

Using the Chevalley basis, we define an involutive antiautomorphism σ of $\mathfrak{G}$ given by $x_{\gamma_i}^\sigma = y_{\gamma_i}$ $\forall i$ and $h^\sigma = h$ $\forall h \in \mathfrak{H}$.

It can be checked that $\beta(u^\sigma) = \beta(u)$ $\forall u \in U(\mathfrak{G})$. Thus for $z \in Z(\mathfrak{G})$, $\chi \in Z(\hat{\mathfrak{G}})$, $\chi(z^\sigma) = \chi(z)$. (In fact, it is even true that $z = z^\sigma$ $\forall z \in Z(\mathfrak{G})$; the proof uses the fact $Z(\mathfrak{G}) \simeq \mathbb{C}[z_1, \ldots, z_l]$.)

In the course of discussion, we use the following lemmas from commutative algebra repeatedly:

Let R be a commutative ring with 1. We then have

LEMMA 4.1. *If $a, b, c \in R$ are three elements such that the sets $\{a, c\}$ and $\{b, c\}$ both generate R as an ideal then so does the set $\{ab, c\}$.*

LEMMA 4.2. *If $\{a_i\}_{1 \leq i \leq r} \in R$ are such that any two (distinct) of them generate R as an ideal then so does the set*

$$\{a_1 a_2 \ldots a_{r-1}, a_1 a_2 \ldots a_{r-2} a_r, \ldots, a_2 \ldots a_r\}.$$

The proofs of these lemmas are quite straightforward.

For $\lambda \in \mathfrak{H}^*$, let $\lambda_t \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{H}, A)$ be given by $\lambda_t(h) = \lambda(h) + t \cdot \varrho(h)\ \forall h \in \mathfrak{H}$. Let $M(\lambda_t)$ be the $\mathfrak{G}_A$-Verma module with "highest weight" λ_t. More precisely, elements of $M(\lambda_t)$ are of the form $\sum_{\pi \in \Pi} a_\pi y_\pi m_{\lambda_t} (a_\pi \in A)$; $\mathfrak{H}$ acts on m_{λ_t} by λ_t, $\mathfrak{n}^+$ acts trivially and $\mathfrak{n}^-$ by left-multiplication. In fact $\{y_\pi m_{\lambda_t} | \pi \in \Pi\}$ is an A-basis of $M(\lambda_t)$.

Let F be a finite dimensional irreducible $\mathfrak{G}$-module. Then $V = M(\lambda_t) \otimes_{\mathbb{C}} F$ is a $\mathfrak{G}$-module under the "usual" action and an A-module by multiplication by elements of A in the "$M(\lambda_t)$-part". These actions commute and so V gets the structure of $\mathfrak{G}_A$-module.

The structure of V has three different features built in it. One arises from a filtration of F, another from the action of $Z(\mathfrak{G})$ on it and a third one coming from a "contravariant" form. We will examine these features and their interrelations. We start with the following:

PROPOSITION 4.3. V has a filtration $V = V_n \supseteq V_{n-1} \supseteq \dots \supseteq V_0 \supseteq (0)$ by $\mathfrak{G}_A$-submodules such that $V_i | V_{i-1} \simeq M((\lambda + \mu_i)_t)\ \forall i$ where μ_is are the weight of F counted with multiplicity.

PROOF. Choose a basis $\{f_j\}_{0 \le j \le n}$ of F such that

(i) f_j is a weight vector of weight μ_j

(ii) $\mu_j - \mu_k = \sum_{\alpha \in \Delta} c_\alpha \cdot \alpha$ with $c_\alpha \ge 0$ (and at least one $c_\alpha \ne 0) \Rightarrow j < k$.

Let V_i be the $\mathfrak{G}_A$-submodule of V generated by $\{m_{\lambda_t} \otimes f_0, m_{\lambda_t} \otimes f_1, \dots, m_{\lambda_t} \otimes f_i\}$. Then we have $V = V_n \supseteq V_{n-1} \supseteq \dots \supseteq V_0 \supseteq \{0\}$. Clearly V_i / V_{i-1} is generated by the image $\overline{m_{\lambda_t} \otimes f_i}$ of $m_{\lambda_t} \otimes f_i \in V_i$. Also, for $\alpha \in \Delta$, $x_\alpha \cdot \overline{m_{\lambda_t} \otimes f_i} = \overline{m_{\lambda_t} \otimes x_\alpha \cdot f_i} = 0$ as $x_\alpha \cdot f_i$ is of weight $\mu_i + \alpha$ and so $x_\alpha \cdot f_i = \sum_{j<i} c_j f_j$, $c_j \in \mathbb{C}$. We now show that

$$(*) \qquad \sum_{\pi \in \Pi} a_\pi^{(i)} y_\pi \overline{m_{\lambda_t} \otimes f_i} = 0 \Rightarrow a_\pi^{(i)} = 0\ \forall \pi \in \Pi.$$

Assume $a_\pi^{(i)} \ne 0$ for some $\pi \in \Pi$. We have,

$$\sum_{\pi \in \Pi} a_\pi^{(i)} \cdot y_\pi (m_{\lambda_t} \otimes f_i) = v_{i-1} \in V_{i-1}.$$

Observe that v_{i-1} can be written as

$$\sum_{\substack{j \le i-1 \\ \pi \in \Pi}} -a_\pi^{(j)} \cdot y_\pi (m_{\lambda_t} \otimes f_j).$$

(This follows from an easy induction on i). Thus

$$\sum_{\substack{j \le i \\ \pi \in \Pi}} a_\pi^{(j)} \cdot y_\pi (m_{\lambda_t} \otimes f_j) = 0$$

and at least one $a_\pi^{(i)} \ne 0$. Choose j_0 least such that $a_{\pi_0}^{(j_0)} \ne 0$ for some $\pi_0 \in \Pi$. (This exists as $a_\pi^{(i)} \ne 0$ for some $\pi \in \Pi$). Now expand the above equation and rewrite it in the form $\sum_k m_k \otimes f_k = 0$, $m_k \in M(\lambda_t)$. Observe that because of the minimality of j_0, $a_{\pi_0}^{(j_0)} \cdot y_{\pi_0} m_{\lambda_t}$ "occurs" in m_{j_0} and hence $m_{j_0} \ne 0$. (Recall that $\{y_\pi m_{\lambda_t} | \pi \in \Pi\}$ is an A-basis of $M(\lambda_t)$). This gives a contradiction and so proves that $a_\pi^{(i)} = 0\ \forall \pi$. This completes the proof of the proposition.

Next we show that it is possible to decompose V into submodules parametrized by $Z(\mathfrak{G})$. We note that it is here that the abundance of invertible elements in A is used..

We will first give some definitions and results for any $\mathfrak{G}_A$-module E.

Let $\varphi: A \to \mathbb{C}$ be the map $\varphi(\sum_{i\geq 0} c_i t^i) = c_0$. For any $z \in Z(\mathfrak{G})$ and $c \in \mathbb{C}$, define

$$E_c(z) = \{e \in E \mid \exists p_1, \ldots, p_k \in \varphi^{-1}(c) \text{ such that } (z-p_1) \ \ldots \ (z-p_k)\cdot e = 0\}.$$

Define

$$E^0(z) = \{e \in E \mid \exists q_1, \ldots, q_k \in A \text{ such that } (z-q_1) \ \ldots \ (z-q_k)e = 0\}.$$

(Note that here we do not demand $\varphi(q_i)$'s to be equal to each other).

Clearly $E_c(z)$, $c \in \mathbb{C}$ and $E^0(z)$ are $\mathfrak{G}_A$-submodules of E. We then have:

PROPOSITION 4.4. $E^0(z) = \oplus_{c\in\mathbb{C}} E_c(z)$.

PROOF. We will first show that the sum is direct. Let $c_1, \ldots, c_r \in \mathbb{C}$ be all distinct and let $e_s \in E_{c_s}(z)$ be such that $e_1 + \ldots + e_r = 0$. For each s, choose $\{p_{s,j}\}_{1\leq j\leq k_s} \in \varphi^{-1}(c_s)$ such that $(z-p_{s,1}) \ \ldots \ (z-p_{s,k_s})e_s = 0$. Consider $R = A[\theta]$, the polynomial ring in indeterminate θ over A. We note that for $p, q \in A$ with $\varphi(p) \neq \varphi(q)$, $\{\theta - p, \theta - q\}$ generate R as an ideal. (The point is that $p-q$ is invertible in A.) Hence applying Lemma 4.1 repeatedly, we see that for $l \neq s$

$$\{\prod_{1\leq n\leq k_l} (\theta - p_{l,n}), \prod_{1\leq j\leq k_s} (\theta - p_{s,j})\}$$

generate R as an ideal. Therefore by Lemma 4.2 $\exists \{g_s(\theta)\}_{1\leq s\leq r} \in R$ such that

$$1 = \sum_{1\leq s\leq r} g_s(\theta)\cdot \prod_{l\neq s} (\prod_{1\leq n\leq k_l} (\theta - p_{l,n})).$$

Replacing θ by z and operating on e_1, we get

$$e_1 = g_1(z)\cdot \prod_{l\geq 2} (\prod_{1\leq n\leq k_l} (z - p_{l,n}))\cdot e_1$$

Also, applying

$$g_1(z)\cdot \prod_{l\geq 2} (\prod_{1\leq n\leq k_l} (z - p_{l,n}))$$

to $e_1 + \ldots \ e_r = 0$ we get,

$$0 = g_1(z)\cdot \prod_{l\geq 2} (\prod_{1\leq n\leq k_l} (z - p_{l,n}))e_1 = e_1.$$

Similarly we show $e_2 = e_3 = \ldots = e_r = 0$. This shows that the sum is direct.

Next we show that $\sum_{c\in\mathbb{C}} E_c(z) = E^0(z)$. Observe first that by every definition $E_c(z) \subseteq E^0(z)$ $\forall c \in \mathbb{C}$. Thus we are left to show $E^0(z) \subseteq \sum_{c\in\mathbb{C}} E_c(z)$. Let $e \in E^0(z)$. Then $\exists q_1, \ldots, q_k \in A$ such that $(z-q_1) \ \ldots \ (z-q_k)e = 0$. Group together q_i's which have the same image under φ. After relabelling, we are in the situation:

there exist $c_1, \dots, c_r \in \mathbb{C}$ all distinct from each other and for each c_s there exists a set $\{p_{s,j}\}_{1 \le j \le k_s}$ such that

$$\prod_{1 \le s \le r} \cdot \prod_{1 \le j \le k_s} (z - p_{s,j}) \cdot e = 0.$$

We again use Lemmas 4.1 and 4.2 to get polynomials $\{g_s(\theta)\}_{1 \le s \le r} \in R$ such that

$$1 = \sum_{1 \le s \le r} g_s(\theta) \cdot \prod_{l \ne s} (\prod_{1 \le n \le k_l} (\theta - p_{l,n})).$$

Thus

$$e = \sum_{1 \le s \le r} g_s(z) \cdot \prod_{l \ne s} (\prod_{1 \le n \le k_l} (z - p_{l,n} 1)) \cdot e.$$

Put

$$e_s = g_s(z) \cdot \prod_{l \ne s} (\prod_{1 \le n \le k_l} (z - p_{l,n})) \cdot e.$$

As

$$\prod_{1 \le l \le r} \cdot \prod_{1 \le n \le k_l} (z - p_{l,n}) \cdot e = 0,$$

it is clear that $e_s \in E_{c_s}(z)$. This completes the proof of the proposition.

We next have:

LEMMA 4.5. *For $z, z' \in Z(\mathfrak{G})$,*

(i) $E_{cd}(dz) = E_c(z)$ $\forall c$, $d \in \mathbb{C}$, $d \ne 0$.

(ii) $E_c(z) \cap E_{c'}(z') \subseteq E_{c+c'}(z+z') \cap E_{cc'}(zz')$ $\forall c$, $c' \in \mathbb{C}$.

(iii) $E^0(z) \cap E^0(z') = \oplus_{c, c' \in \mathbb{C}} E_c(z) \cap E_{c'}(z')$.

(iv) $E^0(z) \cap E^0(z') \subseteq E^0(z+z') \cap E^0(zz')$.

PROOF. (i) is clear from definition.

(ii) Let $c, c' \in \mathbb{C}$. We will first show $E_c(z) \cap E_{c'}(z') \subseteq E_{cc'}(zz')$. Let $e \in E_c(z) \cap$ $\cap E_{c'}(z')$. Then $\exists \{p_i\}_{1 \le i \le r} \in \varphi^{-1}(c)$ and $\{p'_j\}_{1 \le j \le s} \in \varphi^{-1}(c')$ such that

$$(z - p_1) \dots (z - p_r) e = 0 = (z' - p'_1) \dots (z' - p'_s) e.$$

Consider

$$\prod_{\substack{1 \le i \le r \\ 1 \le j \le s}} (zz' - p_i p'_j) = \prod_{\substack{1 \le i \le r \\ 1 \le j \le s}} (z'(z - p_i) + p_i(z' - p'_j)).$$

Fix i and consider

$$\prod_{1 \le j \le s} (z'(z - p_i) + p_i(z' - p'_j)).$$

On expanding out, we see that this product is equal to

$$(z - p_i) \cdot g_i(z, z') + p_i^s \cdot \prod_{1 \le j \le s} (z' - p'_j)$$

for some $g_i(\theta, \theta') A[\theta, \theta']$. Hence

$$\prod_{\substack{1 \le i \le r \\ 1 \le j \le s}} (zz' - p_i p_j')e = \prod_{1 \le i \le r} ((z - p_i) \cdot g_i(z, z') + p_i^s \prod_{1 \le j \le s} (z' - p_j')) \cdot e$$

$$= \prod_{1 \le i \le r} (z - p_i) \cdot g_i(z, z') \cdot e = 0.$$

Thus $e \in E_{cc'}(zz')$ by definition. (Note $p_i p_j' \in \varphi^{-1}(cc')$). The proof of

$$E_c(z) \cap E_{c'}(z') \subseteq E_{c+c'}(z + z')$$

is similar. This proves (ii).

For (iii), note that $E_c(z)$ is a $\mathfrak{G}_A$-submodule of E. Hence by Proposition 4.4 and (ii) above,

$$E^0(z) \cap E^0(z') = \bigoplus_{c \in \mathbb{C}} E_c(z) \cap E^0(z')$$

$$= \bigoplus_{c \in \mathbb{C}} \Big(\bigoplus_{c' \in \mathbb{C}} E_c(z) \cap E_{c'}(z') \Big)$$

$$= \bigoplus_{c, c' \in \mathbb{C}} E_c(z) \cap E_{c'}(z').$$

(iv) follows from (ii) and (iii). This completes the proof of the lemma.

COROLLARY 4.6. For $\chi \in Z(\hat{\mathfrak{G}})$, define

$$E_\chi = \bigcap_{z \in Z(\mathfrak{G})} E_{\chi(z)}(z) \text{ and } E^0 = \bigcap_{z \in Z(\mathfrak{G})} E^0(z).$$

Then

$$E^0 = \bigoplus_{\chi \in Z(\hat{\mathfrak{G}})} E_\chi.$$

PROOF. From lemma 4.5, it is clear that

$$E^0 = \bigcap_{1 \le i \le l} E^0(z_i) \text{ and } E_\chi = \bigcap_{1 \le i \le l} E_{\chi(z_i)}(z_i).$$

(Recall that $\{z_1, \ldots, z_l\}$ is a polynomial basis of $Z(\mathfrak{G})$).

Since we are now involved with finitely many intersections, the corollary follows by repeated application of (iii) of Lemma 4.5.

PROPOSITION 4.7. Let $\eta: E_1 \to E_2$ be a $\mathfrak{G}_A$-module homomorphism of $\mathfrak{G}_A$-modules E_1 and E_2. Then

(i) $\eta((E_1)_\chi) \subseteq (E_2)_\chi \; \forall \chi \in Z(\hat{\mathfrak{G}})$

(ii) $\eta(E_1^0) \subseteq E_2^0$.

Proof is immediate.

We now apply these general results to the special case of V and its submodules V_i as defined in Proposition 4.3. We first prove:

PROPOSITION 4.8. For $0 \le i \le n$, $V_i^0 = V_i$.

PROOF. Clearly it suffices to prove that for any $z \in Z(\mathfrak{G})$, $(V_i)^0(z) = V_i$. Let $v \in V_i$. Then the image $\bar{v}$ of v in $V_i | V_{i-1}$ is of the form

$$\bar{v} = \sum_{\pi \in \Pi} a_\pi y_\pi \overline{(m_{\lambda_t} \otimes f_i)} (a_\pi \in A).$$

As $z \cdot \overline{m_{\lambda_t} \otimes f_i} = (\lambda + \mu_i)_t(\beta(z)) \cdot \overline{m_{\lambda_t} \otimes f_i}$, it is clear that $z \cdot \bar{v} = (\lambda + \mu_i)(\beta(z))\bar{v}$ ie. $(z - (\lambda + \mu_i)_t(\beta(z))) \cdot v \in V_{i-1}$. We note that $\varphi((\lambda + \mu_i)_t(\beta(z))) = (\lambda + \mu_i)(\beta(z)) = \chi_{\lambda + \mu_i}(z)$. (We will use this later on.)

Coming back to the proof, an induction on i shows that

$$\prod_{j \le i} (z - (\lambda + \mu_j)_t(\beta(z))) \cdot v = 0.$$

This shows that $v \in (V_i)^0(z)$. This completes the proof of the proposition.

COROLLARY 4.9. $V_i = \oplus_{\chi \in Z(\mathfrak{G})} (V_i)_\chi \ \forall 0 \le i \le n$.

Proof is immediate using Corollary 4.6 and Proposition 4.8. We note here that $(V_i)_\chi = V_i \cap V_\chi$, $0 \le i \le n$, $\chi \in Z(\hat{\mathfrak{G}})$. We now have two sets of submodules of V viz. $\{V_i\}_{0 \le i \le n}$ and $\{V_\chi\}_{\chi \in Z(\hat{\mathfrak{G}})}$. In Corollary 4.9, we fixed V_i of one set and saw the effect of the other set on it. We now do the opposite. So fix a V_χ. We then have:

PROPOSITION 4.10.

(i) $V_\chi \cap V_i = V_\chi \cap V_{i-1}$ if $\chi \ne \chi_{\lambda + \mu_i}$

(ii) $V_\chi \cap V_i / V_\chi \cap V_{i-1} \simeq V_i / V_{i-1} (\simeq M((\lambda + \mu_i)_t)$ if $\chi = \chi_{\lambda + \mu_i}$.

PROOF. Consider $V_\chi \cap V_i / V_\chi \cap V_{i-1} \to V_i / V_{i-1}$. First of all,

$$V_i / V_{i-1} = (V_i / V_{i-1})^0 = (V_i / V_{i-1})\chi_{\lambda + \mu_i}.$$

(This is so as $V_i / V_{i-1} \simeq M((\lambda + \mu_i)_t)$). Also, $V_\chi \cap V_i / V_\chi \cap V_{i-1}$ is the image of $(V_i)_\chi$ under the map $\eta : V_i \hookrightarrow V_i / V_{i-1}$. Hence by Proposition 4.7, $V_\chi \cap V_i / V_\chi \cap \cap V_{i-1} \subseteq (V_i / V_{i-1})_\chi$ which is $\{0\}$ if $\chi \ne \chi_{\lambda + \mu_i}$. This proves (i). To prove (ii), we use Corollary 4.9. Write $m_{\lambda_t} \otimes f_i = \sum_{\chi' \in Z(\hat{\mathfrak{G}})} v_{\chi'}, v_{\chi'} \in (V_i)_{\chi'}$. Consider the images in V_i / V_{i-1}. As $\overline{m_{\lambda_t} \otimes f_i} \in (V_i / V_{i-1})_\chi$, it is clear that $\bar{v}_{\chi'} = 0 \ \forall \chi' \ne \chi$. Thus $\overline{m_{\lambda_t} \otimes f_i} = \bar{v}_\chi$. Since $\overline{m_{\lambda_t} \otimes f_i}$ generates V_i / V_{i-1}, we have (ii).

COROLLARY 4.11. For $\chi \in Z(\hat{\mathfrak{G}})$, let $0 \le i_1 < i_2 \ldots \le i_s \le n$ be the indices such that $\chi_{\lambda + \mu_{i_j}} = \chi$. Then V_χ has a filtration in which successive subquotients are $\mathfrak{G}_A$-isomorphic to $M((\lambda + \mu_{i_j})_t)$ $1 \le j \le s$.

We now turn our attention to contravariant forms. Let E be any $\mathfrak{G}_A$-module. By a contravariant form B on E we mean an A-bilinear map on $E \times E$ with values in A which satisfies:

$$B(u \cdot e, e') = B(e, u^\sigma e') \ \forall e, e' \in E, u \in \mathfrak{G}.$$

("contravariance" property of B). (Recall that σ is the involutive antiautomorphism of $\mathfrak{G}$ such that $x^{\sigma}_{\gamma_i} = y_{\gamma_i}$ $\forall 1 \le i \le k$. We extend it to $\mathfrak{G}_A$ in a natural way).

As an immediate consequence of the definition of "contravariance" property, we have:

PROPOSITION 4.12. For $\chi \neq \chi' \in Z(\hat{\mathfrak{G}})$, $B(E_{\chi}, E_{\chi'}) = 0$.

PROOF. Choose $z \in Z(\mathfrak{G})$ such that $\chi(z) \neq \chi'(z)$ $(= \chi'(z^{\sigma}))$. Let $e_{\chi} \in E_{\chi}$, $e_{\chi'} \in E_{\chi'}$. Then by definition, $\exists \{p_1, \ldots, p_r\} \in \varphi^{-1}(\chi(z))$ and $\{p'_1, \ldots, p'_s\} \in \varphi^{-1}(\chi'(z^{\sigma}))$ such that

$$\prod_{1 \le i \le r} (z - p_i) \cdot e_{\chi} = 0 = \prod_{1 \le j \le s} (z^{\sigma} - p'_j) \cdot e_{\chi'}.$$

By applying Lemmas 4.1 and 4.2 we get polynomials $g(\theta)$ and $g'(\theta) \in R$ such that

$$1 = g(\theta) \prod_{1 \le i \le r} (\theta - p_i) + g'(\theta) \prod_{1 \le j \le s} (\theta - p'_j).$$

Hence

$$\begin{aligned} e_{\chi} &= g(z) \cdot \prod_{1 \le i \le r} (z - p_i) \cdot e_{\chi} + g'(z) \cdot \prod_{1 \le j \le s} (z - p'_j) \cdot e_{\chi} \\ &= \prod_{1 \le j \le s} (z - p'_j) \cdot (g'(z) \cdot e_{\chi}). \end{aligned}$$

Therefore,

$$\begin{aligned} B(e_{\chi}, e_{\chi'}) &= B\Big(\prod_{1 \le j \le s} (z - p'_j)(g'(z) \cdot e_{\chi}), e_{\chi'}\Big) \\ &= B\Big(g'(z) \cdot e_{\chi}, \prod_{1 \le j \le s} (z^{\sigma} - p'_j) \cdot e_{\chi'}\Big) \\ &= B(g'(z) \cdot e_{\chi}, 0) = \\ &= 0. \end{aligned}$$

This proves the proposition.

Associated to a contravariant form B on E, we define

$$E^r(B) = \{e \in E \mid B(e, E) \subseteq t^r \cdot A\}, r \ge 0.$$

By convention, $E^r(B) = E$ if $r < 0$. Clearly $E^r(B)$ is a $\mathfrak{G}_A$-submodule of E and $E = E^0(B) \supseteq E^1(B) \supseteq \ldots$.

If $p \in A$ then $p \cdot B$ is also a contravariant form on E and it is easy to check that for $p \neq 0$,

(4.13) $\quad E^r(p \cdot B) = E^{r - \mathrm{ord}\,(p)}(B)$ $\forall r \ge 0$.

Here, ord (p) = largest integer k such that $p \in t^k \cdot A$.

From Proposition 4.12, it is clear that

(4.14) $\quad (E^0)^r(B) = \bigoplus_{\chi \in Z(\hat{\mathfrak{G}})} (E_{\chi})^r(B)$.

Note that in general only $(E_\chi)^r(B) \supseteq E_\chi \cap E^r(B)$ may hold. (That is because E may not be equal to E^0.)

We now turn to the special cases of $M(\lambda_t)$, V and V_i's. We associate to each $M(\lambda_t)$ a canonical contravariant from B_{λ_t} in the following way:

Define

$$B_{\lambda_t}(y_\pi \cdot m_{\lambda_t}, y_{\pi'} \cdot m_{\lambda_t}) = (\lambda_t)(\beta(y_\pi^\sigma y_{\pi'})) \in A.$$

Extend B_{λ_t} to the whole of $M(\lambda_t)$ by A-bilinearity. Note that the values of B_{λ_t} on "basic" vectors $\{y_\pi m_{\lambda_t}\}$ are in $\mathbb{C}[t]$ actually.

It can be checked that B_{λ_t} is a contravariant form. (This property is "almost" built in).

The following proposition is easy to prove:

PROPOSITION 4.15.

(i) If π, $\pi' \in \Pi$ such that $\|\pi\| \neq \|\pi'\|$ then $B_{\lambda_t}(y_\pi m_{\lambda_t}, y_{\pi'} m_{\lambda_t}) = 0$. (Recall that $\|\pi\| = \sum_{1 \le i \le k} \pi(i)\gamma_i$).

(ii) If B is any contravariant form an $M(\lambda_t)$ then $\exists! p \in A$ such that $B = p \cdot B_{\lambda_t}$.

Proof of (i) is clear as $\beta(y_\pi^\sigma y_{\pi'}) = 0$ if $\|\pi\| \neq \|\pi'\|$. For (ii), put

$$p = B(m_{\lambda_t}, m_{\lambda_t}) \in A$$

and observe that the contravariance property of B forces $B = p \cdot B_{\lambda_t}$.

Consider the module $V = M(\lambda_t) \otimes_{\mathbb{C}} F$. Now F has a canonical ($\mathbb{C}$)-contravariant form B_F with values in $\mathbb{C}$ and in fact it is non-degenerate. Thus we get a contravariant form B_V on V given by:

$$B_V(m \otimes f, m' \otimes f') = B_{\lambda_t}(m, m') \cdot B_F(f, f') \; \forall m, m' \in M(\lambda_t), f, f' \in F.$$

We will be interested in filtrations $\{M(\lambda_t)^r(B_{\lambda_t})\}_{r \ge 0}$ and $\{V^r(B_V)\}_{r \ge 0}$ on $M(\lambda_t)$ and V respectively. For the sake of simplicity of notation, we will denote these simply by $\{M(\lambda_t)^r\}_{r \ge 0}$ and $\{V^r\}_{r \ge 0}$. These are related by:

(4.16) $\quad V^r = M(\lambda_t)^r \otimes_{\mathbb{C}} F.$

(This is clear since B_F is non-degenerate and has values in $\mathbb{C}$.)

Also, from Corollary 4.9 and Proposition 4.12 it is clear that

(4.17) $\quad V^r = \bigoplus_{\chi \in Z(\mathfrak{G})} V_\chi^r.$

We now consider the relation of $M(\lambda_t)$ (respectively V) with the $\mathfrak{G}$-Verma module $M(\lambda)$ (respectively the $\mathfrak{G}$-module $M(\lambda) \otimes_{\mathbb{C}} F$).

Let $\Phi: M(\lambda_t) \to M(\lambda)$ be defined by

$$\Phi\Big(\sum_{\pi \in \Pi} a_\pi y_\pi m_{\lambda_t}\Big) = \sum_{\pi \in \Pi} \varphi(a_\pi) y_\pi m_\lambda$$

(m_λ is a generator of $M(\lambda)$) Φ is then a $\mathfrak{G}$-module homomorphism onto $M(\lambda)$. In fact $\Phi(aum) = \varphi(a) \cdot u \cdot \Phi(m) \; \forall a \in A$, $u \in \mathfrak{G}$, $m \in M(\lambda_t)$. Define $M(\lambda)^r = \Phi(M(\lambda_t)^r)$.

Then $\{M(\lambda)^r\}_{r\geq 0}$ is a filtration of $\mathfrak{G}$-submodules of $M(\lambda)$. Consider now the map $\Phi\otimes Id: M(\lambda_t)\otimes_{\mathbb{C}}F\to M(\lambda)\otimes_{\mathbb{C}}F$. By (4.16),

$$(\Phi\otimes Id)(V^r)=(\Phi\otimes Id)(M(\lambda_t)^r\otimes F)=M(\lambda)^r\otimes F.$$

Also from the way V_χ is defined ($\chi\in Z(\hat{\mathfrak{G}})$), it is clear that $(\Phi\otimes Id)(V_\chi)=$ $=(M(\lambda)\otimes F)_\chi$. Combining, we get

$$(4.18)\qquad (\Phi\otimes Id)(V_\chi^r)=(M(\lambda)^r\otimes_{\mathbb{C}}F)_\chi.$$

To end this section, we compare our situation with one considered by Jantzen ([3-§ 5]). He uses $\mathbb{C}[t]$-coefficients to define a module $\tilde{M}(\lambda_t)$. It can be easily seen that $\tilde{M}(\lambda_t)\hookrightarrow M(\lambda_t)$. In fact, $\{y_\pi m_{\lambda_t}\mid \pi\in\Pi\}$ is a $\mathbb{C}[t]$-basis for $\tilde{M}(\lambda_t)$ (and, as we know, a $\mathbb{C}[[t]]$-basis for $M(\lambda_t)$). As observed earlier, $B_{\lambda_t}(y_\pi m_{\lambda_t}, y_{\pi'}m_{\lambda_t})\in\mathbb{C}[t]$ which is a p.i.d. and so $\exists$ two $\mathbb{C}[t]$-bases $\{\tilde{m}_j\}$ and $\{\tilde{m}_j'\}$ of $\tilde{M}(\lambda_t)$ which will be $\mathbb{C}[[t]]$-bases of $\tilde{M}(\lambda_t)$ as well and such that

$$B_{\lambda_t}(\tilde{m}_i,\tilde{m}_j)=\delta_{i,j}\cdot t^{n_i}\cdot q_i \text{ with } n_i\geq 0,\ \varphi(q_i)\neq 0.$$

So $\{\tilde{m}_i\mid n_i\geq r\}\cup\{t^{r-n_j}\tilde{m}_j\mid n_j<r\}$ is a $\mathbb{C}[t]$-basis for $\tilde{M}(\lambda_t)^r$ and a $\mathbb{C}[[t]]$-basis for $M(\lambda_t)^r$. Thus

$$\tilde{M}(\lambda_t)^r=\tilde{M}(\lambda_t)\cap M(\lambda_t)^r \text{ and } \Phi(\tilde{M}(\lambda_t)^r)=\Phi(M(\lambda_t)^r)=M(\lambda)^r.$$

Thus our filtration $\{M(\lambda)^r\}_{r\geq 0}$ coincides with the one obtained by Jantzen using "$\mathbb{C}[t]$-coefficients".

§ 5. PROOF OF THEOREM 1.1

Recall the hypothesis of the theorem: λ_0, μ_0 are given to be dominant integral elements of $\mathfrak{H}^*$; F is the finite dimensional irreducible $\mathfrak{G}$-module with highest weight μ_0. For $x\in W$, let $\lambda_x=x(\lambda_0+\varrho)-\varrho$ and $\mu_x=x(\mu_0)$. We will use the notation of § 4 with λ_x in place of λ.

Let $\chi=\chi_{\lambda_x+\mu_x}$. We first observe that $\chi_{\lambda_x+\mu_i}=\chi$ iff $\mu_i=\mu_x$ and the multiplicity of the weight μ_x is one (i.e. $\exists$ precisely one i, say i_0, such that $\chi=\chi_{\lambda_x+\mu_i}$). By Corollary 4.11, we get

$$V_\chi \xrightarrow[\eta_x]{\sim} M(\lambda_x+\mu_x)_t).$$

Let Φ' be the canonical map: $M(\lambda_x+\mu_x)_t)\to M(\lambda_x+\mu_x)$ (defined in the same way as Φ). It is now easy to check that $\Phi'\circ\eta_x$ is in fact zero on ker $(\Phi\otimes Id)$ and so we get a ! map $\psi_x:(M(\lambda_x)\otimes F)_\chi\to M(\lambda_x+\mu_x)$ making the following diagram commutative:

$$(5.1)\qquad \begin{array}{ccc} V_\chi & \xrightarrow[\eta_x]{\sim} & M(\lambda_x+\mu_x)_t) \\ \big\downarrow{\scriptstyle \Phi\otimes Id} & & \big\downarrow{\scriptstyle \Phi'} \\ (M(\lambda_x)\otimes F)_\chi & \xrightarrow[\psi_x]{\sim} & M(\lambda_x+\mu_x) \end{array}$$

In fact, ψ_x can be seen to be an isomorphism. The contravariant form B_V on V_χ gives a contravariant form $B_V \circ (\eta_x^{-1} \times \eta_x^{-1})$ on $M(\lambda_x + \mu_x)_t)$ and so by (ii) of Proposition 4.15, $\exists\,!\, p \in A$ such that

$$(5.2) \qquad B_V \circ (\eta_x^{-1} \times \eta_x^{-1}) = p \cdot B_{(\lambda_x + \mu_x)_t}.$$

Hence by (4.13), η_x takes V_χ^r isomorphically onto $M((\lambda_x + \mu_x)_t)^{r\text{-ord}\ p}$ $\forall r \geq 0$. By (4.18)

$$(\Phi \times Id)(V_\chi^r) = (M(\lambda)^r \otimes_{\mathbb{C}} F)_\chi.$$

Also, $\Phi'(M(\lambda_x + \mu_x)_t^{r\text{-ord}\ p}) = M(\lambda_x + \mu_x)^{r\text{-ord}\ p}$ by definition. Combining, we get: ψ_x maps $(M(\lambda_x)^r \otimes F)_\chi$ isomorphically onto $M(\lambda_x + \mu_x)^{r\text{-ord}\ p}$ $\forall r \geq 0$. It is now left to show that ord $p = 0$ i.e. $\varphi(p) \neq 0$.

If ord $(p) > 0$ then it can be easily seen that

$$\psi_x((M(\lambda_x)^1 \otimes F)_\chi) = M(\lambda_x + \mu_x) = \psi_x((M(\lambda_x) \otimes F)_\chi).$$

Since ψ_x is an isomorphism, one has

$$(M(\lambda_x) \otimes F)_\chi = (M(\lambda_x)^1 \otimes F)_\chi.$$

ie.

$$\Phi_{\lambda_0 + \mu_0}^{\lambda_0}(M(\lambda_x)) = \Phi_{\lambda_0 + \mu_0}^{\lambda_0}(M(\lambda_x)^1).$$

Since $\Phi_{\lambda_0 + \mu_0}^{\lambda_0}$ is exact and $M(\lambda_x)/M(\lambda_x)^1 \simeq L(\lambda_x)$ we get $\Phi_{\lambda_0 + \mu_0}^{\lambda_0}(L(\lambda_x)) = 0$. However, it is known (e.g. [3-§ 2.4]) that $\Phi_{\lambda_0 + \mu_0}^{\lambda_0}(L(\lambda_x)) \simeq L(\lambda_x + \mu_x)$. This contradiction proves that ord $(p) = 0$ and this completes the proof of the theorem.

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