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On super edge-magic deficiency of volvox and dumbbell graphs

Muhammad Imran[a,*], Hafiz Usman Afzal[b], A.Q. Baig[c]

[a] *Department of Mathematics, School of Natural Sciences(SNS), National University of Sciences and Technology (NUST), Sector H-12, Islamabad, Pakistan*
[b] *Department of Mathematics, Government College University Lahore, Pakistan*
[c] *Department of Mathematics, COMSATS Institute of Information Technology, Attock, Pakistan*

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Abstract

Let $G = (V, E)$ be a finite, simple and undirected graph of order p and size q. A super edge-magic total labeling of a graph G is a bijection $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$, where the vertices are labeled with the numbers $1, 2, \ldots, p$ and there exists a constant t such that $f(x) + f(xy) + f(y) = t$, for every edge $xy \in E(G)$. The super edge-magic deficiency of a graph G, denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic total labeling, or it is ∞ if there exists no such n.

In this paper, we are dealing with the super edge-magic deficiency of volvox and dumbbell type graphs.

Keywords: Super edge-magic total; Deficiency; Cycle; Volvox; Dumbbell

1. Introduction

In this paper, we consider finite, simple and undirected graphs. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively, where $|V(G)| = p$ and $|E(G)| = q$. An *edge-magic total labeling* of a graph G is a bijection $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$, where there exists a constant t such that $f(x) + f(xy) + f(y) = t$, for every edge $xy \in E(G)$. The constant t is called the *magic constant* and a graph that admits an edge magic total labeling is called an *edge-magic total graph*. An edge-magic total labeling λ is called *super edge-magic total* if the vertices are labeled with the smallest possible numbers, i.e. $1, 2, \ldots, p$.

The graph labeling has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to the mathematical challenges of graph labeling, but also for the wide range of its application, for instance X-ray, crystallography, coding theory, radar, astronomy, circuit design, network design and communication design. In fact Bloom and Golomb studied applications of graph labelings to other branches of science and it is possible to find part of this work in [1] and [2].

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* Corresponding author.
E-mail addresses: imrandhab@gmail.com (M. Imran), huafzal@gmail.com (H.U. Afzal), aqbaig1@gmail.com (A.Q. Baig).

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The concept of edge-magic total labeling was given by Kotzig and Rosa [3] in 1970. They proved that for any graph G there exists an edge-magic total graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This fact leads to the concept of *edge-magic total deficiency* of a graph G [3], which is the minimum nonnegative integer n such that $G \cup nK_1$ is edge-magic total. The edge-magic deficiency of G is denoted by $\mu(G)$. In particular,

$$\mu(G) = \min\{n \geq 0 : G \cup nK_1 \text{ is edge-magic}\}.$$

In the same paper, Kotzig and Rosa gave the upper bound of the edge-magic deficiency of a graph G with n vertices,

$$\mu(G) \leq F_{n+2} - 2 - n - \frac{1}{2}n(n-1)$$

where F_n is the nth Fibonacci number.

Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno et al. [4] defined a similar concept for the super edge-magic total labelings. The *super edge-magic deficiency* of a graph G, denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic total labeling, or ∞ if there exists no such n. More precisely, if

$$M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge-magic graph}\},$$

then

$$\mu_s(G) = \begin{cases} \min M(G), & \text{if } M(G) \neq \emptyset, \\ \infty, & \text{if } M(G) = \emptyset. \end{cases}$$

It is easy to see that for every graph G, $\mu(G) \leq \mu_s(G)$.

In [5,4] Figueroa-Centeno et al. showed the exact values of the super edge-magic deficiencies of several classes of graphs, such as cycles, complete graphs, 2-regular graphs and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. In particular, they proved that

$$\mu_s(nK_2) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}. \end{cases}$$

In [6] Ngurah, Simanjuntak and Baskoro proved some upper bound for the super edge-magic deficiency of fans, double fans and wheels. In [7] Figueroa-Centeno *et al.* proved

$$\mu_s(P_m \cup K_{1,n}) = \begin{cases} 1, & \text{if } m = 2 \text{ and } n \text{ is odd or} \\ & m = 3 \text{ and } n \not\equiv 0 \pmod 3, \\ 0, & \text{otherwise}. \end{cases}$$

In the same paper, they proved that

$$\mu_s(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \text{if } m \text{ is a multiple of } n+1 \text{ or} \\ & n \text{ is a multiple of } m+1, \\ 1, & \text{otherwise}. \end{cases}$$

They also conjectured that every forest with two components has the super edge-magic deficiency less or equal to 1.

For a positive integer n, let $St(n)$ be a star with n leaves. Lee and Kong [8] use $St(n_1, n_2, \ldots, n_k)$ to denote the disjoint union of the k stars $St(n_1), St(n_2), \ldots, St(n_k)$. They proved that the following graphs are super edge-magic: $St(m, n)$ where $n \equiv 0 \pmod{(m+1)}$, $St(1, 1, n)$, $St(1, 2, n)$, $St(1, n, n)$, $St(2, 2, n)$, $St(2, 3, n)$, $St(1, 1, 2, n)$ for $n \geq 2$, $St(1, 1, 3, n)$, $St(1, 2, 2, n)$ and $St(2, 2, 2, n)$. They conjectured that $St(n_1, n_2, \ldots, n_k)$ is super edge-magic when k is odd.

It is known that if a graph G with p vertices and q edges is super edge-magic, then $q \leq 2p - 3$, see [9].

In proving the results in this paper, we frequently use the following proposition.

Lemma 1 (*[10]*)**.** *A graph G with p vertices and q edges is super edge-magic total if and only if there exists a bijective function $\lambda : V(G) \to \{1, 2, \ldots, p\}$ such that the set $S = \{\lambda(x)+\lambda(y) | xy \in E(G)\}$ consists of q consecutive integers. In such a case, λ extends to a super edge-magic total labeling of G.*

In this paper we are dealing with super edge-magic deficiency of volvox and dumbbell type graphs.

2. Main results

Volvox graphs

In this section we are dealing with super edge-magic deficiency of volvox graphs. Volvox is one of the best known chlorophytes and is the most developed in a series of genera that forms spherical colonies. Each mature volvox colony is composed of numerous flagellate cells similar to chlamydomonas and embedded in the surface of a hollow sphere or cenobium containing an extracellular matrix made of a gelatinous glycoprotien [11].

For m and n both are odd, we define the volvox graph $G \cong mC_n + (mn-3)e \cup (m-1)K_1$ as follows, where e is the arbitrary number of edges and K_1 denotes an isolated vertex.

$$V(G) = \{x_i^l : 1 \le i \le m, 1 \le l \le k\} \cup \{y_i^l : 1 \le i \le m, 2 \le l \le k-1\} \cup \{v_i : 1 \le i \le m-1\}$$

and

$$\begin{aligned} E(G) = {} & \{x_i^l x_i^{l+1} : 1 \le i \le m, 1 \le l \le k-1\} \cup \{y_i^l y_i^{l+1} : 1 \le i \le m, 2 \le l \le k-2\} \\ & \cup \{x_i^1 y_i^2, x_i^k y_i^{k-1} : 1 \le i \le m\} \cup \{x_i^l y_i^l : 1 \le i \le m, 2 \le l \le k-1\} \\ & \cup \{x_i^l y_i^{l-1} : 1 \le i \le m, 3 \le l \le k\} \cup \left\{x_i^1 x_{i+1}^1 : 1 \le i \le \frac{m-1}{2}\right\} \\ & \cup \left\{x_i^l y_{i-1}^l : m \le l \le \frac{m+3}{2}\right\} \cup \left\{x_i^1 x_{i+2}^1 : 1 \le i \le \frac{m-1}{2}\right\} \\ & \cup \left\{x_i^1 x_{i+1}^2 : 1 \le i \le \frac{m-1}{2}\right\} \cup \{y_i^1 y_{i+1}^{1+1} : 1 \le i \le m-1, k-1 \le l \le k\}. \end{aligned}$$

We have the following theorem.

Theorem 1. *For n and m odd, we have the super edge-magic deficiency of* $G \cong mC_n + (mn-3)e$ *is*

$$\mu_s(G) \le m-1.$$

Proof. If $p = |V(G)|$ and $q = |E(G)|$ then $p = mn + m - 1$ and $q = 2mn - 3$. Now, define a labeling $f : V(G) \to \{1, 2, \dots, mn+m-1\}$ as follows:

$$f(x_i^l) = \begin{cases} 1 + 2(i-1) & \text{when } 1 \le i \le m,\ l = 1 \\ 2(i + m(l-2)) & \text{when } 1 \le i \le m,\ 2 \le l \le k, \end{cases}$$

$$f(y_i^l) = 2m(l-1) + 2i - 1, \quad 1 \le i \le m,\ 2 \le l \le k,$$

and

$$f(v_i) = m(n-1) + 2i : 1 \le i \le m-1.$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $3, 4, \dots, 2mn-1$. Therefore by Lemma 1, f extends to a super edge-magic total labeling with magic constant $t = 3mn + m - 1$. ∎

In the next theorem, we are dealing the graph $G \cong mC_n + (mn-3)e$ when m even and n odd. For our convenience, we define the volvox graph $G \cong mC_n + (mn-3)e \cup (m-1)K_1$ as follows:

$$V(G) = \{x_i^l : 1 \le i \le m, l = 1, k\} \cup \{x_i^l : 1 \le i \le 2m, 2 \le l \le k-1\} \cup \{v_i : 1 \le i \le m-1\}$$

and

$$\begin{aligned} E(G) = {} & \{x_i^l x_i^{l+1} : 1 \le i \le m, 1 \le l \le k\} \cup \{x_i^1 x_i^2 : 1 \le i \le m\} \\ & \cup \{y_i^l y_i^{l+1} : 1 \le i \le m, 2 \le l \le k-1\} \cup \{x_i^k y_i^k : 1 \le i \le m\} \\ & \cup \left\{y_i^l x_i^{l+1} : 1 \le i \le \frac{m}{2}, 2 \le l \le k-1\right\} \cup \left\{x_i^l y_i^l : 1 \le i \le \frac{m}{2}, 3 \le l \le k-1\right\} \\ & \cup \left\{x_i^1 x_{i+1}^1, x_i^1 x_{i+1}^2 : 1 \le i \le \frac{m}{2}\right\} \cup \left\{y_i^l x_{i+1}^l : 1 \le i \le \frac{m}{2}, 2 \le l \le k\right\} \end{aligned}$$

$$\cup\left\{y_i^l x_{i+1}^{l+1} : 1 \le i \le \frac{m}{2}, 2 \le l \le k-1\right\} \cup \left\{y_i^l x_{i+1}^l : \frac{m+2}{2} \le i \le m-1, l = k\right\}$$

$$\cup\{y_i^{k-1} y_{i+1}^k : 1 \le i \le m-1\} \cup \left\{x_i^1 x_{i+2}^1 : 1 \le i \le \frac{m-2}{2}\right\}.$$

We have also the following theorem.

Theorem 2. *For m even and n odd, we have*

$$\mu_s(mC_n + (mn-3)e) \le m-1.$$

Proof. If $p = |V(G)|$ and $q = |E(G)|$ then $p = mn + m - 1$ and $q = 2mn - 3$. Now, define a labeling $f : V(G) \to \{1, 2, \dots, mn + m - 1\}$ as follows:

$$f(x_i^l) = \begin{cases} 1 + 2(i-1), & 1 \le i \le m, l = 1; \\ 2i + 2m(l-2), & 1 \le i \le m \text{ and } 2 \le l \le k, \end{cases}$$

$$f(y_i^l) = 2m(l-1) + 2i - 1, \quad 1 \le i \le m,\ 2 \le l \le k,$$

and

$$f(v_i) = m(n-1) + 2i, \quad 1 \le i \le m-1.$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $3, 4, \dots, 2mn - 1$. Therefore by Lemma 1, f extends to a super edge-magic total labeling with magic constant $t = 3mn + m - 1$. ■

Dumbbell type graphs

In this section we are dealing with super edge magic deficiency of dumbbell type graphs.

Theorem 3. *For n even, m a positive integer, the dumbbell type graph DB_1 defined as below admits a super edge-magic total labeling. i.e. $\mu_s(DB_1) = 0$.*

Proof. Let us define the vertex and edge sets of DB_1 as follows:

$$V(DB_1) = \{x_i : 1 \le i \le 2(m+1)\} \cup \{y_i : 1 \le i \le n+m-2\} \cup \{z_i : 1 \le i \le n+m-2\}$$

and

$$\begin{aligned} E(DB_1) = &\left\{y_i y_{i+1} : 1 \le i \le \frac{n}{2} - 2, \frac{n}{2} \le i \le n+m-3\right\} \\ &\cup \left\{z_i z_{i+1} : 1 \le i \le \frac{n}{2} + m - 2, \frac{n}{2} + m \le i \le n+m-3\right\} \\ &\cup \{x_1 y_1, x_1 z_1, x_{2(m+1)} y_{n+m-2}, x_{2(m+1)} z_{n+m-2}\} \cup \left\{y_i x_{2i-n+2} : \frac{n}{2} \le i \le \frac{n}{2} + m - 1\right\} \\ &\cup \left\{z_i x_{2i-n+2} : \frac{n}{2} \le i \le \frac{n}{2} + m - 1\right\} \cup \left\{y_i x_{2i-n+3} : \frac{n}{2} \le i \le \frac{n}{2} + m - 1\right\} \\ &\cup \left\{z_i x_{2i-n+3} : \frac{n}{2} \le i \le \frac{n}{2} + m - 1\right\} \cup \left\{x_2 y_{\frac{n}{2}-1}, x_2 z_{\frac{n}{2}-1}, x_{2m+1} y_{\frac{m}{2}+n}, x_{2m+1} z_{\frac{m}{2}+n}\right\} \\ &\cup \left\{y_i z_{i+1} : 1 \le i \le \frac{n}{2} - 2, \frac{n}{2} + m \le i \le n+m-3\right\} \\ &\cup \left\{x_1 y_2, x_2 z_{\frac{n}{2}-2}, x_{2m+1} y_{\frac{n}{2}+m+1}, z_{m+n-3} x_{2(m+1)}\right\} \\ &\cup \left\{z_i y_{i+2} : 1 \le i \le \frac{n}{2} - 3, m + \frac{n}{2} \le i \le n+m-4\right\} \{x_i x_{i+1} : 2 \le i \le 2m\}. \end{aligned}$$

If $p = |V(DB_1)|$ and $q = |E(DB_1)|$ then $p = 2(2m + n - 1)$ and $q = 4(2m + n - 1) - 3$. Now, define a labeling $f : V(DB_1) \to \{1, 2, \dots, 2(2m + n - 1)\}$ as follows:

$$f(x_i) = \begin{cases} 1, & i = 1; \\ 2i + n - 4, & 2 \le i \le 2m, i \equiv 0 \pmod 2; \\ 2i + n - 3, & 3 \le i \le 2m+1, i \equiv 1 \pmod 2; \\ 2(2m + n - 1), & i = 2(m+1). \end{cases}$$

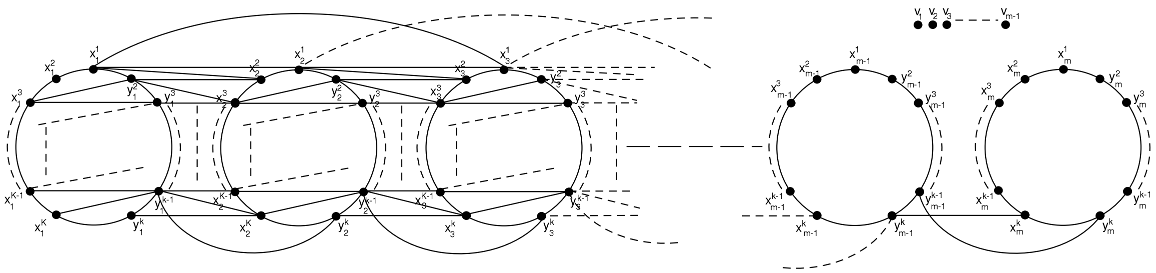

Fig. 1. An illustration of graph $G \cong mC_n + (mn-3)e \cup (m-1)K_1$ for m and n both odd.

$$f(y_i) = \begin{cases} 2i, & 1 \le i \le \frac{n}{2} - 1; \\ 4i - n + 2, & \frac{n}{2} \le i \le \frac{n}{2} + m - 1; \\ 2(i+m), & \frac{n}{2} + m \le i \le n + m - 2. \end{cases}$$

$$f(z_i) = \begin{cases} 2i + 1, & 1 \le i \le \frac{n}{2}; \\ 4i - n + 1, & \frac{n}{2} + 1 \le i \le \frac{n}{2} + m; \\ 2(i+m) + 1, & \frac{n}{2} + m + 1 \le i \le n + m - 2. \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $3, 4, \ldots, 4(2m+n-1)-1$. Therefore by Lemma 1, f extends to a super edge-magic total labeling with magic constant $6(2m+n-1)$. ∎

Theorem 4. *For n even, m a positive integer, the dumbbell type graph DB_2 defined as below has a super edge-magic total labeling. i.e. $\mu_s(DB_2) = 0$.*

Proof. Let us define the vertex and edge sets of DB_2 as follows:

$$V(DB_2) = \{x_i : 1 \le i \le 2(m+1)\} \cup \{y_i : 1 \le i \le n+m-2\} \cup \{z_i : 1 \le i \le n+m-2\}$$

and

$$\begin{aligned} E(DB_2) = & \left\{ y_i y_{i+1} : 1 \le i \le \frac{n}{2} - 2, \frac{n}{2} \le i \le n+m-3 \right\} \\ & \cup \left\{ z_i z_{i+1} : 1 \le i \le \frac{n}{2} + m - 2, \frac{n}{2} + m \le i \le n+m-3 \right\} \\ & \cup \{x_1 y_1, x_1 z_1, x_{2(m+1)} y_{n+m-2}, x_{2(m+1)} z_{n+m-2}\} \cup \left\{ y_i x_{2i-n+2} : \frac{n}{2} \le i \le \frac{n}{2} + m - 1 \right\} \\ & \cup \left\{ z_i x_{2i-n+2} : \frac{n}{2} \le i \le \frac{n}{2} + m - 1 \right\} \cup \left\{ y_i x_{2i-n+3} : \frac{n}{2} \le i \le \frac{n}{2} + m - 1 \right\} \\ & \cup \left\{ z_i x_{2i-n+3} : \frac{n}{2} \le i \le \frac{n}{2} + m - 1 \right\} \cup \left\{ x_2 y_{\frac{n}{2}-1}, x_2 z_{\frac{n}{2}-1}, x_{2m+1} y_{\frac{m}{2}+n}, x_{2m+1} z_{\frac{m}{2}+n} \right\} \\ & \cup \left\{ y_i z_{i+1} : 1 \le i \le \frac{n}{2} - 2, \frac{n}{2} + m \le i \le n+m-3 \right\} \\ & \cup \left\{ x_1 y_2, x_2 z_{\frac{n}{2}-2}, x_{2m+1} y_{\frac{n}{2}+m+1}, z_{m+n-3} x_{2(m+1)} \right\} \\ & \cup \left\{ z_i y_{i+2} : 1 \le i \le \frac{n}{2} - 3, m + \frac{n}{2} \le i \le n+m-4 \right\} \\ & \{x_i x_{i+1} : 3 \le i \le 2m-1, i \equiv 1 \pmod 2\} \left\{ y_i z_i : \frac{n}{2} \le i \le \frac{n}{2} + m - 1 \right\}. \end{aligned}$$ ∎

Labeling scheme is same as designed in Theorem 1 (see Figs. 1–4).

Theorem 5. *For n odd, m a positive integer, the dumbbell type graph DB_3 defined as below admits a super edge-magic total labeling. i.e. $\mu_s(DB_3) = 0$.*

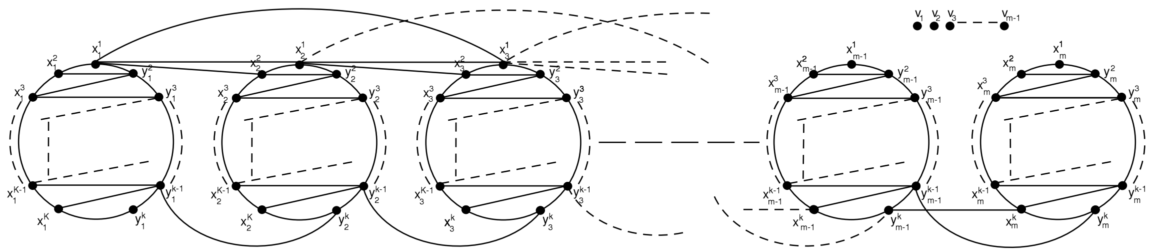

Fig. 2. An illustration of graph $G \cong mC_n + (mn-3)e \cup (m-1)K_1$ for m even and n odd.

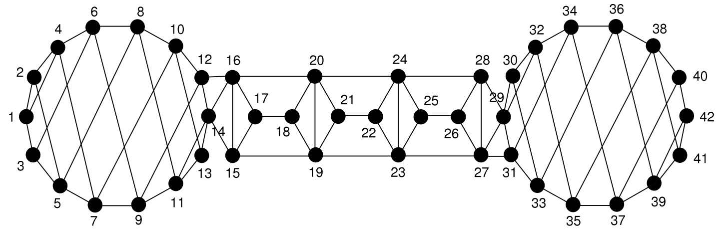

Fig. 3. Dumbbell graphs DB_1, even cycle.

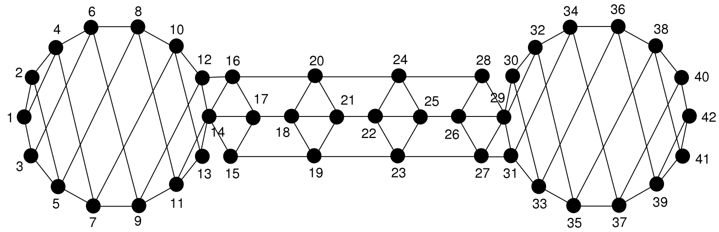

Fig. 4. Dumbbell graphs DB_2, even cycle.

Proof. Let us define the vertex and edge sets of DB_3 as follows:

$$V(DB_3) = \{x_i : 1 \le i \le 2m\} \cup \{y_i : 1 \le i \le n+m-1\} \cup \{z_i : 1 \le i \le n+m-1\}$$

and

$$\begin{aligned}
E(DB_3) = &\left\{y_i y_{i+1} : 1 \le i \le \frac{n-1}{2} + m, \frac{n+1}{2} + m \le i \le n+m-2\right\} \\
&\cup \left\{z_i z_{i+1} : 1 \le i \le \frac{n-3}{2}, \frac{n+1}{2} \le i \le n+m-2\right\} \\
&\cup \left\{x_1 y_{\frac{n-1}{2}}, x_1 z_{\frac{n-1}{2}}, x_{2m} y_{\frac{n+1}{2}+m}, x_{2m} z_{\frac{n+1}{2}+m}\right\} \\
&\cup \{y_i z_i : i = 1, n+m-1\} \cup \left\{z_i y_{i+1} : 1 \le i \le \frac{n-3}{2}, \frac{n+1}{2} + m \le i \le n+m-2\right\} \\
&\cup \left\{y_i z_{i+2} : 1 \le i \le \frac{n-5}{2}, \frac{n+1}{2} + m \le i \le n+m-3\right\} \\
&\cup \left\{x_1 y_{\frac{n-3}{2}}, x_{2m} z_{m+\frac{n+3}{2}}\right\} \cup \{x_i x_{i+1} : 1 \le i \le 2m-1\} \\
&\cup \left\{y_i x_{2i-n} : \frac{n+1}{2} \le i \le \frac{n-1}{2} + m\right\} \cup \left\{y_i x_{2i-n+1} : \frac{n+1}{2} \le i \le \frac{n-1}{2} + m\right\} \\
&\cup \left\{z_i x_{2i-n} : \frac{n+1}{2} \le i \le \frac{n-1}{2} + m\right\} \cup \left\{z_i x_{2i-n+1} : \frac{n+1}{2} \le i \le \frac{n-1}{2} + m\right\}.
\end{aligned}$$

If $p = |V(DB_3)|$ and $q = |E(DB_3)|$ then $p = 2(2m + n - 1)$ and $q = 4(4m + n - 1) - 3$. Now, define a labeling $f : (DB_3) \to \{1, 2, \ldots, 2(2m + n - 1)\}$ as follows:

$$f(x_i) = \begin{cases} 2i + n - 2, & 1 \le i \le 2m - 1, i \equiv 1 \pmod 2; \\ 2i + n - 1, & 2 \le i \le 2m, i \equiv 0 \pmod 2; \end{cases}$$

$$f(y_i) = \begin{cases} 2i, & 1 \le i \le \frac{n+1}{2}; \\ 2i - n - 1, & \frac{n+3}{2} \le i \le \frac{n+1}{2} + m; \\ 2(i + m), & \frac{n+3}{2} + m \le i \le n + m - 3. \end{cases}$$

$$f(z_i) = \begin{cases} 2i - 1, & 1 \le i \le \frac{n-1}{2}; \\ 4i - n, & \frac{n+1}{2} \le i \le \frac{n-1}{2} + m; \\ 2(i + m) - 1, & \frac{n-1}{2} + m \le i \le n + m - 2. \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $3, 4, \ldots, 4(2m + n - 1) - 1$. Therefore by Lemma 1, f extends to a super edge-magic total labeling with magic constant $6(2m+n-1)$. ■

Theorem 6. *For n odd, m a positive integer, the dumbbell type graph DB_4 defined as below admits a super edge-magic total labeling. i.e. $\mu_s(DB_4) = 0$.*

Proof. Let us define the vertex and edge sets of DB_4 as follows:

$$V(DB_4) = \{x_i : 1 \le i \le 2m\} \cup \{y_i : 1 \le i \le n + m - 1\} \cup \{z_i : 1 \le i \le n + m - 1\}$$

and

$$\begin{aligned} E(DB_4) = & \left\{y_i y_{i+1} : 1 \le i \le \frac{n-1}{2} + m, \frac{n+1}{2} + m \le i \le n + m - 2\right\} \\ & \cup \left\{z_i z_{i+1} : 1 \le i \le \frac{n-3}{2}, \frac{n+1}{2} \le i \le n + m - 2\right\} \\ & \cup \left\{x_1 y_{\frac{n-1}{2}}, x_1 z_{\frac{n-1}{2}}, x_{2m} y_{\frac{n+1}{2}+m}, x_{2m} z_{\frac{n+1}{2}+m}\right\} \\ & \cup \left\{z_i y_{i+1} : 1 \le i \le \frac{n-3}{2}, \frac{n+1}{2} + m \le i \le n + m - 2\right\} \\ & \cup \left\{y_i z_{i+2} : 1 \le i \le \frac{n-5}{2}, \frac{n+1}{2} + m \le i \le n + m - 2\right\} \\ & \cup \left\{x_1 y_{\frac{n-3}{2}}, x_{2m} z_{m+\frac{n+3}{2}}\right\} \cup \{x_i x_{i+1} : 2 \le i \le 2m - 2, i \equiv 0 \pmod 2\} \\ & \cup \left\{y_i x_{2i-n} : \frac{n+1}{2} \le i \le \frac{n-1}{2} + m\right\} \cup \left\{y_i x_{2i-n+1} : \frac{n+1}{2} \le i \le \frac{n-1}{2} + m\right\} \\ & \cup \left\{z_i x_{2i-n} : \frac{n+1}{2} \le i \le \frac{n-1}{2} + m\right\} \\ & \cup \left\{z_i x_{2i-n+1} : \frac{n+1}{2} \le i \le \frac{n-1}{2} + m\right\} \\ & \cup \left\{y_i z_i : \frac{n+1}{2} \le i \le \frac{n-1}{2} + m, i = 1, n + m - 1\right\}. \end{aligned}$$

Labeling scheme is same as designed in Theorem 3. ■

3. Concluding remarks

In this paper, we have determined an upper bound for super edge-magic deficiency of volvox graphs. We have also determined the exact value of super magic deficiency of some dumbbell type graphs. We encourage the researchers to try to determine the super edge-magic deficiency of other graphs for further research. In fact, it seems to be a very challenging problem to find the exact value for the super edge-magic deficiency of families of graphs.

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