On the Poisson Transform on Symmetric Spaces of Real Rank One

Alexandru D. Ionescu


In this paper we investigate $L^2$ boundedness properties of the Poisson transform associated to a symmetric space of real rank one and prove a related Plancherel-type theorem.

1. INTRODUCTION

The Poisson transform associated to a symmetric space is the natural analog of the operator

$$P_2 f(x) = \int_{\mathbb{S}^{n-1}} f(u) e^{i x \cdot u} du$$

that maps suitable functions $f$ defined on the sphere $\mathbb{S}^{n-1}$ to functions defined on the Euclidean space $\mathbb{R}^n$. As in the case of Euclidean spaces, the Poisson transform associated to a symmetric space is closely connected with the structure of the eigenspaces of the Laplacian (more generally, with the eigenspaces of certain invariant differential operators on symmetric spaces of high real rank). For example, it was proved by Helgason [2] that all eigenfunctions of the Laplacian on the hyperbolic spaces are obtained as Poisson transforms of certain analytic functionals. A suitable generalization of this theorem to arbitrary symmetric spaces was proved in [4]. A characterization of certain Poisson transforms of $L^q$ functions has been

1 This research was supported by NSF Grant DMS 97-29992. I thank R. Strichartz for explaining the problem to me during a short visit to Cornell University. I also thank E. M. Stein for pointing out the paper [5] and for several useful discussions on the subject.

In [10, Sects. 3, 4] Strichartz addressed the question of $L^2$ boundedness properties of the Poisson transform on Euclidean spaces and on hyperbolic spaces. Lemma 4.2 in [10] states that the Poisson transform associated to a hyperbolic space establishes a bijection between $L^2(S^{n-1})$ and a suitable subspace of an eigenspace of the Laplacian and that it preserves certain $L^2$-type norms. However, as explained in [11, Sect. 2], the proof of this lemma, and hence the proof of Theorem 4.3 following this lemma, is not correct. The purpose of this paper is to prove these conjectures in the more general setting of symmetric spaces of real rank one.

2. NOTATION AND THE MAIN THEOREMS

Most of our notation related to semisimple Lie groups and symmetric spaces is standard and can be found, for example, in [3]. Let $G$ be a noncompact connected semisimple Lie group with finite center, $\mathfrak{g}$ the Lie algebra of $G$, $\theta$ a Cartan involution of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition. Let $\mathfrak{K} = \exp \mathfrak{k}$ be a maximal compact subgroup of $G$ and let $X = G/\mathfrak{K}$ be an associated symmetric space. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, $A = \exp \mathfrak{a}$ the corresponding subgroup of $G$, $\mathcal{M}$ the centralizer of $A$ in $\mathfrak{K}$, $\Sigma$ the set of nonzero roots of the pair $(\mathfrak{g}, \mathfrak{a})$, and $W$ the associated Weyl group. We fix once and for all a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and let $\Sigma^+$ denote the corresponding set of positive roots. For any root $\alpha \in \Sigma$ let $\mathfrak{g}_\alpha$ be the root space associated to $\alpha$, $n = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$, and $N = \exp n$.

The Poisson transform is defined for every $\lambda \in \mathfrak{a}^*_C$ (the complex dual of $\mathfrak{a}$) and every integrable function $f$ on $\mathfrak{K}/\mathfrak{M}$ by the formula

$$P_\mathcal{M} f(z) = \int_{\mathfrak{K}/\mathfrak{M}} e^{(i\lambda, \theta) A(z, b)} f(b) \, db,$$  \hspace{1cm} (2.1)

where $\lambda = \frac{1}{2} \sum_{\alpha \in \Sigma^+} (\dim \mathfrak{g}_\alpha) \alpha$ and $A(z, b) \in \mathfrak{a}$ is a suitable analog on symmetric spaces of the usual scalar product on Euclidean spaces. If $g \in G$ has the decomposition

$$g = k(g) \exp(H(g)) \, n(g)$$  \hspace{1cm} (2.2)

in the Iwasawa decomposition $G = \mathfrak{K} A N$ then one has by definition

$$A(g \mathfrak{K}, k \mathfrak{H}) = -H(g^{-1} k)$$
for any $g \in G$ and $k \in K$. The measure $db$ on $\mathbb{K}/M$ in (2.1) is induced by a
Haar measure on the compact group $K$ (see (2.6) for normalizations of the
various Haar measures).

We will assume from now on that the group $G$ has real rank one i.e.
$\dim \mathfrak{a} = 1$. In this case it is well known that the set $\Sigma$ of nonzero roots is
either of the form $\{-x, x\}$ or of the form $\{-2x, -x, x, 2x\}$. Let $H_0$ be the
unique element of $\mathfrak{a}$ with the property that $\pi(H_0) = 1$ and normalize the
Killing form $B$ on the Lie algebra $\mathfrak{g}$ such that $|H_0| = d(\exp(H_0), K, K) = 1$.

For every locally integrable function $F$ on $X$ let

\[ M(F) = \left( \limsup_{R \to \infty} \frac{1}{R} \int_{B(0, R)} |F(z)|^2 \, dz \right)^{1/2}, \tag{2.3} \]

where $\mathfrak{h} = e\mathfrak{k}$ is the origin of the symmetric space $X$. The “norm” $M$ defined in (2.3) is to be thought of as a substitute of the usual $L^2$ norm on
the space $X$; this substitution is useful since in general eigenfunctions of the
Laplacian are not elements of $L^2(X)$. Finally, for any $\lambda \in \mathfrak{a}^*$ (the real dual
of $\mathfrak{a}$) we define the space

\[ \mathcal{E}^2_* (X) = \{ F \in \mathcal{E}^2_* (X) : \Delta F = -(|\lambda|^2 + |\rho|^2) F \text{ and } M(F) < \infty \}, \]

where $\Delta$ is the Laplacian on $X$, $|\lambda| = |\lambda(H_0)|$ and $|\rho| = \rho(H_0)$.

The main theorem we prove in this paper is the following:

**Theorem 1.** If $\lambda \in \mathfrak{a}^* \setminus \{0\}$ then the Poisson transform $\mathcal{P}_\lambda$ establishes a
bijection between $L^2(\mathbb{K}/M)$ and $\mathcal{E}^2_* (X)$ and for any $f \in L^2(\mathbb{K}/M)$

\[ M(\mathcal{P}_\lambda f) = C_0 |\mathfrak{c}(\lambda)| \cdot ||f||_{L^2(\mathbb{K}/M)}. \tag{2.4} \]

The function $\mathfrak{c}$ in (2.4) is the Harish-Chandra function

\[ \mathfrak{c}(\lambda) = \int_{\mathfrak{g}} e^{-i(\lambda + \rho) H(\bar{\eta})} \, d\bar{\eta}, \tag{2.5} \]

where $\bar{\eta} = \exp(\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha})$ and the meaning of $H(\bar{\eta})$ is the same as in
(2.2). It is well known that the integral above defining the function $\mathfrak{c}$ converges absolutely if $\Re(\lambda(H_0)) > 0$ and extends meromorphically to all
$\lambda \in \mathfrak{a}_+$. In addition, if $m_1 = \dim \mathfrak{g}_+$, $m_2 = \dim \mathfrak{g}_{2+}$ and we normalize the
Haar measures on $K$, $M$, $\bar{\mathfrak{g}}$, and $G$ such that

\[ \int_{\mathbb{K}} 1 = 1, \quad \int_{\mathbb{K}} \eta = 1, \quad \int_{\mathbb{K}} \bar{\eta} = 1, \quad \int_{\mathbb{K}} \bar{\eta} = 1, \tag{2.6} \]
\[
\begin{align*}
\int_{K} 1 \, dk &= 1;
\int_{M} 1 \, dm &= 1;
\int_{N} e^{-2p(H_0)} \, \, d\bar{n} &= 1;
\int_{G} F(g) \, dg &= \frac{1}{2} \int_{K} \int_{K} F(k_1 \exp(tH_0) k_2) (\sinh t)^{m_1}
\times (\sinh 2t)^{m_2} \, dt \, dk_1 \, dk_2 \quad (2.6)
\end{align*}
\]

(the last normalization is related to the Cartan decomposition \( G = \mathfrak{g}(\exp \mathfrak{a}_+) \mathbb{K} \)), then the constant \( C_0 \) in (2.4) is equal to 1.

As an immediate consequence on Theorem 1 we obtain the following conjecture of W. Bray [1, Sect. 4.2]:

**Corollary 2.** If \( \lambda \in \mathfrak{a}^* \setminus \{0\} \) then \( \mathfrak{e}^*_0(M) \) is a Banach space.

Let \( \Phi_\lambda = \partial_\lambda 1 \) be an elementary spherical function on \( \mathfrak{X} \); it is well known that \( A \Phi_\lambda = -(|\lambda|^2 + |\rho|^2) \Phi_\lambda \) for any \( \lambda \in \mathfrak{a}^* \). Thus for any function \( f \in C_c^\infty(\mathfrak{X}) \) the "projection" function

\[
\text{Pr}_\lambda f(z) = f \ast \Phi_\lambda(z) = \int_{\mathfrak{g}} f(g \cdot 0) \Phi_\lambda(g^{-1} \cdot z) \, dg
\]

has the property that

\[
A(\text{Pr}_\lambda f) = -(|\lambda|^2 + |\rho|^2) \text{Pr}_\lambda f. \quad (2.7)
\]

It is known (see, for example, [3, Chap. III]) that if \( f \in C_c^{\infty}(\mathfrak{X}) \) then

\[
\text{Pr}_\lambda f(z) = \int_{\mathfrak{a}^*} \mathfrak{g}(\lambda, b) \mathfrak{a}^* f(\lambda, b) \, db,
\]

where \( \mathfrak{g} \) is the Fourier transform of \( f \). Thus, by Plancherel theorem, the function \( \text{Pr}_\lambda f \) is well defined for any \( f \in L^2(\mathfrak{X}) \) and a.e. \( \lambda \in \mathfrak{a}^* \) and satisfies (2.7). Let \( \mathfrak{a}^*_+ = \{ \lambda \in \mathfrak{a}^* : \lambda(H_0) > 0 \} \). Our last theorem is the analog of Theorem 4.3 in [10].

**Theorem 3.** Let \( F_\lambda(z) \) be a measurable function on \( \mathfrak{a}^*_+ \times \mathfrak{X} \) with the property that \( A F_\lambda = -(|\lambda|^2 + |\rho|^2) F_\lambda \) for a.e. \( \lambda \in \mathfrak{a}^*_+ \). Then there exists \( f \in L^2(\mathfrak{X}) \) such that \( F_\lambda = \text{Pr}_\lambda f \) a.e. if and only if

\[
\int_{\mathfrak{a}^*_+} |M(F_\lambda)|^2 |e(\lambda)|^{-4} \, d\lambda < \infty.
\]
Furthermore, we have
\[ \|f\|_{L^2(\mathbb{X})}^2 = C_1 \int_{\mathbb{R}^*} |M(F_\lambda)|^2 |e(\lambda)|^{-4} d\lambda. \]

As explained in [10, Sect. 4], Theorem 3 follows easily from Theorem 1 and Plancherel theorem. Thus we only need to give a proof of Theorem 1, to which we turn in the next section.

3. PROOF OF THEOREM 1

As explained in [1, Sect. 4.2, 11, Sect. 2], the main difficulty in proving Theorem 1 is obtaining the following uniform estimate:

**Proposition 4.** If \( \lambda \in \mathfrak{a}^* \setminus \{0\} \) and \( f \in L^2(\mathbb{K}/\mathbb{M}) \) then
\[
\left( \sup_{R} \frac{1}{R} \int_{|n_n| \sim R} |\mathcal{P}_z f(z)|^2 \, dz \right)^{1/2} \leq C(\lambda) \|f\|_{L^2(\mathbb{K}/\mathbb{M})}. \quad (3.1)
\]

**Remark.** It was also conjectured in [1, Sect. 4.2; 10, Lemma 4.2] that the constant \( C(\lambda) \) in (3.1) is controlled by \( C \cdot |\lambda| \) for some suitable constant \( C \). Our method does not allow us to obtain this uniform estimate as \( |\lambda| \to \infty \). A rough estimate on the function \( C(\lambda) \) is \( C(\lambda) \leq C(|\lambda| + |\lambda|^{-1}) \), which is good enough for our purposes.

**Proof of Proposition 4.** The group \( \mathbb{G} \) has the Iwasawa decomposition \( \mathbb{G} = N(\exp a) \mathbb{K} \), so the symmetric space \( \mathbb{X} \) can be identified with \( N \times \mathbb{K} \) using the map \( (n, s) \mapsto n(\exp(sH_0)) \cdot 0 \). The corresponding change of measure is \( dz = C \cdot e^{2|s|} d\mu ds \). In addition, it follows from [3, Chap. II, Theorem 6.1] that the ball \( B(0, R) \) is included in the set \( \{ n(\exp(sH_0)) \cdot 0 : n \in N, |s| \leq R \} \). Thus
\[
\frac{1}{R} \int_{|n_n| \sim R} |\mathcal{P}_z f(z)|^2 \, dz \leq C \frac{1}{R} \int_{-R}^{R} \int_{\mathbb{N}} |\mathcal{P}_z f(n(\exp(sH_0)) \cdot 0)|^2 \, d\nu e^{2|s|} \, ds
\]
which shows that it suffices to prove that for any \( s \in \mathbb{R} \)
\[
\left( \int_{\mathbb{N}} |\mathcal{P}_z f(n(\exp(sH_0)) \cdot 0)|^2 \, d\nu \right)^{1/2} \leq C(\lambda) e^{-|s|} \|f\|_{L^2(\mathbb{K}/\mathbb{M})}. \quad (3.2)
\]
We will now transfer the integration over $K/M$ in the definition (2.1) of the Poisson transform to integration over $\hat{N}$. The general formula (see [3, p. 102]) is
\[
\int_{K/M} g(b) \, db = \int_{\hat{N}} g(k(\hat{m}) \| \hat{\delta}) \, e^{-2\pi i H(\hat{m})} \, d\hat{m}
\]
(3.3)
for any integrable function $g$, where the functions $k(\hat{m})$ and $H(\hat{m})$ have the same meaning as in (2.2). Recall also that for any $H \in \mathfrak{a}$ one has $(\exp(H)) \hat{\delta}(\exp(-H)) = \hat{N}$ and $(\exp(H)) \hat{\delta}(\exp(-H)) = \hat{\delta}$. For any $\hat{m} \in \hat{N}$ let $\delta_{\hat{m}} : \hat{N} \to \hat{N}$ be the dilation of $\hat{N}$ given by $\delta_{\hat{m}}(\hat{n}) = (\exp(iH)) \hat{n}$. Then
\[
A(\hat{m}(\exp(sH_0)) \cdot k(\hat{m}) \| \hat{\delta}) = -H((\exp(sH_0)))^{-1}k(\hat{m}) + H(\hat{m}) + sH_0.
\]
If we combine this last equality, the transfer formula (3.3), and the definition (2.1) of the Poisson transform we get
\[
\hat{P}_{sL_2} f(\hat{m}(\exp(sH_0)) \cdot 0, k(\hat{m}) \| \hat{\delta}) = e^{i(s\hat{m})} \hat{H}_s \int_{\hat{N}} e^{-i(\hat{m} \cdot \hat{\delta})(sH_0))} f(k(\hat{m}) \| \hat{\delta}) \, e^{i(sH_0))} \, d\hat{m}.
\]
Recall that the map $\hat{n}_1 \to \hat{n}_2 = \delta_{\hat{m}}(\hat{n}_1)$ is a dilation of $\hat{N}$ with $d\hat{n}_2 = e^{-2\pi i s} d\hat{n}_1$. Thus
\[
\hat{P}_{sL_2} f(\delta_{\hat{m}}(\hat{n})(\exp(sH_0)) \cdot 0) = e^{i(s\hat{m})} \hat{H}_s \int_{\hat{N}} e^{-i(\hat{m} \cdot \hat{\delta})(sH_0))} U_{sL_2} f(\hat{m}) \, d\hat{m},
\]
where $U_{sL_2} f(\hat{m}) = f(k(\delta_{\hat{m}}(\hat{n})) \| \hat{\delta}) \, e^{i(sH_0)}$. Using (3.3) one has $\| f \|_{L_2(K/M)} = \| U_{sL_2} f \|_{L_2(\hat{N})}$, therefore (3.2) and the proposition follow once we prove that convolution with the kernel
\[
K_{\hat{m}}(\hat{n}) = e^{-(\hat{m} \cdot \hat{\delta})(sH_0))}
\]
defines a bounded operator on $L_2(\hat{N})$. 

\[\boxed{\text{518 ALEXANDRU D. IONESCU}}\]
Lemma 5. If \( \lambda \in \mathfrak{a}^* \backslash \{0\} \), \( g \in L^2(\widehat{\mathfrak{g}}) \) and \( K_\lambda \) is defined as above then
\[
|g \ast K_\lambda|_{L^2(\mathfrak{g})} \leq C(\lambda) \|g\|_{L^2(\mathfrak{g})}.
\]

Proof of Lemma 5. Convolution on \( \widehat{\mathfrak{g}} \) is defined by the usual formula
\[
a \ast b(n) = \int_{\widehat{\mathfrak{g}}} a(\tilde{m}) b(n \tilde{m}^{-1}) \, d\tilde{m} = \int_{\widehat{\mathfrak{g}}} a(n \tilde{m}^{-1}) b(\tilde{m}) \, d\tilde{m}
\]
for any suitable functions \( a \) and \( b \). In the simplest case when \( G = \mathrm{SO}(n, 1) \), the symmetric space \( \mathbb{X} \) is the hyperbolic space \( \mathbb{H}^n \) and \( \mathfrak{g} \) can be identified with \( \mathbb{R}^{n-1} \) with the usual group structure. Using this identification it follows from [3, Chap. II, Theorem 6.1] that \( K_\lambda(X) = (1 + |X|^2)^{-(1-\epsilon)/2} \) where \( \epsilon \) is a fixed constant and \( \gamma = \lambda/|H_0|/|H_0| \). This is a kernel of the Calderon–Zygmund type and the \( L^2 \) boundedness of the operator defined by this kernel is related to the oscillation induced by the exponent \( \gamma \in \mathbb{R} \). In the general case, the nilpotent group \( \mathfrak{h} \) can be identified with \( \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \) via a natural map \( \tilde{n} = n(X, Y) \), where \( m_1 = \dim g_{-s} \), \( m_2 = \dim g_{-2s} \), \( X \in \mathbb{R}^{m_1} \), and \( Y \in \mathbb{R}^{m_2} \). The group \( \mathfrak{h} \) is equipped with natural dilations \( \delta_s \) defined in the proof of Proposition 4 which are group automorphisms; one also has \( \delta_s(\tilde{n}(X, Y)) = \tilde{n}(e^{-\epsilon}X, e^{-2\gamma}Y) \). Thus one can define a norm function \( |n| \) on \( \mathfrak{h} \)
\[
|n(X, Y)| = p(X, Y) = |X| + |Y|^{\frac{1}{2}}
\]
with the property that \( |\delta_s(\tilde{n})| = e^{-\epsilon} |n| \) for any \( s \in \mathbb{R} \), \( n \in \mathfrak{h} \). The homogeneous dimension of the group \( \mathfrak{h} \) is \( d = m_1 + 2m_2 = 2 |\rho| \) and the kernel \( K_\lambda \) is given by the formula
\[
K_\lambda(X, Y) = [(1 + c_1 |X|^2)^{2} + c_2 |Y|^2]^{(1-\epsilon)/4},
\]
where \( c_1 \), \( c_2 \) are some suitable constants and \( \gamma = \lambda/|H_0|/|H_0| \) (see [3, Chap. II, Theorem 6.1]). Convolution operators defined by kernels of this type have been studied in [5, Sect. I]. It is more convenient however to obtain our estimate as a consequence of [9, Theorem 4, p. 623]. One clearly has
\[
|K_\lambda(X, Y)| \leq C p(X, Y)^{-d};
\]
\[
\frac{\partial}{\partial x_i} K(X, Y) \leq C(1 + |\lambda|) p(X, Y)^{-d-1}, \quad i = 1 \ldots m_1;
\]
\[
\frac{\partial}{\partial y_j} K(X, Y) \leq C(1 + |\lambda|) p(X, Y)^{-d-2}, \quad j = 1 \ldots m_2.
\]
which are the regularity conditions (61) in [9, p. 622]. Thus it only remains to prove that
\[ \left| \int_{R^m \times R^m} K_\varepsilon(X, Y) \Phi(\varepsilon X, \varepsilon^2 Y) \ dX \ dY \right| \leq C(\lambda) \] (3.4)
for any \( \varepsilon > 0 \) and any normalized bump function \( \Phi \) satisfying the conditions described in [9, p. 622]. Assume that \( m_2 \geq 1 \) (only straightforward modifications are needed in the easier case \( m_2 = 0 \)) and for any \( u, v \geq 0 \) let
\[ \phi(u, v) = \int_{S^{m_1-1} \times S^{m_2-1}} \Phi(u \omega X, v \omega Y) \ d\omega X \ d\omega Y, \]
where the usual measures on the spheres \( S^{m_1-1} \) and \( S^{m_2-1} \) are normalized to have total masses 1. Inequality (3.4) is equivalent to
\[ \left| \int_{R_+ \times R_+} u^{m_1-1} v^{m_2-1} [(1 + c_1 u^2)^2 + c_2 v^2]^{-(1-\eta)/2} \ d^4 \phi(\varepsilon u, \varepsilon^2 v) \ du \ dv \right| \leq C(\lambda), \]
where the function \( \phi: R_+ \times R_+ \) is supported inside the set \( \{(u, v): 0 \leq u, v \leq 1\} \) and satisfies the estimates \( |\phi(u, v)| \leq 1, |\frac{\partial}{\partial u} \phi(u, v)| \leq 1, \) and \( |\frac{\partial}{\partial v} \phi(u, v)| \leq 1 \). The change of variable \( v = c_2^{-1/2}(1 + c_1 u^2) w \) shows that it suffices to prove that
\[ \left| \int_{R_+} u^{m_1-1}(1 + c_1 u^2)^{-m_2/2} (1 + c_1 u^2)^{-\eta/2} \psi(\varepsilon, u) \ du \right| \leq C(\lambda), \] (3.5)
where
\[ \psi(\varepsilon, u) = \int_{R_+} w^{m_2-1}(1 + w^2)^{-(1-\eta)/2} \ d^4 \phi(\varepsilon u, \varepsilon^2 c_2^{-1/2}(1 + c_1 u^2) w) \ dv. \]
Clearly \( \psi(\varepsilon, u) = 0 \) if \( \varepsilon u \geq 1 \); since \( m_1 \geq 1 \) one also has \( |\psi(\varepsilon, u)| \leq C \) and \( |\frac{\partial}{\partial \varepsilon} \psi(\varepsilon, u)| \leq C \varepsilon \). Finally, the inequality (3.5) follows if one makes the change of variable \( u = c_1^{-1/2}(\varepsilon' - 1)^{1/2} \), integrates by parts in \( \varepsilon' \), and uses the estimates on the function \( \psi \) and its derivative. The constant \( C(\lambda) \) is controlled by \( C|\gamma| \).

One can now apply [9, Theorem 4, p. 623] to complete the proof of Lemma 5 and Proposition 4 follows.

We will now show how to apply Proposition 4 to complete the proof of Theorem 1. Our approach follows closely the original idea in [10] (see also [1, Sect. 4]), the only difference being that our setting, that of symmetric spaces of real rank one, is more general. Let \( \mathcal{K} \) denote the set of equivalence classes of unitary irreducible representations of \( \mathbb{K} \) and for each \( \delta \in \mathcal{K} \)
let \( V_\delta \) be a vector space with inner product \( \langle \cdot, \cdot \rangle \) on which a representation of class \( \delta \) is realized. Let \( V_\delta^m = \{ v \in V_\delta; \delta(m) v = v \) for all \( m \in M \} \) and let \( \mathfrak{V}_\delta \) denote the set of elements \( \delta \in \mathfrak{K} \) for which \( \dim V_\delta^m \neq 0 \). In this case \( \dim V_\delta^m = 1 \) since \( X \) has real rank one (see [2, Sect. 4] and the reference given there). For any \( \delta \in \mathfrak{V}_\delta \) let \( d(\delta) = \dim V_\delta \) and fix \( \{ v_j, 1 \leq j \leq d(\delta) \} \) an orthonormal basis of \( V_\delta \) such that \( v_1 \in V_\delta^m \). Let \( Y_{\delta,j}(k M) = d(\delta)^{1/2} \langle v_j, \delta(k) v_1 \rangle \). By Peter–Weyl theorem and the orthogonality relations of Schur, the set of functions \( \{ Y_{\delta,j}; \delta \in \mathfrak{V}_\delta, 1 \leq j \leq d(\delta) \} \) is an orthonormal basis of \( L^2(\mathfrak{K}/M) \). Lemma 4.2 in [2] states that

\[
\mathcal{P}_\delta(Y_{\delta,j})(k(\exp(sH_0)) \cdot 0) = Y_{\delta,j}(k M) \Phi_{\lambda,\delta}(\exp(sH_0) \cdot 0),
\]

where

\[
\Phi_{\lambda,\delta}(\cdot) = \int_{-\infty}^{\infty} e^{i\langle k, \cdot \rangle + \rho A(s, k M)} \langle v_1, \delta(k) v_1 \rangle dk
\]

is a generalized spherical function.

**Proposition 6.** If \( \lambda \in \mathfrak{a}^* \setminus \{0\} \) and \( \delta \in \mathfrak{V}_\delta \) then

\[
M(\mathcal{P}_\delta(Y_{\delta,j})) = |e(\lambda)|.
\]

**Proof of Proposition 6.** By (3.6) and (2.6) it suffices to prove that if \( \lambda \in \mathfrak{a}^* \setminus \{0\} \) and \( \delta \in \mathfrak{V}_\delta \) is a fixed representation then

\[
\lim_{R \to \infty} \int_0^R \big| \Phi_{\lambda,\delta}(\exp(sH_0) \cdot 0) \big|^2 (\sinh s)^m (\sinh 2s)^m ds = 2 |e(\lambda)|^2.
\]

The function \( \phi(s) = \Phi_{\lambda,\delta}(\exp(sH_0) \cdot 0) \) satisfies the differential equation

\[
\phi''(s) + \left[ m_1 (\coth s) + 2m_2 (\coth 2s) \right] \phi'(s)
+ [d_1 (\sinh s)^{-2} + d_2 (\sinh 2s)^{-2} + \lambda (H_0)^2 + \rho (H_0)^2] \phi(s) = 0,
\]

where \( d_1 \) and \( d_2 \) are two numbers that depend on the representation \( \delta \) (see [2, Lemma 4.4]). A standard argument involving expansions in power series of \( e^{-s} \) and identifications of coefficients (see, for example, [3, Chap. III, Theorem 2.7]) shows that if \( s > 0 \) and \( \lambda \in \mathfrak{a}^* \) then

\[
\Phi_{\lambda,\delta}(\exp(sH_0) \cdot 0) = C_1(\lambda, \delta) e^{i(\lambda - \rho) H_0} \sum_{n=0}^{\infty} \Gamma_{\alpha,\delta}(\lambda) e^{-2\pi n}
+ C_{-1}(\lambda, \delta) e^{i(-\lambda - \rho) H_0} \sum_{n=0}^{\infty} \Gamma_{\alpha,\delta}(-\lambda) e^{-2\pi n},
\]

where \( \mathcal{P}_\delta(Y_{\delta,j})(k(\exp(sH_0)) \cdot 0) = Y_{\delta,j}(k M) \Phi_{\lambda,\delta}(\exp(sH_0) \cdot 0), \)

and \( M(\mathcal{P}_\delta(Y_{\delta,j})) = |e(\lambda)|. \)
where \( \Gamma_{\alpha, \delta} = 1 \), and for any \( n \geq 1 \)

\[
\Gamma_{n, \delta}(\lambda) = \sum_{j=0}^{n-1} \Gamma_j(\lambda) 
\times \frac{[m_1/2 + m_2 \epsilon'_j + (\rho - \tilde{j}) H_0 - (n-j)(d_1 + d_2 \epsilon'_j/2)]}{n[n - \tilde{j}(H_0)]},
\]

The numbers \( d_1 \) and \( d_2 \) are the same as in (3.8) and \( \epsilon'_j = 1 \) if \( n \equiv j \pmod{2} \) and \( \epsilon'_j = 0 \) otherwise. A simple inductive argument shows that there exist two numbers \( C(\delta) \) and \( D(\delta) \) such that for any \( n \geq 1 \) and \( \lambda \in \mathfrak{a}^* \) one has

\[
|\Gamma_{n, \delta}(\lambda)| \leq C(\delta) n^{D(\delta)}. \tag{3.10}
\]

The functions \( C_1(\lambda, \delta) \) and \( C_{-1}(\lambda, \delta) \) are computed in [2, p. 335]. One has

\[
\begin{align*}
C_1(\lambda, \delta) &= c(\lambda); \\
C_{-1}(\lambda, \delta) &= c(-\lambda) Q(\lambda),
\end{align*}
\]

where \( c \) is the Harish-Chandra defined in (2.5) and \( Q(\lambda) \) is a certain function of \( \lambda \) involving the \( \Gamma \) function; it is important to notice that for any \( \lambda \in \mathfrak{a}^* \) one has \( |Q(\lambda)| = 1 \). Therefore, it follows from (3.9) and (3.10) that

\[
\Phi_{\lambda, \delta}(\exp(sH_0) \cdot 0) = e^{-|s|} \{ c(\lambda) e^{i\tilde{j} H_0} + c(-\lambda) Q(\lambda) e^{-i\tilde{j} H_0} + E(\lambda, \delta, s) \}, \tag{3.11}
\]

where \( |E(\lambda, \delta, s)| \leq C_{\lambda, \delta} e^{-2|s|} \) if \( s \geq 1 \). The Harish-Chandra function \( c \) is given by the formula (see [8, Sect. 3])

\[
c(\lambda) = \frac{\Gamma(i\tilde{j}(H_0)) \Gamma(i\tilde{j}(H_0) + m_1/2)}{\Gamma(i\tilde{j}(H_0) + m_1/2) \Gamma(i\tilde{j}(H_0) + m_1/2 + m_2/2)}, \tag{3.12}
\]

which shows that \( |c(\lambda)| = |c(-\lambda)| \) if \( \lambda \in \mathfrak{a}^* \). Thus (3.7) follows from (3.11), the estimate on the error term \( |E(\lambda, \delta, s)| \) following (3.11), and the simple observation that \( |\Phi_{\lambda, \delta}(\exp(sH_0) \cdot 0)| \leq C \) for \( s \in [0, 1] \).

The main formula (2.4) now follows from Proposition 4, Proposition 6, and (3.6) by an easy limiting argument. It is clear from (3.12) that \( c(\lambda) \neq 0 \) if \( \lambda \in \mathfrak{a}^* \), therefore the Poisson transform is injective on \( L^2(\mathfrak{g}/\mathfrak{h}) \) for any \( \lambda \in \mathfrak{a}^* \setminus \{0\} \) (of course, this also follows from the well known fact that if \( \lambda \in \mathfrak{a}^* \) then \( \lambda \) is simple [3, Chap. II, Lemma 6.6]). In order to prove that
our operator is onto we start with a function $F \in \mathcal{E}(\mathbb{K})$ and expand it as in [2, Proposition 5.3],

$$F(z) = \sum_{\delta \in \mathcal{K}_M} \left( \sum_{j=1}^{d_\delta} a_{\delta, j} Z_{\lambda_\delta, j}(z) \right),$$

where

$$Z_{\lambda_\delta, j} = \mathcal{P}_\lambda(Y_{\delta, j}).$$

It follows from (3.7) and [2, Proposition 5.3] that $\sum_{\delta, j} |a_{\delta, j}|^2 \leq [ |M(F)|^2 / |\mathcal{E}(\mathbb{L})| ]^2$ thus one can define a function $f = \sum_{\delta, j} a_{\delta, j} Y_{\delta, j} \in L^2(\mathbb{K}/\mathbb{M})$ and the representation $F = \mathcal{P}_\lambda f$ is immediate.

REFERENCES