Convergence rate of some semi-groups to their invariant probability

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Abstract

Let us consider the following stochastic differential equation:

$$X_t = x + B_t - \frac{1}{2}\int_0^t b(X_s)\,\mathrm{d}s, \tag{E}$$

where $(B_t)_{t\geqslant 0}$ is a d-dimensional brownian motion starting at 0 and b a function from $\mathbb{R}^d$ to $\mathbb{R}^d$ which is a gradient field. We aim at studying the convergence rate of the semi-group associated to (E) to its invariant probability.

0. Introduction

0.1. Assumptions

Throughout this paper, it will be assumed that

(H1) (i) b is a gradient field on $\mathbb{R}^d$, that is: there is a function $V:\mathbb{R}^d \to \mathbb{R}$ such that $b = \nabla V$

(ii) Moreover, we suppose that $\int_{\mathbb{R}^d} \mathrm{e}^{-V(x)}\,\mathrm{d}x < +\infty$.

Without loss of generality, we can set $\int_{\mathbb{R}^d} \mathrm{e}^{-V(x)}\,\mathrm{d}x = 1$.

We denote by μ the probability measure on $\mathbb{R}^d$ with density $\mathrm{e}^{-V(x)}$ and by L^p, the L^p space associated to the measure μ. For any measurable function f, $\|f\|_p$ denotes the L^p norm of f:

$$\|f\|_p = \left[\int_{\mathbb{R}^d} |f(x)|^p \mu(\mathrm{d}x)\right]^{1/p}.$$

$\langle\, ,\rangle$ is the inner product associated to the measure μ, defined by

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)g(x)\mu(\mathrm{d}x).$$

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(H2) b is locally lipschitz-continuous, that is

For any compact set K, $\exists l_K > 0$, $\forall x$, $\forall y \in K$, $|b(x) - b(y)| \leqslant l_K |x - y|$.

(H3) For any $x \in \mathbb{R}^d$, $b(x)$ is an outward vector, i.e, $\forall x \in \mathbb{R}^d$, $b(x).x \geqslant 0$.

Under (H2) and (H3), it is known that (E) admits a unique strong solution (see Karatzas and Shreve (1991), Theorem 2.5, p. 287).

If $(P_t)_{t \geqslant 0}$ is the semi-group associated to (E) and L its generator, then for any bounded measurable function f

$$P_t f(x) = E_x f(X_t) \tag{0.1}$$

and for any function f of class C^2 with compact support ($f \in C_c^2(\mathbb{R}^d)$)

$$Lf = \tfrac{1}{2}(\Delta f - b.\nabla f). \tag{0.2}$$

Moreover, it is well known that

(i) μ is the unique invariant probability measure for $(P_t)_{t \geqslant 0}$;

(ii) for any p, $1 \leqslant p \leqslant +\infty$, any $t \geqslant 0$, P_t is bounded in L^p and for any measurable function f

$$\|P_t f\|_p \leqslant \|f\|_p; \tag{0.3}$$

(iii) L is semi-bounded symmetric operator: for any f, $g \in C_c^2(\mathbb{R}^d)$

$$\langle Lf, g\rangle = \langle f, Lg\rangle = -\frac{1}{2} \int_{\mathbb{R}^d} \nabla f(x).\nabla g(x) \mu(\mathrm{d}x). \tag{0.4}$$

Therefore, Friedrichs' extension theorem shows that L admits a self-adjoint extension;

(iv) 0 is an eigenvalue of L and the associated eigenspace is the set of μ a.s. constant functions.

The spectral decomposition of self-adjoint operators leads us to the following result (see Bakry and Emery, 1995, pp. 177–206; Bakry, 1994):

$$\lim_{t \to \infty} \|P_t f - \mu(f)\|_2 = 0. \tag{0.5}$$

0.2. The objectives

The purpose of this paper consists in an estimation of the rate of the convergence of P_t to μ. We consider the uniform convergence on compact sets of $\mathbb{R}^d$, or the uniform convergence on $\mathbb{R}^d$ in the case (C1) below.

Our results depend on the behaviour of the vector field b at infinity. Three different cases are considered:

(C1) $|b(x)| \underset{|x|\to\infty}{\sim} C|x|^{\alpha}, \quad \alpha > 1.$

(C2) $|b(x)| \underset{|x|\to\infty}{\sim} C|x|^{\alpha}, \quad -1 < \alpha \leqslant 1.$

(C3) $|b(x)| \underset{|x|\to\infty}{\sim} \dfrac{C}{x}, \quad C > 1,\ d = 1.$

The corresponding results are

(R1) $\exists \sigma > 0, \exists C > 0, \exists t_0 > 0$ such that for any $t \geqslant t_0$ and for any $f \in L^2$

$$\sup_{x\in\mathbb{R}^d} |P_t f(x) - \mu(f)| \leqslant C e^{-\sigma t} \|f\|_2.$$

(R2) (i) If $|\alpha| < 1$, $\exists \sigma > 0$, $\exists t_0 > 0$ such that for any compact set K, $\exists C(K) > 0$ such that for any $f \in L_\infty$ and $t \geqslant t_0$

$$\sup_{x\in K} |P_t f(x) - \mu(f)| \leqslant C(K) e^{-\sigma t^{\rho}} \|f\|_\infty \quad \text{with } \rho = (\alpha + 1)/2.$$

(ii) If $\alpha = 1$, $\forall \rho < 1$, $\exists \sigma > 0$, $\exists t_0 > 0$ such that for any compact set K, $\exists C(K) > 0$ such that for any $f \in L_\infty$ and $t \geqslant t_0$

$$\sup_{x\in K} |P_t f(x) - \mu(f)| \leqslant C(K) e^{-\sigma t^{\rho}} \|f\|_\infty.$$

(R3) $\exists \gamma > 0$, $\exists t_0 > 0$ such that for any compact set K, $\exists C(K) > 0$ such that for any $f \in L_\infty$ and $t \geqslant t_0$

$$\sup_{x\in K} |P_t f(x) - \mu(f)| \leqslant C(K) \frac{1}{t^{\gamma}} \|f\|_\infty.$$

0.3. Outline

The sequel is organised as follows:
Section 1 deals with the case (R1), $\alpha > 1$. This is a very favorable situation since the semi-group $(P_t)_{t\geqslant 0}$ benefits of the very strong property of ultracontractivity. Exploiting results of Kavian et al. (1993), the proof of (R1) is almost straightforward.

In Section 2, we describe a general approach for $-1 \leqslant \alpha \leqslant 1$. The ultracontractivity property being not true, this situation is much more delicate and a direct evaluation of $P_t f - \mu(f)$ seems out of reach. To avoid this difficulty, the diffusion $(X_t)_{t\geqslant 0}$ is approximated by the diffusion $(X_t^a)_{t\geqslant 0}$ associated to the same diffusion equation (E) but reflected on the ball centered at zero and of radius a. On one hand, it may be proved that the semi-group $(P_t^a)_{t\geqslant 0}$ (and respectively the corresponding stationary distribution μ^a) are close to $(P_t)_{t\geqslant 0}$ (respectively μ) for a large enough. On the other hand, $(X_t^a)_{t\geqslant 0}$ being restricted to a bounded state, it is easy to evaluate the convergence rate of P_t^a to μ^a. Thus, this approach provides a way of estimating $P_t f - \mu(f)$.

Section 3, can be viewed as an application of the methodology of Section 2 and corresponding to the specific case $-1 < \alpha \leqslant 1$.

Again, Section 4 relies upon Section 2 and corresponding to the case $\alpha = -1$.

1. The ultracontractive case: $\alpha > 1$

In this section, assumption H3 is strengthened into H3′ (H1 and H2 remaining unchanged):

$$\text{H3}' \quad \begin{cases} |b(x)| \underset{|x|\to\infty}{\sim} C|x|^{\alpha}, \\ b(x) \text{ is a radial function } \exists k\colon \mathbb{R}_+ \to \mathbb{R} \text{ such that } b(x) = k(|x|)\frac{x}{|x|}. \end{cases}$$

Under H3′, the semi-group $(P_t)_{t\geqslant 0}$ is ultracontractive (see Kavian et al., 1993 for further information). Namely,

Definition 1.1. The semi-group $(P_t)_{t\geqslant 0}$ defined on L^1 is said ultracontractive if

$$\forall t>0,\ \exists C_t>0,\ \forall f\in L^1, \quad \|P_t f\|_{\infty} \leqslant C_t\|f\|_1. \tag{1.6}$$

The ultracontractivity property leads to the main theorem of this section.

Theorem 1.1. $\exists\sigma>0,\ \exists C_1>0,\ \exists t_0>0$ *such that*

$$\forall t\geqslant t_0,\ \forall f\in L^2,\ \|P_t f - \mu(f)\|_{\infty} \leqslant C_1 e^{-\sigma t}\|f\|_1.$$

Remark 1.1. Obviously, the same relation holds with $\|f\|_2$ in place of $\|f\|_1$.

Proof of Theorem 1.1. First, we need two technical lemmas.

Lemma 1.2. *Under* H3′, P_t *is a Hilbert–Schmidt operator.*

Proof. For any function f, it can be shown (see Durrett, 1984) that $P_t f$ can be expressed through:

$$P_t f(x) = \int_{\mathbb{R}^d} p_t(x,y) f(y)\mu(\mathrm{d}y), \tag{1.7}$$

where p_t is a continuous function of x and y.

It is enough to check that

$$\forall t>0, \quad \sup_{x,y} |p_t(x,y)| \leqslant C_t. \tag{1.8}$$

For any $x_0\in\mathbb{R}$, let us consider a sequence (f_n), $f_n\in L^1$ where

(i) $\forall x\in\mathbb{R}^d$, $f_n(x)\geqslant 0$,

(ii) $\int_{\mathbb{R}^d} f_n(x)\mu(\mathrm{d}x) = 1$ ($\|f_n\|_1 = 1$),

(iii) $\forall g$ bounded and continuous, $\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x) g(x)\mu(\mathrm{d}x) = g(x_0)$.

Applying the ultracontractivity property to f_n, it turns out:

$$\|P_t f_n\|_{\infty} \leqslant C_t\|f_n\|_1 = C_t \tag{1.9}$$

(iii) gives

$$\lim_{n\to\infty} P_t f_n(x) = p_t(x,x_0). \tag{1.10}$$

Eqs. (1.9) and (1.10) yield inequality (1.8). The proof is completed. □

Lemma 1.3. *There exists an orthonormal basis* $(h_n)_{n\geqslant 0}$ *of* L^2 *such that*

$$(\mathrm{S})\quad \begin{cases} \text{(i) } \forall n\geqslant 0,\ Lh_n = -\lambda_n h_n, \\ \text{(ii) } \forall n\geqslant 0,\ P_t h_n = \mathrm{e}^{-\lambda_n t} h_n, \\ \text{(iii) } \lambda_0 = 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \\ \text{(iv) } h_0 = 1. \end{cases}$$

$\sigma = \lambda_1$ *is called the spectral gap of* L.

Proof. Lemma 1.2 shows that P_t is a compact operator in L^2. Hence, the resolvant of L is also compact entailing the above-mentioned properties. (see Davies, 1989). □

Proof of Theorem 1.1. In two steps: (i) Let us consider the eigenvalues λ_n and eigenvectors h_n of Lemma 1.3.

P_t being a Hilbert–Schmidt operator, it turns out that

$$\forall t>0,\quad \sum_{n=0}^{+\infty} \|P_t h_n\|_2^2 < +\infty,$$

$$\forall t>0,\quad \sum_{n=0}^{+\infty} \mathrm{e}^{-\lambda_n t} = \sum_{n=0}^{+\infty} \|P_{t/2} h_n\|_2^2 < +\infty. \tag{1.11}$$

It is worth noticing that the ultracontractivity property (1.6) and Lemma 1.3 show that the h_n are in L^∞; in fact

$$\forall t>0,\quad \|P_t h_n\|_\infty = \|\mathrm{e}^{-\lambda_n t} h_n\|_\infty \leqslant C_t \tag{1.12}$$

that is,

$$\forall t>0,\quad \|h_n\|_\infty \leqslant \mathrm{e}^{\lambda_n t} C_t. \tag{1.13}$$

(ii) Let f be a function of L^2. Again by Lemma 1.3 we see that

$$\mu(f) = \langle f, 1\rangle = \langle f, h_0\rangle = \langle f, h_0\rangle\, h_0.$$

The decomposition of $P_t f - \mu(f)$ on the orthonormal basis $(h_n)_{n\geqslant 0}$ yields

$$\begin{aligned} P_t f - \mu(f) &= \sum_{n\geqslant 0} \langle f, h_n\rangle\, \mathrm{e}^{-\lambda_n t} h_n - \langle f, h_0\rangle\, h_0 \\ &= \sum_{n\geqslant 1} \langle f, h_n\rangle\, \mathrm{e}^{-\lambda_n t} h_n, \end{aligned} \tag{1.14}$$

hence

$$\|P_t f - \mu(f)\|_\infty \leqslant \sum_{n\geqslant 1} |\langle f, h_n\rangle|\, \mathrm{e}^{-\lambda_n t} \|h_n\|_\infty. \tag{1.15}$$

For $t_0>0$ and $t>3t_0$, inequality (1.12) applied to t_0 leads to

$$\|P_t f - \mu(f)\|_\infty \leqslant C_{t_0}^2 \|f\|_1 \sum_{n\geqslant 1} \mathrm{e}^{-\lambda_n (t-2t_0)}.$$

Since $\lambda_n \geqslant \sigma$ for $n \geqslant 1$ (Lemma 1.3) and $t > 3t_0$, it turns out that

$$\|P_t f - \mu(f)\|_\infty \leqslant C_{t_0}^2 e^{3\sigma t_0} \|f\|_1 e^{-\sigma t} \sum_{n \geqslant 1} e^{-\lambda_n t_0}.$$

Equation (1.11) indicates that the sum $\sum_{n \geqslant 1} e^{-\lambda_n t_0}$ is finite. Thus

$$\|P_t f - \mu(f)\|_\infty \leqslant C_1 e^{-\sigma t} \|f\|_1$$

which achieves the proof. □

2. The case $-1 \leqslant \alpha \leqslant 1$: general situation

This case is more delicate than the former since we do not know whether the semi-group is compact or not. To avoid this difficulty, we introduce another semi-group, $(P_t^a)_{t \geqslant 0}$ wich is compact and approaches P_t for large values of a. It will be seen later that for any f in L^2

$$\forall t > 0, \quad \lim_{a \to \infty} \|P_t f - P_t^a f\|_2 = 0. \tag{2.1}$$

Then, an estimation of $P_t^a f - \mu(f)$ will be given for large values of a and t. Let us begin with the construction and the main properties of $(P_t^a)_{t \geqslant 0}$.

2.1. The semi-group $(P_t^a)_{t \geqslant 0}$

Let us consider the diffusion reflected on Ω^a, the ball centered at zero with radius a. This diffusion is associated to the equation ($\mathrm{E^a}$) defined below:

$$\begin{aligned} &X_t^a = x + B_t - \frac{1}{2} \int_0^t b(X_s^a)\, ds - \int_0^t n(X_s^a)\, dl_s^a, \\ &\text{where } n(x) = \frac{x}{\|x\|} \text{ is the outward normal,} \\ &\forall t \geqslant 0, \quad X_t \in \Omega^a, \\ &l_t^a = \int_0^t 1_{X_s^a \in \partial\Omega^a}\, dl_s. \end{aligned} \tag{$\mathrm{E^a}$}$$

Let $(P_t^a)_{t \geqslant 0}$ be the semi-group associated to X_t^a and L^a its generator, i.e

$$P_t^a f(x) = E_x f(X_t^a). \tag{2.2}$$

L^a is defined on the domain

$$\mathscr{D}(L^a) = \{f \in C^2, \ \frac{\partial f}{\partial n} = 0 \text{ on } \partial\Omega^a\}, \tag{2.3}$$

$$\forall f \in \mathscr{D}(L^a), \quad L^a f = \tfrac{1}{2}(\Delta f - b.\nabla f). \tag{2.4}$$

Let μ^a be the restriction of μ to Ω^a, that is, for any μ-measurable set $A \subset \Omega^a$

$$\mu^a(A) = \frac{\mu(A)}{\mu(\Omega^a)}. \tag{2.5}$$

Let $L_a^p = L^p(\mu^a)$, $\langle , \rangle_a$ the inner product on L_a^2, and $\|.\|_{p,a}$ the norm on L_a^p:

$$\forall f \in L_a^p, \quad \|f\|_{p,a}^p = \int_{\mathbb{R}^d} |f(x)|^p \mu^a(\mathrm{d}x).$$

Then, it is known that

(i) L^a is a semi-bounded symmetric operator on $\mathscr{D}(L^a)$ and

$$\langle f, L^a g\rangle_a = \langle L^a f, g\rangle_a = -\frac{1}{2}\int_{\Omega^a} \nabla f.\nabla g \ \mu^a(\mathrm{d}x). \tag{2.6}$$

(ii) $(P_t^a)_{t\geqslant 0}$ is a compact semi-group.

(iii) L^a admits an orthonormal basis of eigenvectors, $(h_n^a)_{n\geqslant 0}$ on L_a^2 such that

- (i) $\forall n \geqslant 0,\ L^a h_n^a = -\lambda_n^a h_n^a$,
- (ii) $\forall n \geqslant 0,\ P_t^a h_n^a = \mathrm{e}^{-\lambda_n^a t} h_n^a$,
- (iii) $\lambda_0^a = 0 < \lambda_1^a \leqslant \lambda_2^a \leqslant \cdots$,
- (iv) $h_0^a = 1$.

And the spectral gap is given by $\sigma^a = \lambda_1^a$. These properties being recalled, we carry on with a crucial property of μ^a:

Lemma 2.1. *μ^a defined by* Eq. (2.5), *is the unique invariant probability measure of* $(P_t^a)_{t\geqslant 0}$.

Proof. Let ν be an invariant measure for P_t^a. ν satisfies

$$\forall f \in \mathscr{D}(L^a), \quad \nu(P_t^a f) = \nu(f). \tag{2.7}$$

Evaluating the t derivative on $t = 0$, we have

$$\nu(L^a f) = 0.$$

which can also be written as

$$\int_{\Omega^a} (\Delta f - b.\nabla f)(x)\nu(\mathrm{d}x) = 0. \tag{2.8}$$

By Eq. (2.7) and the regularising property of the semi-group P_t^a, ν is a measure with a regular density. So we can use Green–Stockes formula, and Eq. (2.8) becomes

$$\int_{\Omega^a} f(\Delta\nu + \mathrm{div}(b\nu))\,\mathrm{d}x + \int_{\partial\Omega^a} \left[\nu\frac{\partial f}{\partial n} - f\left(\frac{\partial \nu}{\partial n} + \nu(b.n)\right)\right]\mathrm{d}\sigma = 0. \tag{2.9}$$

Since $f \in \mathscr{D}(L^a)$, $\partial f/\partial n = 0$ on $\partial\Omega^a$, Eq. (2.9) turns into:

$$\forall f \in \mathscr{D}(L^a), \quad \int_{\Omega^a} f(\Delta\nu - \mathrm{div}(b\nu))\,\mathrm{d}x - \int_{\partial\Omega^a} f\left[\frac{\partial \nu}{\partial n} + \nu(b.n)\right]\mathrm{d}\sigma = 0. \tag{2.10}$$

Hence, v is a solution of

$$\operatorname{div}(\nabla v + bv) = 0 \quad \text{on } \Omega^a,$$
$$\frac{\partial v}{\partial n} + v(b.n) = 0 \quad \text{on } \partial\Omega^a.$$

This system admits a unique solution of mass 1. It is now enough to check that μ^a is solution of this system. This last point being left to the reader. □

From now on, it is assumed that $d = 1$ *and* $\Omega^a = [-a, a]$.

As in previous section, the spectral gap σ^a plays a major role in the determination of the rate of convergence. Thus, it is of prime interest to derive a lower bound of it.

Proposition 2.2. *Setting* $\Phi(a) := \int_{-a}^{a} e^{-V(x)}\,dx \int_{-|x|}^{|x|} e^{V(y)}\,dy$, *then*

$$\sigma^a \geqslant \frac{1}{2\Phi(a)}.$$

Proof. The proof of this proposition is based on the following lemma.

Lemma 2.3. *Let* f *be an eigenfunction of* L^a *associated to the eigenvalue* $-\lambda$, $\lambda \neq 0$. *Setting*, $g := f - f(0)$, *then*

(i) $\langle g, L^a g\rangle_a = \langle f, L^a f\rangle_a$,

(ii) $\|g\|_{2,a}^2 = \|f\|_{2,a}^2 + f^2(0)$,

(iii) $\|g\|_{2,a}^2 \leqslant \Phi(a)\|g'\|_{2,a}^2$.

Proof. (i) Remembering that constant functions are eigenfunctions of L^a associated to the eigenvalue 0 (property (iii) of reflected diffusion), we can write $L^a\,[f(0)] = 0$ and an elementary calculation gives (i).

(ii) $\|g\|_{2,a} = \langle f - f(0), f - f(0)\rangle_a = \|f\|_{2,a}^2 + f(0)^2 - 2\langle f, f(0)\rangle_a$. By orthogonality of f and $f(0)$, (ii) is proved.

(iii) Notice that

$$\|g\|_{2,a}^2 = \frac{1}{\mu(\Omega^a)} \int_{-a}^{a} g^2(x) e^{-V(x)}\,dx \leqslant \frac{1}{\mu(\Omega^a)} \int_{-a}^{a} e^{-V(x)} \left(\int_{-|x|}^{|x|} |g'(t)|\,dt \right)^2 dx.$$

Schwarz inequality shows that

$$\begin{aligned}
\left(\int_{-|x|}^{|x|} |g'(t)|\,dt \right)^2 &= \left(\int_{-|x|}^{|x|} |g'(t)| e^{V(t)/2} e^{-V(t)/2}\,dt \right)^2 \\
&\leqslant \left(\int_{-|x|}^{|x|} (g'(t))^2 e^{-V(t)}\,dt \right) \left(\int_{-|x|}^{|x|} e^{V(t)}\,dt \right) \\
&\leqslant \left(\int_{-a}^{a} (g'(t))^2 e^{-V(t)}\,dt \right) \left(\int_{-|x|}^{|x|} e^{V(t)}\,dt \right)
\end{aligned}$$

hence,

$$\|g\|_{2,a}^2 \leqslant \frac{1}{\mu(\Omega^a)} \int_{-a}^{a} (g'(t))^2 \mathrm{e}^{-V(t)}\,\mathrm{d}t \int_{-a}^{a} \mathrm{e}^{-V(x)}\,\mathrm{d}x \int_{-|x|}^{|x|} \mathrm{e}^{V(t)}\,\mathrm{d}t$$
$$= \|g'\|_{2,a}^2 \Phi(a). \qquad \square$$

We now turn on Proposition 2.2.

Proof of Proposition 2.2. Let f be an eigenfunction of L^a associated to the nonzero eigenvalue, $-\lambda, (\lambda > 0)$ and let $g = f - f(0)$. Obviously,

$$|\langle f, L^a f\rangle_a| = \lambda \|f\|_{2,a}^2. \tag{2.11}$$

A glance at Eq. (2.6) shows that

$$|\langle f, L^a f\rangle_a| = \tfrac{1}{2}\|f'\|_{2,a}^2 = \tfrac{1}{2}\|g'\|_{2,a}^2.$$

From properties (iii) and (ii) of Lemma 2.3, we obtain

$$\frac{1}{2}\|g'\|_{2,a}^2 \geqslant \frac{1}{2\Phi(a)}\|g\|_{2,a}^2 = \frac{1}{2\Phi(a)}(\|f\|_{2,a}^2 + f^2(0)) \geqslant \frac{1}{2\Phi(a)}\|f\|_{2,a}^2. \tag{2.12}$$

From inequalities (2.11) and (2.12) it is seen that

$$\lambda\|f\|_{2,a}^2 \geqslant \frac{1}{2\Phi(a)}\|f\|_{2,a}^2,$$

thus

$$\lambda \geqslant \frac{1}{2\Phi(a)}. \qquad \square$$

Now, the tools are in hand to present the general method of estimating $P_t(f) - \mu(f)$.

2.2. *Estimation of $P_t f - \mu(f)$*

Let f be a function in L^2. For any x real, it is clear that

$$|P_t f(x) - \mu(f)| \leqslant |P_t f(x) - P_t^a f(x)| + |P_t^a f(x) - \mu^a(f)| + |\mu^a(f) - \mu(f)| \tag{2.13}$$

and we have to deal with three terms to evaluate $P_t f - \mu(f)$.

Theorem 2.4. (i) *For any function f in L^2 and $t > 0$*

$$\|P_t^a(f) - \mu^a(f)\|_{2,a} \leqslant \mathrm{e}^{-\sigma^a t}\|f\|_{2,a}. \tag{2.14}$$

(ii) *For any compact set K, $\exists C_1(K) > 0$, $\exists a_0(K) > 0$ such that, for any f in L^2 and any $t \geqslant 1$*

$$\forall a \geqslant a_0(K), \quad \sup_{x \in K} |P_t^a f(x) - \mu^a(f)| \leqslant C(K)\mathrm{e}^{-\sigma^a t/2}\|f\|_{2,a}. \tag{2.15}$$

(iii) *For any* f *in* L^∞

$$|\mu(f) - \mu^a(f)| \leqslant 2(1-\mu(\Omega^a))\|f\|_\infty. \tag{2.16}$$

(iv) $\forall x \in \mathbb{R},\ \forall f \in L^\infty$

$$|P_t f(x) - P_t^a f(x)| \leqslant 2\|f\|_\infty P_x(T_a \leqslant t), \tag{2.17}$$

where T_a *is the first passage time on level* a *of the process* $(|X_t|)_{t\geqslant 0}$, *namely*

$$T_a = \inf\{t \geqslant 0,\ |X_t| = a\}.$$

Proof. The proofs of items (iii) and (iv) are quite easy and left to the reader. In order to prove (i) and (ii), we need the following lemma.

Lemma 2.5. *Let* h *be a differentiable function defined on* $\mathbb{R}$. $\forall K$ *compact*, $\exists C(K) > 0$, $\exists a_0(K) > 0$ *such that* $\forall a \geqslant a_0(K)$

$$\sup_{x\in K} |h(x)|^2 \leqslant C(K)(\|h\|_{2,a}^2 + \|h'\|_{2,a}^2). \tag{2.18}$$

Proof. Let us define $\delta_0(K) = \sup\{|x|,\ x \in K\}$, δ_1 such that $\mu([-\delta_1, \delta_1]) = \frac{1}{2}$, $\delta = \delta(K) = \max(\delta_1, \delta_0(K))$, and $C_1(K) = \int_{-\delta}^{\delta} e^{V(t)}\,dt$.

For any x and y in $[-\delta, \delta]$, $h(x) = h(y) + \int_x^y h'(t)\,dt$.

Using $(a+b)^2 \leqslant 2a^2 + 2b^2$, it turns out that

$$(h(x))^2 \leqslant 2(h(y))^2 + 2\left(\int_x^y h'(t)\,dt\right)^2.$$

An appeal to Schwarz inequality gives

$$\begin{aligned}\left(\int_x^y h'(t)\,dt\right)^2 &= \left(\int_x^y h'(t)e^{-V(t)/2}e^{V(t)/2}\,dt\right)^2\\ &\leqslant \left|\int_x^y (h'(t))^2 e^{-V(t)}\,dt\right| \left|\int_x^y e^{V(t)}\,dt\right|\\ &\leqslant \mu([-\delta,\delta])\|h'\|_{2,\delta}^2 \left|\int_x^y e^{V(t)}\,dt\right|\\ &\leqslant C_1(K)\mu([-\delta,\delta])\|h'\|_{2,\delta}^2\end{aligned}$$

hence,

$$(h(x))^2 \leqslant 2(h(y))^2 + 2C_1(K)\mu([-\delta,\delta])\|h'\|_{2,\delta}^2.$$

Integrating this last inequality with the measure $\mu^\delta(dy)$, it comes:

$$\sup_{x\in K}(h(x))^2 \leqslant 2\|h\|_{2,\delta}^2 + 2C_1(K)\mu([-\delta,\delta])\|h'\|_{2,\delta}^2.$$

Noticing that for any $a \geqslant \delta$

$$\mu([-\delta,\delta])\|h\|_{2,\delta}^2 = \int_{-\delta}^{\delta} h^2(t)\mathrm{e}^{-V(t)}\,\mathrm{d}t \leqslant \int_{-a}^{a} h^2(t)\mathrm{e}^{-V(t)}\,\mathrm{d}t \leqslant \|h\|_{2,a}^2,$$

$$\sup_{x\in K}(h(x))^2 \leqslant \frac{2}{\mu([-\delta,\delta])}\|h\|_{2,a}^2 + 2C_1(K)\|h'\|_{2,a}^2.$$

Setting, $C(K) = \max(4, 2C_1(K))$, the proof is achieved. □

Let us come back to the *proof of Theorem* 2.4.

(i) Let $f \in L^2$. Setting

$$h = P_t^a f - \mu^a(f) = \sum_{n\geqslant 1}\langle f, h_n^a\rangle_a \mathrm{e}^{-\lambda_n^a t} h_n^a. \tag{2.19}$$

Using the spectral gap property, inequality (i) of Theorem 2.4 is straightforward:

$$\|P_t^a f - \mu^a(f)\|_{2,a}^2 = \|h\|_{2,a}^2 \leqslant \mathrm{e}^{-2\sigma^a t}\|f\|_{2,a}^2. \tag{2.20}$$

(ii) Eq. (2.6) shows that

$$\forall n,m, \quad \langle (h_n^a)', (h_m^a)'\rangle_a = -2\langle h_n^a, L^a h_m^a\rangle_a = 2\delta_{n,m}\lambda_n^a,$$

where $\delta_{n,m}$ denotes the Kronecker symbol.

Since $x \leqslant e^x$, $\forall x \geqslant 0$, for $n \geqslant 1$ the spectral gap property yields

$$2\lambda_n^a \mathrm{e}^{-2\lambda_n^a t} \leqslant 2\frac{\mathrm{e}^{-\lambda_n^a t}}{t} \leqslant 2\frac{\mathrm{e}^{-\sigma^a t}}{t}.$$

From Eq. (2.19) it finally obtains

$$\|h'\|_{2,a}^2 = \sum_{n\geqslant 1} 2\lambda_n^a |\langle f, h_n^a\rangle_a|^2 \mathrm{e}^{-2\lambda_n^a t} \leqslant 2\frac{\mathrm{e}^{-\sigma^a t}}{t}\|f\|_{2,a}^2. \tag{2.21}$$

Bearing in mind Lemma 2.5, an appeal to inequalities (2.20), (2.21) gives for $t > 0$, $x \in K$ and $a \geqslant a_0(K)$:

$$|P_t^a f(x) - \mu^a(f)|^2 \leqslant C(K)\mathrm{e}^{-\sigma^a t}\left(\mathrm{e}^{-\sigma^a t} + \frac{2}{t}\right)\|f\|_{2,a}^2.$$

Therefore, for any $t \geqslant 1$, it turns out that for $a \geqslant a_0(K)$

$$\sup_{x\in K}|P_t^a f(x) - \mu^a(f)| \leqslant [3C(K)]^{1/2}\,\mathrm{e}^{-\sigma^a t/2}\|f\|_{2,a}$$

which completes the proof of (ii). □

3. The case $-1 < \alpha \leqslant 1$

Again in this section, we confine ourselves to $d = 1$.

For the sake of simplicity we assume that b is odd. The general case is more intricate but can be handled in the same manner.

The assumptions on the behaviour of b are

$$(\mathrm{C}_2') \quad \begin{cases} b \text{ is an odd function: } \forall x \in \mathbb{R},\ b(-x) = -b(x), \\ b(x) = Cx^\alpha(1+\varepsilon(x)) \text{ where } \lim_{x\to\infty} \varepsilon(x) = 0 \text{ for } x \geqslant 0. \end{cases}$$

The main theorem of this section is the following.

Theorem 3.1. (i) *If* $|\alpha| < 1$, $\exists \sigma > 0$, $\exists t_0 > 0$ *such that for any compact set* K, *one can find a constant* $C_2(K) > 0$ *for which for any* $f \in L^\infty$, $t \geqslant t_0$:

$$\sup_{x\in K} |P_t f(x) - \mu(f)| \leqslant C_2(K)\mathrm{e}^{-\sigma t^\rho} \|f\|_\infty, \tag{3.1}$$

where $\rho = (\alpha+1)/2$.

(ii) If $\alpha = 1$, $\forall \rho < 1$, $\exists \sigma > 0$, $\exists t_0 > 0$ *such that for any compact set* K, *one can find a constant* $C_2(K) > 0$ *for which for any* $f \in L^\infty$, $t \geqslant t_0$:

$$\sup_{x\in K} |P_t f(x) - \mu(f)| \leqslant C_2(K)\mathrm{e}^{-\sigma t^\rho} \|f\|_\infty. \tag{3.2}$$

Proof. The proof of this theorem relies heavily upon the results of Theorem 2.4. Our first need is to improve each of the inequalities (2.14)–(2.17) in the specific case $-1 < \alpha \leqslant 1$. We proceed in three steps:

- First, we give a precise estimation of the spectral gap σ^a via $\Phi(a)$ (see Section 2, property (iii) of L^a).
- Then, we deal with the term $P_x(T_a \leqslant t)$.
- Finally we prove Theorem 3.1.

In view of Proposition 2.2, $\Phi(a) \geqslant 1/2\sigma^a$, where $\Phi(a) = \int_{-a}^{a} \mathrm{e}^{-V(x)}\,\mathrm{d}x \int_{-|x|}^{|x|} \mathrm{e}^{V(y)}\mathrm{d}y$. Hence, an upper bound of $\Phi(a)$ gives a lower bound for σ^a.

Lemma 3.2. (i) *if* $-1 < \alpha < 1$, $\exists C_3 > 0$, $\exists C_4 > 0$ *such that*

$$\Phi(a) \leqslant C_3 + C_4 a^{1-\alpha}, \tag{3.3}$$

(ii) *if* $\alpha = 1$, $\exists C_3' > 0$, $\exists C_4' > 0$ *such that*

$$\Phi(a) \leqslant C_3' + C_4' \ln(a). \tag{3.4}$$

Proof. Let us consider $h(x) = x^{-\alpha}\mathrm{e}^{V(x)}$, then

$$h'(x) = \left(\frac{V'(x)}{x^\alpha} - \frac{\alpha}{x^{\alpha+1}}\right)\mathrm{e}^{V(x)}.$$

Due to (C_2'), it comes that for $\alpha > -1$

$$\lim_{x\to+\infty} \frac{V'(x)}{x^\alpha} - \frac{\alpha}{x^{\alpha+1}} = C.$$

Therefore, there exists x_0 such that

$$\forall x \geqslant x_0, \quad \mathrm{e}^{V(x)} \leqslant \frac{2}{C} h'(x),$$

by integrating it appears that

$$\int_{x_0}^{x} e^{V(y)} dy \leqslant \frac{2}{Cx^\alpha} e^{V(x)}.$$

Now, observe that for $a \geqslant x_0$

$$\frac{1}{4}\Phi(a) = \int_0^{x_0} e^{-V(x)}\, dx \int_0^x e^{V(y)} dy + \int_{x_0}^{a} e^{-V(x)}\, dx \int_0^{x_0} e^{V(y)} dy$$
$$+ \int_{x_0}^{a} e^{-V(x)}\, dx \int_{x_0}^{x} e^{V(y)} dy$$

hence, $\Phi(a) \leqslant C_3 + (8/C)\int_{x_0}^a dx/x^\alpha$, and then if $\|\alpha| < 1$

$$\Phi(a) \leqslant C_3 + C_4 a^{1-\alpha}.$$

If $\alpha = 1$, it turns out that: $\Phi(a) \leqslant C_3 + C_4' \ln(a)$. □

Now, let us turn on to $P_x(T_a \leqslant t)$.

In this purpose, we shall classically use the behaviour of eigenfunctions of L. Let β be a real, $\beta > 0$ and f_β a function such that

$$Lf_\beta = \beta f_\beta, \quad \text{where } Lf = \tfrac{1}{2}(f'' - bf').$$

Notice that f_β is not in L^2. Indeed, as was seen in the introduction (property (iii)), the eigenvalues associated to L^2 functions were non-positive.

It is known that $M_t = f(X_t)\exp(-\int_0^t (Lf(X_s)/f(X_s))\, ds)$ is a local martingale on Ω^a. If f is bounded and strictly positive on compacts sets, the stopping theorem can be applied (see Revuz and Yor, 1994), using the Markov inequality it turns out that

$$P_x(T_a \leqslant t) \leqslant e^{\beta t} E_x e^{-\beta T_a} \leqslant e^{\beta t} \frac{f_\beta(x)}{\inf(f_\beta(a), f_\beta(-a))}. \tag{3.5}$$

The inequality (3.5) with a judicious choice of f_β, yields an adequate upper bound for $P_x(T_a \leqslant t)$. f_β will be chosen as follows:

Proposition 3.3. *Let g be the unique solution of the differential equation* (D_1'):

$$(D_1') \quad \begin{aligned} &Lg - g = 0, \\ &g(0) = 1, \\ &g'(0) = 0. \end{aligned}$$

For any $\beta \in]0,1]$, for any constant $0 < k < C/(\alpha+1)$ (where C is given in (C_2')), one can find an even function f_β and a positive real k_1, independant of β, such that

(i) $\forall x \in \mathbb{R}, \ f_\beta(x) \geqslant 1/\beta$,

(ii) $\forall x \in \mathbb{R}, \ f_\beta(x) \leqslant g(x)/\beta$,

(iii) $\exists x_k \in \mathbb{R}$ *independant of β such that*

$$\forall x \geqslant x_k, \quad f_\beta(x) \geqslant k_1 e^{k x^{\alpha+1}}. \tag{3.6}$$

In order to prove this proposition, we use the following classical lemma which is an easy consequence of the maximum principle.

Lemma 3.4. *Let h be a C^2 function, $\beta > 0$ and x_0 real. Assume that*

$$\forall x \geqslant x_0, \; Lh(x) - \beta h(x) \geqslant 0,$$

$$h(x_0) \geqslant 0,$$

$$h'(x_0) \geqslant 0.$$

Then,

$$\forall x \geqslant x_0, \quad h(x) \geqslant 0.$$

Proof of Proposition 3.3. Let f_β be the unique solution of Eq. (D_β):

$$(D_\beta) \quad \begin{aligned} & Lf_\beta(x) - \beta f_\beta(x) = 0, \\ & f_\beta(0) = \frac{1}{\beta}, \\ & f'_\beta(0) = 0. \end{aligned}$$

As b being odd, f_β is an even function.

(i) Setting

$$\hat{f}_\beta = f_\beta - \frac{1}{\beta},$$

it is obvious that $\hat{f}_\beta$ is solution of (D'_β):

$$(D'_\beta) \quad \begin{aligned} & L\hat{f}_\beta(x) - \beta \hat{f}_\beta(x) = 1, \\ & \hat{f}_\beta(0) = 0, \\ & \hat{f}'_\beta(0) = 0. \end{aligned}$$

Consequently, according to Lemma 3.4 $\forall x \in \mathbb{R}$, $\hat{f}_\beta(x) \geqslant 0$ which proves (i).
(ii) Let g be the function defined by (D'_1), Proposition 3.3 and set,

$$g_\beta = \frac{g}{\beta} - f_\beta.$$

We easily check that g_β satisfies

$$Lg_\beta(x) - \beta g_\beta(x) = (1 - \beta) g(x) \geqslant 0,$$

$$g_\beta(0) = 0,$$

$$g'_\beta(0) = 0.$$

Consequently, according to Lemma 3.4

$$\forall x \in \mathbb{R}, \quad g_\beta(x) \geqslant 0$$

which proves (ii).

(iii) First notice that there is a positive real x_0, independent of β such that

$$\forall x \geqslant x_0, \quad f'_\beta(x) \geqslant \tfrac{1}{2}. \tag{3.7}$$

Indeed, since $Lf = \dfrac{1}{2}\mathrm{e}^V(\mathrm{e}^{-V} f')'$

$$f'_\beta(x) = 2\beta \mathrm{e}^{V(x)} \int_0^x f_\beta(t)\mathrm{e}^{-V(t)}\,\mathrm{d}t.$$

From (i) we obtain

$$\forall x \geqslant 0, \quad f'_\beta(x) \geqslant 2\mathrm{e}^{V(x)} \int_0^x \mathrm{e}^{-V(t)}\,\mathrm{d}t.$$

(C'_2) shows that $V(x) \geqslant 0$ for x large enough, moreover $\int_0^{+\infty} \mathrm{e}^{-V(x)}\,\mathrm{d}x = \frac{1}{2}$, thus

$$\exists x_0, \ \forall x \geqslant x_0, \quad f'_\beta(x) \geqslant \tfrac{1}{2}.$$

Consider, $h_k(x) = M\mathrm{e}^{kx^{\alpha+1}}$, where $M > 0$. A trite calculation gives

$$Lh_k(x) - \beta h_k(x) = \tfrac{1}{2}k(\alpha+1)x^{\alpha-1}h_k(x)[\alpha + x^{\alpha+1}(k(\alpha+1) - C(1+\varepsilon(x)))].$$

From the conditions $k < C/(\alpha+1)$ and $\alpha+1 > 0$, it is obvious that there exists ${x_k}^* > 0$ such that

$$\forall x \geqslant {x_k}^*, \quad Lh_k \leqslant 0,$$

hence $Lh_k - \beta h_k \leqslant 0, \forall x \geqslant {x_k}^*$, setting $x_k = \max({x_k}^*, x_0)$, we can choose $M = k_1$ in order to have

$$h_k(x_k) \leqslant 1,$$

$$h'_k(x_k) \leqslant \frac{1}{2}.$$

Then, it is readily seen that $f = f_\beta - h_k$ obeys

$$Lf(x) - \beta f(x) \geqslant 0, \quad \forall x \geqslant x_k,$$

$$f(x_k) \geqslant \frac{1}{\beta} - h_k(x_k) \geqslant 0,$$

$$f'(x_k) \geqslant \frac{1}{2} - h'_k(x_k) \geqslant 0,$$

hence $f_\beta(x) \geqslant h_k(x), \forall x \geqslant x_k$ by Lemma 3.4. □

Proposition 3.3, entails the following result:

Corollary 3.5. *For any K compact of $\mathbb{R}$ and any $0 < k < C/(\alpha+1)$, one can find a constant $M_1(K)$ and a positive real x_k, such that*

$$\forall t \geqslant 0, \ \forall x \in K, \ \forall a \geqslant x_k, \quad P_x(T_a \leqslant t) \leqslant \frac{M_1(K)}{\beta \mathrm{e}^{ka^{\alpha+1}}}\mathrm{e}^{\beta t}.$$

Proof. Immediate from Proposition 3.3 and inequality (3.5). □

We are now ready for the proof of Theorem 3.1

Proof of Theorem 3.1. The proof relies upon Theorem 2.4. We begin with the case $-1<\alpha<1$. From inequality (3.3) and Proposition 2.2, it is known that

$$\Phi(a)\leqslant C_3+C_4a^{1-\alpha} \quad \text{and} \quad \sigma^a\geqslant\frac{1}{2\Phi(a)}.$$

An appeal to Eq. (2.15) gives

$$\sup_{x\in K}|P_t^a f(x)-\mu^a(f)|\leqslant C(K)\mathrm{e}^{-t/4(C_3+C_4a^{1-\alpha})}\|f\|_{2,a}. \tag{3.8}$$

Assumption (C_2') and Eq. (2.16) entail

$$|\mu(f)-\mu^a(f)|\leqslant C_5\mathrm{e}^{-C_6a^{\alpha+1}}\|f\|_\infty. \tag{3.9}$$

A look at Corollary 3.5 and Eq. (2.17) shows that for any $\beta\in]0.1[$

$$\forall x\in K,\ \forall a>x_k,\quad |P_t^a f(x)-P_tf(x)|\leqslant 2M_1(K)\frac{\mathrm{e}^{\beta t}}{\beta\mathrm{e}^{ka^{\alpha+1}}}\|f\|_\infty. \tag{3.10}$$

Choosing $a=t^\lambda$ with $\lambda<1/(1-\alpha)$ and $\beta=1/t$, Eqs. (3.8)–(3.10) lead to

$$\sup_{x\in K}|P_tf(x)-\mu(f)|\leqslant C'(K)\mathrm{e}^{-C_7t^{1-\lambda(1-\alpha)}}\|f\|_\infty+C_5\mathrm{e}^{-C_6t^{\lambda(\alpha+1)}}\|f\|_\infty$$
$$+M_1'(K)t\mathrm{e}^{-kt^{\lambda(\alpha+1)}}\|f\|_\infty.$$

The optimal rate is reached for $\lambda=\frac{1}{2}$. It turns out that for any $\rho<(\alpha+1)/2$, one can find $\sigma>0$ such that for t large enough

$$\sup_{x\in K}|E_xf(X_t)-\mu(f)|\leqslant C_2(K)\|f\|_\infty\mathrm{e}^{-\sigma t^\rho},$$

which completes (i) when $-1<\alpha<1$.

If $\alpha=1$, Eqs. (2.15) and (3.4) induce a change in Eq. (3.8) and we have

$$\sup_{x\in K}|P_t^a f(x)-\mu^a(f)|\leqslant C(K)\mathrm{e}^{-t/4(C_3'+C_4'\ln(a))}\|f\|_{2,a}. \tag{3.11}$$

It is easily checked that the preceding conclusion remains true, that is, for any $\rho<1$, there is a $\sigma>0$ such that for t large enough

$$\sup_{x\in K}|E_xf(X_t)-\mu(f)|\leqslant C_k\|f\|_\infty\mathrm{e}^{-\sigma t^\rho}.$$

The proof of (i) is now completed. □

Remark 3.1. The case $\alpha=0$ is interesting. We can prove that $\mathrm{e}^{-kt^{1/2}}$ is the otimal rate. Namely, if we consider the SDE: $X_t=x+B_t-\int_0^t\operatorname{sgn}(X_s)\,\mathrm{d}s$, the stationary probability being $\mu(\mathrm{d}x)=\mathrm{e}^{-2|x|}\,\mathrm{d}x$, we can prove the following result: $\exists M>0$, such that for any compact set K ($K\neq\emptyset$),

$$\sup_{f\in L^1}\sup_{x\in K}\frac{|P_tf(x)-\mu(f)|}{\|f\|_1}\geqslant M\mathrm{e}^{-t^{1/2}}.$$

The proof will be achieved in the appendix.

4. The case $\alpha = -1$

Again we confine ourselves to $d = 1$.

It is assumed that b satisfies (C_2''):

$$(C_2'')\quad \begin{cases} b \text{ is odd: } \forall x \in \mathbb{R},\ b(-x) = -b(x), \\ \forall x > 0,\ b(x) = (1 + \varepsilon(x))C/x \text{ with } C > 1, \text{ and } \lim_{|x|\to\infty} \varepsilon(x) = 0. \end{cases}$$

The main result of this section is Theorem 4.1.

Theorem 4.1. *Let $\rho < (C-1)/2$, there exists $t_0 > 0$ such that for any compact set K we can find $C_3(K) > 0$ such that for any $t \geqslant t_0$ and for any $f \in L^\infty$*

$$\sup_{x\in K} |P_t f(x) - \mu(f)| \le \frac{C_3(K)}{t^\rho} \|f\|_\infty. \tag{4.1}$$

The method is exactly the same as the previous case $-1 < \alpha \leqslant 1$:

- We first provide a lower bound for the spectral gap,
- Then we give an upper bound for $P_x(T_a \leqslant t)$,
- Finally, Theorem 4.1 is readily proved.

Lemma 4.2. *There exist two constants D_1 and D_2 such that*

$$\Phi(a) \leqslant D_1 + D_2 a^2. \tag{4.2}$$

Proof. The proof is left to the reader. The method is exactly the same as the proof of Lemma 3.2 □

In order to give an estimation of $P_x(T_a \leqslant t)$, we use the same method as in previous section.

For any $\beta > 0$, we are interested in the function f_β such that

$$L f_\beta = \beta f_\beta.$$

As was seen before in Eq. (3.5)

$$P_x(T_a \leqslant t) \leqslant e^{\beta t} \frac{f_\beta(x)}{\inf(f_\beta(a), f_\beta(-a))}.$$

A judicious choice of the function f_β provides an estimation of $P_x(T_a \leqslant t)$. f_β is choosen according to:

Proposition 4.3. *Let g be the unique solution of the differential equation*

$$(D_1')\quad \begin{aligned} &Lg - g = 0, \\ &g(0) = 1, \\ &g'(0) = 0. \end{aligned}$$

For any $\beta \in]0,1]$, for any constant $0 < k < C + 1$ (where C is given in (C_2'')), one can find a function f_β and a positive real k_1, independant of β such that

(i) $\forall x \in \mathbb{R},\ f_\beta(x) \geqslant 1/\beta$

(ii) $\forall x \in \mathbb{R},\ f_\beta(x) \leqslant g(x)/\beta$

(iii) $\exists x_k \in \mathbb{R}$ *independant of* β *such that*

$$\forall x \geqslant x_k, \quad f_\beta(x) \geqslant k_1 x^k e^{\beta x}. \tag{4.3}$$

Proof. Let us choose f_β exactly as in Proposition 3.3. Assertions (i) and (ii) are then obvious.

Recall moreover that with this choice of f_β, there is a constant x_0 independant of β such that $\forall x \geqslant x_0,\ f'_\beta(x) \geqslant \frac{1}{2}$.

(iii) Setting $h_\beta(x) = Mx^k e^{\beta x},\ M > 0,\ k < C+1$, a trite calculation gives

$$Lh_\beta(x) - \beta h_\beta(x) = \frac{1}{2} h_\beta(x) \left[-\frac{k}{x^2}(xb(x) - (k-1)) - \beta \left(2 - \beta + \frac{xb(x) - 2k}{x} \right) \right].$$

Using (C''_2) and $k < C+1$ we get, $\exists x_k > x_0,\ \forall x \geqslant x_k,\ Lh_\beta(x) - \beta h_\beta(x) \leqslant 0$. Let us choose M such that

$$h_\beta(x_k) \leqslant 1,$$

$$h'_\beta(x_k) \leqslant \frac{1}{2}.$$

Setting $f = f_\beta - h_\beta$, it is obvious that

$$Lf(x) - \beta f(x) \geqslant 0, \quad \forall x \geqslant x_k,$$

$$f(x_k) \geqslant 0,$$

$$f'(x_k) \geqslant 0.$$

Lemma 3.4 gives: $\forall x \geqslant x_k,\ f_\beta(x) \geqslant Mx^k e^{\beta x}$. □

Proof of Theorem 4.1. Thanks to the previous results, it is now easy to conclude.

Using estimation of the spectral gap (inequality (4.2)) and Proposition 2.2, inequalities (2.15) and (2.16) become

$$\sup_{x \in K} |P_t^a f(x) - \mu^a(f)| \leqslant C(K) e^{-t/4(D_1 + D_2 a^2)} \|f\|_{2,a}, \tag{4.4}$$

$$\forall \delta < C - 1,\ \exists k_\delta > 0, \quad |\mu^a(f) - \mu(f)| \leqslant k_\delta a^{-\delta} \|f\|_\infty. \tag{4.5}$$

Inequalities (2.17) and (4.3) give the existence of $x_K > 0$ and $M'_K > 0$ such that for any $\beta \in]0, 1]$

$$\forall x \in K,\ \forall a \geqslant x_K, \quad |P_t f(x) - P_t^a f(x)| \leqslant \frac{M'_K e^{\beta t}}{\beta a^k e^{\beta a}} \|f\|_\infty. \tag{4.6}$$

Now we set $\beta = 1/t$ and $a = [t/\ln t]^{1/2}$, then for any $\rho < (C-1)/2$ we can choose $\delta < C-1$ such that Eqs. (4.4)–(4.6) yield

$$\forall x \in K, \quad |P_t f(x) - \mu(f)| \leqslant \frac{M''_K}{t^\rho} \|f\|_\infty.$$

(Note that the constant D_2 in Eqs. (4.4) or (4.2) can be chosen close to $1/2(C+1)$) and the proof of (i) is completed. □

Appendix A.

Proof of Remark 3.1. For the the S.D.E $X_t = x + B_t - \int_0^t \mathrm{sgn}(X_s)\,\mathrm{d}s$ with stationary distribution $\mu(\mathrm{d}x) = \exp(-2|x|)\,\mathrm{d}x$, the following inequality holds: $\exists M>0$ such that for any compact K $(K \neq \emptyset)$

$$\sup_{f\in L^1}\sup_{x\in K}\frac{|P_t f(x)-\mu(f)|}{\|f\|_1} \geqslant M\mathrm{e}^{-t^{1/2}}.$$

Proof. For any $f \in L^1(\mu)$, we show that

$$P_t f(0) = \int_{-\infty}^{+\infty}\frac{f(x)}{\sqrt{2\pi}}\left(\psi\left(\frac{|x|-t}{\sqrt{t}}\right)+\frac{1}{\sqrt{t}}\exp\left(-\frac{(|x|-t)^2}{2t}\right)\right)\mu(\mathrm{d}x), \tag{A.1}$$

where $\psi(z) := \int_z^{+\infty}\exp(-u^2/2)\,\mathrm{d}u$.

The Girsanov formula (Karatzas and Shreve (1991), p.191, (5.6)) yields:

$$P_t f(0) = E_0\left(f(B_t)\exp\left(-\int_0^t \mathrm{sgn}(B_s)\,\mathrm{d}B_s - \frac{t}{2}\right)\right).$$

Bearing in mind the Tanaka formula (Karatzas and Shreve (1991), p. 205)

$$|B_t| = \int_0^t \mathrm{sgn}(B_s)\,\mathrm{d}B_s + L_t,$$

where L_t is the local time of (B_t) at 0. It turns out that,

$$P_t f(0) = E_0\left[f(B_t)\exp\left(L_t - |B_t| - \frac{t}{2}\right)\right]. \tag{A.2}$$

For $x\geqslant 0$, $y\geqslant 0$, the density of $(|B_t|, L_t)$ is given by (Revuz and Yor, 1994, p. 227, Example (2.18))

$$\eta(t,x,y) = \sqrt{\frac{2}{\pi t^3}}(x+y)\exp\left(-\frac{(x+y)^2}{2t}\right).$$

Following (Benachour et al., 1996, p. 47), after some easy calculations, we get Eq. (A.1).

From Eq. (A.1), it is easily seen with Lebesgue's theorem that

$$\lim_{t\to\infty} P_t f(0) = \int_{-\infty}^{+\infty} f(x)\mu(\mathrm{d}x) = \mu(f).$$

By difference, it obtains

$$P_t f(0) - \mu(f) = \int_{-\infty}^{+\infty}\frac{f(x)}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{t}}\mathrm{e}^{-(t-|x|)^2/2t} - \psi\left(\frac{t-|x|}{\sqrt{t}}\right)\right)\mathrm{d}\mu. \tag{A.3}$$

For f symmetric Eq. (A.3) turns into

$$P_t f(0) - \mu(f) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x)(h'(x) - h(x))\mathrm{e}^{-2x}\,\mathrm{d}x,$$

where $h(x) = \psi((t-x)/\sqrt{t})$. Integrating by parts we get

$$P_t f(0) - \mu(f) = \sqrt{\frac{2}{\pi}} \left(-f(0)\psi(\sqrt{t}) + \int_0^{+\infty} \mathrm{e}^{-2x}(f(x) - f'(x))\psi\left(\frac{t-x}{\sqrt{t}}\right) \mathrm{d}x \right) \tag{A.4}$$

if $\lim_{x\to\infty} f(x)\mathrm{e}^{-2x} = 0$.

For $f(x) = f_\lambda(x) = \mathrm{e}^{\lambda|x|}$ with $\lambda < 2$ Eq. (A.4) gives

$$P_t f(0) - \mu(f) = \sqrt{\frac{2}{\pi}} \left(-\psi(\sqrt{t}) + (1-\lambda)\sqrt{t} \int_{-\infty}^{\sqrt{t}} \psi(x)\mathrm{e}^{-(2-\lambda)(t - x\sqrt{t})}\,\mathrm{d}x \right).$$

Choosing $\lambda = 2 - 1/\sqrt{t}$ and setting $f_t(x) = \mathrm{e}^{\lambda x}$, it turns out that $\|f_t\|_1 = 2\sqrt{t}$ and

$$\frac{P_t f_t(0) - \mu(f_t)}{\|f_t\|_1} = \frac{1}{\sqrt{2\pi}} \left(-\frac{\psi(\sqrt{t})}{\sqrt{t}} + \left(\frac{1}{\sqrt{t}} - 1\right) \mathrm{e}^{-\sqrt{t}} \int_{-\infty}^{\sqrt{t}} \psi(x)\mathrm{e}^{x}\,\mathrm{d}x \right)$$

and so, there exists a constant $M > 0$ such that

$$\frac{|P_t f_t(0) - \mu(f_t)|}{\|f_t\|_1} \geqslant M\mathrm{e}^{-\sqrt{t}}$$

entailing

$$\sup_{f\in L^1} \sup_{x\in K} \frac{|P_t f(x) - \mu(f)|}{\|f\|_1} \geqslant M\mathrm{e}^{-t^{1/2}},$$

which completes the proof. □

- Let us remark that the optimal lower band in the preceding calculus is obtained for $f_\lambda(x) = \mathrm{e}^{\lambda|x|}$, $(\lambda \nearrow 2)$. This is not surprising insofar f_λ is the eigenvector of $L = \frac{1}{2}f'' - \operatorname{sgn} f'$ therefore

$$\begin{aligned} Lf_\lambda &= \frac{\lambda^2}{2}\mathrm{e}^{\lambda x} - \lambda\mathrm{e}^{\lambda x} \quad (x > 0) \\ &= \lambda\left(\frac{\lambda}{2} - 1\right)\mathrm{e}^{\lambda x}, \end{aligned}$$

  and the eigenvalue $\rho(\lambda) = \lambda((\lambda/2) - 1)$ goes to 0 as $\lambda \to 2$.
- Let us remark that we did not prove

$$\sup_{f\in L^2} \sup_{x\in K} \frac{|P_t f(x) - \mu(f)|}{\|f\|_2} \geqslant C\mathrm{e}^{-t^{1/2}}$$

  but this inequality only with $\sup_{f\in L^1}$.

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