SHEAVES OF INTEGRAL DOMAINS ON STONE SPACES*

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It is well known in the theory of ring representations on Stone spaces, that if $R$ is a commutative ring with 1, it is representable as a sheaf of fields (local rings) on a Stone space iff $R$ is a von Neumann regular (exchange) ring. These results make use of the Pierce representation of $R$. The question of necessary and sufficient conditions on $R$ to guarantee representability as a sheaf of integral domains is answered in this article. The appropriate condition on $R$ is that of being an 'almost weak Baer' ring, where this means that $\text{Ann}(a)$ is generated by its idempotent elements for all $a \in R$. Two examples from rings of continuous functions distinguish this property from several closely related ring theoretic conditions.

Introduction

In [5, Chapter V], Johnstone presents a survey of the theory of representations of rings (commutative with identity). Such results generally determine when

$$ R \text{ is isomorphic to the ring of global sections of a sheaf of } \_\_ \text{ rings on a } \_\_ \text{ space.} $$

Indeed, if a sheaf of local rings on a coherent space (i.e. spectral, in the sense of [3]) is desired, then every ring can be so represented by the Zariski spectrum. If the space is to be a Stone space (i.e. Hausdorff, as well as coherent, or equivalent totally disconnected, compact and Hausdorff), and one is willing to possibly weaken local to indecomposable (i.e. the only idempotents are 0 and 1), then every ring can be so represented by the Pierce spectrum [11]. However, if a stronger condition on the sheaf of rings (on a Stone space) is desired, it is often necessary to put some restriction on the ring $R$. For example, $R$ is isomorphic to the ring of global sections of a sheaf of local rings on a Stone space if and only if $R$ is an exchange ring, i.e. every element of $R$ is the sum of an idempotent and an invertible element [8]. If a sheaf of fields is desired, then $R$ must be a (von Neumann) regular ring [5]. In both these

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cases the Pierce representation provides the sheaf of rings. Moreover, it is precisely
the exchange (respectively, regular) property that must be added to the already in-
decomposable stalks of the Pierce spectrum to obtain the desired local rings (respec-
tively, fields).

Kennison [6] has characterized those rings which are representable by a sheaf of
integral domains without placing restrictions on the type of space. The goal of this
paper is to present a necessary and sufficient condition for $R$ to be represented as
a sheaf of integral domains on a Stone space. As in the above cases, the representa-
tion will be via the Pierce spectrum. Thus, we begin by introducing a condition
(almost weak Baer) that makes an indecomposable ring into an integral domain. In
Section 2, we prove that $R$ is isomorphic to the ring of global sections of a sheaf
of integral domains if and only if $R$ is an almost weak Baer ring. We conclude, in
Section 3, with a few examples to distinguish this property from some other closely
related classes of rings.

1. Almost weak Baer rings

Throughout this paper $R$ denotes a commutative ring with identity. Recall that
$R$ is called weak Baer if $\text{Ann}(r)$ is a principal ideal generated by an idempotent, for
all $r \in R$. We shall say that $R$ is almost weak Baer if whenever $rs = 0$, there is a
decomposition $R \cong R_1 \times R_2$ such that $r$ maps to 0 in $R_1$ and $s$ maps to 0 in $R_2$, or
equivalently if there exist idempotents $e_1, e_2 \in R$ such that $e_1 + e_2 = 1$, $re_1 = 0$, and
$se_2 = 0$. It is not difficult to show that every weak Baer ring is almost weak Baer.
An example of an almost weak Baer ring that is not weak Baer will be presented
in Section 3. The following lemma is easily established.

**Lemma 1.1.** $R$ is an integral domain if and only if $R$ is an indecomposable almost
weak Baer ring. □

Recall that the set $E(R)$ of idempotents of $R$ is a Boolean algebra with $e \vee f =
e + f - ef$, and $e \wedge f = ef$. The following lemma provides a simplification of ideals
generated by idempotents, and leads to an alternative characterization of almost
weak Baer rings which is more closely related to the above definition of weak Baer.

**Lemma 1.2.** Let $A$ be an ideal generated by its idempotents. Then

$$A = \{re | e \in A \text{ and } e^2 = e\}.$$  

**Proof.** Since the set of idempotents of $A$ is closed under $\vee$, it suffices to
show that if $r = se + tf$, where $e, f \in E(R) \cap A$, then $r = r(e \vee f)$. But, $r(e \vee f) =
(se + tf)(e + f - ef) = se + tf = r$. □

**Proposition 1.3.** $R$ is almost weak Baer if and only if $\text{Ann}(r)$ is generated by its
idempotents, for all $r$.  

Proof. If \( R \) is almost weak Baer, and \( s \in \text{Ann}(r) \), then \( rs = 0 \) and so there exist idempotents \( e_1, e_2 \in R \) such that \( e_1 + e_2 = 1 \), \( re_1 = 0 \), and \( se_2 = 0 \). Then \( s = se_1 + se_2 = se_1 \) and \( e_1 \in \text{Ann}(r) \). It follows that \( \text{Ann}(r) \) is generated by its idempotents.

Conversely, if \( \text{Ann}(r) \) is generated by its idempotents and \( rs = 0 \), then \( s \in \text{Ann}(r) \), and so, by Lemma 1.2, there exists an idempotent \( e_1 \in \text{Ann}(r) \) such that \( s = se_1 \). Taking \( e_2 = 1 - e_1 \), the desired result follows. □

These results provide us with a simpler description of annihilators in almost weak Baer rings. Recall that \( R \) is said to satisfy weak de Morgan's law (WDML) if \( \text{Ann}(rs) = \text{Ann}(r) + \text{Ann}(s) \), for all \( r, s \in R \). This term was introduced by Niefield and Rosenthal [10], in their investigation of algebraic analogues of de Morgan's laws. Although these rings have been referred to in the literature as MP rings, we shall use WDML, as we wish to emphasize the annihilator condition. In [1], Artico and Marconi show that \( R \) satisfies WDML if and only if every prime ideal of \( R \) contains a unique minimal prime. They also show that \( R \) is weak Baer if and only if \( R \) satisfies WDML and the minimal spectrum of \( R \) is compact.

Lemma 1.4. If \( R \) is an almost weak Baer ring, then \( R \) satisfies WDML.

Proof. Let \( r, s \in R \). It suffices to show that \( \text{Ann}(rs) \subseteq \text{Ann}(r) + \text{Ann}(s) \), since the other containment holds in any case. If \( x \in \text{Ann}(rs) \), then \( (xr)s = 0 \), and so there exist idempotents \( e_1, e_2 \in R \) such that \( e_1 + e_2 = 1 \), \( (xr)e_1 = 0 \), and \( se_2 = 0 \). Thus, \( x = xe_1 + xe_2 \), where \( xe_1 \in \text{Ann}(r) \), and \( xe_2 \in \text{Ann}(s) \) are desired. □

Note that the converse of Lemma 1.4 does not hold, as will be seen in Section 3 when we present some examples.

Next we present a characterization of weak Baer rings related to the one in [1], but without mention of the minimal spectrum.

Proposition 1.5. \( R \) is a weak Baer ring if and only if \( R \) satisfies WDML and \( \text{Ann}(r) \) is finitely generated, for all \( r \in R \).

Proof. If \( R \) is weak Baer, then \( \text{Ann}(r) \) is clearly finitely generated. That \( R \) satisfies WDML follows immediately from Lemma 1.4.

For the converse, let \( r \in R \) and suppose \( \text{Ann}(r) = (x_1, \ldots, x_n) \). We shall show that \( \text{Ann}(r) \oplus \text{Ann}^2(r) = R \). Since WDML implies \( R \) has no nilpotents [10], \( \text{Ann}(r) \cap \text{Ann}^2(r) = 0 \). Also,

\[
\text{Ann}(r) + \text{Ann}^2(r) = \text{Ann}(r) + \text{Ann}(x_1, \ldots, x_n)
= \text{Ann}(r) + [\text{Ann}(x_1) \cap \cdots \cap \text{Ann}(x_n)]
\supseteq [\text{Ann}(r) + \text{Ann}(x_1)] \cdot \cdots \cdot [\text{Ann}(r) + \text{Ann}(x_n)]
= \text{Ann}(rx_1) \cdot \cdots \cdot \text{Ann}(rx_n) = \text{Ann}(0) \cdot \cdots \cdot \text{Ann}(0) = R
\]
where the second to last equality follows from WDML. □

**Corollary 1.6.** If $R$ is a Noetherian ring, then $R$ is weak Baer if and only if $R$ is almost weak Baer.

We conclude this section with a lemma about WDML which we shall need in the next section.

**Lemma 1.7.** Suppose $R$ satisfies WDML. If $A$ is an ideal of $R$ such that $A = \{ r \in R \mid \text{Ann}(r) \not\subseteq A \}$, then $A$ is a prime ideal.

**Proof.** Suppose $rs \in A$. Then $\text{Ann}(rs) \not\subseteq A$. Since by WDML, $\text{Ann}(rs) = \text{Ann}(r) + \text{Ann}(s)$, it follows that $\text{Ann}(r) \not\subseteq A$ or $\text{Ann}(s) \not\subseteq A$. Thus, $r \in A$ or $s \in A$. □

2. Sheaves of integral domains

We begin with a brief discussion of sheaves of rings on a topological space $X$. First, if $P$ is a property that a ring may have (e.g. the property of being an integral domain, a field, a local ring, or an indecomposable ring), then what do we mean when we say 'a sheaf of $P$-rings'? One possibility is a sheaf of rings whose stalks are $P$-rings. Another possibility is that the sheaf of rings satisfies $P$ in the internal logic of the topos $\text{Sh}(X)$ of set-valued sheaves on $X$, when considered as a ring object in this category. As pointed out in [4, 9], one must be careful in how a particular property is defined in this internal logic, as classically equivalent definitions may not be equivalent in the intuitionistic logic of sheaves. It turns out that if $P$ can be given by so-called 'coherent' axioms, then a ring object in $\text{Sh}(X)$ is a sheaf of $P$-rings in the internal sense if and only if each stalk is a $P$-ring. The use of such coherent axioms is essentially due to Joyal, and Makkai and Reyes [7]. See also [5] for a detailed discussion, as well as the expression of the field, local ring and indecomposable ring properties in terms of coherent axioms. Note that an integral domain can be defined by the coherent axioms $\forall x, y (xy = 0 \Rightarrow x = 0 \lor y = 0)$ and $(0 = 1) \Rightarrow \bot$

Now, in [5], also those coherent axioms are considered which are preserved by direct image functors. There, a class of coherent axioms is singled out which are called **regular axioms**. Such an axiom is of the form $\forall x_1, \ldots, x_n (\phi \Rightarrow \psi)$, where $\phi$ and $\psi$ are built up from atomic formulae (i.e. $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$, where $f$ and $g$ are words in the ring operations) by using the connectives $\land$ and $\exists$, and $x_1, \ldots, x_n$ contain the free variables of $\phi$ and $\psi$. Examples of rings which can be described by these axioms are regular rings and exchange rings. An additional example is almost weak Baer rings, for this condition can be expressed as

$$\forall r, s (rs = 0 \Rightarrow \exists e, f (e^2 = e \land f^2 = f \land e + f = 1 \land re = 0 \land sf = 0)).$$
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Note that this axiom is satisfied (internally) by an integral domain defined as above. We shall make use of the following proposition which summarizes the results of [5, V 1.12-1.13].

**Proposition 2.1.** If $X$ is a Stone space, then the global section functor $\text{Sh}(X) \to \text{Sets}$ preserves the validity of regular axioms. 

Before presenting our main theorem, we shall give a brief description of the *Pierce representation* of a ring $R$. This construction is due to Pierce and was first introduced in his monograph [11]. For further details we refer the reader to [5, V-2].

The *Pierce spectrum* of $R$ is the topological space $\text{Spec}_p(R)$ described as follows. The points of $\text{Spec}_p(R)$ are the prime (hence, maximal) ideals of the Boolean algebra $E(R)$ of idempotents of $R$. The open subsets are of the form $U_I = \{P \mid I \nsubseteq P\}$, where $I$ is any ideal of $E(R)$. Moreover, it can be shown that $\text{Spec}_p(R)$ is a Stone space. Now, there is a sheaf of rings on $\text{Spec}_p(R)$ called the *Pierce sheaf* of $R$, and denoted by $\mathcal{R}$. The ring of sections over a principal ideal $\downarrow(e) = \{f \in E(R) \mid f \leq e\}$ is given by $R[e^{-1}] \cong R/(1-e)$. Furthermore, if $P$ is a point of $\text{Spec}_p(R)$, then the stalk of $\mathcal{R}$ at $P$ is the ring $R/(P)$, where $(P)$ denotes the ideal of $R$ generated by $P$. It is precisely this sheaf of rings on a Stone space that is used in the representation theorems discussed in the introduction.

**Theorem 2.2.** The following are equivalent:

(a) $R$ is an almost weak Baer ring;

(b) The Pierce sheaf $\mathcal{R}$ is a sheaf of integral domains;

(c) $R$ is isomorphic to the ring of global sections of a sheaf of integral domains on a Stone space.

**Proof.** (a) $\Rightarrow$ (b). Since the stalks of $\mathcal{R}$ are of the form $R/(P)$, where $P$ is a prime ideal of $E(R)$, it suffices to show that $(P)$ is a prime ideal of $R$ for each such $P$. By Lemma 1.7, it suffices to show that $(P) = \{r \in R \mid \text{Ann}(r) \nsubseteq (P)\}$. Suppose $r \in (P)$. Then (by Lemma 1.2) $r = re$ for some $e \in P$. Since $1-e \in \text{Ann}(r)$, and $e \in P$, it follows that $\text{Ann}(r) \nsubseteq P$. Conversely, suppose $\text{Ann}(r) \nsubseteq (P)$. Since $\text{Ann}(r)$ is generated by idempotents, there is an $e \in \text{Ann}(r)$ such that $e \notin (P)$, and hence $e \notin P$. Thus, $1-e \in P$. Since $re=0$, we know $r(1-e)=r$, and so $r \in (P)$.

(b) $\Rightarrow$ (c). This follows from the above remarks.

(c) $\Rightarrow$ (a). Since being an almost weak Baer ring can be expressed by a regular axiom which is satisfied by all integral domains, by Proposition 2.1, $R$ must be an almost weak Baer ring. 

3. Two examples

We conclude this paper by presenting two examples, a ring which satisfies WDML but is not almost weak Baer, and a ring which is almost weak Baer but not weak
Baer. In particular, we shall consider rings of the form $C(X)$, the ring of continuous real-valued functions on a topological space $X$.

Recall that $X$ is called an $F$-space if every finitely generated ideal of $C(X)$ is principal. If $X$ is compact, then $X$ is an $F$-space iff every ideal of the form

$$O_p = \{ f \in C(X) \mid f|_V = 0 \text{ for some neighborhood } V \text{ of } p \}$$

is prime, where $p \in X$ [2, 14.25]. Since every prime ideal of $C(X)$ contains a unique ideal of the form $O_p$ [2, 7.15], if such ideals are prime, then every prime ideal contains a unique minimal prime, i.e. $C(X)$ satisfies WDML [1]. Thus, if $X$ is a compact $F$-space, then $C(X)$ satisfies WDML. As usual, $\beta$ denotes the Stone-Čech compactification.

**Example 3.1.** $C(\beta \mathbb{R}^+ \! \setminus \! \mathbb{R}^+)$ satisfies WDML but is not an almost weak Baer ring, where $\mathbb{R}^+$ denotes the non-negative reals with the usual topology.

**Proof.** First, $\beta \mathbb{R}^+ \! \setminus \! \mathbb{R}^+$ is a compact, connected $F$-space [2, p. 211], and hence, $C(\beta \mathbb{R}^+ \! \setminus \! \mathbb{R}^+)$ satisfies WDML. Now, $X$ is connected iff $C(X)$ is indecomposable [2, 1B]. Thus, $C(\beta \mathbb{R}^+ \! \setminus \! \mathbb{R}^+)$ is indecomposable. If $C(\beta \mathbb{R}^+ \! \setminus \! \mathbb{R}^+)$ were an almost weak Baer ring, then by Lemma 1.1, it would be an integral domain, which is impossible since $\beta \mathbb{R}^+ \! \setminus \! \mathbb{R}^+$ has more than one point. □

Recall that if $X$ is a Stone space, then the points of the Pierce spectrum of $C(X)$ are in one-to-one correspondence with the points of $X$, and the stalk of the Pierce sheaf of $C(X)$ at $p \in X$ is given by $C(X)/O_p$ [5, p. 184]. Thus, if $X$ is a Stone space and an $F$-space, then the Pierce sheaf is an integral domain, and so by Theorem 2.2, $C(X)$ is an almost weak Baer ring. On the other hand, it is not difficult to show that $C(X)$ is weak Baer iff $\text{coz}(f)$ is open, for all $f$. Such spaces are called basically disconnected [2]. This leads to the following example.

**Example 3.2.** $C(\beta \mathbb{N} \! \setminus \! \mathbb{N})$ is an almost weak Baer ring which is not weak Baer.

**Proof.** First, $\beta \mathbb{N} \! \setminus \! \mathbb{N}$ is a compact $F$-space [2, 14.27] and also totally disconnected [2, 6S4]. Thus, $\beta \mathbb{N} \! \setminus \! \mathbb{N}$ is a Stone space and an $F$-space and so $C(\beta \mathbb{N} \! \setminus \! \mathbb{N})$ is an almost weak Baer ring. Now, $\beta \mathbb{N} \! \setminus \! \mathbb{N}$ is not basically disconnected [2, 6W3] and so $C(\beta \mathbb{N} \! \setminus \! \mathbb{N})$ is not weak Baer. □

**References**


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