Journal of Pure and Applied Algebra 216 (2012) 1245–1252

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

A family of Koszul self-injective algebras with finite Hochschild cohomology

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ARTICLE INFO

Article history:
Received 12 May 2011
Received in revised form 5 November 2011
Available online 22 December 2011
Communicated by S. Iyengar

Dedicated to E.L. Green on the occasion of his 65th birthday.

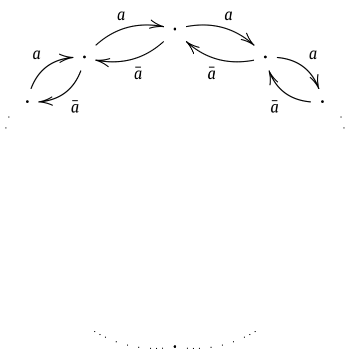
MSC: 16D50; 16E40; 16S37

ABSTRACT

This paper presents an infinite family of Koszul self-injective algebras whose Hochschild cohomology ring is finite-dimensional. Moreover, for each $N \geqslant 5$ we give an example where the Hochschild cohomology ring has dimension N. This family of algebras includes and generalizes the 4-dimensional Koszul self-injective local algebras of [R.-O. Buchweitz, E.L. Green, D. Madsen, Ø. Solberg, Finite Hochschild cohomology without finite global dimension, Math. Res. Lett. 12 (2005) 805–816] which were used to give a negative answer to Happel's question, in that they have infinite global dimension but finite-dimensional Hochschild cohomology.

0. Introduction

Let K be a field. Throughout this paper we suppose $m \geqslant 1$, and let $\mathcal{Q}$ be the quiver with m vertices, labelled $0, 1, \ldots, m-1$, and $2m$ arrows as follows:



Let a_i denote the arrow that goes from vertex i to vertex $i+1$, and let $\bar{a}_i$ denote the arrow that goes from vertex $i+1$ to vertex i, for each $i = 0, \ldots, m-1$ (with the obvious conventions modulo m). We denote the trivial path at the vertex i by e_i. Paths are written from left to right.

We define Λ to be the algebra $K\mathcal{Q}/I$ where I is the ideal of $K\mathcal{Q}$ generated by $a_i a_{i+1}$, $\bar{a}_{i-1}\bar{a}_{i-2}$ and $a_i\bar{a}_i - \bar{a}_{i-1}a_{i-1}$, for $i = 0, \ldots, m-1$, where the subscripts are taken modulo m. These algebras are Koszul self-injective special biserial algebras

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0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2011.12.011

and as such play an important role in various aspects of the representation theory of algebras. In particular, for m even, this algebra occurred in the presentation by quiver and relations of the Drinfeld double of the generalized Taft algebras studied in [4], and in the study of the representation theory of $U_q(\mathfrak{sl}_2)$, for which, see [3,10,14,15].

For $m \geqslant 1$ and for each $\mathbf{q} = (q_0, q_1, \ldots, q_{m-1}) \in (K^*)^m$, we define $\Lambda_{\mathbf{q}} = K\mathcal{Q}/I_{\mathbf{q}}$, where $I_{\mathbf{q}}$ is the ideal of $K\mathcal{Q}$ generated by

$$a_ia_{i+1},\ \bar{a}_{i-1}\bar{a}_{i-2},\ q_ia_i\bar{a}_i - \bar{a}_{i-1}a_{i-1} \quad \text{for } i = 0, \ldots, m-1.$$

These algebras are socle deformations of the algebra Λ, with $\Lambda_{\mathbf{q}} = \Lambda$ when $\mathbf{q} = (1, 1, \ldots, 1)$, and were studied in [13]. We are assuming each q_i is non-zero since we wish to study self-injective algebras. Indeed, the algebra $\Lambda_{\mathbf{q}}$ is a Koszul self-injective socle deformation of Λ, and the K-dimension of $\Lambda_{\mathbf{q}}$ is $4m$.

In the case $m = 1$, the algebras $\Lambda_{\mathbf{q}}$ were studied in [2], where they were used to answer negatively a question of Happel, in that their Hochschild cohomology ring is finite-dimensional but they are of infinite global dimension when $q \in K^*$ is not a root of unity. In this paper we show, for all $m \geqslant 1$, that the algebras $\Lambda_{\mathbf{q}}$, where $\mathbf{q} = (q_0, q_1, \ldots, q_{m-1}) \in (K^*)^m$, all have a finite-dimensional Hochschild cohomology ring when $q_0q_1\cdots q_{m-1}$ is not a root of unity. Thus, for each non-zero element of K which is not a root of unity, we have generalized the 4-dimensional algebra of [2] to an infinite family of algebras which all give a negative answer to Happel's question. This also complements the paper of Bergh and Erdmann [1] in which they extended the example of [2] by producing a family of local algebras of infinite global dimension for which the Hochschild cohomology ring is finite-dimensional. We remark that the algebras of [1,2] are local algebras with a 5-dimensional Hochschild cohomology ring. In this paper we give, for each $N \geqslant 5$, a finite-dimensional algebra with $m = N - 4$ simple modules and of infinite global dimension whose Hochschild cohomology ring is N-dimensional.

For a finite-dimensional K-algebra $\mathcal{A}$ with Jacobson radical $\mathfrak{r}$, the Hochschild cohomology ring of $\mathcal{A}$ is given by $\mathrm{HH}^*(\mathcal{A}) = \mathrm{Ext}^*_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}) = \oplus_{n\geqslant 0} \mathrm{Ext}^n_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A})$ with the Yoneda product, where $\mathcal{A}^e = \mathcal{A}^{\mathrm{op}} \otimes_K \mathcal{A}$ is the enveloping algebra of $\mathcal{A}$. Since all tensors are over the field K we write $\otimes$ for $\otimes_K$ throughout. We denote by $\mathcal{N}$ the ideal of $\mathrm{HH}^*(\mathcal{A})$ which is generated by all homogeneous nilpotent elements. Thus $\mathrm{HH}^*(\mathcal{A})/\mathcal{N}$ is a commutative K-algebra.

The Hochschild cohomology ring modulo nilpotence of $\Lambda_{\mathbf{q}}$, where $\mathbf{q} = (q_0, q_1, \ldots, q_{m-1}) \in (K^*)^m$, was explicitly determined in [13], where it was shown that $\mathrm{HH}^*(\Lambda_{\mathbf{q}})/\mathcal{N}$ is a commutative finitely generated K-algebra of Krull dimension 2 when $q_0\cdots q_{m-1}$ is a root of unity, and is K otherwise. Note that, by setting $\mathbf{z} = (q_0q_1\cdots q_{m-1}, 1, \ldots, 1)$, we have an isomorphism $\Lambda_{\mathbf{q}} \cong \Lambda_{\mathbf{z}}$ induced by $a_i \mapsto q_0q_1\cdots q_ia_i,\ \bar{a}_i \mapsto \bar{a}_i$. However, for ease of notation, we will consider the algebra in the form $\Lambda_{\mathbf{q}} = K\mathcal{Q}/I_{\mathbf{q}}$ with $\mathbf{q} = (q_0, q_1, \ldots, q_{m-1}) \in (K^*)^m$. It was shown by Erdmann and Solberg in [6, Proposition 2.1] that, if $q_0q_1\cdots q_{m-1}$ is a root of unity, then the finite generation condition **(Fg)** holds, so that $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$ is a finitely generated Noetherian K-algebra. (See [5,6,11] for more details on the finite generation condition **(Fg)** and the rich theory of support varieties for modules over algebras which satisfy this condition.)

The aim of this paper is to determine $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$ for each $m \geqslant 1$ in the case where $q_0q_1\cdots q_{m-1}$ is not a root of unity, and in particular to show that this ring is finite-dimensional. Thus we set $\zeta = q_0q_1\cdots q_{m-1} \in K^*$ and assume that ζ is not a root of unity.

1. The projective resolution of $\Lambda_{\mathbf{q}}$

A minimal projective bimodule resolution for Λ was given in [12, Theorem 1.2]. Since $\Lambda_{\mathbf{q}}$ is a Koszul algebra, we again use the approach of [7] and [8] and modify the resolution for Λ from [12] to give a minimal projective bimodule resolution (P^*, ∂^*) for $\Lambda_{\mathbf{q}}$.

We recall from [9], that the multiplicity of $\Lambda_{\mathbf{q}}e_i\otimes e_j\Lambda_{\mathbf{q}}$ as a direct summand of P^n is equal to the dimension of $\mathrm{Ext}^n_{\Lambda_{\mathbf{q}}}(S_i, S_j)$, where S_i, S_j are the simple right $\Lambda_{\mathbf{q}}$-modules corresponding to the vertices i, j respectively. Thus the projective bimodules P^n are the same as those in the minimal projective bimodule resolution for Λ, and we have, for $n \geqslant 0$, that

$$P^n = \bigoplus_{i=0}^{m-1}\left[\bigoplus_{r=0}^{n} \Lambda_{\mathbf{q}}e_i \otimes e_{i+n-2r}\Lambda_{\mathbf{q}}\right].$$

Write $\mathfrak{o}(\alpha)$ for the trivial path corresponding to the origin of the arrow α, so that $\mathfrak{o}(a_i) = e_i$ and $\mathfrak{o}(\bar{a}_i) = e_{i+1}$. We write $\mathfrak{t}(\alpha)$ for the trivial path corresponding to the terminus of the arrow α, so that $\mathfrak{t}(a_i) = e_{i+1}$ and $\mathfrak{t}(\bar{a}_i) = e_i$. Recall that a non-zero element $r \in K\mathcal{Q}$ is said to be uniform if there are vertices v, w such that $r = vr = rw$. We then write $v = \mathfrak{o}(r)$ and $w = \mathfrak{t}(r)$.

In [8], the authors give an explicit inductive construction of a minimal projective resolution of $\mathcal{A}/\mathfrak{r}$ as a right $\mathcal{A}$-module, for a finite-dimensional K-algebra $\mathcal{A}$. For $\mathcal{A} = K\Gamma/I$ and finite-dimensional, they define g^0 to be the set of vertices of Γ, g^1 to be the set of arrows of Γ, and g^2 to be a minimal set of uniform relations in the generating set of I, and then show that there are subsets g^n, $n \geqslant 3$, of $K\Gamma$, where $x \in g^n$ are uniform elements satisfying $x = \sum_{y\in g^{n-1}} yr_y = \sum_{z\in g^{n-2}} zs_z$ for unique $r_y, s_z \in K\Gamma$, which can be chosen in such a way that there is a minimal projective $\mathcal{A}$-resolution of the form

$$\cdots \to Q^4 \to Q^3 \to Q^2 \to Q^1 \to Q^0 \to \mathcal{A}/\mathfrak{r} \to 0$$

having the following properties:

(1) for each $n \geqslant 0$, $Q^n = \coprod_{x\in g^n} \mathfrak{t}(x)\mathcal{A}$,

(2) for each $x \in g^n$, there are unique elements $r_j \in K\Gamma$ with $x = \sum_j g_j^{n-1} r_j$,
(3) for each $n \geqslant 1$, using the decomposition of (2), for $x \in g^n$, the map $Q^n \to Q^{n-1}$ is given by

$$\mathfrak{t}(x)a \mapsto \sum_j r_j \mathfrak{t}(x)a \quad \text{for all } a \in \mathcal{A},$$

where the elements of the set g^n are labelled by $g^n = \{g_j^n\}$. Thus the maps in this minimal projective resolution of $\mathcal{A}/\mathfrak{r}$ as a right $\mathcal{A}$-module are described by the elements r_j which are uniquely determined by (2).

For our algebra $\Lambda_{\mathbf{q}}$, we now define sets g^n in the path algebra $K\mathcal{Q}$ which we will use to label the generators of P^n.

Definition 1.1. For the algebra $\Lambda_{\mathbf{q}}$, $i = 0, 1, \ldots, m-1$ and $r = 0, 1, \ldots, n$, define $g_{0,i}^0 = e_i$ and, inductively for $n \geqslant 1$,

$$g_{r,i}^n = g_{r,i}^{n-1} a_{i+n-2r-1} + (-1)^n q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r} g_{r-1,i}^{n-1} \bar{a}_{i+n-2r}$$

with the conventions that $g_{-1,i}^{n-1} = 0$ and $g_{n,i}^{n-1} = 0$ for all n, i, and that $q_{i-r+1}q_{i-r+2}\cdots q_{i+n-2r} = 1$ if $r = n$.

Define $g^n = \bigcup_{i=0}^{m-1} \{g_{r,i}^n \mid r = 0, \ldots, n\}$.

It is easy to see, for $n = 1$, that $g_{0,i}^1 = a_i$ and $g_{1,i}^1 = -\bar{a}_{i-1}$, whilst, for $n = 2$, we have $g_{0,i}^2 = a_i a_{i+1}$, $g_{1,i}^2 = q_i a_i \bar{a}_i - \bar{a}_{i-1} a_{i-1}$ and $g_{2,i}^2 = -\bar{a}_{i-1}\bar{a}_{i-2}$. Thus

$$\begin{aligned} g^0 &= \{e_i \mid i = 0, \ldots, m-1\}, \\ g^1 &= \{a_i, -\bar{a}_i \mid i = 0, \ldots, m-1\}, \\ g^2 &= \{a_i a_{i+1}, q_i a_i \bar{a}_i - \bar{a}_{i-1} a_{i-1}, -\bar{a}_{i-1}\bar{a}_{i-2} \text{ for all } i\}, \end{aligned}$$

so that g^2 is a minimal set of uniform relations in the generating set of $I_{\mathbf{q}}$.

Moreover, $g_{r,i}^n \in e_i(K\mathcal{Q})e_{i+n-2r}$, for $i = 0, \ldots, m-1$ and $r = 0, \ldots, n$. Since the elements $g_{r,i}^n$ are uniform elements, we may define $\mathfrak{o}(g_{r,i}^n) = e_i$ and $\mathfrak{t}(g_{r,i}^n) = e_{i+n-2r}$. Then

$$P^n = \bigoplus_{i=0}^{m-1} \left[\bigoplus_{r=0}^{n} \Lambda_{\mathbf{q}} \mathfrak{o}(g_{r,i}^n) \otimes \mathfrak{t}(g_{r,i}^n) \Lambda_{\mathbf{q}} \right].$$

To describe the map $\partial^n \colon P^n \to P^{n-1}$, we need some notation and the following lemma, the proof of which is an easy induction and is left to the reader.

Lemma 1.2. *For the algebra $\Lambda_{\mathbf{q}}$, for $n \geqslant 1$, $i = 0, 1, \ldots, m-1$ and $r = 0, 1, \ldots, n$, we have:*

$$\begin{aligned} g_{r,i}^n &= g_{r,i}^{n-1} a_{i+n-2r-1} + (-1)^n \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r}}_{n-r \text{ terms}} g_{r-1,i}^{n-1} \bar{a}_{i+n-2r} \\ &= (-1)^r \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_i}_{r \text{ terms}} a_i g_{r,i+1}^{n-1} + (-1)^r \bar{a}_{i-1} g_{r-1,i-1}^{n-1} \end{aligned}$$

with the conventions that $g_{-1,i}^n = 0$ and $g_{n,i}^{n-1} = 0$ for all n, i, and that $q_{i-r+1}q_{i-r+2}\cdots q_{i+n-2r} = 1$ if $r = n$ and $q_{i-r+1}q_{i-r+2}\cdots q_i = 1$ if $r = 0$. Thus

$$g_{0,i}^n = g_{0,i}^{n-1} a_{i+n-1} = a_i g_{0,i+1}^{n-1} \quad \text{and} \quad g_{n,i}^n = (-1)^n g_{n-1,i}^{n-1} \bar{a}_{i-n} = (-1)^n \bar{a}_{i-1} g_{n-1,i-1}^{n-1}.$$

In order to define ∂^n for $n \geqslant 1$ in a minimal projective bimodule resolution (P^*, ∂^*) of $\Lambda_{\mathbf{q}}$, we use the following notation. In describing the image of $\mathfrak{o}(g_{r,i}^n) \otimes \mathfrak{t}(g_{r,i}^n)$ under ∂^n in the projective module P^{n-1}, we use subscripts under $\otimes$ to indicate the appropriate summands of the projective module P^{n-1}. Specifically, let $\otimes_r$ denote a term in the summand of P^{n-1} corresponding to $g_{r,-}^{n-1}$, and $\otimes_{r-1}$ denote a term in the summand of P^{n-1} corresponding to $g_{r-1,-}^{n-1}$, where the appropriate index $-$ of the vertex may always be uniquely determined from the context. Indeed, since the relations are uniform along the quiver, we can also take labelling elements defined by a formula independent of i, and hence we omit the index i when it is clear from the context. Recall that nonetheless all tensors are over K.

The algebra $\Lambda_{\mathbf{q}}$ is Koszul, so we now use [7] to give a minimal projective bimodule resolution (P^*, ∂^*) of $\Lambda_{\mathbf{q}}$. We define the map $\partial^0 \colon P^0 \to \Lambda_{\mathbf{q}}$ to be the multiplication map. For $n \geqslant 1$, we define the map $\partial^n : P^n \to P^{n-1}$ as follows:

$$\begin{aligned} \partial^n \colon \mathfrak{o}(g_{r,i}^n) \otimes \mathfrak{t}(g_{r,i}^n) \mapsto\ & (e_i \otimes_r a_{i+n-2r-1} + (-1)^n \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_{i+n-2r}}_{n-r \text{ terms}} e_i \otimes_{r-1} \bar{a}_{i+n-2r}) \\ & + (-1)^n ((-1)^r \underbrace{q_{i-r+1} q_{i-r+2} \cdots q_i}_{r \text{ terms}} a_i \otimes_r e_{i+n-2r} + (-1)^r \bar{a}_{i-1} \otimes_{r-1} e_{i+n-2r}). \end{aligned}$$

Using our conventions, the degenerate cases $r = 0$ and $r = n$ simplify to

$$\partial^n : \mathfrak{o}(g^n_{0,i}) \otimes \mathfrak{t}(g^n_{0,i}) \mapsto e_i \otimes_0 a_{i+n-1} + (-1)^n a_i \otimes_0 e_{i+n}$$

where the first term is in the summand corresponding to $g^{n-1}_{0,i}$ and the second term is in the summand corresponding to $g^{n-1}_{0,i+1}$, whilst

$$\partial^n : \mathfrak{o}(g^n_{n,i}) \otimes \mathfrak{t}(g^n_{n,i}) \mapsto (-1)^n e_i \otimes_{n-1} \bar{a}_{i-n} + \bar{a}_{i-1} \otimes_{n-1} e_{i-n},$$

with the first term in the summand corresponding to $g^{n-1}_{n-1,i}$ and the second term in the summand corresponding to $g^{n-1}_{n-1,i-1}$. The following result shows that (P^*, ∂^*) is a complex. The proof is a matter of applying the two different recursive formulae for $g^n_{r,i}$, and the details are left to the reader.

Lemma 1.3. *We have $\partial^n \circ \partial^{n+1} = 0$.*

The next theorem is now immediate from [7, Theorem 2.1].

Theorem 1.4. *With the above notation, (P^*, ∂^*) is a minimal projective bimodule resolution of $\Lambda_{\mathbf{q}}$.*

2. The Hochschild cohomology ring of $\Lambda_{\mathbf{q}}$

We consider the complex $\mathrm{Hom}_{\Lambda^e_{\mathbf{q}}}(P^n, \Lambda_{\mathbf{q}})$. All our homomorphisms are $\Lambda^e_{\mathbf{q}}$-homomorphisms and so we write $\mathrm{Hom}(-,-)$ for $\mathrm{Hom}_{\Lambda^e_{\mathbf{q}}}(-,-)$. We start by computing the dimension of the space $\mathrm{Hom}(P^n, \Lambda_{\mathbf{q}})$ for each $n \geqslant 0$. For $m \geqslant 3$, we write $n = pm + t$ where $p \geqslant 0$ and $0 \leqslant t \leqslant m-1$.

Lemma 2.1. *Suppose $m \geqslant 3$ and $n = pm + t$ where $p \geqslant 0$ and $0 \leqslant t \leqslant m-1$. Then*

$$\dim_K \mathrm{Hom}(P^n, \Lambda_{\mathbf{q}}) = \begin{cases} (4p+2)m & \text{if } t \neq m-1 \\ (4p+4)m & \text{if } t = m-1. \end{cases}$$

If $m = 1$ or $m = 2$ then

$$\dim_K \mathrm{Hom}(P^n, \Lambda_{\mathbf{q}}) = 4(n+1).$$

The proof is as for the non-deformed case (with $q_0 = q_1 = \cdots = q_{m-1} = 1$) in [12, Lemma 1.7] and where $N = 1$, and so is omitted.

Applying $\mathrm{Hom}(-, \Lambda_{\mathbf{q}})$ to the resolution (P^*, ∂^*) gives the complex $(\mathrm{Hom}(P^n, \Lambda_{\mathbf{q}}), d^n)$ where $d^n : \mathrm{Hom}(P^n, \Lambda_{\mathbf{q}}) \to \mathrm{Hom}(P^{n+1}, \Lambda_{\mathbf{q}})$ is induced by the map $\partial^{n+1} : P^{n+1} \to P^n$. The nth Hochschild cohomology group $\mathrm{HH}^n(\Lambda_{\mathbf{q}})$ is then given by $\mathrm{HH}^n(\Lambda_{\mathbf{q}}) = \mathrm{Ker}\, d^n / \mathrm{Im}\, d^{n-1}$. We start by calculating the dimensions of $\mathrm{Ker}\, d^n$ and $\mathrm{Im}\, d^{n-1}$. We consider the cases $m \geqslant 3$ and $m = 2$ separately, and recall that the Hochschild cohomology of $\Lambda_{\mathbf{q}}$ in the case $m = 1$ was fully determined in [2].

We keep to the notational conventions of [12]. So far, we have simplified notation by denoting the idempotent $\mathfrak{o}(g^n_{r,i}) \otimes \mathfrak{t}(g^n_{r,i})$ of the summand $\Lambda_{\mathbf{q}}\mathfrak{o}(g^n_{r,i}) \otimes \mathfrak{t}(g^n_{r,i})\Lambda_{\mathbf{q}}$ of P^n uniquely by $e_i \otimes_r e_{i+n-2r}$ where $0 \leqslant i \leqslant m-1$. However, even this notation with subscripts under the tensor product symbol becomes cumbersome in computations. Thus we now recall the additional conventions of [12, 1.3] which we keep throughout the rest of the paper. Specifically, since $e_{i+n-2r} \in \{e_0, e_1, \ldots, e_{m-1}\}$, it would be usual to reduce the subscript $i + n - 2r$ modulo m. However, to make it explicitly clear to which summand of the projective module P^n we are referring and thus to avoid confusion, whenever we write $e_i \otimes e_{i+k}$ for an element of P^n, we will always have $i \in \{0, 1, \ldots, m-1\}$ and consider $i + k$ as an element of $\mathbb{Z}$, in that $r = (n-k)/2$ and $e_i \otimes e_{i+k} = e_i \otimes_{\frac{n-k}{2}} e_{i+k}$ and thus lies in the $\frac{n-k}{2}$-th summand of P^n. We do not reduce $i + k$ modulo m in any of our computations. In this way, when considering elements in P^n, our element $e_i \otimes e_{i+k}$ corresponds uniquely to the idempotent $\mathfrak{o}(g^n_{r,i}) \otimes \mathfrak{t}(g^n_{r,i})$ of P^n with $r = (n-k)/2$, for each $i = 0, 1, \ldots, m-1$.

With this notation and for future reference, we note that an element $f \in \mathrm{Hom}(P^n, \Lambda_{\mathbf{q}})$ is determined by its image on each $e_i \otimes e_j$ that generates a summand of P^n. Now $f(e_i \otimes e_j) \in e_i \Lambda_{\mathbf{q}} e_j$ and hence can only be non-zero if $i = j$ or if $i = j \pm 1$. For $m \geqslant 3$ and $f \in \mathrm{Hom}(P^n, \Lambda_{\mathbf{q}})$ we may write:

$$\begin{cases} f(e_i \otimes e_{i+\alpha m}) = \sigma^\alpha_i e_i + \tau^\alpha_i \bar{a}_{i-1} a_{i-1}, \\ f(e_i \otimes e_{i+\beta m-1}) = \lambda^\beta_i \bar{a}_{i-1}, \\ f(e_i \otimes e_{i+\gamma m+1}) = \mu^\gamma_i a_i, \end{cases}$$

with coefficients σ^α_i, τ^α_i, λ^β_i and μ^γ_i in K, and appropriate ranges of integers α, β and γ. Specifically, for $\Lambda_{\mathbf{q}} e_i \otimes e_{i+\alpha m} \Lambda_{\mathbf{q}}$ to be a summand of P^n, we require $i + \alpha m = i + n - 2r$ for some $0 \leqslant r \leqslant n$. Similarly we require $i + \beta m - 1 = i + n - 2r$ and $i + \gamma m + 1 = i + n - 2r$ for some $0 \leqslant r \leqslant n$. The precise ranges of α, β and γ for the case $m \geqslant 3$ are as follows. (We have four cases based on the parity of t and of m, where $n = pm + t$ with $0 \leqslant t \leqslant m-1$.)

If both t and m are even, then we only need α. We have $2p + 1$ values of α with $-p \leqslant \alpha \leqslant p$.

If t is even and m is odd, then we have $p+1$ values of α with $-p \leqslant \alpha \leqslant p$ and $\alpha \equiv p \mod 2$. For $t \leqslant m-2$ we also have p values of β and γ with $-p+1 \leqslant \beta \leqslant p-1$, $-p+1 \leqslant \gamma \leqslant p-1$ and $\beta \equiv \gamma \equiv p+1 \mod 2$. If $t = m-1$ then we get $p+1$ values of β and γ with $-p+1 \leqslant \beta \leqslant p+1$, $-p-1 \leqslant \gamma \leqslant p-1$ and $\beta \equiv \gamma \equiv p+1 \mod 2$.

If t is odd and m is even, then we have no values for α. For $t \leqslant m-2$ we have $2p+1$ values of β and γ with $-p \leqslant \beta \leqslant p$ and $-p \leqslant \gamma \leqslant p$. If $t = m-1$ then we get $2p+2$ values of β and γ with $-p \leqslant \beta \leqslant p+1$ and $-p-1 \leqslant \gamma \leqslant p$.

If t is odd and m is odd, then we have p values of α with $-p+1 \leqslant \alpha \leqslant p-1$ and $\alpha \equiv p+1 \mod 2$. We also have $p+1$ values of β and γ with $-p \leqslant \beta \leqslant p$, $-p \leqslant \gamma \leqslant p$ and $\beta \equiv \gamma \equiv p \mod 2$.

We consider the case $m = 2$ in Section 3 and now determine $\operatorname{Ker} d^n$ when $m \geqslant 3$.

Let $f \in \operatorname{Hom}(P^n, \Lambda_{\mathbf{q}})$ and suppose $f \in \operatorname{Ker} d^n$ so that $d^n(f) = f \circ \partial^{n+1} \in \operatorname{Hom}(P^{n+1}, \Lambda_{\mathbf{q}})$. Assume $m \geqslant 3$ and write $n = pm + t$ with $0 \leqslant t \leqslant m-1$. We evaluate $d^n(f)$ at $e_i \otimes e_{i+n+1-2r}$ for $r = 0, \ldots, n+1$. We have three separate cases for r to consider.

We first consider $r = 0$. Then, for each $i = 0, \ldots, m-1$ we have:

$$d^n(f)(e_i \otimes e_{i+n+1}) = \begin{cases} (q_i\lambda_i^{p+1} - (-1)^n\lambda_{i+1}^{p+1})a_i\bar{a}_i & \text{if } t = m-1 \\ (\sigma_i^p - (-1)^n\sigma_{i+1}^p)a_i & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus if $f \in \operatorname{Ker} d^n$ and $t = m-1$ this gives the condition

$$\lambda_{i+1}^{p+1} = (-1)^n q_i \lambda_i^{p+1} = (-1)^{2n} q_i q_{i-1} \lambda_{i-1}^{p+1} = \cdots = (-1)^{mn} q_i q_{i-1} q_{i-2} \cdots q_{i-m+1} \lambda_{i+1}^{p+1}$$

and hence $\lambda_{i+1}^{p+1} = (-1)^{mn}\zeta\lambda_{i+1}^{p+1}$. But we assumed that ζ is not a root of unity and thus there are no non-trivial solutions for λ_{i+1}^{p+1}, that is, $\lambda_i^{p+1} = 0$ for all i.

If $f \in \operatorname{Ker} d^n$ and $t = 0$ this gives, after iteration, the condition $\sigma_{i+1}^p = (-1)^{mn}\sigma_{i+1}^p$ and so to get non-trivial solutions for σ_{i+1}^p we need $(-1)^{mn} = 1$. Now note that each σ_i^p is determined by the others, so we need only determine one of them, say σ_0^p. Then we will have a free choice for σ_0^p if mn is even or $\operatorname{char} K = 2$, but $\sigma_0^p = 0$ (and hence $\sigma_i^p = 0$ for all i) if mn is odd and $\operatorname{char} K \neq 2$.

So if $r = 0$ then, for f to be in $\operatorname{Ker} d^n$, we have the conditions:

$$\begin{cases} \lambda_i^{p+1} = 0 & \text{if } t = m-1 \\ \sigma_i^p = 0 & \text{if } t = 0 \text{ and } (-1)^{mn} \neq 1 \\ \sigma_i^p = (-1)^{in}\sigma_0^p & \text{if } t = 0 \text{ and } (-1)^{mn} = 1 \end{cases}$$

for all $i = 0, \ldots, m-1$.

We next consider $r = n+1$. A similar analysis to the $r = 0$ case yields the conditions:

$$\begin{cases} \mu_i^{-p-1} = 0 & \text{if } t = m-1 \\ \sigma_i^{-p} = 0 & \text{if } t = 0 \text{ and } (-1)^{mn} \neq 1 \\ \sigma_i^{-p} = (-1)^{in}\sigma_0^{-p} & \text{if } t = 0 \text{ and } (-1)^{mn} = 1 \end{cases}$$

for all $i = 0, \ldots, m-1$.

We now do the generic case for r with $1 \leqslant r \leqslant n$. We have

$$d^n(f)(e_i \otimes e_{i+n+1-2r}) = \begin{cases} \sigma_i^\alpha a_i - (-1)^{n+r} q_{i-r+1}q_{i-r+2}\cdots q_i \sigma_{i+1}^\alpha a_i & \text{if } n-2r = \alpha m \\ (q_i\lambda_i^\beta - (-1)^n q_{i-r+1}q_{i-r+2}\cdots q_{i+n-2r+1}\mu_i^\beta)a_i\bar{a}_i \\ \quad -(-1)^{n+r}(q_{i-r+1}q_{i-r+2}\cdots q_i\lambda_{i+1}^\beta + q_i\mu_{i-1}^\beta)a_i\bar{a}_i & \text{if } n-2r = \beta m - 1 \\ -(-1)^n q_{i-r+1}q_{i-r+2}\cdots q_{i+n-2r+1}\sigma_i^\alpha \bar{a}_{i-1} - (-1)^{n+r}\sigma_{i-1}^\alpha \bar{a}_{i-1} & \text{if } n-2r = \alpha m - 2 \\ 0 & \text{otherwise.} \end{cases}$$

For $n-2r = \alpha m$ we get a similar situation to the $r = 0$ and $t = m-1$ case. After iteration $\sigma_i^\alpha = (-1)^{mn+mr}\zeta^r\sigma_i^\alpha$. Hence (by assumption on ζ) $\sigma_i^\alpha = 0$ for all i and all α with $n-2r = \alpha m$.

For $n-2r = \beta m - 1$, the condition that f is in $\operatorname{Ker} d^n$ yields m equations in the $2m$ variables $\lambda_i^\beta, \mu_i^\beta$ where $i = 0, \ldots, m-1$. These may be rewritten so that all the λ_i^β are in terms of $\lambda_0^\beta, \mu_0^\beta, \ldots, \mu_{m-1}^\beta$. We may then write λ_0^β in terms of $\mu_0^\beta, \ldots, \mu_{m-1}^\beta$, provided that the coefficient of λ_0^β is non-zero. Specifically, if $r \neq 1$, then the equations give

$$\lambda_0^\beta = (-1)^{(n+r)m}(\zeta^{-1})^{r-1}\lambda_0^\beta + \text{ terms in } \mu_0^\beta, \ldots, \mu_{m-1}^\beta.$$

Since ζ is not a root of unity, it follows that we may write λ_0^β in terms of $\mu_0^\beta, \ldots, \mu_{m-1}^\beta$. On the other hand, suppose $r = 1$. Here the original equations reduce to

$$\lambda_{i+1}^\beta = (-1)^{n+1}\lambda_i^\beta + \underbrace{q_{i+1}q_{i+2}\cdots q_{i+n-1}}_{n-1 \text{ terms}}\mu_i^\beta - \mu_{i-1}^\beta.$$

If n is even and $\operatorname{char} K \neq 2$ then we can again write λ_0^β in terms of $\mu_0^\beta, \ldots, \mu_{m-1}^\beta$. However, if n is odd or $\operatorname{char} K = 2$ then adding these equations together gives

$$\sum_{i=0}^{m-1}((q_{i+1}\cdots q_{i+n-1}) - 1)\mu_i^\beta = 0$$

so that there is a dependency among the μ_i^β but λ_0^β is a free variable if $n \neq 1$. (If $n = 1$ then both sides are zero so there is no dependency.)

Finally, we consider the case where $n - 2r = \alpha m - 2$. Here we have the condition:

$$(-1)^n q_{i-r+1}q_{i-r+2}\cdots q_{i+n-2r+1}\sigma_i^\alpha = -(-1)^{n+r}\sigma_{i-1}^\alpha.$$

This is similar to the $n - 2r = \alpha m$ case and we deduce that all the σ_i^α are zero since ζ is not a root of unity.

Hence, if $1 \leqslant r \leqslant n$ and f is in $\operatorname{Ker} d^n$, we have:

$$\begin{cases} \sigma_i^\alpha = 0 & \text{if } n-2r = \alpha m \text{ or if } n-2r = \alpha m - 2 \\ \dim \operatorname{sp}\{\lambda_0^\beta, \ldots, \lambda_{m-1}^\beta, \mu_0^\beta, \ldots, \mu_{m-1}^\beta\} = m & \text{if } n-2r = \beta m - 1 \text{ and either } r \neq 1 \text{ or } n \neq 1 \\ \dim \operatorname{sp}\{\lambda_0^\beta, \ldots, \lambda_{m-1}^\beta, \mu_0^\beta, \ldots, \mu_{m-1}^\beta\} = m+1 & \text{if } n-2r = \beta m - 1, r = 1 \text{ and } n = 1. \end{cases}$$

We now combine this information to determine $\dim \operatorname{Ker} d^n$.

Proposition 2.2. *For $m \geqslant 3$,*

$$\dim \operatorname{Ker} d^n = \begin{cases} m+1 & \text{if } n = 0 \text{ or } n = 1 \\ (2p+1)m & \text{if } n \geqslant 2. \end{cases}$$

Proof. We first do the cases $n = 0, 1$. If $n = 0$ then $r = 0, 1$ and $\alpha = 0$. Moreover $(-1)^{mn} = 1$, so $\sigma_i^0 = \sigma_0^0$ for all i. Thus $\dim \operatorname{Ker} d^0 = m + 1$. If $n = 1$ then we have $r = 0, 1, 2$ and so $n - 2r = 1, -1, -3$ respectively. The only condition comes from the $r = 1$ case, where we have free variables $\lambda_0^0, \mu_0^0, \ldots, \mu_{m-1}^0$. Thus $\dim \operatorname{Ker} d^0 = m + 1$.

For $n = pm + t \geqslant 2$ there are 4 cases depending on the parity of t and of m. We consider the case where both t and m are even, and leave the other cases to the reader. Here we need only consider the possible values of σ_i^α and τ_i^α with $-p \leqslant \alpha \leqslant p$. We have that all σ_i^α are zero. (Note that if $t = 0$ so $n = pm$ then the $r = 1$ case where $n - 2 = pm - 2$ shows that all the σ_i^p are zero and the $r = n$ case where $n - 2n = -pm$ shows that all the σ_i^{-p} are zero.) Hence the only contribution to the kernel is from the τ_i^α and thus $\dim \operatorname{Ker} d^n = (2p + 1)m$. □

Using the rank-nullity theorem we now get the dimension of $\operatorname{Im} d^{n-1}$.

Proposition 2.3. *For $m \geqslant 3$ and $n = pm + t$ we have*

$$\dim \operatorname{Im} d^{n-1} = \begin{cases} 0 & \text{if } n = 0 \\ m-1 & \text{if } n = 1 \text{ or } n = 2 \\ (2p+1)m & \text{if } n \geqslant 3. \end{cases}$$

We come now to our main results where we determine the Hochschild cohomology ring of the algebra $\Lambda_{\mathbf{q}}$ when ζ is not a root of unity.

Theorem 2.4. *For $m \geqslant 3$,*

$$\dim \operatorname{HH}^n(\Lambda_{\mathbf{q}}) = \begin{cases} m+1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 0 & \text{if } n \geqslant 3. \end{cases}$$

Thus $\operatorname{HH}^*(\Lambda_{\mathbf{q}})$ is a finite-dimensional algebra of dimension $m + 4$.

Theorem 2.5. *For $m \geqslant 3$, we have*

$$\operatorname{HH}^*(\Lambda_{\mathbf{q}}) \cong K[x_0, x_1, \ldots, x_{m-1}]/(x_i x_j) \times_K \bigwedge(u_1, u_2)$$

where $\times_K$ denotes the fibre product over K, $\bigwedge(u_1, u_2)$ is the exterior algebra on the generators u_1 and u_2, the x_i are in degree 0, and the u_i are in degree 1.

Proof. Since $\mathrm{HH}^0(\Lambda_{\mathbf{q}})$ is the centre $Z(\Lambda_{\mathbf{q}})$, it is clear that $\mathrm{HH}^0(\Lambda_{\mathbf{q}})$ has K-basis $\{1, x_0, \ldots, x_{m-1}\}$ where $x_i = a_i\bar{a}_i$. Thus $\mathrm{HH}^0(\Lambda_{\mathbf{q}}) = K[x_0, x_1, \ldots, x_{m-1}]/(x_ix_j)$.

Define bimodule maps $u_1, u_2 : P^1 \to \Lambda_{\mathbf{q}}$ by

$$u_1 : \quad \begin{cases} \mathfrak{o}(g^1_{0,i}) \otimes \mathfrak{t}(g^1_{0,i}) & \mapsto \quad a_i \quad \text{for all } i = 0, 1, \ldots, m-1 \\ \text{else} & \mapsto \quad 0, \end{cases}$$

$$u_2 : \quad \begin{cases} \mathfrak{o}(g^1_{0,m-1}) \otimes \mathfrak{t}(g^1_{0,m-1}) & \mapsto \quad a_{m-1} \\ \mathfrak{o}(g^1_{1,0}) \otimes \mathfrak{t}(g^1_{1,0}) & \mapsto \quad \bar{a}_{m-1} \\ \text{else} & \mapsto \quad 0. \end{cases}$$

It is straightforward to show that these maps are in $\operatorname{Ker} d^1$ and that they represent linearly independent elements in $\mathrm{HH}^1(\Lambda_{\mathbf{q}})$ which we also denote by u_1 and u_2. Hence $\{u_1, u_2\}$ is a K-basis for $\mathrm{HH}^1(\Lambda_{\mathbf{q}})$.

In order to show that u_1u_2 represents a non-zero element of $\mathrm{HH}^2(\Lambda_{\mathbf{q}})$, we define bimodule maps $\mathcal{L}^0(u_2) : P^1 \to P^0$ and $\mathcal{L}^1(u_2) : P^2 \to P^1$ by

$$\mathcal{L}^0(u_2) : \quad \begin{cases} \mathfrak{o}(g^1_{0,m-1}) \otimes \mathfrak{t}(g^1_{0,m-1}) & \mapsto \quad a_{m-1} \otimes e_0 \\ \mathfrak{o}(g^1_{1,0}) \otimes \mathfrak{t}(g^1_{1,0}) & \mapsto \quad \bar{a}_{m-1} \otimes e_{m-1} \\ \text{else} & \mapsto \quad 0, \end{cases}$$

$$\mathcal{L}^1(u_2) : \quad \begin{cases} \mathfrak{o}(g^2_{0,m-1}) \otimes \mathfrak{t}(g^2_{0,m-1}) & \mapsto \quad a_{m-1}\mathfrak{o}(g^1_{0,0}) \otimes \mathfrak{t}(g^1_{0,0}) \\ \mathfrak{o}(g^2_{1,0}) \otimes \mathfrak{t}(g^2_{1,0}) & \mapsto \quad \bar{a}_{m-1}\mathfrak{o}(g^1_{0,m-1}) \otimes \mathfrak{t}(g^1_{0,m-1}) \\ \mathfrak{o}(g^2_{1,m-1}) \otimes \mathfrak{t}(g^2_{1,m-1}) & \mapsto \quad -q_{m-1}a_{m-1}\mathfrak{o}(g^1_{1,0}) \otimes \mathfrak{t}(g^1_{1,0}) \\ \mathfrak{o}(g^2_{2,0}) \otimes \mathfrak{t}(g^2_{2,0}) & \mapsto \quad -\bar{a}_{m-1}\mathfrak{o}(g^1_{1,m-1}) \otimes \mathfrak{t}(g^1_{1,m-1}) \\ \text{else} & \mapsto \quad 0. \end{cases}$$

Then the following diagram is commutative

$$\begin{array}{ccccc} P^2 & \xrightarrow{\partial^2} & P^1 & & \\ \downarrow{\scriptstyle \mathcal{L}^1(u_2)} & & \downarrow{\scriptstyle \mathcal{L}^0(u_2)} & \searrow{\scriptstyle u_2} & \\ P^1 & \xrightarrow{\partial^1} & P^0 & \longrightarrow & \Lambda_{\mathbf{q}} \end{array}$$

where $P^0 \to \Lambda_{\mathbf{q}}$ is the multiplication map. Thus the element $u_1u_2 \in \mathrm{HH}^2(\Lambda_{\mathbf{q}})$ is represented by the map $u_1 \circ \mathcal{L}^1(u_2) : P^2 \to \Lambda_{\mathbf{q}}$, that is, by the map

$$\begin{cases} \mathfrak{o}(g^2_{1,0}) \otimes \mathfrak{t}(g^2_{1,0}) & \mapsto \quad \bar{a}_{m-1}a_{m-1} \\ \text{else} & \mapsto \quad 0. \end{cases}$$

Since this map is not in $\operatorname{Im} d^1$, it follows that u_1u_2 is non-zero in $\mathrm{HH}^2(\Lambda_{\mathbf{q}})$ and hence $\mathrm{HH}^2(\Lambda_{\mathbf{q}}) = \mathrm{sp}\{u_1u_2\}$.

From the lifting $\mathcal{L}^1(u_2)$ it is easy to see that u_2^2 represents the zero element in $\mathrm{HH}^2(\Lambda_{\mathbf{q}})$, and a similar calculation shows that u_1^2 also represents the zero element in $\mathrm{HH}^2(\Lambda_{\mathbf{q}})$. (Note that although it is immediate from the graded commutativity of $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$ that $u_1^2 = 0 = u_2^2$ in $\mathrm{HH}^2(\Lambda_{\mathbf{q}})$ when $\operatorname{char} K \neq 2$, this direct calculation is required when $\operatorname{char} K = 2$.)

Thus we have elements u_1 and u_2 in $\mathrm{HH}^1(\Lambda_{\mathbf{q}})$ which are annihilated by all the $x_i \in \mathrm{HH}^0(\Lambda_{\mathbf{q}})$ and with $u_1^2 = 0 = u_2^2$ and $u_1u_2 = -u_2u_1$ (with the latter by the graded-commutativity of $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$). Thus

$$\mathrm{HH}^*(\Lambda_{\mathbf{q}}) \cong K[x_0, x_1, \ldots, x_{m-1}]/(x_ix_j) \times_K \bigwedge(u_1, u_2)$$

where $\times_K$ denotes the fibre product over K, $\bigwedge(u_1, u_2)$ is the exterior algebra on the generators u_1 and u_2, the x_i are in degree 0, and the u_i are in degree 1. □

3. The case $m = 2$

We assume that $m = 2$ throughout this section. Recall from Lemma 2.1 that $\dim_K \operatorname{Hom}(P^n, \Lambda_{\mathbf{q}}) = 4(n+1)$. For $f \in \operatorname{Hom}(P^n, \Lambda_{\mathbf{q}})$ we may write:

$$\begin{cases} f(e_i \otimes e_{i+2\alpha}) = \sigma_i^\alpha e_i + \tau_i^\alpha \bar{a}_{i-1}a_{i-1} & \text{if } n \text{ even} \\ f(e_i \otimes e_{i+2\beta+1}) = \lambda_i^\beta \bar{a}_{i-1} + \mu_i^\beta a_i & \text{if } n \text{ odd,} \end{cases}$$

with coefficients σ_i^α, τ_i^α, λ_i^β and μ_i^β in K. The choices of α and β are:

$$\begin{cases} -p \leq \alpha \leq p & \text{if } n \text{ is even} \\ -p-1 \leq \beta \leq p & \text{if } n \text{ is odd,} \end{cases}$$

which gives $n+1$ values in each case. A similar analysis to before yields the following result.

Proposition 3.1. *For $m = 2$ and $n = 2p + t$ with $t = 0, 1$, we have*

$$\dim \operatorname{Ker} d^n = \begin{cases} 3 & \text{if } n = 0 \text{ or } n = 1 \\ 2(2p+1) & \text{if } n \geqslant 2, \end{cases}$$

and

$$\dim \operatorname{Im} d^n = \begin{cases} 1 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 2(2p+3) & \text{if } n \geqslant 2 \text{ and } n \text{ odd} \\ 2(2p+1) & \text{if } n \geqslant 2 \text{ and } n \text{ even}. \end{cases}$$

Noting that $\dim \mathrm{HH}^0(\Lambda_{\mathbf{q}}) = 3 = m + 1$, we combine these results with Theorem 2.4 to give the following theorem.

Theorem 3.2. *For $m \geqslant 2$,*

$$\dim \mathrm{HH}^n(\Lambda_{\mathbf{q}}) = \begin{cases} m+1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 0 & \text{if } n \geqslant 3. \end{cases}$$

Thus $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$ is a finite-dimensional algebra of dimension m+4.

It can be verified directly that the proof of Theorem 2.5 also holds when $m = 2$. Hence we have the following result which describes the ring structure of $\mathrm{HH}^*(\Lambda_{\mathbf{q}})$ when $m = 2$ and ζ is not a root of unity.

Theorem 3.3. *For $m = 2$, we have*

$$\mathrm{HH}^*(\Lambda_{\mathbf{q}}) \cong K[x_0, x_1]/(x_i x_j) \times_K \bigwedge(u_1, u_2)$$

where $\times_K$ denotes the fibre product over K, $\bigwedge(u_1, u_2)$ is the exterior algebra on the generators u_1 and u_2, and the elements x_0, x_1 are in degree 0 and u_1, u_2 in degree 1.

We end by remarking that we have exhibited self-injective algebras whose Hochschild cohomology ring is of arbitrarily large, but nevertheless finite, dimension. The case $m = 1$ was studied in [2] where it was shown that the Hochschild cohomology ring is 5-dimensional when ζ is not a root of unity. Thus, for all $m \geqslant 1$, we now have self-injective algebras whose Hochschild cohomology ring is $(m + 4)$-dimensional. Hence, for each $N \geqslant 5$ we have an algebra with $N - 4$ simple modules, of dimension $4(N - 4)$ and with infinite global dimension whose Hochschild cohomology ring is N-dimensional.

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