Thermoelastic contact instability of a functionally graded layer and a homogeneous half-plane

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ABSTRACT

The conductive heat transfer between two elastic bodies in the static contact can cause the system to be unstable due to the interaction between the thermoelastic distortion and pressure-dependent thermal contact resistance. This paper investigates the thermoelastic contact instability of a functionally graded material (FGM) layer and a homogeneous half-plane using the perturbation method. The FGM layer and half-plane are exposed to a uniform heat flux and are pressed together by a uniform pressure. The material properties of the FGM layer vary exponentially along the thickness direction. The characteristic equation governing the thermoelastic stability behavior is obtained to determine the stability boundary. The effects of the gradient index, layer thickness and material combination on the critical heat flux are discussed in detail through a parametric study. Results indicate that the thermoelastic stability behavior can be modified by adjusting the gradient index of the FGM layer.

1. Introduction

When two elastic bodies are pressed together or sliding against each other, the heat flux across the interface or the frictional heat generation can cause the thermoelastic distortion, and hence modify the contact pressure distribution and the content of the contact area. This feedback process is generally found to be unstable and is known as the thermoelastic instability. Thermoelastic instability can be identified as two distinct categories. The first category is described in the sliding contact system involving the frictional heat, and also called as the frictionally-excited thermoelastic instability (Barber, 1969). The second category is found in the static contact system, where the heat flux transfers across the contact interface, due to the pressure-dependent thermal contact resistance (Thomas and Probert, 1970; Yigit and Barber, 1994). Here, we call it as the static thermoelastic instability. The behavior of the thermoelastic instability is of considerable importance in many industrial settings, such as castings, moulding, valves, pistons, thermostats, thermal expansion of railways, cylinder heads, etc (Lee and Dinwiddie, 1998). The thermoelastic instability problems of these two categories have been concerned by many investigators, which will be reviewed briefly.

1.1. Frictionally-excited thermoelastic instability

If the sliding speed is sufficiently high, both steady-state and transient solutions can be unstable in the sense that an arbitrarily small perturbation in the initial condition can cause large changes in the subsequent behavior and finally result in the frictionally excited thermoelastic instability (Dow and Burton, 1972). Burton et al. (1973) developed the perturbation method to investigate the contact stability between two sliding tubes. With this method, Lee and Barber (1993) investigated the frictionally-excited thermoelastic instability in automotive disk brake systems. The system was modeled as a layer with the finite thickness sliding between two half-plane pressed by a uniform pressure. Yi et al. (1999) used the finite element method to reduce the problem of the thermoelastic instability for a brake disk to an eigenvalue problem for the critical speed. They also explored the effect of increasing geometric complexity on the critical speeds and the associated mode shapes. Yi et al. (2000) proposed the Fourier reduction method to obtain a remarkably efficient solution of the frictional thermoelastic instability problem for systems with the axisymmetric geometry. Lee (2000) studied the frictionally-excited thermoelastic instability in automotive drum brake systems with one side frictional heating model. The effect of the friction coefficient and brake material properties on the critical speed was examined. Decuzzi et al. (2001) considered the frictionally-excited thermoelastic instability in multi-disk clutches and brakes by introducing a two...
1.2. Static thermoelastic instability

When the heat is conducted across an interface between two bodies in the static contact, the system could also show unstable due to the thermoelastic distortion and pressure-dependent thermal contact resistance. Barber and his co-authors have achieved some pioneering works in the static thermoelastic instability. Barber (1978) gave an assumption of the thermal contact resistance varying with the contact pressure to avoid certain problems in which a cooled rigid punch indents an elastic half-space have no steady state solution. Barber (1987) analyzed the stability of nominally uniform contact between two elastic half-planes by assuming the pressure-dependent thermal contact resistance. Afferrante and Barber (1990) presented the stability criterion for the thermoelastic contact of two dissimilar materials, and found that the material combinations can be classified into five categories depending on the ratios of the thermal conductivities, diffusivities and distortivities. By extending Zhang and Barber’s results, Ye and Barber (1991) studied the contact stability between a layer and a half-plane taking into account the effect of the layer thickness. Li and Barber (1997) discussed the thermoelastic stability of a system consisting of two layers in contact by using the perturbation method. Moreover, Schade and his co-authors conducted the thermoelastic stability of two bonded half-plane (Schade et al., 2000) and a layer bonded to a half-plane (Schade and Karr, 2002).

Specially, the coupled problem between the frictionally-excited thermoelastic instability and static thermoelastic instability for the homogeneous materials has been investigated by Ciavarella and his co-workers. Ciavarella et al. (2003) considered a thermoelastic rod sliding against a rigid plane with both the frictional heating and thermal contact resistance. Afferrante and Ciavarella (2004a,b) studied the combined effect of pressure-dependent thermal contact resistance and frictional heating in the context of an elastic conducting half-plane sliding against a rigid perfect conductor wall or two half-planes sliding out-of-plane. Afferrante and Ciavarella (2004c) extended the Aldo model to the case of frictional sliding. They found that when the solution was unique, it was always stable. Ciavarella and Barber (2005) presented the stability boundary for the thermoelastic contact of a rectangular elastic block sliding against a rigid wall. This geometry was considered as intermediate between the idealized “Aldo” rod model and continuum solution for the elastic half-plane.

Functionally graded materials (FGMs) are usually a mixture of two distinct material phases with continuously varying volume fractions of constituent materials, hence their effective material properties change in a continuous and smooth manner. FGMs used as coatings or interfacial zones can reduce the magnitude of residual and thermal stresses, mitigate stress concentration, and increase fracture toughness (Suresh and Mortensen, 1997). In the past few years, many experimental and numerical results have shown that a properly controlled material property gradient in FGMs can lead to a significant improvement in the resistance to the contact deformation and damage (Suresh, 2001; Ke and Wang, 2006; Ke and Wang, 2007; Guler and Erdogan, 2004; Guler and Erdogan, 2006; Guler and Erdogan, 2007; El-Borgi et al., 2006; Elloumi et al., 2010; Liu et al., 2012). Recently, many theoretical and numerical results have shown that FGMs have the potential application to improve thermoelastic stability behaviors in brake disk system of cars. Jang and his co-authors presented comprehensive works on the frictionally-excited thermoelastic instability of FGMs, including a stationary FGM layer between two sliding homogeneous layers (Jang and Ahn, 2007), an FGM half-plane sliding against a homogeneous half-plane (Lee and Jang, 2009a) and an FGM layer sliding against two homogeneous half-plane (Lee and Jang, 2009b). Their studies showed that FGMs have a significant effect on the contact stability in the frictional sliding system. An optimal gradient index of FGMs can lead to a maximum critical speed, and therefore increase the stability of the system. Hernik (2009) employed the functionally graded A356R-based composite in brake disk structure to prevent the loss of global stability in contrast with homogeneous A356R composite and stainless steel ASTM321 brake disk which guaranties safety and durability of the braking system. So far, no work was reported on the static thermoelastic instability of FGMs induced by the thermal contact resistance. However, similar with the effect on the frictionally-excited thermoelastic instability, we believe that FGMs can also used to modify the thermoelastic contact stability of the system conducting the heat flux and thus motivate us to investigate the static thermoelastic contact stability of FGMs.

In this paper, the static thermoelastic contact instability of an FGM layer and a homogeneous half-plane under the plane strain state is investigated by using the perturbation method. The FGM layer and homogeneous half-plane are pressed together by a uniform pressure and transmit a uniform heat flux in the thickness direction. The material properties of the FGM layer vary exponentially along the thickness direction. A pressure-dependent thermal contact resistance at the interface is considered because of the imperfect contact between the FGM layer and the half-plane. The characteristic equation is obtained to determine the stability boundary for three types of material combinations. A parametric study is presented to examine the effects of the gradient index, layer thickness and material combination on the critical heat flux.

2. Formulation of the thermoelastic instability problem

Fig. 1 shows the contact between an FGM layer (0< y < h) and a homogeneous half-plane (y = 0) at their common place y = 0. The FGM layer and homogeneous half-plane are pressed together by a uniform pressure p_0 and transmit a uniform heat flux q_0 = q_0 in the positive y direction. The thermoelastic properties of the FGM layer vary along the thickness direction according to the exponential forms as

\[ \mu(y) = \mu_0 e^{\beta y}, \quad \beta = \ln(\mu_l/\mu_h)/h, \]

\[ k(y) = k_0 e^{\delta y}, \quad \delta = \ln(k_l/k_h)/h, \]

\[ \alpha(y) = \alpha_0 e^{\gamma y}, \quad \gamma = \ln(\alpha_l/\alpha_h)/h. \]

![Fig. 1. An FGM layer on a homogeneous half-plane pressed by a uniform pressure and transmitting heat.](image-url)
\[ c(y) = a_0 e^{by}, \quad e = \ln(c_1/c_2)/h, \quad (1d) \]

\[ \rho(y) = \rho_0 e^{by}, \quad \zeta = \ln(\rho_1/\rho_0)/h, \quad (1e) \]

where \( \mu(y), k(y), \lambda(y), c(y) \) and \( \rho(y) \) are the shear modulus, thermal conductivity coefficient, thermal expansion coefficient, specific heat, density, respectively; \( h \) is the thickness of the FGM layer; \( \beta, \gamma, \varepsilon \) and \( \zeta \) are the gradient indexes; the Poisson’s ratio \( \nu \) is assumed as constant for simplicity; subscripts “b” and “t” refer to the bottom and top of the FGM layer, respectively.

### 2.1. Temperature perturbation

The stability of the system can be examined by finding the condition under which a small perturbation in the temperature and stress fields can grow exponentially with time. Therefore, the temperature perturbation can be written in the form as

\[ T_j(x, y, t) = f_j(y)e^{\varepsilon_1x + \varepsilon_2y}, \quad j = 1, 2, \quad (2) \]

where \( \varepsilon = \sqrt{-1}; \) subscripts “1” and “2” indicate the FGM layer and half-plane; \( f_j(y) \) are complex functions of the real variable \( y; m \) is the wave number; and \( b \) is the exponential growth rate, which may be (i) negative for the stable perturbation, (ii) positive for the unstable perturbation, and (iii) zero for the threshold of instability.

The temperature perturbation must satisfy the transient heat conduction equation

\[ \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} + \frac{c_1}{\lambda_1} \frac{\partial T_1}{\partial x} = 0, \quad (3) \]

for the FGM layer, and

\[ \frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} + \frac{c_2}{\lambda_2} \frac{\partial T_2}{\partial x} = 0, \quad (4) \]

for the homogenous half-plane, where \( \lambda_j = k_j/\rho_j c_j (j = 1, 2) \) are the thermal diffusivity coefficient of the FGM layer and half-plane, respectively. Note that the thermal diffusivity coefficient is not constant but changed with the location in the FGM layer. For the mathematical convenience, it is assumed that the gradient indexes of the thermal conductivity, density and specific heat have the relation of \( \delta = \varepsilon + \zeta \) in order to obtain a constant thermal diffusivity coefficient. This assumption makes it possible to solve Eq. (3) analytically. More importantly, our previous study (Liu et al., 2011) also indicated that the graded variation of the thermal diffusivity coefficient has a slight effect on the thermoelastic fields.

Substituting Eq. (2) into Eq. (3) and solving for \( f_1(y) \), we obtain the temperature perturbation for the FGM layer

\[ T_1(x, y, t) = (C_{11} e^{b_1 x} + C_{12} e^{b_2 x}) e^{\varepsilon_1y + \varepsilon_2 y}, \quad (5) \]

where \( C_{11} \) and \( C_{12} \) are unknown constants to be determined,

\[ \eta_{11} = \frac{1}{2} \left[ \delta - \sqrt{\delta^2 + 4 \left( m^2 + \frac{b}{\lambda_1} \right)} \right], \]

\[ \eta_{12} = \frac{1}{2} \left[ \delta + \sqrt{\delta^2 + 4 \left( m^2 + \frac{b}{\lambda_2} \right)} \right]. \quad (6) \]

Similarly, substituting Eq. (2) into Eq. (4) and solving for \( f_2(y) \) yield the temperature perturbation for the homogenous half-plane, which satisfies the regularity condition at infinity: \( T_2 \to 0 \) as \( y \to \infty \). Thus, we have

\[ T_2(x, y, t) = C_{21} e^{b_1 x} e^{\varepsilon_1y + \varepsilon_2y}, \quad (7) \]

where \( C_{21} \) is an unknown constant, and

\[ \eta_{21} = \sqrt{m^2 + \frac{b}{\lambda_2}}. \quad (8) \]

### 2.2. Thermoelastic stress and displacement fields

For both FGM layer and homogenous half-plane, the governing equations of the linear isotropic elastic solid under the plane strain state are given by

\[ \nabla^2 u_j + \frac{2}{\Theta_j - 1} \left( \frac{\partial^2 u_{yj}}{\partial y^2} + \frac{\partial^2 u_{xj}}{\partial x^2} \right) + \frac{\beta}{\Theta_j - 1} \left( \frac{\partial u_{yj}}{\partial y} + \frac{\partial u_{xj}}{\partial x} \right) \]

\[ = \frac{4\lambda_j e^{\varepsilon_2 y}}{\Theta_j - 1} \frac{\partial T_j}{\partial x}, \quad (9) \]

\[ \nabla^2 u_{yy} + \frac{2}{\Theta_j - 1} \left( \frac{\partial^2 u_{yy}}{\partial y^2} + \frac{\partial^2 u_{xx}}{\partial x^2} \right) + \frac{\beta}{\Theta_j - 1} \left[ (1 + \Theta_j) \frac{\partial u_{yy}}{\partial y} + (3 - \Theta_j) \frac{\partial u_{xx}}{\partial x} \right] \]

\[ = \frac{4\lambda_j e^{\varepsilon_2 y}}{\Theta_j - 1} \left( (\beta + \gamma) T_j + \frac{\partial T_j}{\partial y} \right), \quad (10) \]

where \( j = 1, 2; u_{yy} = u_j(x, y, t) \) and \( u_{xj} = u_j(x, y, t) \) are the displacements in the \( x \) and \( y \) directions, respectively; and \( \lambda_1 = \lambda_2 (1 + \nu_1), \quad \lambda_2 = \lambda_2 (1 + \nu_2), \quad \Theta_j = 3 - 4\nu_j, \quad (11) \)

with \( \nu_1 \) and \( \nu_2 \) correspond to the Poisson’s ratio of the FGM layer and the homogenous half-plane, respectively.

Similarly, the displacement fields induced by the temperature perturbation can be taken the forms as

\[ u_j(x, y, t) = U_j(y)e^{b_1 x}, \quad (12) \]

\[ u_{yy}(x, y, t) = U_{yy}(y)e^{b_1 x}, \quad (13) \]

where \( U_j(y) \) and \( U_{yy}(y) \) are complex functions of the real variable \( y \).

#### 2.2.1. The FGM layer

For the FGM layer, the stress fields are defined by

\[ \sigma_{x1} = \frac{(\Theta_1 + 1)\mu_j e^{b_j y}}{\Theta_1 - 1} \left( \frac{\partial u_{y1}}{\partial y} - \frac{\partial u_{x1} + 3 \partial u_{y2}}{\partial x} \right) - \frac{4\lambda_2 \xi_j e^{(\beta + \gamma)y}}{\Theta_1 - 1} T_1, \quad (14) \]

\[ \sigma_{y1} = \mu_j e^{b_j y} \left( \frac{\partial u_{x1}}{\partial x} + \frac{\partial u_{y1}}{\partial y} \right). \quad (15) \]

Substituting Eqs. (5), (12), and (13) into Eqs. (9) and (10), we obtain

\[ U_{x1}(y) + \beta U_{y1}(y) = \frac{(\Theta_1 + 1)m^2}{\Theta_1 - 1} U_{x1}(y) + \text{im} \left[ \beta U_{x1}(y) + \frac{2}{\Theta_1 - 1} U_{y1}(y) \right] \]

\[ = \frac{4i m \xi_j e^{b_2 y}}{\Theta_1 - 1} \left( C_{11} e^{b_1 x} + C_{12} e^{b_2 x} \right), \quad (16) \]

\[ \Theta_1 + 1 U_{x1}(y) + \frac{\beta}{\Theta_1 - 1} U_{y1}(y) - m^2 U_{x1}(y) \]

\[ + \frac{i \text{im}}{\Theta_1 - 1} \left[ \beta (3 - \Theta_1) U_{x1}(y) + 2 U_{y1}(y) \right] \]

\[ = \frac{4\lambda_2 \xi_j e^{(\beta + \gamma)y}}{\Theta_1 - 1} \left[ (\beta + \gamma + \eta_{12}) C_{11} e^{b_1 x} + (\beta + \gamma + \eta_{12}) C_{12} e^{b_2 x} \right]. \quad (17) \]

The solutions of Eqs. (16) and (17) are composed of the homogeneous solution and particular solution. We can write them as

\[ U_{x1}(y) = \sum_{i=1}^{4} A_{1i} e^{b_1^i y} + A_{15} e^{b_{11} y} + A_{16} e^{b_{12} y}, \quad (18) \]

\[ U_{y1}(y) = \sum_{i=1}^{4} B_{1i} e^{b_1^i y} + B_{15} e^{b_{11} y} + B_{16} e^{b_{12} y}. \quad (19) \]
where \( A_{1}, A_{12}, \ldots, A_{16} \) and \( B_{1}, B_{12}, \ldots, B_{16} \) are the unknowns to be determined from boundary conditions, and

\[
\begin{align*}
 f_{11} &= \frac{1}{2} \left[ -\beta - \sqrt{4\beta^{2} + 4\beta^{2} - 4\beta^{2}} \sqrt{3 - \Theta_{1}} \Theta_{1} + 1 \right], \\
 f_{12} &= \frac{1}{2} \left[ -\beta - \sqrt{4\beta^{2} + 4\beta^{2} + 4\beta^{2}} \sqrt{3 - \Theta_{1}} \Theta_{1} + 1 \right], \\
 f_{13} &= \frac{1}{2} \left[ -\beta + \sqrt{4\beta^{2} + 4\beta^{2} - 4\beta^{2}} \sqrt{3 - \Theta_{1}} \Theta_{1} + 1 \right], \\
 f_{14} &= \frac{1}{2} \left[ -\beta + \sqrt{4\beta^{2} + 4\beta^{2} + 4\beta^{2}} \sqrt{3 - \Theta_{1}} \Theta_{1} + 1 \right], \\
 B_{11} &= i\gamma A_{11}, \\
 B_{12} &= i\gamma A_{16}, \\
 B_{13} &= \frac{N_{1} A_{15}}{imM_{1}}, \\
 B_{14} &= \frac{N_{2} A_{16}}{imM_{2}}, \\
 M_{j} &= P_{j} \left( \Theta_{j} + 1 \right) \Theta_{j} + 1 \left[ \Theta_{j} + \Theta_{j} \right] + m^{2} P_{j} Q_{j} + \left[ \Theta_{j} \right] \left( \Theta_{j} \right), \\
 N_{j} &= \Theta_{j} + 1 \left[ \Theta_{j} \right] + 2 \left[ \Theta_{j} \right], \\
 J &= \frac{4imz_{1} M_{1}}{\Theta_{1} - 1} C_{11}, \\
 A_{15} &= \frac{4imz_{1} M_{2}}{\Theta_{1} - 1} C_{12}, \\
 A_{16} &= \frac{4imz_{2} M_{2}}{\Theta_{1} - 1} C_{12}.
\end{align*}
\]

Hence, the displacements of the FGM layer can be further expressed as

\[
\begin{align*}
 u_{31}(x, y, t) &= \sum_{l=1}^{4} A_{11} e^{i\omega_{l} t} + A_{15} e^{i\omega_{1} y} + A_{16} e^{i\omega_{2} y} \right| e^{imx}, \\
 u_{32}(x, y, t) &= \sum_{l=1}^{4} i\gamma A_{11} e^{i\omega_{l} t} + \frac{N_{1} A_{15}}{imM_{1}} A_{12} e^{i\omega_{1} y} + \frac{N_{2} A_{16}}{imM_{2}} A_{12} e^{i\omega_{2} y} \right| e^{imx}.
\end{align*}
\]

Substituting Eqs. (28) and (29) into Eqs. (14) and (15) yields the stress fields of the FGM layer

\[
\begin{align*}
 \sigma_{31}(x, y, t) &= \left\{ \sum_{l=1}^{4} i\gamma A_{11} e^{i\omega_{l} t} + \frac{N_{1} A_{15}}{imM_{1}} A_{12} e^{i\omega_{1} y} + \frac{N_{2} A_{16}}{imM_{2}} A_{12} e^{i\omega_{2} y} \right| e^{imx}, \\
 \sigma_{32}(x, y, t) &= \left\{ \sum_{l=1}^{4} i\gamma A_{11} e^{i\omega_{l} t} + \frac{N_{1} A_{15}}{imM_{1}} A_{12} e^{i\omega_{1} y} + \frac{N_{2} A_{16}}{imM_{2}} A_{12} e^{i\omega_{2} y} \right| e^{imx}.
\end{align*}
\]

2.2.2. The homogeneous half-plane

For the homogeneous half-plane, the stress fields are given by

\[
\sigma_{21}(x, y, t) = \left\{ \sum_{l=1}^{4} i\gamma A_{11} e^{i\omega_{l} t} + \frac{N_{1} A_{15}}{imM_{1}} A_{12} e^{i\omega_{1} y} + \frac{N_{2} A_{16}}{imM_{2}} A_{12} e^{i\omega_{2} y} \right| e^{imx}.
\]

2.3. Boundary conditions

The heat flux and traction at the upper surface of the FGM layer are prescribed to be uniform and hence the perturbations in the quantities are zero, i.e.,
\(\sigma_{y1}(x, h, t) = 0, \quad \sigma_{y2}(x, \dot{h}, t) = 0, \quad q_1(x, h, t) = 0.\) (43)

At the interface \(y = 0,\) the contact is frictionless and the heat flux, stress and displacement are continuous, \(\sigma_{y1}(0, t) = \sigma_{y2}(0, t) = 0, \quad \sigma_{y1}(0, t) = \sigma_{y2}(0, t).\) (44)

\(u_{y1}(0, t) = u_{y2}(0, t), \quad q_1(x, 0, t) = q_2(x, 0, t).\) (45)

where

\[q_1(x, y, t) = -k_1\frac{\partial T_1(x, y, t)}{\partial y} = ik_1\Theta_1 - \frac{4mz_1}{M_1} + \begin{cases} A_{15}\eta_{11}\left(\Theta_2(\gamma + 1) + m(3 - \Theta_1)\right)A_{15} + \\ A_{15}\Theta_1 - \frac{4mz_1}{M_1} + A_{16}\eta_{12}\left(\Theta_2(\gamma + 1) + m(3 - \Theta_1)\right)A_{16} + \\ \frac{4mz_1}{M_1} \\ \end{cases}A_{32}\eta_{21}\left(\Theta_2(\gamma + 1) + m(3 - \Theta_1)\right)A_{32}e^{ib+imx},\] (46)

\[q_2(x, y, t) = -k_2\frac{\partial T_2(x, y, t)}{\partial y} = ik_2\eta_{21}\left(\frac{\Theta_2(\gamma + 1)}{\Theta_2(\gamma + 1) + m(3 - \Theta_1)\right)A_{32}\eta_{21}\left(\Theta_2(\gamma + 1) + m(3 - \Theta_1)\right)A_{32}e^{ib+imx}.\] (47)

The above boundary conditions can lead to a system of eight equations, which are not sufficiently to be solved for nine unknowns \(A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{21}, A_{22}\) and \(A_{23}.\) For the non-trivial solution, the determinant of the coefficient matrix of these nine equations must be zero. Thus, we can obtain the characteristic equation for the exponential growth rate \(b\) as

\[\det\left(\mathbf{g}_{11}, \mathbf{g}_{21}, \mathbf{g}_{31}, \mathbf{g}_{41}, \mathbf{g}_{51}, \mathbf{g}_{61}, \mathbf{g}_{71}, \mathbf{g}_{81}, \mathbf{g}_{91}\right)^{T} = 0,\] (55)

where \(i = 1, 2, \ldots, 9;\) the superscript \(^T\) denotes the transposition of a matrix, and \(\{g_{11}\} - \{g_{91}\}\) are given in the Appendix A.

Introduce the following dimensionless parameters

\[\beta = \frac{b}{m}, \quad \delta = \frac{\delta}{m}, \quad \gamma = \frac{\gamma}{m}, \quad r_1 = \frac{r_2}{m}, \quad r_2 = \frac{r_3}{m}, \quad r_3 = \frac{r_4}{m}, \quad r_4 = \frac{r_5}{m}, \quad r_5 = \frac{r_6}{m}, \quad \theta_1 = \frac{\theta_2}{m}, \quad \theta_2 = \frac{\theta_3}{m}.\] (56)

\[f_{11} = \frac{f_1}{m}, \quad f_{21} = \frac{f_2}{m}, \quad \eta_{11} = \frac{\eta_{11}}{m}, \quad \eta_{12} = \frac{\eta_{12}}{m}, \quad \eta_{21} = \frac{\eta_{21}}{m}.\] (57)

\[P_j = \frac{P_j}{m^2}, \quad Q_j = \frac{Q_j}{m}, \quad N_j = mN_j, \quad M_j = m^2M_j, \quad (j = 1, 2).\] (58)

\[z = \frac{b}{m^2}, \quad H = mh, \quad R = mR, k_0, \quad Q' = -4\tilde{a}_1q_0\Gamma, \] (59)

where

\[\theta_1 = \frac{\tilde{a}_1}{k_0}, \quad \theta_2 = \frac{\tilde{a}_2}{k_0}.\] (60)

Note that \(\theta_j (j = 1, 2)\) are generally defined as the distortivity because they relate the thermoelastic distortion to the local heat flux in the steady-state thermal conduction problems (Dundurs, 1974). With the help of these dimensionless parameters, the characteristic Eq. (55) can be written in the dimensionless form as

\[R' + C_1(H, z)Q' + C_1(H, z) = 0,\] (61)

with

\[C_1(H, z) = \frac{1}{\eta_{12}\eta_{13}e^{b(z + \eta_{12})} - \eta_{11}\eta_{14}e^{b(z + \eta_{14})} - \eta_{11}\eta_{12}e^{b(z + \eta_{12})} - \eta_{12}\eta_{13}e^{b(z + \eta_{13})}},\] (62)

\[C_2(H, z) = \frac{D_1(H, z)}{D_2(H, z)},\] (63)

where \(D_1(H, z)\) and \(D_2(H, z)\) are given in the Appendix B. If we set the gradient index as zero, Eq. (61) can be reduced to the characteristic equation for the thermoelastic instability between a homogeneous layer and a homogeneous half-plane reported by Yeo and Barber (1991).

### 3. Stability criterion

The system will be unstable if the characteristic Eq. (61) has a solution, either positive or complex with a positive real part, for the dimensionless exponential growth rate \(z = b/m^2\theta_1.\) This
instability will be evident when the first root of the characteristic equation passes into the right half complex plane, either through the origin or by crossing the imaginary axis (Yeo and Barber, 1991).

When the instability occurs for real roots, or when the first root of the characteristic equation passes through the origin, the stability criterion is determined by setting $\Re(z) = 0$. Then, Eq. (61) can be expressed as a linear relation between $R'$ and $Q'$, that is

$$Q' = -\frac{R' + C_1(H,0)}{C_2(H,0)}.$$  \hspace{1cm} (64)

When the instability occurs for complex roots, or when the first root of the characteristic equation crosses the imaginary axis, the stability criterion is determined by setting $z = iw$, where $w$ is real. Separating Eq. (61) into real and imaginary parts, we obtain the following two real equations:

$$R' + \Re(C_2)Q' + \Re(C_1) = 0, \hspace{1cm} (65)$$

$$\Im(C_2)Q' + \Im(C_1) = 0, \hspace{1cm} (66)$$

where

$$Q' = \frac{-\Im(C_1)}{\Im(C_2)}, \hspace{1cm} (67)$$

$$R' = -\Re(C_2)Q' - \Re(C_1). \hspace{1cm} (68)$$

Eqs. (65) and (66) can be solved to determine the relation between $Q'$ and $R'$ at the stability boundary.

4. Results and discussion

This section will present the thermoelastic contact instability between an FGM layer and a homogeneous half-plane. To compare the stability boundaries for various material combinations, we utilize the same classification system as that of Yeo and Barber (1991). They classify the systems to five distinct types of material combinations, which are dependent on ratios of the thermoelastic properties of materials. In the present analysis, the first three types of material combinations, i.e. Type 1, Type 2 and Type 3, are discussed. If the dimensionless ratios of material properties satisfy $r_1 > 1$ and $0 < r_5 < 1/r_1$ or $r_1 < 1$ and $0 < 1/r_5 < r_1$, the system is classified to Type 1; if $1/r_1 < r_5 < 1$ and $r_1 > 1$ or $r_1 < r_5 < 1$ and $r_1 < 1$, the system is Type 2; and if $1 < r_5 < r_1$ or $1 > r_5 > r_1$, the system is Type 3.

In order to quantitatively analyze the effect of different gradient indexes, it is assumed that the gradient indexes have the same value, i.e. $\beta = \gamma = \delta = n$ in the following analysis. The material at the bottom of the FGM layer is chosen as the nodular cast iron. The materials of the homogeneous half-plane are chosen as the SiC sintered for Type 1 ($\theta_1 > \theta_2$), the brass for Type 2 ($\theta_1 > \theta_2$), and the magnesium alloy for Type 3 ($\theta_1 > \theta_2$). Note that the materials with large value of $\beta$ generally correspond to materials with great distortion. $\theta_1$ represents the distortivity of the bottom material of the FGM layer. The thermoelastic properties of these materials are listed in Table 1 (Zhang and Barber, 1990). The FGM layer with positive/negative gradients indicates that the material properties exponentially increase/decrease from the bottom of the layer. The effects of the gradient index, dimensionless thickness of the layer, thermal contact resistance and classification of material properties on the critical heat flux are analyzed in Figs. 3–10.

If we set the gradient indexes of the FGM layer as zero, the present problem can be directly reduced to the thermoelastic stability problem of a homogeneous layer and a homogeneous half-plane as discussed in Yeo and Barber (1991). Fig. 2 presents the stability boundaries as a function of $R'$ with different thickness $H$ for the case of an aluminum alloy layer and a copper half-plane. The results given by Yeo and Barber (1991) are also plotted in Fig. 2 for a direct comparison. It is seen that the present results agree very well with Yeo and Barber’s results.

4.1. Type 1 material combination

Fig. 3 shows the stability boundaries for different values of the gradient index $n$ with the dimensionless thickness $H = 1.0$. Since $R'$ can take any positive value, it is convenient to condense the infinite range by plotting $Q'$ against $1/(1+R')$, which is always between 0 and 1. It is showed that for the same value of $n$, bigger the contact resistance $R'$ is, more stable the system is. Note that $n = 0$ corresponds to the contact between a homogeneous layer and a homogeneous half-plane. For larger values of $R'$, stability boundaries can be determined from the linear real root criterion (64), and the value of the critical heat flux $Q'$ increases with the increase of $n$. For smaller values of $R'$, the instability may occur for both real and complex roots and stability boundaries need to be determined from Eqs. (65) and (66) with lower values of $Q'$, and the value of the critical heat flux increases with the decrease of $n$. These results indicate that we can increase the critical heat flux and change the stability boundaries, and hence modify thermoelastic stability behavior of systems by adjusting the gradient index of the FGM layer.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Aluminium alloy</th>
<th>Copper</th>
<th>Nodular cast iron</th>
<th>SiC sintered</th>
<th>Brass</th>
<th>Magnesium alloy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$ (GPa)</td>
<td>27.273</td>
<td>45.489</td>
<td>64.122</td>
<td>172.414</td>
<td>38.372</td>
<td>16.667</td>
</tr>
<tr>
<td>$\lambda$ (C$^{-1}$×10$^{-6}$)</td>
<td>22.0</td>
<td>17.0</td>
<td>13.7</td>
<td>4.4</td>
<td>19.0</td>
<td>26.0</td>
</tr>
<tr>
<td>$k$ (W/mK)</td>
<td>173.0</td>
<td>381.0</td>
<td>48.9</td>
<td>130.0</td>
<td>78.0</td>
<td>95.0</td>
</tr>
<tr>
<td>$\lambda$ (mm$^2$/s)</td>
<td>67.16</td>
<td>101.93</td>
<td>16.05</td>
<td>35.48</td>
<td>21.35</td>
<td>45.11</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.32</td>
<td>0.33</td>
<td>0.31</td>
<td>0.16</td>
<td>0.33</td>
<td>0.35</td>
</tr>
</tbody>
</table>

**Table 1** Thermoelastic properties of selected materials.

![Fig. 2. Stability boundaries as a function of $R'$ with different thickness $H$ for an aluminum alloy on a copper half-plane: comparison with existing results.](image-url)
4.2. Type 2 material combination

Fig. 4 presents the effect of the gradient index \( n \) on the critical heat flux \( Q^* \) for different values of the dimensionless thickness \( H \) with \( R' = 1.0 \). Obviously, the system exhibits instability for both directions of the heat flux. When the heat flux transmits into the more distortive material (i.e. FGM layer, \( Q^* > 0 \)), the critical heat flux \( Q^* \) increases first, then decreases with the increase of the gradient index \( n \) from \(-2.0\) to \(2.0\) for a given \( H \). It is also observed that the thick FGM layer is more likely to be unstable than the thin one. When the heat flux transmits into the less distortive material (i.e. half-plane, \( Q^* < 0 \)), only the thinner FGM layer with positive and large values of the gradient index can exhibit instability. For the negative gradient index \( n \), the system is unstable only for the positive heat flux.

Fig. 5 considers the effect of dimensionless thickness \( H \) on the critical heat flux \( Q^* \) for different values of the gradient index \( n \) with \( R' = 1.0 \). The critical heat flux decreases rapidly as the thickness \( H \) increases from 0 to 2, and then it changes slightly as \( H \) increases from 2 to 6. Furthermore, \( Q^* \to \infty \) as \( H \to 0 \), that’s to say, a very large heat flux is needed to cause instability for a very thinner FGM layer. The similar phenomenon was also reported by Yeo and Barber (1991).

Fig. 6 plots the stability boundaries for different values of \( n \) with \( H = 1.0 \). It is found that the gradient index has a significant effect on the stability behavior of the Type 2 material combination. For \( n < 0 \), the system exhibits both the complex and real root instabilities with the positive heat flux. For \( n = 0 \), the system only exhibits the complex root instability with the positive heat flux, and the stability boundaries are determined from Eqs. (65) and (66). For \( n > 0 \), the system can occur the real root instability with the negative heat flux, and the complex root instability with the positive heat flux. Similar with Type 1 material combination, the absolute value of the critical heat flux increases with the increase of the thermal contact resistance.

Fig. 7 examines the effect of the gradient index \( n \) on the critical heat flux \( Q^* \) for different values of \( H \) with \( R' = 1.0 \). It is shown that the contact for Type 2 material combination may exhibit instability at both directions of the heat flux. When heat flux transmits into the more distortive material (i.e. FGM layer, \( Q^* > 0 \)), \( Q^* \) increases first, then decreases with the increase of gradient index for the thinner layer, and the maximum value of \( Q^* \) occurs when the gradient index ranges from \(-1\) to \(0\). However, for the thicker layer, the critical heat flux can reach very large values for the positive gradient index. This result indicates that the system is very stable for a thicker layer with the positive gradient index. When the heat flux transmits into the less distortive material (i.e. half-plane, \( Q^* < 0 \)), the instability can only occur for the positive gradient index, and the absolute value of \( Q^* \) increases as the gradient index decreases.

Fig. 8 shows the effect of dimensionless thickness \( H \) on the critical heat flux \( Q^* \) for different values of \( n \) with \( R' = 1.0 \). For the smaller value of the gradient index (say \(-0.2 \leq n \leq 0.2\)), instability occurs only for the positive heat flux. It is observed that the minimum \( Q^* (= Q^*_0) \), which always occurs for a layer dimensionless thickness approximated at \( H = 1.3 \), decreases with the increase
of the gradient index. With the increase of $H$, the critical heat flux first decreases rapidly to the minimum value, next increases to the local maximum value, and then decreases to a stable value at larger $H$. It is similar with Type 1 material combination that Type 2 material combination has $Q^* \rightarrow 1$ as $H \rightarrow 0$.

4.3. Type 3 material combination

Fig. 9 depicts the effect of the gradient index $n$ on the critical heat flux $Q^*$ for different values of $H$ with $R' = 1.0$. Similar with Type 1 and Type 2 material combinations, Type 3 material combination permits the instability for both directions of the heat flux.
When the heat flux transmits into the more distortive material (i.e. half-plane, $Q^* < 0$), the larger gradient index can lead to the smaller absolute value of $Q^*$, which changes slightly as $n > 1.0$. When the heat flux transmits into the less distortive material (i.e. FGM layer, $Q^* > 0$), the critical heat flux first increases, and then decreases with the increase of gradient index for the thinner FGM layer. For the thicker layer, the instability can only occur for the FGM layer with negative gradient index. Fig. 10 presents the effect of dimensionless thickness $H$ on the critical heat flux $Q^*$ for different values of $n$ with $R^* = 1.0$. For the positive heat flux, it is observed that the minimum critical heat flux $Q_g$ occurs for the layer thickness approximated at $H = 0.7$. Also, the contact is very stable when the gradient index $n > 0.0$ and the FGM layer thickness $H > 1.1$. For the negative heat flux, the absolute value of $Q^*$ decreases as $H$ increases, and is almost unchanged for the larger $H$ ($H > 2.0$).

Fig. 11 presents effect of the gradient index $n$ on the critical heat flow $Q^*$ for different material combinations with $H = 1.0$ and $R^* = 1.0$. For positive heat flux, type 3 material combination leads to the maximum critical heat flux. It is implied that type 3 material combination has the best performance on thermoelastic instability among these material combinations. However, for the negative heat flux, type 1 will not exhibit instability. Furthermore, the performance of type 2 on thermoelastic instability is better than that of type 3.

Finally, we recollect the results of stability behavior in terms of the gradient index for different material combinations. Table 2 tabulates the stability behavior for the thermoelastic instability between an FGM layer and a homogeneous half-plane with $R^* = 1$ and $H = 1$. In Table 2, Re and $C$ indicate that the system exhibits the instability with a real root and a complex root, respectively. Obviously, the stability behavior is very sensitive to the gradient index of FGMs. For all three types of material combinations, the system can exhibit either complex root or real root instability for a certain range of gradient index $n$.

### 5. Conclusions

The thermoelastic contact instability of an FGM layer and a homogeneous half-plane under the plane strain state is investigated by using the perturbation method. The FGM layer and homogeneous half-plane are pressed together by a uniform pressure and transmit a uniform heat flux in the thickness direction. The characteristic equation is obtained to determine the stability boundary for three types of material combinations. The effects of the gradient index, layer thickness and material combination on the critical heat flux and stability boundaries are discussed in detail. The results of the present analysis are validated by reducing the problem to the contact between a homogeneous layer and a homogeneous half-plane. It is found that:

1. For all three material combinations, the systems can occur the instability at both directions of the heat flux.
2. The system with the thin FGM layer is generally more stable than that with thick FGM layer.
3. Through the investigation of the static thermoelastic instability on FGMs, we could adjust the gradient index to increase the critical heat flux and change the stability boundaries, and hence modify the thermoelastic stability behavior of systems.
4. Type 3 material combination has the best performance on thermoelastic instability among these material combinations for the positive heat flow, while type 1 material combination has the best performance for the negative heat flux.

### Acknowledgements

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### Appendix A

\[
\{g_{11}\} = \begin{cases} 
\epsilon_{i1}^{A1}(f_{11} - ms_1), \epsilon_{i2}^{A1}(f_{12} - ms_2), \epsilon_{i12}^{A1}(f_{13} - ms_3), \\
\end{cases}
\]

\[
e^{i\eta L_j 1}(f_{14} - ms_4), e^{i\eta L_j 12}(N_1 + \eta_{11} + \eta), \\
\]

\[
e^{i\eta L_j 12}(N_2 + \eta_{12} + \eta) 0,0,0, \end{cases}
\]

\[
\{g_{21}\} = \begin{cases} 
\epsilon_{i1}^{A2}L_1, \epsilon_{i2}^{A2}L_2, \epsilon_{i1}^{A2}L_3, \epsilon_{i1}^{A2}L_4, e^{i\eta L_j 12}(W_1, e^{i\eta L_j 12})W_2, 0,0,0, \\
\end{cases}
\]

\[
\{g_{31}\} = \begin{cases} 
0,0,0,0, \epsilon_{i1}^{A2}L_1, \epsilon_{i2}^{A2}L_2, 0,0,0, \end{cases}
\]

\[
\{g_{31}\} = \begin{cases} 
0,0,0,0, \epsilon_{i1}^{A2}L_1, \epsilon_{i2}^{A2}L_2, 0,0,0, \end{cases}
\]

---

### Table 2

<table>
<thead>
<tr>
<th>Material combination</th>
<th>Gradient Index $n$</th>
<th>Critical heat flux</th>
<th>Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1</td>
<td>$-2.0 \leq n \leq 0.2$</td>
<td>$Q^* &gt; 0$</td>
<td>Real</td>
</tr>
<tr>
<td>Type 2</td>
<td>$-2.0 \leq n \leq -0.5$</td>
<td>$Q^* &gt; 0$</td>
<td>Real</td>
</tr>
<tr>
<td></td>
<td>$0.2 \leq n \leq 2.0$</td>
<td>$Q^* &lt; 0$</td>
<td>Complex</td>
</tr>
<tr>
<td></td>
<td>$0.3 \leq n \leq 2.0$</td>
<td>$Q^* &lt; 0$</td>
<td>Real</td>
</tr>
<tr>
<td>Type 3</td>
<td>$-2.0 \leq n \leq -0.2$</td>
<td>$Q^* &gt; 0$</td>
<td>Real</td>
</tr>
<tr>
<td></td>
<td>$-0.2 \leq n \leq 2.0$</td>
<td>$Q^* &lt; 0$</td>
<td>Complex</td>
</tr>
<tr>
<td></td>
<td>$-0.7 \leq n \leq 0.3$</td>
<td>$Q^* &lt; 0$</td>
<td>Complex</td>
</tr>
<tr>
<td></td>
<td>$0.3 \leq n \leq 2.0$</td>
<td>$Q^* &lt; 0$</td>
<td>Real</td>
</tr>
</tbody>
</table>
\[
\{g_{N}\} = \left\{ f_{11} - m_{s} \frac{1}{2} + f_{12} - m_{s} \frac{2}{2}, f_{13} - m_{s} \frac{3}{2}, f_{14} - m_{s} \frac{4}{2}, \frac{N_{1}}{M_{1}}, \frac{N_{2}}{M_{2}}, \eta_{11}, \frac{\gamma_{1}}{M_{2}}, \eta_{12}, \frac{\gamma_{0}}{M_{2}}, \frac{0}{M_{2}}, \frac{0}{M_{2}}, \frac{0}{M_{2}}, \frac{0}{M_{2}} \right\}^{T},
\]

(A4)

\[
\{g_{S}\} = \left\{ 0, 0, 0, 0, 0, 0, 2m_{1} - 1, - \Theta_{2}, 2 \eta_{21} \right\}^{T},
\]

(A5)

\[
\{g_{\mu}\} = \left\{ \frac{\mu_{b}}{\Theta_{2} - 1}L_{1}, \frac{\mu_{b}}{\Theta_{2} - 1}L_{2}, \frac{\mu_{b}}{\Theta_{2} - 1}L_{3}, \frac{\mu_{b}}{\Theta_{2} - 1}L_{4}, \mu_{b}W_{1}, \mu_{b}W_{2}, 2m_{u} - 2(\Theta_{2} + 1)_{2}m_{u} \right\}^{T},
\]

(A6)

\[
\{g_{s}\} = \left\{ s_{1}, s_{2}, s_{3}, s_{4}, -N_{1}, N_{2}, 1, - \Theta_{2}, \eta_{21} \right\}^{T},
\]

(A7)

\[
\{g_{0}\} = \left\{ 0, 0, 0, 0, \frac{\nu_{b}(\Theta_{2} - 1)_{2} \eta_{11}}{\nu_{s}M_{1}}, \frac{\nu_{b}(\Theta_{2} - 1)_{2} \eta_{12}}{\nu_{s}M_{2}}, \frac{0, 0, \nu_{b}(\Theta_{2} + 1)_{2}(\eta_{21} - m_{2})}{\nu_{s}M_{2}} \right\}^{T},
\]

(A8)

References


Lee, S., Jang, Y.H., 2009b. Frictionally excited thermoelastic instability in a thin layer of functionally graded material sliding between two half-planes. Wear 267, 1715–1722.


