Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Nuclear Physics B 776 [PM] (2007) 405–430

NUCLEAR PHYSICS B

Topological twisting of conformal supercharges

Jeong-Hyuck Park [a,*], Dimitrios Tsimpis [b]

[a] *Department of Physics, Sogang University, C.P.O. Box 1142, Seoul 100-611, South Korea*
[b] *Max-Planck-Institut für Physik, Föhringer Ring 6, 80805 München, Germany*

Received 25 October 2006; received in revised form 13 February 2007; accepted 11 April 2007

Available online 18 April 2007

Abstract

Putting a twisted version of $\mathcal{N}=4$ super-Yang–Mills on a curved four-dimensional manifold generically breaks all conformal supersymmetries. In the special case where the four-manifold is a cone, we show that exactly two conformal supercharges remain unbroken. We construct an off-shell formulation of the theory such that the two unbroken conformal supercharges combine into a family of topological charges parameterized by $\mathbb{CP}^1$. The resulting theory is topological in the sense that it is independent of the metric on the three-dimensional base of the cone.

Keywords: Topological field theory; Super-Yang–Mills; Superconformal symmetry

1. Introduction

Topological quantum field theory (TQFT) provides an intriguing link between physics and mathematics. The study of TQFT was initiated with [1], wherein a quantum field-theoretical representation of Donaldson invariants was given. A quantum field theory is called topological if all vacuum expectation values (VEVs) of a certain set of operators ('observables') are metric-independent. In particular, TQFTs of cohomological type are constructed as follows. Let us assume there is a nilpotent symmetry of the action Q, such that $Q^2=0$. It follows that, at least formally, one can deform the Lagrangian by adding an arbitrary Q-exact term without affecting the partition function or the VEVs of observables (which are defined as elements in the cohomology of Q). Since Q is a symmetry of the action, the Lagrangian can be expressed as

* Corresponding author.
E-mail addresses: park@sogang.ac.kr (J.-H. Park), tsimpis@mppmu.mpg.de (D. Tsimpis).

0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2007.04.008

a sum of a Q-exact and a Q-closed piece. The theory is therefore independent of any coupling constant in the Q-exact piece. Moreover, if the energy–momentum tensor is Q-exact all VEVs of observables are metric-independent and the theory is topological.

One way of constructing TQFTs is by 'twisting' theories with extended supersymmetry in Euclidean space. Roughly-speaking, twisting can be thought of as an embedding of the rotation group in the global R-symmetry group, thereby changing the spins of the fields in the parent theory. For the resulting theory to be topological on a general manifold, there should exist at least one scalar among the twisted supersymmetry generators. The TQFT of [1] was obtained, by twisting, from $\mathcal{N}=2$ supersymmetric Yang–Mills theory in four dimensions. Similarly, $\mathcal{N}=4$ SYM in four dimensions can be twisted in three inequivalent ways to obtain a TQFT [2–4]. It has been subsequently noted that all three twisted theories can be thought of as world-volume theories on D3-branes wrapping supersymmetric cycles in string-theory compactifications [5]. Two of the twists have been extensively studied in [2], in connection to instanton invariants. The third twist, which is the focus of the present paper, has been studied in a series of papers [4,6–10], and has been recently found to be relevant to the geometric Langlands program [11].

Four-dimensional $\mathcal{N}=4$ SYM in flat space is superconformal: in addition to sixteen ordinary supercharges it possesses sixteen conformal supercharges. The aim of the present paper is to explore the effect of the twisting on the conformal supercharges. On a general four-manifold all conformal supercharges will be broken. However, in the special case where the manifold is a cone over a three-dimensional base, we find that the twisting leads to exactly two unbroken conformal supercharges, S_0, S_5. We construct an off-shell formulation of the theory such that the two twisted conformal supercharges combine into a family of topological charges $S_z := z_1 S_0 + z_2 S_5$, $z := z_2/z_1$, parameterized by $\mathbb{CP}^1$. In this formalism the components of the energy–momentum tensor along the directions of the base of the cone are manifestly S_z-exact—since S_z is nilpotent off-shell and independent of the metric on the base of the cone. It follows that (at least formally) the theory is topological in the sense that it is independent of the metric on the base of the cone.

There are two important caveats in this statement. Firstly, as is usually the case in the path-integral formulation of TQFTs, we shall assume that the path-integral measure is well defined and S_z-invariant. Secondly, in order to define the functional integration on a cone we have to impose boundary conditions on the fields. As we explain in more detail in Sections 3.5, 3.6, there is a certain freedom in the choice of boundary conditions. Generic boundary conditions will break superconformal invariance. Requiring that the scalar fields vanish at infinity is sufficient for the superconformal invariance to be preserved.

The aim of this paper is to furnish a new tool for the study of topological properties of three-manifolds (viewed as bases of four-dimensional cones). We believe that the topological off-shell theory we construct reveals a rich enough structure to warrant further study. Of course the construction of the off-shell formalism should be viewed as just the first step; eventually we would like to identify the set of observables of the theory (for this, a more complete study of the boundary conditions will be necessary) and their relevance to three-dimensional topological invariants. We will leave this second nontrivial step for future investigation.

The organization of this paper is as follows. In Section 2 we review some well-known facts about ten-dimensional SYM and its reduction to $\mathcal{N}=4$ SYM in four dimensions. In particular, we point out that in flat space the conformal supersymmetries can be viewed as similarity transformations of the ordinary supersymmetries, with respect to a certain inversion map.

Section 3 explains the twisting of the $\mathcal{N}=4$ super-Yang–Mills on which we focus in this paper. We obtain the twisted Lagrangian as well as the ordinary and conformal scalar supersymmetry transformations. The superconformal algebra of the twisted SYM in flat space contains two ordinary scalar supercharges and two conformal scalar supercharges. We show that the conformal scalar supersymmetries survive on a curved four-manifold, if and only if the manifold is (locally) a cone over an arbitrary three-dimensional base. We identify the most general linear combination of the four scalar supercharges which is nilpotent, and thus can be used to define a cohomological structure. Some important 'vanishing theorems' are presented in Sections 3.5 and 3.6.

In Section 4, we show that there exists an off-shell formulation of the theory such that the two twisted conformal supercharges combine into a family of nilpotent operators parameterized by $\mathbb{CP}^1$. Moreover we show that (at least formally) the theory is topological in the sense that it is independent of the metric on the base of the cone.

We conclude with a discussion of open questions in Section 5. Appendices A–C contain some useful proofs and formulae.

2. Review of super-Yang–Mills

In order to establish our conventions and notation, we now give a brief review of super-Yang–Mills in ten Euclidean dimensions and its reduction to four-dimensional $\mathcal{N}=4$ super-Yang–Mills.

By Wick-rotating the ordinary super-Yang–Mills in ten dimensions, one obtains a Euclidean theory which is still formally supersymmetric in that it is invariant under the same set of fermionic transformations as the original theory. However, the minimal spinor representation in ten Euclidean dimensions is no longer Majorana, resulting in a complex action. Of course this phenomenon is not new: it occurs routinely in string theory in a variety of different contexts.

There are essentially two ways to deal with this problem. The point-of-view we take here is to simply ignore the fact that the action is complex, as for example in [11]. This would naively seem to double the fermionic degrees of freedom. However, the complex-conjugate fermions do not appear in the Lagrangian. Therefore in the path-integral one integrates over only half of the (complexified) fermionic degrees of freedom—resulting in the same counting as for the Minkowskian theory. Another way to state this is that the action is not required to be real—it is only required to be holomorphic. The other way to handle the Wick rotation is by imposing reality conditions on the fields. These reality conditions are sometimes introduced in an ad hoc manner. A systematic way to derive them is by time-like reduction of a higher-dimensional theory, as demonstrated in [12].

2.1. Super-Yang–Mills in ten dimensions

Consider the 32×32 gamma matrices in ten-dimensional Euclidean space

$$\Gamma^M\Gamma^N+\Gamma^N\Gamma^M=2\delta^{MN},\qquad 1\leqslant M,N\leqslant 10. \tag{2.1}$$

These are Hermitian $(\Gamma^M)^\dagger=\Gamma^M$ and satisfy

$$\left(\Gamma^M\right)^T=\left(\Gamma^M\right)^*=\mathcal{C}\Gamma^M\mathcal{C}^\dagger,\qquad \mathcal{C}=\mathcal{C}^T=\left(\mathcal{C}^\dagger\right)^{-1}. \tag{2.2}$$

We define the ten-dimensional chirality operator by

$$\Gamma^{(11)}:=-i\Gamma^{12\cdots 10}=\left(\Gamma^{(11)}\right)^\dagger=\left(\Gamma^{(11)}\right)^{-1}=-\mathcal{C}^\dagger\left(\Gamma^{(11)}\right)^T\mathcal{C}. \tag{2.3}$$

The main Fierz identity is

$$\big(\mathcal{C}\Gamma^M P_\pm\big)_{\hat\alpha\hat\beta}(\mathcal{C}\Gamma_M P_\pm)_{\hat\gamma\hat\delta}+\big(\mathcal{C}\Gamma^M P_\pm\big)_{\hat\beta\hat\gamma}(\mathcal{C}\Gamma_M P_\pm)_{\hat\alpha\hat\delta}+\big(\mathcal{C}\Gamma^M P_\pm\big)_{\hat\gamma\hat\alpha}(\mathcal{C}\Gamma_M P_\pm)_{\hat\beta\hat\delta}=0, \tag{2.4}$$

where $\hat\alpha$, $\hat\beta$, $\hat\gamma$, $\hat\delta$ are the 32 component spinorial indices, and $P_\pm$ are the chiral/anti-chiral projection matrices

$$P_+=\frac{1}{2}\big(1+\Gamma^{(11)}\big), \qquad P_-=\frac{1}{2}\big(1-\Gamma^{(11)}\big). \tag{2.5}$$

Super-Yang–Mills in ten Euclidean dimensions is given by:

$$\mathcal{L}_{10\mathrm{D}}=\mathrm{tr}\left[\frac{1}{4}F_{MN}F^{MN}+\frac{1}{2}\bar\Psi\Gamma^M D_M\Psi\right], \tag{2.6}$$

where the gaugino Ψ is a chiral spinor

$$\Gamma^{(11)}\Psi=+\Psi, \tag{2.7}$$

and $\bar\Psi$ is defined by[1]

$$\bar\Psi:=\Psi^T\mathcal{C}, \qquad \bar\Psi\Gamma^{(11)}=-\bar\Psi. \tag{2.8}$$

In our conventions the gauge field strength and the gauge-covariant derivative read

$$F_{MN}=\partial_M A_N-\partial_N A_M-i[A_M,A_N], \qquad D_M\Psi=\nabla_M\Psi-i[A_M,\Psi], \tag{2.9}$$

so that gauge transformations are

$$A_M\to gA_Mg^{-1}+ig\partial_Mg^{-1}, \qquad F_{MN}\to gF_{MN}g^{-1}, \qquad \Psi\to g\Psi g^{-1}. \tag{2.10}$$

The sixteen ordinary supersymmetries are given by

$$\delta A_M=\bar\Psi\Gamma_M\varepsilon_+=-\bar\varepsilon_+\Gamma_M\Psi, \qquad \delta\Psi=\frac{1}{2}F_{MN}\Gamma^{MN}\varepsilon_+, \tag{2.11}$$

where the supersymmetry parameter ε_+ is a constant Weyl spinor, and the subscript denotes positive chirality in ten dimensions.

2.2. Dimensional reduction and superconformal symmetry

We now consider the dimensional reduction to four-dimensional $\mathcal{N}=4$ super-Yang–Mills. Let us denote the four-dimensional coordinates by x^μ, $\mu=1,2,3,4$. We also set $\Phi_I:=A_I$, $5\leqslant I\leqslant 10$. Four-dimensional $\mathcal{N}=4$ super-Yang–Mills is then given by

$$\begin{aligned}\mathcal{L}_{4\mathrm{D}}=\mathrm{tr}\bigg[&\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}D_\mu\Phi_I D^\mu\Phi^I-\frac{1}{4}[\Phi_I,\Phi_J]\big[\Phi^I,\Phi^J\big]\\&+\frac{1}{2}\bar\Psi\Gamma^\mu D_\mu\Psi-i\frac{1}{2}\bar\Psi\Gamma^I[\Phi_I,\Psi]\bigg].\end{aligned} \tag{2.12}$$

In addition to the sixteen ordinary supersymmetries (2.11), there are sixteen conformal supersymmetries

$$\delta A_M=\bar\Psi\Gamma_M X\varepsilon_-=-\bar\varepsilon_- X\Gamma_M\Psi, \qquad \delta\Psi=\left(\frac{1}{2}F_{MN}\Gamma^{MN}X-2\Phi_I\Gamma^I\right)\varepsilon_-, \tag{2.13}$$

[1] Note that although the gaugino is complex in ten Euclidean dimensions, its complex conjugate does not appear in the Lagrangian.

where $X := x^\mu \Gamma_\mu$ and the conformal supersymmetry parameter ε_- is a Weyl spinor of negative chirality, $\Gamma^{(11)}\varepsilon_- = -\varepsilon_-$.[2]

Both ordinary and conformal supersymmetries can be collected in terms of a single 32-component spinor

$$\varepsilon = \varepsilon_+ + \varepsilon_-, \quad \varepsilon_\pm = P_\pm \varepsilon. \tag{2.14}$$

All 32 supersymmetries are then given by

$$\begin{aligned} \delta A_M &= \bar{\Psi}\Gamma_M\big(1 + x^\mu \Gamma_\mu\big)\varepsilon = -\bar{\varepsilon}\big(1 + x^\mu \Gamma_\mu\big)\Gamma_M \Psi, \\ \delta\Psi &= P_+\left[\frac{1}{2}F_{MN}\Gamma^{MN}\big(1 + x^\mu \Gamma_\mu\big) - 2\Phi_I \Gamma^I\right]\varepsilon, \end{aligned} \tag{2.15}$$

which implies

$$\delta\bar{\Psi} = \bar{\varepsilon}\left[-\frac{1}{2}\big(1 + x^\mu \Gamma_\mu\big)F_{MN}\Gamma^{MN} - 2\Phi_I \Gamma^I\right]P_-. \tag{2.16}$$

Accordingly, the Lagrangian (2.12) transforms into a total derivative

$$\delta\mathcal{L}_{\mathrm{4D}} = \partial_\mu \operatorname{tr}\left[F^{\mu N}\delta A_N - \frac{1}{2}\bar{\Psi}\Gamma^\mu \delta\Psi\right], \tag{2.17}$$

leaving the action invariant.

From the corresponding Noether current

$$J^\mu = \operatorname{tr}\big(\bar{\Psi}\Gamma^\mu \delta\Psi\big) = -\operatorname{tr}\big(\delta\bar{\Psi}\Gamma^\mu \Psi\big), \tag{2.18}$$

we obtain a 32-component supercurrent

$$\begin{aligned} \mathcal{Q}^\mu &= \operatorname{tr}\left[\left(\frac{1}{2}\big(1 + x^\nu \Gamma_\nu\big)F_{KL}\Gamma^{KL} + 2\Phi_I \Gamma^I\right)\Gamma^\mu \Psi\right], \\ \bar{\mathcal{Q}}^\mu &= \operatorname{tr}\left[\bar{\Psi}\Gamma^\mu\left(-\frac{1}{2}F_{KL}\Gamma^{KL}\big(1 + x^\nu \Gamma_\nu\big) + 2\Phi_I \Gamma^I\right)\right] = \big(\mathcal{Q}^\mu\big)^T \mathcal{C}, \end{aligned} \tag{2.19}$$

where we have set

$$J^\mu := -\bar{\mathcal{Q}}^\mu \varepsilon = +\bar{\varepsilon}\mathcal{Q}^\mu. \tag{2.20}$$

The chiral and anti-chiral parts $P_+\mathcal{Q}^\mu$, $P_-\mathcal{Q}^\mu$ correspond to the conformal and ordinary supersymmetries respectively.

It is worthwhile to note the existence of an inversion map $\mathcal{I}$ defined in flat background [13–15]

$$\begin{aligned} &x^\mu \xrightarrow{\mathcal{I}} x'^\mu = \frac{x^\mu}{x^2}, \\ &A_\mu(x) \xrightarrow{\mathcal{I}} A'_\mu(x) = \frac{1}{x^2}\left(\delta_\mu{}^\nu - 2\frac{x_\mu x^\nu}{x^2}\right)A_\nu(x'), \\ &\Phi_J(x) \xrightarrow{\mathcal{I}} \Phi'_J(x) = \frac{1}{x^2}\Phi_J(x'), \\ &\Psi(x) \xrightarrow{\mathcal{I}} \Psi'(x) = i\frac{x^\mu \Gamma_\mu}{(x^2)^2}\Psi(x'). \end{aligned} \tag{2.21}$$

2 For simplicity the spinors are given in ten-dimensional notation.

The inversion map is an involution: $\mathcal{I} = \mathcal{I}^{-1}$. The four-dimensional $\mathcal{N} = 4$ super-Yang–Mills action (2.12) is invariant under the action of $\mathcal{I}$. The ordinary and conformal supersymmetry transformations, (2.11) and (2.13), are related to each other by the similarity transformation [14]:

$$\text{conformal supersymmetry} = \mathcal{I} \circ (\text{ordinary supersymmetry}) \circ \mathcal{I}. \tag{2.22}$$

The inversion map flips the chirality of the fermion, which is consistent with the fact that the two supersymmetry parameters ε_+ and ε_- have opposite chiralities. Note, however, that this symmetry is broken in a generic curved background.

3. Twisted $\mathcal{N} = 4$ super-Yang–Mills in four dimensions

We now come to the description of the twist. In the following subsections, we also discuss the unbroken conformal supercharges and derive certain important 'vanishing theorems'.

3.1. Description of the twist

Under Spin(10) $\to$ Spin(2) $\times$ Spin(4) $\times$ Spin(4), the ten-dimensional gamma matrices can be decomposed as

$$\begin{aligned} &\Gamma^\mu = \tau^1 \otimes \gamma^\mu \otimes 1, \qquad &&\Gamma^{\mu+4} = \tau^2 \otimes 1 \otimes \gamma^\mu, \\ &\Gamma^9 = \tau^1 \otimes \gamma^{(5)} \otimes 1, \qquad &&\Gamma^{10} = \tau^2 \otimes 1 \otimes \gamma^{(5)}, \end{aligned} \tag{3.1}$$

where τ^i, $i = 1, 2, 3$, are 2×2 Pauli matrices

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \qquad \tau^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.2}$$

and γ^μ, $\mu = 1, 2, 3, 4$, are four-dimensional gamma-matrices

$$\begin{aligned} &\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu}, \qquad &&(\gamma^\mu)^\dagger = \gamma^\mu, \\ &(\gamma^\mu)^T = C\gamma^\mu C^{-1}, \quad &&C = -C^T = (C^\dagger)^{-1}, \\ &\gamma^{(5)} := \gamma^1\gamma^2\gamma^3\gamma^4, \qquad &&(\gamma^{(5)})^T = C\gamma^{(5)}C^{-1}. \end{aligned} \tag{3.3}$$

Taking (2.2) and (2.3) into account, it follows that

$$\mathcal{C} = \tau^1 \otimes C \otimes C, \qquad \Gamma^{(11)} = \tau^3 \otimes 1 \otimes 1, \tag{3.4}$$

so that the ten-dimensional chirality coincides with the chirality in the Spin(2) part.

The fermions $\Psi_{\pm\alpha\beta}$ carry three indices for Spin(2) $\times$ Spin(4) $\times$ Spin(4). The focus of the present paper is the twist considered in [3,4,6–9,11]: it amounts to replacing the four-dimensional rotation group by the diagonal subgroup of Spin(4) $\times$ Spin(4). Accordingly, the twisted ten-dimensional chiral fermion (2.7) admits the following expansion

$$\Psi_+ = \frac{1}{2}\left(\eta + \psi_\mu\gamma^\mu + \frac{1}{2}\chi_{\mu\nu}\gamma^{\mu\nu} + \omega_\mu\gamma^\mu\gamma^{(5)} + \zeta\gamma^{(5)}\right)C^{-1}, \qquad \Psi_- = 0. \tag{3.5}$$

It follows that the fermions decompose into a pair of anticommuting scalars η, ζ, a pair of vectors ψ_μ, ω_μ and a two-form $\chi_{\mu\nu} = -\chi_{\nu\mu}$. The chiral and anti-chiral supersymmetry parameters ε_+, ε_- in (2.14) admit similar expansions, as we shall see later in (3.20). In particular, there will be two ordinary scalar and two conformal scalar supercharges.

For the bosons we set

$$A^{\pm}_{\mu} := A_{\mu} \pm i\phi_{\mu}, \quad \phi_{\mu} := \Phi_{4+\mu},$$
$$\varphi := \Phi_9 + i\Phi_{10}, \qquad \bar{\varphi} := \Phi_9 - i\Phi_{10}. \tag{3.6}$$

These obey the following reality conditions:

$$\left(A^{+}_{\mu}\right)^{\dagger} = A^{-}_{\mu}, \qquad (\phi_{\mu})^{\dagger} = \phi_{\mu}, \qquad (\varphi)^{\dagger} = \bar{\varphi}, \tag{3.7}$$

while there is no analogous constraint for fermions.[3]

3.2. Twisted Lagrangian

Taking (3.1), (3.5), (3.6) into account, it is straightforward to rewrite the $\mathcal{N}=4$ super-Yang–Mills (2.12) in terms of the anticommuting fields $\eta, \zeta, \psi_{\mu}, \omega_{\mu}, \chi_{\mu\nu}$ and the bosons $A^{\pm}_{\mu}, \phi_{\mu}, \varphi, \bar{\varphi}$. The resulting action defines our twisted $\mathcal{N}=4$ super-Yang–Mills in four-dimensions[4]:

$$\mathcal{S}_{\text{twisted}} = \int \mathrm{d}^4x\, \mathcal{L}_{\text{twisted}}, \quad \mathcal{L}_{\text{twisted}} = \mathcal{L}_{\text{top}} + \sqrt{g}L_g, \tag{3.8}$$

where

$$\mathcal{L}_{\text{top}} = \epsilon^{\kappa\lambda\mu\nu}\operatorname{tr}\left[\frac{1}{2}\omega_{\kappa}\mathcal{D}^{+}_{\lambda}\chi_{\mu\nu} - i\frac{1}{8}\varphi\{\chi_{\kappa\lambda}, \chi_{\mu\nu}\}\right],$$
$$L_g = \operatorname{tr}\Bigg[\frac{1}{4}F^{+}_{\mu\nu}F^{-\mu\nu} - \frac{1}{2}h^2 + h\mathcal{D}_{\mu}\phi^{\mu} + \frac{1}{4}\mathcal{D}^{+}_{\mu}\varphi\mathcal{D}^{-\mu}\bar{\varphi} + \frac{1}{4}\mathcal{D}^{-}_{\mu}\varphi\mathcal{D}^{+\mu}\bar{\varphi} + \frac{1}{8}[\varphi,\bar{\varphi}]^2$$
$$+ \eta\mathcal{D}^{+}_{\mu}\psi^{\mu} + \zeta\mathcal{D}^{-}_{\mu}\omega^{\mu} + \chi^{\mu\nu}\mathcal{D}^{-}_{\mu}\psi_{\nu} + i\varphi\{\eta,\zeta\} - i\bar{\varphi}\left\{\psi_{\mu},\omega^{\mu}\right\}\Bigg]. \tag{3.9}$$

In the above, we have introduced an auxiliary bosonic scalar h. We have coupled the action to a generic curved background with metric $g_{\mu\nu}$.[5] Moreover, $\epsilon^{\kappa\lambda\mu\nu}$ is the totally antisymmetric tensor density, such that $\epsilon^{1234} = 1$ and $\epsilon_{1234} = g$. All derivatives are covariant with respect to both diffeomorphisms and gauge transformations:

$$\mathcal{D}_{\mu}\phi^{\nu} = \nabla_{\mu}\phi^{\nu} - i\left[A_{\mu}, \phi^{\nu}\right], \qquad \mathcal{D}^{\pm}_{\mu}\eta = \nabla_{\mu}\eta - i\left[A^{\pm}_{\mu}, \eta\right],$$
$$\mathcal{D}^{\pm}_{\mu}\psi_{\nu} = \nabla_{\mu}\psi_{\nu} - i\left[A^{\pm}_{\mu}, \psi_{\nu}\right], \qquad \mathcal{D}^{\pm}_{\mu}\chi_{\nu\lambda} = \nabla_{\mu}\chi_{\nu\lambda} - i\left[A^{\pm}_{\mu}, \chi_{\nu\lambda}\right]. \tag{3.10}$$

Note that $\mathcal{L}_{\text{top}}$ is manifestly metric-independent, as follows from the fact that the Christoffel connection is torsion-free.

Moreover, we have defined the field strengths

$$F^{\pm}_{\mu\nu} := \partial_{\mu}A^{\pm}_{\nu} - \partial_{\nu}A^{\pm}_{\mu} - i\left[A^{\pm}_{\mu}, A^{\pm}_{\nu}\right] = F_{\mu\nu} + i[\phi_{\mu},\phi_{\nu}] \pm i(\mathcal{D}_{\mu}\phi_{\nu} - \mathcal{D}_{\nu}\phi_{\mu}). \tag{3.11}$$

---

[3] If we started with super-Yang–Mills in ten-dimensional Minkowski instead of Euclidean spacetime, the fermions would have to satisfy the Majorana condition. This would result in reality conditions for $\eta, \zeta, \psi_{\mu}, \omega_{\mu}, \chi_{\mu\nu}$.

[4] N. Nekrasov has pointed out that this action can be obtained by dimensional reduction of five-dimensional twisted super-Yang–Mills [17].

[5] For the coupling of the topological field theory to gravity see e.g. [18].

Note that the superscript $\pm$ above does *not* denote (anti) self-duality. It is also useful to define a covariant tensor $\mathcal{H}_{\mu\nu}$ by

$$\mathcal{H}_{\mu\nu} := \nabla_\mu A^-_\nu - \nabla_\nu A^+_\mu - i\big[A^+_\mu, A^-_\nu\big], \tag{3.12}$$

such that

$$\begin{aligned}\mathcal{H}_{\mu\nu} &= F_{\mu\nu} - i[\phi_\mu, \phi_\nu] - i\mathcal{D}_\mu\phi_\nu - i\mathcal{D}_\nu\phi_\mu = F^+_{\mu\nu} - 2i\mathcal{D}^+_\mu\phi_\nu \\ &= F^-_{\mu\nu} - 2i\mathcal{D}^-_\nu\phi_\mu,\end{aligned} \tag{3.13}$$

and

$$\mathcal{D}_\mu\phi^\mu = \frac{1}{2}i\mathcal{H}_\mu{}^\mu. \tag{3.14}$$

The field strengths can be seen to obey the following generalized Bianchi identities

$$\begin{aligned}&\mathcal{D}^+_\lambda F^-_{\mu\nu} + \mathcal{D}^-_\nu\mathcal{H}_{\lambda\mu} - \mathcal{D}^-_\mu\mathcal{H}_{\lambda\nu} = 0, \qquad \mathcal{D}^-_\lambda F^+_{\mu\nu} + \mathcal{D}^+_\mu\mathcal{H}_{\nu\lambda} - \mathcal{D}^+_\nu\mathcal{H}_{\mu\lambda} = 0, \\ &cD^+_\lambda F^+_{\mu\nu} + \mathcal{D}^+_\mu F^+_{\nu\lambda} + \mathcal{D}^+_\nu F^+_{\lambda\mu} = 0, \qquad \mathcal{D}^-_\lambda F^-_{\mu\nu} + \mathcal{D}^-_\mu F^-_{\nu\lambda} + \mathcal{D}^-_\nu F^-_{\lambda\mu} = 0.\end{aligned} \tag{3.15}$$

It is important to note that the identities above are valid on any curved manifold. Moreover, taking the following identities into account[6]

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\phi_\lambda + R^\rho{}_{\lambda\mu\nu}\phi_\rho + i[F_{\mu\nu}, \phi_\lambda] = 0, \qquad [\mathcal{D}_\mu, \mathcal{D}_\nu]\phi^\nu + R_{\mu\nu}\phi^\nu + i\big[F_{\mu\nu}, \phi^\nu\big] = 0, \tag{3.16}$$

the bosonic part of the Lagrangian containing $F_{\mu\nu}$ and ϕ_μ can be rewritten as

$$\begin{aligned}&\mathrm{tr}\bigg(\frac{1}{4}F^+_{\mu\nu}F^{-\mu\nu} + \frac{1}{2}\big(\mathcal{D}_\mu\phi^\mu\big)^2\bigg) \\ &\quad = \nabla_\mu\,\mathrm{tr}\bigg(\frac{1}{2}\phi^\mu\mathcal{D}^\nu\phi_\nu - \frac{1}{2}\phi^\nu\mathcal{D}_\nu\phi^\mu\bigg) \\ &\qquad + \mathrm{tr}\bigg(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\phi_\nu\mathcal{D}^\mu\phi^\nu - \frac{1}{4}[\phi_\mu, \phi_\nu]\big[\phi^\mu, \phi^\nu\big] + \frac{1}{2}R_{\mu\nu}\phi^\mu\phi^\nu\bigg).\end{aligned} \tag{3.17}$$

The Ricci term on the right-hand side is necessary to ensure the invariance of the action under the unbroken scalar supersymmetries.

The equations of motion are

$$\begin{aligned}&\mathcal{D}^+_\nu F^{-\mu\nu} + i\mathcal{D}^{-\mu}h - i\frac{1}{2}\big[\varphi, \mathcal{D}^{-\mu}\bar\varphi\big] - i\frac{1}{2}\big[\bar\varphi, \mathcal{D}^{-\mu}\varphi\big] \\ &\quad + 2i\big\{\eta, \psi^\mu\big\} - i\epsilon^{\mu\nu\kappa\lambda}\{\omega_\nu, \chi_{\kappa\lambda}\} = 0, \\ &\mathcal{D}^-_\nu F^{+\mu\nu} - i\mathcal{D}^{+\mu}h - i\frac{1}{2}\big[\varphi, \mathcal{D}^{+\mu}\bar\varphi\big] - i\frac{1}{2}\big[\bar\varphi, \mathcal{D}^{+\mu}\varphi\big] + 2i\big\{\zeta, \omega^\mu\big\} + 2i\big\{\psi_\nu, \chi^{\mu\nu}\big\} = 0, \\ &\mathcal{D}^+_\mu\mathcal{D}^{-\mu}\varphi + \mathcal{D}^-_\mu\mathcal{D}^{+\mu}\varphi + \big[\varphi, [\varphi, \bar\varphi]\big] + 4i\big\{\psi_\mu, \omega^\mu\big\} = 0, \\ &\mathcal{D}^+_\mu\mathcal{D}^{-\mu}\bar\varphi + \mathcal{D}^-_\mu\mathcal{D}^{+\mu}\bar\varphi - \big[\bar\varphi, [\varphi, \bar\varphi]\big] - 4i\{\eta, \zeta\} + i\frac{1}{2\sqrt{g}}\epsilon^{\kappa\lambda\mu\nu}\{\chi_{\kappa\lambda}, \chi_{\mu\nu}\} = 0, \\ &h - \mathcal{D}_\mu\phi^\mu = 0,\end{aligned}$$

[6] In our conventions the Riemann tensor is given by $R^\lambda{}_{\kappa\mu\nu} := \partial_\mu\Gamma^\lambda_{\nu\kappa} - \partial_\nu\Gamma^\lambda_{\mu\kappa} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\kappa} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\kappa}$.

$$
\begin{aligned}
&\mathcal{D}^{-\mu}\psi^{\nu} - \mathcal{D}^{-\nu}\psi^{\mu} - \frac{1}{\sqrt{g}}\epsilon^{\mu\nu\kappa\lambda}\left(\mathcal{D}^{+}_{\kappa}\omega_{\lambda} + i\frac{1}{2}[\chi_{\kappa\lambda},\varphi]\right) = 0,\\
&\mathcal{D}^{+}_{\mu}\eta - \mathcal{D}^{-\nu}\chi_{\mu\nu} - i[\omega_{\mu},\bar{\varphi}] = 0,\\
&\mathcal{D}^{-}_{\mu}\zeta - i[\psi_{\mu},\bar{\varphi}] + \frac{1}{2\sqrt{g}}\epsilon_{\mu}{}^{\nu\kappa\lambda}\mathcal{D}^{+}_{\nu}\chi_{\kappa\lambda} = 0,\\
&\mathcal{D}^{+}_{\mu}\psi^{\mu} + i[\zeta,\varphi] = 0,\\
&\mathcal{D}^{-}_{\mu}\omega^{\mu} + i[\eta,\varphi] = 0.
\end{aligned}
\tag{3.18}
$$

Finally, the energy–momentum tensor is given by[7]

$$
\begin{aligned}
T^{\mu\nu} &= \frac{2}{\sqrt{g}}\frac{\delta\mathcal{S}_{\text{twisted}}}{\delta g_{\mu\nu}}\\
&= g^{\mu\nu}\,\mathrm{tr}\bigg[\frac{1}{4}F^{+}_{\kappa\lambda}F^{-\kappa\lambda} - \frac{1}{2}h^2 - \phi^{\kappa}\mathcal{D}_{\kappa}h + \frac{1}{4}\mathcal{D}^{+}_{\kappa}\varphi\mathcal{D}^{-\kappa}\bar{\varphi} + \frac{1}{4}\mathcal{D}^{-}_{\kappa}\varphi\mathcal{D}^{+\kappa}\bar{\varphi}\\
&\quad + \frac{1}{8}[\varphi,\bar{\varphi}]^2 + \psi^{\kappa}\mathcal{D}^{+}_{\kappa}\eta + \omega^{\kappa}\mathcal{D}^{-}_{\kappa}\zeta + \chi^{\kappa\lambda}\mathcal{D}^{-}_{\kappa}\psi_{\lambda} + i\varphi\{\eta,\zeta\} - i\bar{\varphi}\{\psi_{\mu},\omega^{\mu}\}\bigg]\\
&\quad + \mathrm{tr}\bigg[-F^{+(\mu}{}_{\kappa}F^{-\nu)\kappa} + 2\phi^{(\mu}\mathcal{D}^{\nu)}h - \frac{1}{2}\mathcal{D}^{+(\mu}\varphi\mathcal{D}^{-\nu)}\bar{\varphi} - \frac{1}{2}\mathcal{D}^{-(\mu}\varphi\mathcal{D}^{+\nu)}\bar{\varphi}\\
&\quad - 2\psi^{(\mu}\mathcal{D}^{+\nu)}\eta - 2\omega^{(\mu}\mathcal{D}^{-\nu)}\zeta + 2\chi^{\kappa(\mu}\mathcal{D}^{-\nu)}\psi_{\kappa}\\
&\quad - 2\chi^{\kappa(\mu}\mathcal{D}^{-}_{\kappa}\psi^{\nu)} + 2i\bar{\varphi}\{\psi^{(\mu},\omega^{\nu)}\}\bigg],
\end{aligned}
\tag{3.19}
$$

where the brackets denote symmetrization with weight one.

3.3. Superconformal symmetry

Upon twisting, as in the case of the Yang–Mills gaugino (3.5), the chiral ordinary supersymmetry and the anti-chiral conformal supersymmetry parameters, ε_+ and ε_- respectively, decompose as

$$
\begin{aligned}
\varepsilon_+ &= \frac{1}{2}\left(\varepsilon_0 + \varepsilon_{\mu}\gamma^{\mu} + \frac{1}{2}\varepsilon_{\mu\nu}\gamma^{\mu\nu} + \varepsilon'_{\mu}\gamma^{\mu}\gamma^{(5)} + \varepsilon_5\gamma^{(5)}\right)C^{-1},\\
\varepsilon_- &= \frac{1}{2}\left(\tilde{\varepsilon}_0 + \tilde{\varepsilon}_{\mu}\gamma^{\mu} + \frac{1}{2}\tilde{\varepsilon}_{\mu\nu}\gamma^{\mu\nu} + \tilde{\varepsilon}'_{\mu}\gamma^{\mu}\gamma^{(5)} + \tilde{\varepsilon}_5\gamma^{(5)}\right)C^{-1}.
\end{aligned}
\tag{3.20}
$$

The corresponding supercharges Q_-, S_+ have the opposite chiralities from (2.20), (3.4), and decompose similarly

$$
\begin{aligned}
Q_- &= \frac{1}{2}\left(Q_0 + Q_{\mu}\gamma^{\mu} + \frac{1}{2}Q_{\mu\nu}\gamma^{\mu\nu} + \tilde{Q}_{\mu}\gamma^{\mu}\gamma^{(5)} + Q_5\gamma^{(5)}\right)C^{-1},\\
S_+ &= \frac{1}{2}\left(S_0 + S_{\mu}\gamma^{\mu} + \frac{1}{2}S_{\mu\nu}\gamma^{\mu\nu} + \tilde{S}_{\mu}\gamma^{\mu}\gamma^{(5)} + S_5\gamma^{(5)}\right)C^{-1}.
\end{aligned}
\tag{3.21}
$$

[7] In deriving (3.19), one has to take into account the variation of the Christoffel connection in the covariant derivative ∇_{μ}.

Altogether, there are $(1+4+6+4+1)\times 2 = 32$ components, all of which are unbroken in a flat background. In curved backgrounds, however, in order to have unbroken supersymmetries it is necessary that the corresponding supersymmetry parameters should be covariantly constant. Generically, this requirement can only be met for the scalar parameters $\varepsilon_0, \varepsilon_5, \tilde{\varepsilon}_0, \tilde{\varepsilon}_5$. Hence, in a generic curved background all the non-scalar supersymmetries $Q_\mu, \tilde{Q}_\mu, Q_{\mu\nu}, S_\mu, \tilde{S}_\mu, S_{\mu\nu}$ are broken.[8]

In addition, one has to check that in putting the theory on a curved background the action remains supersymmetric. This is not obvious: in passing from flat to curved background there may be curvature terms arising from the commutator of two covariant derivatives $[\nabla_\mu, \nabla_\nu]$, spoiling the invariance. In the case at hand, one can verify that the two ordinary scalar supercharges Q_0 and Q_5 of (3.22) indeed give rise to the two unbroken topological symmetries of the twisted super-Yang–Mills (3.9), as shown in [3,4,6–9,11]. Explicitly the action of Q_0, Q_5 is given by:

$$
\begin{aligned}
&\left[Q_0, A^+_\mu\right] = 0, && \left[Q_5, A^+_\mu\right] = -2\omega_\mu,\\
&\left[Q_0, A^-_\mu\right] = -2\psi_\mu, && \left[Q_5, A^-_\mu\right] = 0,\\
&[Q_0, \varphi] = 0, && [Q_5, \varphi] = 0,\\
&[Q_0, \bar{\varphi}] = -2\zeta, && [Q_5, \bar{\varphi}] = -2\eta,\\
&[Q_0, h] = [\varphi, \zeta], && [Q_5, h] = -[\varphi, \eta],\\
&\{Q_0, \chi_{\mu\nu}\} = F^+_{\mu\nu}, && \{Q_5, \chi_{\mu\nu}\} = -\frac{1}{2\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}F^-_{\kappa\lambda} = -\left(\star F^-\right)_{\mu\nu},\\
&\{Q_0, \psi_\mu\} = 0, && \{Q_5, \psi_\mu\} = \mathcal{D}^-_\mu\varphi,\\
&\{Q_0, \omega_\mu\} = \mathcal{D}^+_\mu\varphi, && \{Q_5, \omega_\mu\} = 0,\\
&\{Q_0, \eta\} = ih + i\frac{1}{2}[\varphi, \bar{\varphi}], && \{Q_5, \eta\} = 0,\\
&\{Q_0, \zeta\} = 0, && \{Q_5, \zeta\} = -ih + i\frac{1}{2}[\varphi, \bar{\varphi}],\\
&[Q_0, g_{\mu\nu}] = 0, && [Q_5, g_{\mu\nu}] = 0.
\end{aligned}
\tag{3.22}
$$

In contrast to the above ordinary scalar supercharges, the conformal scalar supercharges S_0 and S_5 are not straightforwardly realized in the twisted super-Yang–Mills as Noether symmetries, because the conformal supersymmetry transformations in flat space involve the space coordinate x^μ explicitly, see (2.13). In a curved background, x^μ should be replaced by a vector field v^μ satisfying the following relation[9]:

$$\nabla_\mu v^\nu = \delta_\mu{}^\nu. \tag{3.23}$$

Indeed, the condition $\partial_\mu x^\nu = \delta_\mu{}^\nu$ is the only property of x^μ which is required for the invariance of the action in a flat background. Moreover, it can be checked explicitly that the condition (3.23) suffices for the conformal scalar supercharges S_0 and S_5 to survive as symmetries of the action in a curved background, provided the vector field v^μ exists. It is straightforward to see that any cone

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\hat{s}^2, \qquad \mathrm{d}\hat{s}^2 = \hat{g}_{ij}(y)\,\mathrm{d}y^i\,\mathrm{d}y^j, \quad i, j = 1, 2, 3, \tag{3.24}$$

[8] Manifolds which admit covariantly-constant tensors are restricted to be special-holonomy.

[9] See [19] for a different treatment.

admits such a vector field:

$$v = r\frac{\partial}{\partial r}. \tag{3.25}$$

Conversely, as we show in Appendix A, any manifold admitting such a vector field is a cone, at least locally. Hence, the conformal supercharges S_0 and S_5 remain unbroken in a cone background and, similarly to the case of the ordinary supercharges Q_0 and Q_5, they give rise to a family of TQFTs, as we shall demonstrate explicitly in Section 4.

Using the vector field (3.23), the conformal scalar supersymmetries are given by

$$\begin{aligned}
&[S_0, A^+_\mu] = 2\chi_{\mu\nu}v^\nu, && [S_5, A^+_\mu] = 2\zeta v_\mu,\\
&[S_0, A^-_\mu] = 2\eta v_\mu, && [S_5, A^-_\mu] = -2(\star\chi)_{\mu\nu}v^\nu,\\
&[S_0, \varphi] = -2\omega_\mu v^\mu, && [S_5, \varphi] = -2\psi_\mu v^\mu,\\
&[S_0, \bar\varphi] = 0, && [S_5, \bar\varphi] = 0,\\
&[S_0, h] = 2iv^\mu\mathcal{D}^+_\mu\eta + 4i\eta + [\omega_\mu v^\mu, \bar\varphi], && [S_5, h] = -2iv^\mu\mathcal{D}^-_\mu\zeta - 4i\zeta\\
& && \qquad - [\psi_\mu v^\mu, \bar\varphi],\\
&\{S_0, \chi_{\mu\nu}\} = \frac{1}{\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}v_\kappa\mathcal{D}^-_\lambda\bar\varphi, && \{S_5, \chi_{\mu\nu}\} = -v_\mu\mathcal{D}^+_\nu\bar\varphi + v_\nu\mathcal{D}^+_\mu\bar\varphi,\\
&\{S_0, \psi_\mu\} = v^\lambda\mathcal{H}_{\lambda\mu} + iv_\mu h + i\frac{1}{2}v_\mu[\varphi, \bar\varphi] - 2i\phi_\mu, && \{S_5, \psi_\mu\} = -(\star F^+)_{\mu\nu}v^\nu,\\
&\{S_0, \omega_\mu\} = -(\star F^-)_{\mu\nu}v^\nu, && \{S_5, \omega_\mu\} = -\mathcal{H}_{\mu\nu}v^\nu - iv_\mu h\\
& && \qquad + i\frac{1}{2}v_\mu[\varphi, \bar\varphi] + 2i\phi_\mu,\\
&\{S_0, \eta\} = 0, && \{S_5, \eta\} = v^\mu\mathcal{D}^-_\mu\bar\varphi + 2\bar\varphi,\\
&\{S_0, \zeta\} = v^\mu\mathcal{D}^+_\mu\bar\varphi + 2\bar\varphi, && \{S_5, \zeta\} = 0,\\
&[S_0, g_{\mu\nu}] = 0, && [S_5, g_{\mu\nu}] = 0.
\end{aligned} \tag{3.26}$$

It is worth noting that there exists a discrete symmetry $\mathcal{Z}$ given by[10]

$$\begin{pmatrix} A^+_\mu \\ A^-_\mu \\ \varphi \\ \bar\varphi \\ h \\ \chi_{\mu\nu} \\ \psi_\mu \\ \omega_\mu \\ \eta \\ \zeta \end{pmatrix} \xrightarrow{\mathcal{Z}} \begin{pmatrix} A^-_\mu \\ A^+_\mu \\ \varphi \\ \bar\varphi \\ -h \\ -(\star\chi)_{\mu\nu} \\ \omega_\mu \\ \psi_\mu \\ \zeta \\ \eta \end{pmatrix}, \tag{3.27}$$

[10] Note the similarity with the 'Hermitian-conjugation' defined in [4].

under which the Lagrangian (3.9) is invariant. Furthermore $\mathcal{Z}$ acts on the scalar supercharges as

$$Q_0 = \mathcal{Z} \circ Q_5 \circ \mathcal{Z}, \qquad S_0 = \mathcal{Z} \circ S_5 \circ \mathcal{Z}, \qquad \mathcal{Z}^2 = \text{identity}. \tag{3.28}$$

Apart from the four scalar supersymmetries, the discrete symmetry $\mathcal{Z}$ and the diffeomorphisms of the three-dimensional base, there exist another two (bosonic) symmetries of the twisted $\mathcal{N}=4$ super-Yang–Mills in a generic cone background. One is the U(1) R-symmetry or the SO(2) rotation on the (Φ_9, Φ_{10}) plane

$$\begin{aligned}
&[R, A^+_\mu] = 0, && [R, A^-_\mu] = 0,\\
&[R, \varphi] = +i\varphi, && [R, \bar\varphi] = -i\bar\varphi,\\
&[R, h] = 0, && [R, \chi_{\mu\nu}] = -i\frac{1}{2}\chi_{\mu\nu},\\
&[R, \psi_\mu] = +i\frac{1}{2}\psi_\mu, && [R, \omega_\mu] = +i\frac{1}{2}\omega_\mu,\\
&[R, \eta] = -i\frac{1}{2}\eta, && [R, \zeta] = -i\frac{1}{2}\zeta,\\
&[R, g_{\mu\nu}] = 0.
\end{aligned} \tag{3.29}$$

The other is a ‘dilatation’ symmetry[11]

$$\begin{aligned}
&[\mathfrak{D}, A^+_\mu] = v^\lambda\nabla_\lambda A^+_\mu + A^+_\mu, && [\mathfrak{D}, A^-_\mu] = v^\lambda\nabla_\lambda A^-_\mu + A^-_\mu,\\
&[\mathfrak{D}, \varphi] = v^\mu\nabla_\mu\varphi, && [\mathfrak{D}, \bar\varphi] = v^\mu\nabla_\mu\bar\varphi + 2\bar\varphi,\\
&[\mathfrak{D}, h] = v^\mu\nabla_\mu h + 2h, && [\mathfrak{D}, \chi_{\mu\nu}] = v^\lambda\nabla_\lambda\chi_{\mu\nu} + 2\chi_{\mu\nu},\\
&[\mathfrak{D}, \psi_\mu] = v^\lambda\nabla_\lambda\psi_\mu + \psi_\mu, && [\mathfrak{D}, \omega_\mu] = v^\lambda\nabla_\lambda\omega_\mu + \omega_\mu,\\
&[\mathfrak{D}, \eta] = v^\mu\nabla_\mu\eta + 2\eta, && [\mathfrak{D}, \zeta] = v^\mu\nabla_\mu\zeta + 2\zeta,\\
&[\mathfrak{D}, g_{\mu\nu}] = 0.
\end{aligned} \tag{3.30}$$

Since in flat space the inversion map (2.21) transforms the conformal supersymmetries to the ordinary supersymmetries and vice versa (2.22), one might wonder whether the theory defined by the conformal supercharges S_0, S_5 is truly inequivalent to the one given by the ordinary supercharges Q_0, Q_5. In a flat background, a straightforward computation shows that the conformal scalar supercharges are related by the inversion map to the ordinary vector supercharges. Schematically,[12]

$$S_0 \sim x^\mu Q_\mu, \qquad S_5 \sim x^\mu \tilde{Q}_\mu. \tag{3.32}$$

Hence the conformal scalar supercharges are not equivalent to the ordinary scalar supercharges.

[11] Considering the asymmetric scaling dimensions of φ and $\bar\varphi$ in (3.30), one may wish to regard $\mathfrak{D} + R$ as the dilation operator instead.

[12] More precisely:

$$\begin{aligned}
&[S_0 - x^\mu Q_\mu, h] = 4i\eta, && [S_0 - x^\mu Q_\mu, \psi_\nu] = -2i\phi_\nu, && [S_0 - x^\mu Q_\mu, \zeta] = 2\bar\varphi,\\
&[S_5 - x^\mu \tilde{Q}_\mu, h] = -4i\zeta, && [S_5 - x^\mu \tilde{Q}_\mu, \omega_\nu] = 2i\phi_\nu, && [S_5 - x^\mu \tilde{Q}_\mu, \eta] = 2\bar\varphi,
\end{aligned} \tag{3.31}$$

with all other commutators identically zero.

3.4. Superconformal algebra

All scalar supercharges are nilpotent off-shell

$$Q_0^2 = 0, \qquad Q_5^2 = 0, \qquad S_0^2 = 0, \qquad S_5^2 = 0. \tag{3.33}$$

Moreover, among the scalar supercharges there are two pairs which anticommute off-shell:

$$\{Q_0, S_5\} = 0, \qquad \{Q_5, S_0\} = 0. \tag{3.34}$$

All other pairs of supercharges only anticommute on-shell. More specifically, up to the equations of motion (3.18) we have

$$\begin{aligned} &\{Q_0, Q_5\} \equiv -2\mathfrak{L}_{\varphi}, && \{S_0, S_5\} \equiv 2\mathfrak{L}_{v^2\bar{\varphi}}, \\ &\{Q_0, S_0\} \equiv -2(\mathfrak{D} - \mathfrak{L}_{v\cdot A^+}), && \{Q_5, S_5\} \equiv -2(\mathfrak{D} - \mathfrak{L}_{v\cdot A^-}), \end{aligned} \tag{3.35}$$

where $\mathfrak{L}_{v\cdot A^\pm}$ denotes infinitesimal gauge transformations generated by $v^\mu A^\pm_\mu$, while $\mathfrak{D}$ corresponds to the dilatation (3.30). In the present paper '$\equiv$' denotes an on-shell equality, i.e. an equality up to equations of motion. The dilation operator commutes with all the scalar supercharges off-shell

$$[\mathfrak{D}, Q_0] = 0, \qquad [\mathfrak{D}, Q_5] = 0, \qquad [\mathfrak{D}, S_0] = 0, \qquad [\mathfrak{D}, S_5] = 0. \tag{3.36}$$

The U(1) charges of the supercharges can be read off of the following commutation relations

$$[R, Q_0] = -iQ_0, \qquad [R, Q_5] = -iQ_5, \qquad [R, S_0] = -i\frac{1}{2}S_0, \qquad [R, S_5] = -i\frac{1}{2}S_5. \tag{3.37}$$

Finally, the R-symmetry commutes with the dilatation off-shell

$$[R, \mathfrak{D}] = 0. \tag{3.38}$$

From the superconformal algebra above, we conclude that the most general linear combination of the four scalar supercharges which is nilpotent, is parameterized by four complex numbers subject to one constraint:

$$w_1 Q_0 + w_2 Q_5 + w_3 S_0 + w_4 S_5, \qquad w_1 w_3 + w_2 w_4 = 0. \tag{3.39}$$

In particular, there are four possible pairs of supercharges which are nilpotent:

$$z_1 Q_0 + z_2 Q_5, \qquad z_1 S_0 + z_2 S_5, \qquad z_1 S_0 + z_2 Q_5, \qquad z_1 S_5 + z_2 Q_0. \tag{3.40}$$

In the above, $z_{1,2}$ are arbitrary complex numbers. As in [11] we note that an overall constant is unimportant, hence each pair of supercharges corresponds to a family of topological theories parameterized by $\mathbb{CP}^1$. For simplicity, henceforth we work in the patch $z_1 \neq 0$.[13] The first linear combination in (3.40) was recently investigated by Kapustin and Witten [11] and was shown to be relevant to the Langlands program. The second linear combination is the main focus of the present paper. For completeness, in Appendix B we give all the supersymmetric conditions, i.e. the BPS equations, arising from each linear combination in (3.40).

[13] The parameter $z := z_2/z_1$ is related to the parameter t of [11] through $it = (1-z)/(1+z)$. In particular $|z| = 1$ corresponds to t being real.

3.5. Vanishing theorems—weak version

For the rest of this section we shall focus on the conformal supersymmetry

$$S_z := S_0 + zS_5. \tag{3.41}$$

We shall derive certain important 'vanishing theorems', which follow from a single algebraic identity. Let us define

$$\begin{aligned}
&\mathcal{V}_{\mu\nu}(z) := \frac{1}{\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}v_\kappa\mathcal{D}^-_\lambda\bar{\varphi} - z\big(v_\mu\mathcal{D}^+_\nu\bar{\varphi} - v_\nu\mathcal{D}^+_\mu\bar{\varphi}\big),\\
&\mathcal{V}_\mu(z) := v^\lambda\mathcal{H}_{\lambda\mu} + iv_\mu\mathcal{D}_\lambda\phi^\lambda + i\frac{1}{2}v_\mu[\varphi,\bar{\varphi}] - 2i\phi_\mu - z\big(\star F^+\big)_{\mu\nu}v^\nu,\\
&\hat{\mathcal{V}}_\mu(z) := -\big(\star F^-\big)_{\mu\nu}v^\nu - z\bigg(\mathcal{H}_{\mu\nu}v^\nu + iv_\mu\mathcal{D}_\lambda\phi^\lambda - i\frac{1}{2}v_\mu[\varphi,\bar{\varphi}] - 2i\phi_\mu\bigg),\\
&\mathcal{V}_\eta(z) := z\big(v^\mu\mathcal{D}^-_\mu\bar{\varphi} + 2\bar{\varphi}\big),\\
&\mathcal{V}_\zeta := v^\mu\mathcal{D}^+_\mu\bar{\varphi} + 2\bar{\varphi}.
\end{aligned} \tag{3.42}$$

From the on-shell conformal supersymmetry transformations of the fermions (3.26), we see that an S_z-supersymmetric configuration must satisfy, by definition, the following conditions (BPS equations)

$$\mathcal{V}_{\mu\nu}(z) = \mathcal{V}_\mu(z) = \hat{\mathcal{V}}_\mu(z) = \mathcal{V}_\eta(z) = \mathcal{V}_\zeta = 0. \tag{3.43}$$

Or, in compact notation, $|\mathcal{V}(z)\rangle = 0$. Disregarding the boundary terms we obtain the following algebraic identity involving two arbitrary complex numbers z, w:

$$\begin{aligned}
\big\langle\mathcal{V}(w)\big|\mathcal{V}(z)\big\rangle &= \big\langle\mathcal{V}(z)\big|\mathcal{V}(w)\big\rangle^*\\
&:= \int \mathrm{d}^4x\frac{\sqrt{g}}{2v^2}\,\mathrm{tr}\bigg[\frac{1}{2}\mathcal{V}^{\mu\nu}(z)\mathcal{V}_{\mu\nu}(w)^\dagger + \mathcal{V}^\mu(z)\mathcal{V}_\mu(w)^\dagger\\
&\quad + \hat{\mathcal{V}}^\mu(z)\hat{\mathcal{V}}_\mu(w)^\dagger + \mathcal{V}_\eta(z)\mathcal{V}_\eta(w)^\dagger + \mathcal{V}_\zeta\mathcal{V}_\zeta^\dagger\bigg]\\
&= \int \mathrm{d}^4x\,\big(1 + zw^*\big)\mathcal{L}^{\text{boson}}_{\text{twisted}} + \int \mathrm{d}^4x\frac{1}{8}\epsilon^{\kappa\lambda\mu\nu}\,\mathrm{tr}\big[zF^+_{\kappa\lambda}F^+_{\mu\nu} + w^*F^-_{\kappa\lambda}F^-_{\mu\nu}\big],
\end{aligned} \tag{3.44}$$

where $\mathcal{L}^{\text{boson}}_{\text{twisted}}$ is the bosonic part of our twisted $\mathcal{N} = 4$ super-Yang–Mills Lagrangian (3.9), after eliminating all the auxiliary fields:

$$\mathcal{L}^{\text{boson}}_{\text{twisted}} = \sqrt{g}\,\mathrm{tr}\bigg[\frac{1}{4}F^+_{\mu\nu}F^{-\mu\nu} + \frac{1}{2}\big(\mathcal{D}_\mu\phi^\mu\big)^2 + \frac{1}{4}\mathcal{D}^+_\mu\varphi\mathcal{D}^{-\mu}\bar{\varphi} + \frac{1}{4}\mathcal{D}^-_\mu\varphi\mathcal{D}^{+\mu}\bar{\varphi} + \frac{1}{8}[\varphi,\bar{\varphi}]^2\bigg]. \tag{3.45}$$

Note that $\mathcal{L}^{\text{boson}}_{\text{twisted}}$ is positive semi-definite. Eq. (3.44) is an algebraic identity which holds for field configurations which are smooth enough at the origin of the cone and fall off fast enough at spatial infinity, so that we can neglect the boundary terms.

For $w = z$, the identity (3.44) reduces to

$$\int_{\text{cone}} \mathcal{L}^{\text{boson}}_{\text{twisted}} = \frac{\langle\mathcal{V}(z)|\mathcal{V}(z)\rangle}{1 + |z|^2} - \mathcal{T}(z), \tag{3.46}$$

where we have set

$$\mathcal{T}(z) := \frac{1}{8(1+|z|^2)}\int \mathrm{d}^4x\, \epsilon^{\kappa\lambda\mu\nu}\,\mathrm{tr}\big[zF^+_{\kappa\lambda}F^+_{\mu\nu}+z^*F^-_{\kappa\lambda}F^-_{\mu\nu}\big]$$
$$= \mathrm{Re}\bigg[\int_{\mathrm{cone}}\frac{\mathrm{tr}(zF^+\wedge F^+)}{1+|z|^2}\bigg]. \tag{3.47}$$

Note that $\mathcal{T}(z)$ is real and topological, in that it does not depend on the metric. On the other hand, the choice $w^* = z^{-1}$ gives another identity for the bosonic action:

$$\int_{\mathrm{cone}}\mathcal{L}^{\mathrm{boson}}_{\mathrm{twisted}} = \frac{1}{2}\big\langle\mathcal{V}\big(z^{*-1}\big)\big|\mathcal{V}(z)\big\rangle - \frac{1}{4}\int_{\mathrm{cone}}\mathrm{tr}\big[zF^+\wedge F^+ + z^{-1}F^-\wedge F^-\big]. \tag{3.48}$$

Combining (3.46) and (3.48) one can eliminate the topological quantities to obtain[14]

$$\big(|z|-|z|^{-1}\big)^2\int_{\mathrm{cone}}\mathcal{L}^{\mathrm{boson}}_{\mathrm{twisted}} = \big\langle\mathcal{V}(z)-\mathcal{V}\big(z^{*-1}\big)\big|\mathcal{V}(z)-\mathcal{V}\big(z^{*-1}\big)\big\rangle. \tag{3.49}$$

From the positive semi-definite property of $\mathcal{L}^{\mathrm{boson}}_{\mathrm{twisted}}$ and $\langle\mathcal{V}(z)|\mathcal{V}(z)\rangle$, we have the following inequalities for any bosonic configuration:

$$\int_{\mathrm{cone}}\mathcal{L}^{\mathrm{boson}}_{\mathrm{twisted}} \geqslant -\mathcal{T}(z) \geqslant -\frac{\langle\mathcal{V}(z)|\mathcal{V}(z)\rangle}{1+|z|^2}. \tag{3.50}$$

Hence, given a complex number z, a supersymmetric configuration (for which $|\mathcal{V}(z)\rangle = 0$) minimizes the action within a given topological sector. For an arbitrary bosonic configuration, regarding $\mathcal{T}(z)$ as a function of z, the minimum and the maximum of $\mathcal{T}(z)$ are located generically[15] at two antipodal points on the unit circle:

$$\mathcal{T}(z_{\min}) \leqslant \mathcal{T}(z) \leqslant \mathcal{T}(z_{\max}), \quad |z_{\min}| = |z_{\max}| = 1, \quad z_{\min}+z_{\max} = 0, \tag{3.51}$$

such that

$$\mathcal{T}(z_{\min}) + \mathcal{T}(z_{\max}) = 0. \tag{3.52}$$

At the extremal points, $\int_{\mathrm{cone}} z\,\mathrm{tr}(F^+\wedge F^+)$ becomes real. Hence

$$\mathcal{T}(z_{\min}) = \int_{\mathrm{cone}}\frac{1}{2}z_{\min}\,\mathrm{tr}\big(F^+\wedge F^+\big) = \int_{\mathrm{cone}}\frac{1}{2}z^*_{\min}\,\mathrm{tr}\big(F^-\wedge F^-\big) \leqslant 0. \tag{3.53}$$

Since the left-hand side of (3.46) is independent of z, the above analysis implies that, for any given bosonic configuration, $\langle\mathcal{V}(z)|\mathcal{V}(z)\rangle/(1+|z|^2)$ is bounded both from above and below:

$$\frac{1}{2}\big\langle\mathcal{V}(z_{\min})\big|\mathcal{V}(z_{\min})\big\rangle \leqslant \frac{\langle\mathcal{V}(z)|\mathcal{V}(z)\rangle}{1+|z|^2} \leqslant \frac{1}{2}\big\langle\mathcal{V}(z_{\max})\big|\mathcal{V}(z_{\max})\big\rangle,$$
$$\big\langle\mathcal{V}(z_{\max})\big|\mathcal{V}(z_{\max})\big\rangle = \big\langle\mathcal{V}(z_{\min})\big|\mathcal{V}(z_{\min})\big\rangle + 4\mathcal{T}(z_{\max}). \tag{3.54}$$

In particular, if $\langle\mathcal{V}(z_{\min})|\mathcal{V}(z_{\min})\rangle \neq 0$, $\langle\mathcal{V}(z)|\mathcal{V}(z)\rangle$ cannot vanish for all z. As in [11], this leads to certain vanishing theorems.

[14] Note that $\langle\mathcal{V}(z_1)+\mathcal{V}(z_2)|\mathcal{V}(w)\rangle := \langle\mathcal{V}(z_1)|\mathcal{V}(w)\rangle + \langle\mathcal{V}(z_2)|\mathcal{V}(w)\rangle$, etc.

[15] Unless $\int_{\mathrm{cone}}\mathrm{tr}(F^+\wedge F^+) = 0$, in which case $\mathcal{T}(z)$ is identically zero.

- *Vanishing theorem for generic configurations*:
  A configuration is supersymmetric if and only if $\langle\mathcal{V}(z_{\min})|\mathcal{V}(z_{\min})\rangle = 0$.

Furthermore, from the identities (3.44), (3.46) we obtain the following vanishing theorems for supersymmetric configurations.

- *Vanishing theorems for BPS configurations*:
  (1) The choice $w^* = -z^{-1}$ in (3.44) shows that any BPS state, for which $|\mathcal{V}(z)\rangle = 0$, should satisfy

$$\int_{\text{cone}} \operatorname{tr}\bigl(F^- \wedge F^-\bigr) = \int_{\text{cone}} z^2 \operatorname{tr}\bigl(F^+ \wedge F^+\bigr). \tag{3.55}$$

  Hence, by complex conjugation, we obtain

$$\begin{aligned} &\int_{\text{cone}} \operatorname{tr}\bigl(F^+ \wedge F^+\bigr) = \int_{\text{cone}} \operatorname{tr}\bigl(F^- \wedge F^-\bigr) = 0, \quad \mathcal{T}(z) = 0 \qquad \text{if } |z| \neq 1, \\ &\mathcal{T}(z) = \int_{\text{cone}} \frac{z}{2} \operatorname{tr}\bigl(F^+ \wedge F^+\bigr) = \int_{\text{cone}} \frac{z^*}{2} \operatorname{tr}\bigl(F^- \wedge F^-\bigr): \quad \text{real} \quad \text{if } |z| = 1. \end{aligned} \tag{3.56}$$

  (2) Conversely, when $\int \operatorname{tr}(F^+ \wedge F^+) \neq 0$, the BPS equations $|\mathcal{V}(z)\rangle = 0$ have no solution for $|z| \neq 1$.
  (3) A BPS state, for which $|\mathcal{V}(z)\rangle = 0$, saturates the bound in (3.50)

$$\begin{aligned} \int \mathrm{d}^4 x\, \mathcal{L}^{\text{boson}}_{\text{twisted}}\big|_{\text{BPS}} &= -\mathcal{T}(z) \\ &= \begin{cases} -z \int \mathrm{d}^4 x \frac{1}{8} \epsilon^{\kappa\lambda\mu\nu} \operatorname{tr}(F^+_{\kappa\lambda} F^+_{\mu\nu}) \geqslant 0 & \text{if } |z| = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.57}$$

  (4) Any state which satisfies the BPS equations $|\mathcal{V}(z)\rangle = 0$ for some z such that $|z| \neq 1$, satisfies the BPS equations for all z.
  (5) Any state which satisfies the BPS equations $|\mathcal{V}(z)\rangle = 0$ for some z such that $|z| = 1$, satisfies the BPS equations for all z iff $\mathcal{T}(z) = 0$.

As a corollary we obtain an alternative criterion for a generic configuration to be supersymmetric.

- *Corollary*:
  A configuration is supersymmetric if and only if $|\mathcal{V}(z)\rangle = 0$ for some $|z| = 1$.

Using explicitly the cone coordinates (3.24), the BPS equations read for $|z| = 1$:

$$\begin{aligned} &F^+_{jr} + z\bigl(\star F^+\bigr)_{jr} + 2i\mathcal{D}^+_r \phi_j + i\frac{2}{r}\phi_j = 0, && \mathcal{D}_r \varphi + \frac{2}{r}\varphi = 0, \\ &\mathcal{D}_r \phi_r + \frac{2}{r}\phi_r - \mathcal{D}_i \phi^i = 0, && \mathcal{D}_i \varphi = 0, \\ &[\phi_\mu, \varphi] = 0, && [\varphi, \bar{\varphi}] = 0. \end{aligned} \tag{3.58}$$

In particular if the BPS configuration has a trivial topology $\mathcal{T}(z)=0$ so that $\int\mathcal{L}^{\text{boson}}_{\text{twisted}}=0$, it follows from (3.45) that the supersymmetric configuration satisfies

$$F^+_{\mu\nu}=0,\qquad D_\mu\phi^\mu=0,\qquad \varphi=0,\qquad \mathcal{D}_r\phi_\mu+\frac{1}{r}\phi_\mu=0,\qquad [\phi_r,\phi_i]=0. \tag{3.59}$$

In the special case where the base of the cone is the round sphere, the cone reduces to $\mathbb{R}^4$. From (3.17) it then follows that a BPS configuration with trivial topology satisfies $F_{\mu\nu}=0$, $\varphi=\phi_\mu=0$. These conditions are more restrictive than the usual vacuum equations $F_{\mu\nu}=0$, $\mathcal{D}_\mu\Phi_I=0$, $[\Phi_I,\Phi_J]=0$, where $\Phi_I:=(\phi_\mu,\varphi,\bar{\varphi})$. This is due to our requirement that the fields should decay fast enough at the boundary, and is consistent with the fact that the conformal symmetry remains unbroken if the VEVs of the scalars vanish.

3.6. Vanishing theorems—strong version

Taking Eqs. (3.11), (3.16) and the Bianchi identity for the Riemann tensor into account, we obtain

$$\begin{aligned}
\epsilon^{\kappa\lambda\mu\nu}\operatorname{tr}\big(F^+_{\kappa\lambda}F^+_{\mu\nu}\big)&=\epsilon^{\kappa\lambda\mu\nu}\operatorname{tr}(F_{\kappa\lambda}F_{\mu\nu})-4\epsilon^{\kappa\lambda\mu\nu}\nabla_\kappa\operatorname{tr}\bigg(\phi_\lambda\mathcal{D}_\mu\phi_\nu-i\phi_\lambda F_{\mu\nu}+\frac{2}{3}\phi_\lambda\phi_\mu\phi_\nu\bigg),\\
\epsilon^{\kappa\lambda\mu\nu}\operatorname{tr}\big(F^-_{\kappa\lambda}F^-_{\mu\nu}\big)&=\epsilon^{\kappa\lambda\mu\nu}\operatorname{tr}(F_{\kappa\lambda}F_{\mu\nu})-4\epsilon^{\kappa\lambda\mu\nu}\nabla_\kappa\operatorname{tr}\bigg(\phi_\lambda\mathcal{D}_\mu\phi_\nu+i\phi_\lambda F_{\mu\nu}-\frac{2}{3}\phi_\lambda\phi_\mu\phi_\nu\bigg).
\end{aligned} \tag{3.60}$$

This implies that the topological term is real and it reduces to the usual instanton number

$$\int_{\text{cone}}\operatorname{tr}\big(F^+\wedge F^+\big)=\int_{\text{cone}}\operatorname{tr}\big(F^-\wedge F^-\big)=\int_{\text{cone}}\operatorname{tr}(F\wedge F)\in 16\pi^2\mathbb{Z}, \tag{3.61}$$

provided we can neglect the boundary terms in (3.60). This requirement is automatically satisfied in the case where the VEVs of the scalars vanish at infinity. The latter condition is also sufficient for the conformal symmetry to be unbroken.[16]

Here, we shall not discuss the issue of the boundary conditions any further; instead we shall assume that (3.61) holds and examine the consequences. In this case the minimum and maximum of $\mathcal{T}(z)$ occur at $z=\pm1$, so that

$$\begin{aligned}
&\mathcal{T}(z_{\min})\leqslant\mathcal{T}(z)\leqslant\mathcal{T}(z_{\max}),\quad z^2_{\min}=z^2_{\max}=1,\quad z_{\min}+z_{\max}=0,\\
&\mathcal{T}(z_{\max})=-\mathcal{T}(z_{\min})=\bigg|\int_{\text{cone}}\frac{1}{2}\operatorname{tr}(F\wedge F)\bigg|.
\end{aligned} \tag{3.62}$$

Consequently we have the following strong version of the vanishing theorems of the previous subsections.

- *Vanishing theorems—strong version*:
  (1) A configuration is supersymmetric if and only if $|\mathcal{V}(z)\rangle=0$ for $z=+1$ or $z=-1$.

[16] Also in ordinary (untwisted) super-Yang–Mills in flat space, demanding the action to be finite leads to the condition that $F_{\mu\nu}$, $D_\mu\Phi_I$, $[\Phi_I,\Phi_J]$ should vanish fast enough at infinity. On the other hand, in the twisted theory we are considering, the boundary conditions for a finite action are slightly different from the ones in the untwisted case. Namely, what should vanish at infinity is $F^+_{\mu\nu}$, $\mathcal{D}_\mu\phi^\mu$, $\mathcal{D}^\pm_\mu\varphi$ rather than $F_{\mu\nu}$, $D_\mu\Phi_I$, $[\Phi_I,\Phi_J]$.

(2) For a generic supersymmetric configuration the action saturates the bound

$$\int \mathrm{d}^4x\, \mathcal{L}^{\text{boson}}_{\text{twisted}}\Big|_{\text{BPS}} = \left| \int_{\text{cone}} \frac{1}{2}\operatorname{tr}(F \wedge F) \right| \in 8\pi^2\mathbb{N}. \tag{3.63}$$

In particular, in the case where the cone is the flat $\mathbb{R}^4$, we see from (3.17), (3.63) that the BPS state should satisfy $D_\mu \Phi_I = 0$, $[\Phi_I, \Phi_J] = 0$. Thus, the BPS equations (3.58) reduce to the instanton equations $F_{\mu\nu} \pm (\star F)_{\mu\nu} = 0$, $\phi_\mu = \varphi = 0$.

4. Off-shell formulation

The conformal supersymmetry transformations (3.26) given in the previous section have two unsatisfactory features. Firstly, the anti-commutator relation $\{S_0, S_5\} \equiv 2\mathfrak{L}_{v^2\bar{\varphi}}$ only holds on-shell. Secondly, the transformations depend explicitly on the metric on the base of the cone, $\hat{g}_{ij}(y)$. In this section, we shall construct an off-shell formalism for the superconformal symmetry which is manifestly independent of the metric on the base. Specifically, we shall introduce additional auxiliary fields and off-shell superconformal transformations generated by a new conformal supercharge S'_z, such that the following relation holds:

$$S'_z \equiv S_z = S_0 + zS_5, \tag{4.1}$$

i.e. the supercharge S'_z reduces on-shell to the supercharge S_z defined in (3.41). In addition, S'_z will be shown to be nilpotent off-shell. For simplicity of notation, in the following we suppress the prime in S'_z.

4.1. *Off-shell algebra for conformal supersymmetries*

In addition to h, let us introduce the following new auxiliary fields

$$\begin{aligned} &h^+_{\mu\nu},\ h^-_{\mu\nu},\ \bar{h}^+_{\mu\nu},\ \bar{h}^-_{\mu\nu},\ h^+_\mu,\ h^-_\mu: \quad \text{bosonic},\\ &\xi^+_{\mu\nu},\ \xi^-_{\mu\nu}: \quad \text{fermionic},\end{aligned} \tag{4.2}$$

where all tensor fields are anti-symmetric

$$h^\pm_{\mu\nu} = -h^\pm_{\nu\mu}, \qquad \bar{h}^\pm_{\mu\nu} = -\bar{h}^\pm_{\nu\mu}, \qquad \xi^\pm_{\mu\nu} = -\xi^\pm_{\nu\mu}. \tag{4.3}$$

We require all the auxiliary fields to be orthogonal to the vector v

$$h^\pm_{\mu\nu} v^\nu = 0, \qquad \bar{h}^\pm_{\mu\nu} v^\nu = 0, \qquad h^\pm_\mu v^\mu = 0, \qquad \xi^\pm_{\mu\nu} v^\nu = 0. \tag{4.4}$$

Furthermore, the bosonic auxiliary fields are required to satisfy reality conditions similar to (3.7)

$$\left(h^+_{\mu\nu}\right)^\dagger = \bar{h}^-_{\mu\nu}, \qquad \left(h^-_{\mu\nu}\right)^\dagger = \bar{h}^+_{\mu\nu}, \qquad \left(h^+_\mu\right)^\dagger = h^-_\mu, \qquad (h)^\dagger = h. \tag{4.5}$$

We also introduce a second fermionic two form field $\hat{\chi}_{\mu\nu} = -\hat{\chi}_{\nu\mu}$. Under the discrete symmetry (3.27) we have

$$\begin{aligned} &h^+_{\mu\nu} \xleftrightarrow{\mathcal{Z}} h^-_{\mu\nu}, \qquad \bar{h}^+_{\mu\nu} \xleftrightarrow{\mathcal{Z}} \bar{h}^-_{\mu\nu}, \qquad h^+_\mu \xleftrightarrow{\mathcal{Z}} h^-_\mu,\\ &\xi^+_{\mu\nu} \xleftrightarrow{\mathcal{Z}} \xi^-_{\mu\nu}, \qquad \chi_{\mu\nu} \xleftrightarrow{\mathcal{Z}} \hat{\chi}_{\mu\nu}.\end{aligned} \tag{4.6}$$

Let us also define the projection operator

$$P^{\mu}{}_{\nu} := \delta^{\mu}{}_{\nu} - \frac{v^{\mu} v_{\nu}}{v^2}, \qquad P^{\mu}{}_{\nu} v^{\nu} = 0, \qquad P^{\lambda}{}_{\mu} P^{\mu}{}_{\nu} = P^{\lambda}{}_{\nu}, \tag{4.7}$$

which projects onto subspaces orthogonal to the vector v.

We are now ready to give the off-shell transformations of all fields:

$$\begin{aligned}
&\big[S_z, A^+_{\mu}\big] = 2\chi_{\mu\nu} v^{\nu} + 2z\zeta v_{\mu},\\
&\big[S_z, A^-_{\mu}\big] = 2\eta v_{\mu} + 2z\hat{\chi}_{\mu\nu} v^{\nu},\\
&[S_z, \varphi] = -2(\omega_{\mu} + z\psi_{\mu}) v^{\mu},\\
&[S_z, \bar{\varphi}] = 0,\\
&\{S_z, \chi_{\mu\nu}\} = \bar{h}^-_{\mu\nu} - z\big(v_{\mu}\mathcal{D}^+_{\nu}\bar{\varphi} - v_{\nu}\mathcal{D}^+_{\mu}\bar{\varphi}\big),\\
&\{S_z, \hat{\chi}_{\mu\nu}\} = z\bar{h}^+_{\mu\nu} - v_{\mu}\mathcal{D}^-_{\nu}\bar{\varphi} + v_{\nu}\mathcal{D}^-_{\mu}\bar{\varphi},\\
&\{S_z, \psi_{\mu}\} = v^{\lambda}\mathcal{H}_{\lambda\mu} + i v_{\mu} h + i\frac{1}{2} v_{\mu}[\varphi, \bar{\varphi}] - 2i\phi_{\mu} - z h^+_{\mu},\\
&\{S_z, \omega_{\mu}\} = -h^-_{\mu} + z\bigg(-\mathcal{H}_{\mu\nu} v^{\nu} - i v_{\mu} h + i\frac{1}{2} v_{\mu}[\varphi, \bar{\varphi}] + 2i\phi_{\mu}\bigg),\\
&\{S_z, \eta\} = z\big(v^{\mu}\mathcal{D}^-_{\mu}\bar{\varphi} + 2\bar{\varphi}\big),\\
&\{S_z, \zeta\} = v^{\mu}\mathcal{D}^+_{\mu}\bar{\varphi} + 2\bar{\varphi},\\
&[S_z, h] = 2i v^{\mu}\mathcal{D}^+_{\mu}\eta + 4i\eta + \big[\omega_{\mu} v^{\mu}, \bar{\varphi}\big] - z\big(2i v^{\mu}\mathcal{D}^-_{\mu}\zeta + 4i\zeta + \big[\psi_{\mu} v^{\mu}, \bar{\varphi}\big]\big),\\
&\big[S_z, h^+_{\mu\nu}\big] = \xi^+_{\mu\nu},\\
&\big[S_z, h^-_{\mu\nu}\big] = \xi^-_{\mu\nu},\\
&\big[S_z, \bar{h}^+_{\mu\nu}\big] = -2i\big[v^2\hat{\chi}_{\mu\nu} + v_{\mu}\hat{\chi}_{\nu\kappa} v^{\kappa} - v_{\nu}\hat{\chi}_{\mu\kappa} v^{\kappa}, \bar{\varphi}\big],\\
&\big[S_z, \bar{h}^-_{\mu\nu}\big] = -2iz\big[v^2\chi_{\mu\nu} + v_{\mu}\chi_{\nu\kappa} v^{\kappa} - v_{\nu}\chi_{\mu\kappa} v^{\kappa}, \bar{\varphi}\big],\\
&\big[S_z, h^+_{\mu}\big] = 2v^{\kappa} v^{\lambda}\mathcal{D}^+_{\lambda}\hat{\chi}_{\mu\kappa} + 4\hat{\chi}_{\mu\nu} v^{\nu} - 2v^2 P_{\mu}{}^{\nu}\big(\mathcal{D}^-_{\nu}\zeta - i[\psi_{\nu}, \bar{\varphi}]\big),\\
&\big[S_z, h^-_{\mu}\big] = z\big(2v^{\kappa} v^{\lambda}\mathcal{D}^-_{\lambda}\chi_{\mu\kappa} + 4\chi_{\mu\nu} v^{\nu} - 2v^2 P_{\mu}{}^{\nu}\big(\mathcal{D}^+_{\nu}\eta - i[\omega_{\nu}, \bar{\varphi}]\big)\big),\\
&\big\{S_z, \xi^+_{\mu\nu}\big\} = 2iz\big[v^2\bar{\varphi}, h^+_{\mu\nu}\big],\\
&\big\{S_z, \xi^-_{\mu\nu}\big\} = 2iz\big[v^2\bar{\varphi}, h^-_{\mu\nu}\big].
\end{aligned} \tag{4.8}$$

The nice feature of the above extended algebra is that S_z squares *off-shell* to a gauge transformation:

$$S_z^2 = 2z\mathfrak{L}_{v^2\bar{\varphi}}. \tag{4.9}$$

Moreover, S_z is manifestly independent of the metric on the base of the cone.

Under the discrete symmetry $\mathcal{Z}$ (3.27), (4.6), S_z transforms as

$$S_z \xrightarrow{\mathcal{Z}} zS_{1/z}. \tag{4.10}$$

Note also that S_z preserves the anti-symmetric and the orthogonal properties of the auxiliary fields, (4.3) and (4.4).

On-shell $\chi_{\mu\nu}$ and $\hat{\chi}_{\mu\nu}$ are related by

$$-\hat{\chi}_{\mu\nu} \equiv (\star\chi)_{\mu\nu} = \frac{1}{2\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}\chi_{\kappa\lambda}, \tag{4.11}$$

and the auxiliary fields reduce to their on-shell values

$$\begin{aligned}
&\left(h^{+}_{\mu\nu}\right)_{\text{on}} = \frac{1}{\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}v_{\kappa}\mathcal{D}^{+}_{\lambda}\varphi, \qquad &&\left(h^{-}_{\mu\nu}\right)_{\text{on}} = \frac{1}{\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}v_{\kappa}\mathcal{D}^{-}_{\lambda}\varphi,\\
&\left(\bar{h}^{+}_{\mu\nu}\right)_{\text{on}} = \frac{1}{\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}v_{\kappa}\mathcal{D}^{+}_{\lambda}\bar{\varphi}, \qquad &&\left(\bar{h}^{-}_{\mu\nu}\right)_{\text{on}} = \frac{1}{\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}v_{\kappa}\mathcal{D}^{-}_{\lambda}\bar{\varphi},\\
&\left(h^{+}_{\mu}\right)_{\text{on}} = \left(\star F^{+}\right)_{\mu\nu}v^{\nu}, \qquad &&\left(h^{-}_{\mu}\right)_{\text{on}} = \left(\star F^{-}\right)_{\mu\nu}v^{\nu},
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
\left(\xi^{+}_{\mu\nu}\right)_{\text{on}} &:= -\frac{2}{\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}v_{\kappa}\left(v^{\rho}\mathcal{D}^{+}_{\lambda}\omega_{\rho} + \omega_{\lambda} + zv^{\rho}\mathcal{D}^{+}_{\lambda}\psi_{\rho} + z\psi_{\lambda} - i\left[(\star\hat{\chi})_{\lambda\rho}v^{\rho},\varphi\right]\right),\\
\left(\xi^{-}_{\mu\nu}\right)_{\text{on}} &:= -\frac{2}{\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}v_{\kappa}\left(v^{\rho}\mathcal{D}^{-}_{\lambda}\omega_{\rho} + \omega_{\lambda} + zv^{\rho}\mathcal{D}^{-}_{\lambda}\psi_{\rho} + z\psi_{\lambda} - iz\left[(\star\chi)_{\lambda\rho}v^{\rho},\varphi\right]\right),
\end{aligned} \tag{4.13}$$

which are consistent with (4.3)–(4.6). Note that there is some ambiguity in the expressions of the on-shell values of the fermionic auxiliary fields $\xi^{\pm}_{\mu\nu}$ in (4.13), due to the equations of motion for the fermions. However, the precise expressions in (4.13) will turn out to be necessary for the off-shell construction of the Lagrangian as we shall see below in (4.16).

4.2. *Off-shell Lagrangian*

Let us define

$$\begin{aligned}
V_z := \frac{\sqrt{g}}{2v^2(1-|z|^2)}\operatorname{tr}\Bigg[&-\frac{1}{4}\chi^{\mu\nu}\left\{[S_z,\chi_{\mu\nu}] - \frac{1}{\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}v_{\kappa}\mathcal{D}^{-}_{\lambda}\bar{\varphi} - z\left(\star\bar{h}^{+}\right)_{\mu\nu}\right\}^{\dagger} - \eta[S_z,\eta]^{\dagger}\\
&+\frac{1}{4}\hat{\chi}^{\mu\nu}\left\{[S_z,\hat{\chi}_{\mu\nu}] - z\frac{1}{\sqrt{g}}\epsilon_{\mu\nu}{}^{\kappa\lambda}v_{\kappa}\mathcal{D}^{+}_{\lambda}\bar{\varphi} - \left(\star\bar{h}^{-}\right)_{\mu\nu}\right\}^{\dagger} + \zeta[S_z,\zeta]^{\dagger}\\
&-\omega^{\mu}\left\{[S_z,\omega_{\mu}] + \left(1-|z|^2\right)\left(\star F^{-}\right)_{\mu\nu}v^{\nu} + 2izv_{\mu}\left(h - \mathcal{D}_{\lambda}\phi^{\lambda}\right)\right\}^{\dagger}\\
&+\psi^{\mu}\left\{[S_z,\psi_{\mu}] - \frac{1}{z^*}\left(1-|z|^2\right)\left(\star F^{+}\right)_{\mu\nu}v^{\nu} - 2iv_{\mu}\left(h - \mathcal{D}_{\lambda}\phi^{\lambda}\right)\right\}^{\dagger}\Bigg].
\end{aligned} \tag{4.14}$$

Our main formula is then

$$-\frac{1}{16}\epsilon^{\kappa\lambda\mu\nu}\operatorname{tr}\left[zF^{+}_{\kappa\lambda}F^{+}_{\mu\nu} + \frac{1}{z}F^{-}_{\kappa\lambda}F^{-}_{\mu\nu}\right] + \{S_z,V_z\} = \mathcal{L}_{\text{twisted}} + \frac{\sqrt{g}}{v^2(1-|z|^2)}L_{\text{auxiliary}}, \tag{4.15}$$

where $\mathcal{L}_{\text{twisted}}$ is the original Lagrangian given in (3.8) and $L_{\text{auxiliary}}$ consists of auxiliary terms

$$\begin{aligned}
L_{\text{auxiliary}} = \operatorname{tr}\Bigg[&-\frac{1}{2}\left(1-|z|^2\right)\left\|h^{+}_{\mu} - \left(h^{+}_{\mu}\right)_{\text{on}}\right\|^2 + \frac{1}{4}|z|^2\left\|h^{-}_{\mu\nu} - \left(h^{-}_{\mu\nu}\right)_{\text{on}}\right\|^2\\
&-\frac{1}{4}\left\|h^{+}_{\mu\nu} - \left(h^{+}_{\mu\nu}\right)_{\text{on}}\right\|^2 + \frac{1}{8}(\chi + \star\hat{\chi})^{\mu\nu}\left(\xi^{+}_{\mu\nu} - \left(\xi^{+}_{\mu\nu}\right)_{\text{on}}\right)
\end{aligned}$$

$$- \frac{1}{8} z^* (\hat{\chi} + \star\chi)^{\mu\nu} \big(\xi^-_{\mu\nu} - (\xi^-_{\mu\nu})_{\rm on}\big)$$
$$+ \big(1 - |z|^2\big)(\hat{\chi} + \star\chi)^{\mu\rho} v_\rho \Lambda_{\mu\nu} v^\nu \bigg] + \text{total derivatives}. \tag{4.16}$$

Here $(h^+_\mu)_{\rm on}$, $(h^\pm_{\mu\nu})_{\rm on}$, $(\xi^\pm_{\mu\nu})_{\rm on}$ denote the on-shell expressions of the auxiliary fields given in (4.12), (4.13), and we have set

$$\|T_{\mu_1\mu_2\cdots\mu_n}\|^2 := \frac{1}{n!} T_{\mu_1\mu_2\cdots\mu_n} \big(T^{\mu_1\mu_2\cdots\mu_n}\big)^\dagger. \tag{4.17}$$

We also define

$$\Lambda_{\mu\nu} := \mathcal{D}^+_\mu \omega_\nu - \mathcal{D}^+_\nu \omega_\mu - \frac{1}{\sqrt{g}} \epsilon_{\mu\nu}{}^{\kappa\lambda} \mathcal{D}^-_\kappa \psi_\lambda + i\frac{1}{2}\big[(\chi - \star\hat{\chi})_{\mu\nu}, \varphi\big], \tag{4.18}$$

which corresponds to the equation of motion for $\chi_{\mu\nu}$ (3.18). Thus all the terms in $L_{\rm auxiliary}$ vanish on-shell.

The equations of motion for the auxiliary fields $\xi^\pm_{\mu\nu}$, $h^\pm_\mu$, $h^\pm_{\mu\nu}$ give the on-shell relations (4.11) and (4.12). In particular, a nice feature is that, due to the orthogonal property (4.4), we need to integrate out both $\xi^+_{\mu\nu}$ and $\xi^-_{\mu\nu}$ in order to arrive at the on-shell relation $\hat{\chi}_{\mu\nu} \equiv -(\star\chi)_{\mu\nu}$ (4.11). This is important for a nontrivial path integral measure for the fermionic auxiliary fields. All other equations of motion then consist of those coming from the original Lagrangian (3.18) together with the additional equation

$$\xi^+_{\mu\nu} - \big(\xi^+_{\mu\nu}\big)_{\rm on} = \frac{z^*}{2\sqrt{g}} \epsilon_{\mu\nu}{}^{\kappa\lambda} \big[\xi^-_{\kappa\lambda} - \big(\xi^-_{\kappa\lambda}\big)_{\rm on}\big], \tag{4.19}$$

which is of course consistent with the on-shell relation (4.13).

From the above construction it follows that the components of the energy–momentum tensor along the base are manifestly S_z-exact. Hence the theory is topological, at least formally, in the sense that it is independent of the metric on the base.

5. Discussion

In this paper we focused on the twisting of four-dimensional $\mathcal{N} = 4$ SYM studied in [4,6–9,11]. We have found that putting the theory on a four-dimensional cone leaves two conformal supercharges unbroken. The resulting theory is topological in that it is independent of the metric on the base of the cone.

A few questions still remain open. Firstly one would like to identify the set of observables of the theory, i.e. the set of operators in the cohomology of the topological charge, and their potential relevance to topological invariants of the three-dimensional base of the cone. The role of all possible linear combinations of nilpotent scalar supercharges identified in Section 3, also deserves further study.

The S-duality of $\mathcal{N} = 4$ SYM has significant implications for the twisted version considered in the present paper, and this played a central role in the recent discussion of [11]. It would be interesting to examine this issue further. Another possible direction would be to consider reductions of the theory to lower-dimensional cones. A more thorough discussion of the different possible boundary conditions is postponed for future work.

Acknowledgements

We are grateful to Giuseppe Policastro for useful discussions. J.H.P. benefited from discussions with Nigel Hitchin and Nikita Nekrasov. The research of J.H.P. is supported by the Alexander von Humboldt Foundation and the Center for Quantum Spacetime of Sogang University with grant number R11-2005-021.

Appendix A. Cone geometry

This section contains, among some other useful facts about cone geometry, a proof of the statement that the existence of a vector field v satisfying (3.23)

$$\nabla_\mu v^\nu = \delta_\mu{}^\nu, \tag{A.1}$$

implies that, at least locally, the manifold is a cone:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\hat{s}^2, \qquad \mathrm{d}\hat{s}^2 = \hat{g}_{ij}(y)\,\mathrm{d}y^i\,\mathrm{d}y^j, \quad y^i \neq r, \tag{A.2}$$

and the vector field is given by

$$v = r\frac{\partial}{\partial r}. \tag{A.3}$$

Proof. Let us define coordinates $x^\mu = (r, y^i)$ such that

$$r := \sqrt{v^2}. \tag{A.4}$$

Under diffeomorphisms of the form

$$r \to r' = r, \qquad y^i \to y'^i = f^i(r, y), \tag{A.5}$$

the $g_{ri}(x)$ component of the metric transforms as

$$g_{ri}(x) \to g'_{ri}(x) = \frac{\partial f^j}{\partial y^i} g_{jk}(x')\big(\partial_r f^k(r, y) + \bar{g}^{kl}(x')g_{rl}(x')\big), \tag{A.6}$$

where $\bar{g}^{kl}$ is the inverse of g_{ij}, $\bar{g}^{ij}g_{jk} = \delta^i{}_k$. Given initial conditions $f^k(0, y)$ at $r = 0$, one can uniquely fix the evolution along the r-direction by demanding

$$\partial_r f^k(r, y) = -\bar{g}^{kl}\big(r, f(r, y)\big)g_{rl}\big(r, f(r, y)\big). \tag{A.7}$$

This determines all the higher-order derivatives of $f^k(r, y)$ with respect to r. Thus, it is possible to choose a gauge such that [16]

$$g_{ri} = 0. \tag{A.8}$$

Moreover, taking $\partial_\mu(r^2) = \partial_\mu(v^2) = 2v_\mu$ into account, we arrive at $v_r = r$, $v_i = 0$. Hence, $g_{rr} = g^{rr} = 1$ and the metric is of the form

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\hat{g}_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j. \tag{A.9}$$

Eq. (A.1) now reads

$$\delta_\mu{}^\nu = \nabla_\mu v^\nu = \delta_\mu{}^r\delta_r{}^\nu + r\Gamma^\nu_{\mu r}. \tag{A.10}$$

In particular, by taking the $\nabla_i v^j$ component, we see that $\hat{g}_{ij}$ is independent of the radial coordinate r. Hence the geometry is a cone and the vector field is given by (A.3). All other components of (A.10) can also be seen to hold, since:

$$\Gamma^i{}_{rj} = \frac{1}{r}\delta^i{}_j, \qquad \Gamma^r_{ij} = -r\hat{g}_{ij}, \qquad \Gamma^i{}_{jk} = \frac{1}{2}\hat{g}^{il}(\partial_j\hat{g}_{lk} + \partial_k\hat{g}_{jl} - \partial_l\hat{g}_{jk}),$$
$$\Gamma^r_{rr} = \Gamma^r_{ri} = \Gamma^i_{rr} = 0. \tag{A.11}$$

This completes the proof. □

Let us also mention the following useful relations:

$$R_{rr} = 0, \qquad R_{ri} = R_{ir} = 0, \qquad R_{ij} = \hat{R}_{ij} - 2\hat{g}_{ij}, \tag{A.12}$$

where $\hat{R}_{ij}$ is the Ricci curvature of the three-dimensional base manifold. In particular,

$$v^\mu R_{\mu\nu} = 0. \tag{A.13}$$

Appendix B. BPS equations

It follows from the superconformal algebra of the twisted $\mathcal{N} = 4$ super-Yang–Mills (3.33)–(3.35), that there are four possible pairs of supercharges which are nilpotent, and hence define four different cohomological structures

$$Q_0 + zQ_5, \quad S_0 + zS_5, \quad S_0 + zQ_5, \quad S_5 + zQ_0. \tag{B.1}$$

Here we give the BPS equations for each case, using the cone coordinates (3.24). The equations follow directly from (3.22) and (3.26). Demanding that the fields vanish fast enough at the boundary will generally result in additional constraints, as in (3.59), but we do not consider this issue here.

B.1. $Q_0 + zQ_5$

(1) If $z = 0$:

$$F^+_{\mu\nu} = 0, \qquad \mathcal{D}^+_\mu\varphi = 0, \qquad \mathcal{D}_\mu\phi^\mu + \frac{1}{2}[\varphi, \bar{\varphi}] = 0. \tag{B.2}$$

(2) If $z = \infty$:

$$F^+_{\mu\nu} = 0, \qquad \mathcal{D}^-_\mu\varphi = 0, \qquad \mathcal{D}_\mu\phi^\mu - \frac{1}{2}[\varphi, \bar{\varphi}] = 0. \tag{B.3}$$

(3) If $|z| = 1$:

$$F^+_{\mu\nu} - z(\star F^-)_{\mu\nu} = 0, \qquad \mathcal{D}_\mu\varphi = 0, \qquad \mathcal{D}_\mu\phi^\mu = 0,$$
$$[\phi_\mu, \varphi] = 0, \qquad [\varphi, \bar{\varphi}] = 0. \tag{B.4}$$

(4) If $|z| \neq 0, \infty, 1$:

$$F^+_{\mu\nu} = 0, \qquad \mathcal{D}_\mu\varphi = 0, \qquad \mathcal{D}_\mu\phi^\mu = 0, \qquad [\phi_\mu, \varphi] = 0, \qquad [\varphi, \bar{\varphi}] = 0. \tag{B.5}$$

B.2. $S_0 + zS_5$

(1) If $z = 0$:

$$F^+_{jr} + 2i\mathcal{D}^+_r\phi_j + i\frac{2}{r}\phi_j = 0, \qquad F^+_{ij} = 0,$$
$$\mathcal{D}^-_r\varphi + \frac{2}{r}\varphi = 0, \qquad \mathcal{D}^+_i\varphi = 0,$$
$$\mathcal{D}_r\phi_r + \frac{2}{r}\phi_r - \mathcal{D}_i\phi^i - \frac{1}{2}[\varphi, \bar{\varphi}] = 0. \tag{B.6}$$

(2) If $z = \infty$:

$$F^+_{jr} + 2i\mathcal{D}^+_r\phi_j + i\frac{2}{r}\phi_j = 0, \qquad F^+_{ij} = 0,$$
$$\mathcal{D}^+_r\varphi + \frac{2}{r}\varphi = 0, \qquad \mathcal{D}^-_i\varphi = 0,$$
$$\mathcal{D}_r\phi_r + \frac{2}{r}\phi_r - \mathcal{D}_i\phi^i + \frac{1}{2}[\varphi, \bar{\varphi}] = 0. \tag{B.7}$$

(3) If $|z| = 1$:

$$F^+_{jr} + z(\star F^+)_{jr} + 2i\mathcal{D}^+_r\phi_j + i\frac{2}{r}\phi_j = 0, \qquad \mathcal{D}_r\varphi + \frac{2}{r}\varphi = 0,$$
$$\mathcal{D}_r\phi_r + \frac{2}{r}\phi_r - \mathcal{D}_i\phi^i = 0, \qquad \mathcal{D}_i\varphi = 0,$$
$$[\phi_\mu, \varphi] = 0, \qquad [\varphi, \bar{\varphi}] = 0. \tag{B.8}$$

(4) If $|z| \neq 0, \infty, 1$:

$$F^+_{jr} + 2i\mathcal{D}^+_r\phi_j + i\frac{2}{r}\phi_j = 0, \qquad F^+_{ij} = 0,$$
$$\mathcal{D}_r\varphi + \frac{2}{r}\varphi = 0, \qquad \mathcal{D}_i\varphi = 0,$$
$$\mathcal{D}_r\phi_r + \frac{2}{r}\phi_r - \mathcal{D}_i\phi^i = 0, \qquad [\varphi, \bar{\varphi}] = 0,$$
$$[\phi_\mu, \varphi] = 0. \tag{B.9}$$

B.3. $S_0 + zQ_5$

$$r\mathcal{D}^+_j\varphi + z^*F^+_{jr} = 0, \qquad F^+_{ij} = 0, \qquad r\mathcal{D}^-_r\varphi + 2\varphi + iz^*\left(\mathcal{D}_\mu\phi^\mu - \frac{1}{2}[\varphi, \bar{\varphi}]\right) = 0,$$
$$F^+_{jr} + 2i\mathcal{D}^+_r\phi_j + i\frac{2}{r}\phi_j - z\frac{1}{r}\mathcal{D}^-_j\varphi = 0,$$
$$\mathcal{D}_r\phi_r + \frac{2}{r}\phi_r - \mathcal{D}_i\phi^i - \frac{1}{2}[\varphi, \bar{\varphi}] + z\frac{i}{r}\mathcal{D}^-_r\varphi = 0. \tag{B.10}$$

If $z = 0$ or $z = \infty$ the BPS equations reduce to (B.2) or (B.3), respectively.

B.4. $S_5 + zQ_0$

$$r\mathcal{D}_j^- \varphi + z^* F_{jr}^- = 0, \qquad F_{ij}^+ = 0, \qquad r\mathcal{D}_r^+ \varphi + 2\varphi - iz^*\left(\mathcal{D}_\mu \phi^\mu + \frac{1}{2}[\varphi, \bar{\varphi}]\right) = 0,$$
$$F_{jr}^- - 2i\mathcal{D}_r^- \phi_j - i\frac{2}{r}\phi_j - z\frac{1}{r}\mathcal{D}_j^+ \varphi = 0,$$
$$\mathcal{D}_r \phi_r + \frac{2}{r}\phi_r - \mathcal{D}_i \phi^i + \frac{1}{2}[\varphi, \bar{\varphi}] - z\frac{i}{r}\mathcal{D}_r^+ \varphi = 0. \tag{B.11}$$

If $z = 0$ or $z = \infty$ the BPS equations reduce to (B.3) or (B.2), respectively.

Note that under the discrete symmetry $\mathcal{Z}$ (3.27), Eqs. (B.2), (B.6), (B.10) transform to Eqs. (B.3), (B.7), (B.11), respectively.

Appendix C. Useful identities

Any two-form tensors $A_{\mu\nu}$, $B_{\mu\nu}$ satisfy

$$A^{\mu\nu} v_\nu (\star B)_{\mu\lambda} v^\lambda + (\star A)^{\mu\nu} v_\nu B_{\mu\lambda} v^\lambda = \frac{1}{2} v^2 (\star A)^{\mu\nu} B_{\mu\nu}, \tag{C.1}$$

which implies in particular

$$\left\{\chi^{\mu\nu} v_\nu, (\star\chi)_{\mu\lambda} v^\lambda\right\} = \frac{1}{4} v^2 \left\{\chi^{\mu\nu}, (\star\chi)_{\mu\nu}\right\}. \tag{C.2}$$

Some other useful identities are

$$\frac{1}{\sqrt{g}} \epsilon_{\mu\nu}{}^{\kappa\lambda} v_\kappa v^\rho \chi_{\lambda\rho} = -v^2 (\star\chi)_{\mu\nu} - v_\mu v^\kappa (\star\chi)_{\nu\kappa} + v_\nu v^\kappa (\star\chi)_{\mu\kappa}, \tag{C.3}$$

$$\frac{1}{(v^2)^2} \operatorname{tr}\left[v^2 \phi_\mu v^\mu \mathcal{D}_\nu \phi^\nu + v^2 \phi^\mu v^\nu \mathcal{D}_\mu \phi_\nu + v^2 \phi_\mu \phi^\mu + 2v^\lambda \phi_\lambda v^\mu v^\nu \mathcal{D}_\mu \phi_\nu\right]$$
$$= \nabla_\lambda \operatorname{tr}\left[\frac{\phi^\lambda \phi^\mu v_\mu}{v^2} + \frac{v^\lambda (\phi^\mu v_\mu)^2}{(v^2)^2}\right]. \tag{C.4}$$

For any two complex numbers, z, w, it follows from (3.46), (4.15) that

$$\int_{\text{cone}} \{S_z, V_z\} = \frac{\langle \mathcal{V}(w) | \mathcal{V}(w) \rangle}{1 + |w|^2} + \int_{\text{cone}} \operatorname{tr}\left(a_+ F^+ \wedge F^+ + a_- F^- \wedge F^-\right) + \cdots, \tag{C.5}$$

where we have set

$$a_+ := \frac{z}{4} - \frac{w}{2 + 2|w|^2}, \qquad a_- := \frac{1}{4z} - \frac{w^*}{2 + 2|w|^2}, \tag{C.6}$$

and the ellipses denote the fermionic and auxiliary parts; the latter vanishes on-shell, as we can see from (4.11)–(4.13).

References

[1] E. Witten, Topological quantum field theory, Commun. Math. Phys. 117 (1988) 353.
[2] C. Vafa, E. Witten, A strong coupling test of S duality, Nucl. Phys. B 431 (1994) 3, hep-th/9408074.
[3] J.P. Yamron, Topological actions from twisted supersymmetric theories, Phys. Lett. B 213 (1988) 325.
[4] N. Marcus, The other topological twisting of $N = 4$ Yang–Mills, Nucl. Phys. B 452 (1995) 331, hep-th/9506002.

[5] M. Bershadsky, C. Vafa, V. Sadov, D-branes and topological field theories, Nucl. Phys. B 463 (1996) 420, hep-th/9511222.
[6] M. Blau, G. Thompson, Aspects of $N_T \geqslant 2$ topological gauge theories and D-branes, Nucl. Phys. B 492 (1997) 545, hep-th/9612143.
[7] J.M.F. Labastida, C. Lozano, Mathai–Quillen formulation of twisted $N = 4$ supersymmetric gauge theories in four dimensions, Nucl. Phys. B 502 (1997) 741, hep-th/9702106.
[8] J.M.F. Labastida, C. Lozano, Duality in twisted $N = 4$ supersymmetric gauge theories in four dimensions, Nucl. Phys. B 537 (1999) 203, hep-th/9806032.
[9] C. Lozano, Duality in topological quantum field theories, hep-th/9907123.
[10] M. Unsal, Twisted supersymmetric gauge theories and orbifold lattices, hep-th/0603046.
[11] A. Kapustin, E. Witten, Electric–magnetic duality and the geometric Langlands program, hep-th/0604151.
[12] M. Blau, G. Thompson, Euclidean SYM theories by time reduction and special holonomy manifolds, Phys. Lett. B 415 (1997) 242, hep-th/9706225.
[13] H. Osborn, A.C. Petkou, Implications of conformal invariance in field theories for general dimensions, Ann. Phys. 231 (1994) 311, hep-th/9307010.
[14] J.-H. Park, Superconformal symmetry and correlation functions, Nucl. Phys. B 559 (1999) 455, hep-th/9903230.
[15] J.-H. Park, Superconformal symmetry in higher than two dimensions, PhD thesis, University of Cambridge.
[16] N. Kim, J.-H. Park, Massive super-Yang–Mills quantum mechanics: Classification and the relation to supermembrane, hep-th/0607005.
[17] N. Nekrasov, J.-H. Park, The final twist: Supersymmetric Yang–Mills theory in five dimensions and string homology, in preparation.
[18] A. Karlhede, M. Rocek, Topological quantum field theory and $N = 2$ conformal supergravity, Phys. Lett. B 212 (1988) 51.
[19] P. de Medeiros, C.M. Hull, B.J. Spence, J.M. Figueroa-O'Farrill, Conformal topological Yang–Mills theory and de Sitter holography, JHEP 0208 (2002) 055, hep-th/0111190.