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Restriction categories I: categories of partial maps

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Abstract

Given a category with a stable system of monics, one can form the corresponding category of partial maps. To each map in this category there is, on the domain of the map, an associated idempotent, which measures the degree of partiality. This structure is captured abstractly by the notion of a *restriction category*, in which every arrow is required to have such an associated idempotent. Categories with a stable system of monics, functors preserving this structure, and natural transformations which are cartesian with respect to the chosen monics, form a 2-category which we call $\mathscr{M}\mathsf{Cat}$. The construction of categories of partial maps provides a 2-functor $\mathsf{Par}:\mathscr{M}\mathsf{Cat} \to \mathsf{Cat}$. We show that Par can be made into an equivalence of 2-categories between $\mathscr{M}\mathsf{Cat}$ and a 2-category of restriction categories. The underlying ordinary functor $\mathsf{Par}_0:\mathscr{M}\mathsf{Cat}_0 \to \mathsf{Cat}_0$ of the above 2-functor Par turns out to be monadic, and, from this, we deduce the completeness and cocompleteness of the 2-categories of $\mathscr{M}$-categories and of restriction categories. We also consider the problem of how to turn a formal system of subobjects into an actual system of subobjects. A formal system of subobjects is given by a functor into the category sLat of semilattices. This structure gives rise to a restriction category which, via the above equivalence of 2-categories, gives an $\mathscr{M}$-category. This $\mathscr{M}$-category contains the universal realization of the given formal subobjects as actual subobjects. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

Categories of partial maps lie at the heart of many of the semantic and theoretical issues both in computer science and, indeed, in the more traditional areas of mathematics such as geometry and analysis. While partial recursive functions appear very

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brutally and fundamentally at the foundations of theoretical computer science, we tend to regard the fact that "continuous functions" such as $1/(x - a)$ are not really functions, but rather partial functions, as a minor inconvenience to be surmounted with as much mathematical grace as possible. Yet it is often the delicate handling of these issues of partiality which gives depth to results in computability and algebraic geometry.

Because partial maps are central to so many issues in computer science there has been a considerable effort to develop their theory. Di Paola and Heller [10] introduced the notion of "dominical categories" as an algebraic setting in which one could study partial maps (and computability theory). They approached partiality through "zero morphisms" (maps which are nowhere defined) and the presence of "near products", which are tensor products which behave like products with respect to total maps.

Robinson and Rosolini [22] quickly pointed out that the zero structure was not really necessary to obtain a theory of partiality. They observed that it could be obtained through the "near-product" structure alone. Accordingly they introduced P-categories – categories with a "near-product" structure – as the basis for a theory of partiality (these ideas were also presented in Rosolini's thesis [24]). In more modern terms these are symmetric monoidal categories in which each object has a monoidal natural cocommutative coassociative comultiplication (and possibly an unnatural counit). These categories were considered by Jacobs [14] as the semantics of weakening. A quite different approach was taken by Carboni [4], where the bicategory structure was taken as primitive.

These P-categories are also, essentially, what the first author called "copy categories" in a manuscript [7] which started circulating in about 1995 but was never published. The manuscript never reached publication for two main reasons. First, much of the material on partiality was already available in the above references (and we should here include Mulry's work [18–20]). Secondly, one of the main motivating results, the description of the extensive completion of a distributive category, had also been proved independently by the second author [17] at roughly the same time. Thus, it had been resolved that we should try to pool resources and publish the results jointly … and that event had to wait for a time when we could get together physically!

When eventually we did get together in Sydney in the Australian winter of 1998, it was inevitable that we should completely rework our approach. This new approach may be regarded as a return to the key ideas expressed in the opening sentence of Di Paola and Heller's paper on dominical categories:

> Dominical categories are categories in which the notion of partial morphism and their domains become explicit, with the latter being endomorphisms rather than subobjects of their sources.

In fact, the endomorphisms which express the domains are idempotents. Although the crucial role of the idempotents which express the amount of partiality had long been recognized by those working in this field, there seems to have been some hesitation in directly legislating their presence. Thus, despite the above opening, Di Paolo and

Heller's dominical categories have an axiomatization which does not directly mention this structure: these idempotents arise as a *consequence* of other structure. Similarly, Rosolini and Robinson's P-categories have an axiomatization in which these idempotents play no role, although they certainly arise as a consequence.

We therefore resolved to take these idempotents, which we call *restriction idempotents*, as primitive in our description of partiality, and so came up with the notion of *restriction category*. This departs from previous treatments in that it assumes no more structure on the category of total maps than is needed in order to define a category of partial maps. In the earlier treatments [10, 22, 4], it was always assumed that the category of total maps had (at least) finite products.

While developing the theory of restriction categories we became aware of the relationship of this work to certain aspects of semigroup theory. In particular, we realized that an *inverse monoid* is a special case of a one object restriction category. We therefore have taken the opportunity to describe a (restriction) categorical approach to some of the basic results on inverse monoids. The discussion of inverse categories in Section 2.3.2 is a direct categorical translation of the notion of an inverse monoid. One can compare our development of the relationship between restriction categories and categories with a chosen stable system of monics to the related (but dual) development in [12, Section VII.8] using division categories. Our discussion of the Vagner–Preston representation theorem in Section 3.4 uses an observation of Phil Mulry to provide a (categorically) natural strengthening of this theorem both to cover the more general case of restriction categories and to be a full and faithful representation (into a presheaf category).

In Section 2 we introduce the 2-category rCat of restriction categories, provide a variety of examples of restriction categories, and prove in Theorem 2.3 that the underlying category rCat_0 of rCat is monadic over Cat_0, as well as giving an explicit description of the monad.

In Section 3 we use the notion of a category with a stable system of monics[3] to construct the corresponding category of partial maps and we show that this is a restriction category. This is the key ingredient in the equivalence of 2-categories given in Theorem 3.4 between the 2-category $\mathscr{M}\mathsf{Cat}$ of categories with a chosen stable system of monics, and the full sub-2-category rCat_s of rCat comprising those restriction categories in which the restriction idempotents split. Of particular interest in this presentation is the class of natural transformations taken for the 2-cells in $\mathscr{M}\mathsf{Cat}$: they must be $\mathscr{M}$-cartesian in order to obtain the correspondence. We deduce in Corollary 3.6 the monadicity of $\mathscr{M}\mathsf{Cat}_0$ over Cat_0; here the forgetful functor takes an object of $\mathscr{M}\mathsf{Cat}_0$ to the corresponding category of partial maps.

In Section 4 we consider fibred semilattices, which we think of as equipping a category with a *formal* stable system of monics. The key results here are Theorem 4.2,

[3] A stable system of monics is (essentially) what Roslini called a *dominion* in [24] (this name was also used by Fiore [11]), Robinson and Rosolini called an *admissible class of subobjects* in [22], and Mulry called a *domain structure* in [18].

which establishes an adjunction between rCat and the 2-category sLatFib of fibred semilattices; and Theorem 4.3, which describes a universal way of realizing formal subobjects as actual subobjects.

Some of the main theorems of the paper involve the existence or construction of certain adjunctions. These are summarized in the diagram appearing in Section 5.

In a sequel [8] to this paper we shall consider the relationship of partial map classifiers to restriction categories. This is closely related to more recent work on partiality in which the monad arising from the partial map classifier is taken as primitive; see [3]. In a further sequel [9], we shall consider restriction categories arising as categories of partial maps in a category with extra structure such as products, coproducts, distributivity, extensivity, and so on; and it is in this context that we shall see the precise connection with approaches such as [10, 22]. Also in [9] we shall describe the extensive completion of a distributive category.

We also intend in the future to investigate the role of partiality in algebraic geometry as in Example 14 of Section 2.1.3 below, suggested to us by Terry Bisson.

2. Restriction categories

2.1. Restrictions

Given a function $f : A \to B$ and a subset S of A, one may consider the restriction $f|_S$ of f to S, defined by $f|_S(s) = f(s)$. Our use of the word restriction is derived from this usage. A restriction structure on a category associates to each map in the category an idempotent on its domain. This idempotent must satisfy some simple axioms (see Section 2.1.1) and is used to measure the degree of partiality of the arrow.

A significant motivation behind the study of partial maps is the study of (one object) categories of partial recursive functions. For these categories the restriction operation is defined by modifying the computation of the given partial recursive function so that it returns its input unchanged when the original computation terminates. Notice that, because these categories simply do not have enough subobjects, we cannot directly equate a partial recursive function to a total function on a subobject. This makes it useful to have a more abstract approach to the theory of partial functions.

A more basic example of a restriction category is the category of sets and partial functions. The restriction structure in this case associates to a partial map $f : X \to Y$ the idempotent partial map $\bar{f} : X \to X$ where $\bar{f}(x)$ is x whenever f is defined and is undefined otherwise. This example and others are described in Section 2.1.3.

2.1.1. The definition

A *restriction structure* on a category $\mathbf{X}$ is an assignment of an arrow $\bar{f} : A \to A$ to each $f : A \to B$, such that the following four conditions are satisfied:

(R.1) $f\bar{f} = f$ for all f,

(R.2) $\bar{f}\bar{g} = \bar{g}\bar{f}$ whenever $\operatorname{dom} f = \operatorname{dom} g$,

(R.3) $\overline{g\bar{f}} = \bar{g}\bar{f}$ whenever $\operatorname{dom} f = \operatorname{dom} g$,
(R.4) $\bar{g}f = f\overline{gf}$ whenever $\operatorname{cod} f = \operatorname{dom} g$.
A category with a restriction structure is called a *restriction category*.

It is important to realize that a restriction structure is not a property of a category but rather extra structure: a given category can have more than one restriction structure. In fact, every category has at least one restriction structure, for the assignment $\bar{f} = 1$ can always be made, giving rise to the *trivial restriction structure* on the category. If these were the only restriction structures the theory would not be very interesting!

A rather important fact is that $\bar{f}$ is idempotent. We record this together with some other basic consequences of the definition:

Lemma 2.1. *If* **X** *is a restriction category then*:
(i) $\bar{f}$ *is idempotent*;
(ii) $\bar{f}\overline{gf} = \overline{gf}$;
(iii) $\overline{\bar{g}f} = \overline{gf}$;
(iv) $\bar{\bar{f}} = \bar{f}$;
(v) $\overline{\bar{g}\bar{f}} = \bar{g}\bar{f}$;
(vi) *If* f *is monic then* $\bar{f} = 1$ (*and so in particular* $\bar{1} = 1$);
(vii) $f\bar{g} = f$ *implies* $\bar{f} = \bar{f}\bar{g}$.

Proof. (i) By (R.3) and (R.1) we have $\bar{f}\bar{f} = \overline{f\bar{f}} = \bar{f}$.
(ii) By (R.2), (R.3), and (R.1) we have $\bar{f}\overline{gf} = \overline{gf}\bar{f} = \overline{gf\bar{f}} = \overline{gf}$.
(iii) We use (R.4), (R.3), and (ii) to conclude $\overline{\bar{g}f} = \overline{f\overline{gf}} = \bar{f}\overline{gf} = \overline{gf}$.
(iv) By (iii) we have $\bar{f} = \overline{\bar{f}1} = \overline{\bar{f}}1 = \bar{\bar{f}}$.
(v) By (R.3) we have $\overline{\bar{g}\bar{f}} = \bar{\bar{g}}\bar{f} = \bar{g}\bar{f}$.
(vi) Since $f\bar{f} = f1$, when f is monic we conclude that $\bar{f} = 1$.
(vii) By (R.3) we have $\bar{f} = \overline{f\bar{g}} = \bar{f}\bar{g}$. $\square$

A map f such that $f = \bar{f}$ is called a *restriction idempotent*. Clearly, these are precisely the maps of the form $\bar{f}$ for some f, since by the above lemma $\bar{f} = \bar{\bar{f}}$.

2.1.2. Total maps

A map f in a restriction category is said to be *total* if $\bar{f} = 1$. The total maps form an important subcategory:

Lemma 2.2. *In any restriction category*:
(i) *every monomorphism* (*and so in particular every identity*) *is total*;
(ii) *if f and g are total then gf is total*;
(iii) *if gf is total then f is total*;
(iv) *the total maps form a subcategory.*

Proof. (i) Lemma 2.1(vi).
(ii) If f and g are total then $\overline{gf} = \overline{\bar{g}f} = \bar{f} = 1$.

(iii) If gf is total then $\bar{f} = \bar{f}\,\overline{gf} = \overline{gf} = 1$.
(iv) This is now immediate. □

Note that the third point accords with the intuition that once something becomes undefined it remains undefined!

We shall denote the subcategory of total maps of a restriction category **X** by Total(**X**). The restriction on **X** is trivial (that is, $\bar{f} = 1$ for all arrows f) precisely when the inclusion of Total(**X**) is an isomorphism.

2.1.3. Basic examples

The following are some basic examples of categories with a restriction:

1. The category of sets and partial functions. For a partial map $f : A \to B$ the partial map $\bar{f} : A \to A$ is defined by
$$\bar{f}(a) = \begin{cases} a \text{ whenever } \downarrow f(a), \\ \uparrow \text{ otherwise.} \end{cases}$$
Here $\downarrow f(a)$ is shorthand for "f is defined at a", while $\bar{f}a = \uparrow$ is shorthand for "$\bar{f}$ is not defined at a".
The total maps are exactly the total functions in the usual sense.
2. Partial recursive functions on the natural numbers. The definition of $\bar{f}$ is the same as in the previous example: the reader will easily see that this partial function is still recursive. This provides an example of a *restriction monoid*, that is a restriction category with one object.
The total maps are the total recursive functions.
3. The category of domains (which we take to mean partial orders with bottom element) with strict (that is, bottom-preserving) maps is a restriction category: the restriction $\bar{f} : X \to X$ of a morphism $f : X \to Y$ is defined by
$$\bar{f}(x) = \begin{cases} x \text{ when } f(x) \neq \bot, \\ \bot \text{ otherwise.} \end{cases}$$
The total maps are the bottom-reflecting maps.
4. The category of finite non-empty linear orders with bottom-preserving maps; this is a full subcategory of the category in the previous example, and the restriction structure is taken to be the same.
5. The full subcategory of the category of domains consisting of the ordinal ω; that is, the category of all order-preserving endofunctions of the natural numbers.
6. A *copy category* [7, 9] is a symmetric monoidal category $(\mathscr{A}, \otimes, I)$ with a monoidal "copy" transformation $\Delta : A \to A \otimes A$ which is cocommutative and coassociative. It turns each object into a cosemigroup. We say the copy category is *total* if every object has a counit $! : A \to I$, which need not be natural; total copy categories are the same as P-categories. Given any $f : A \to B$ we define $\bar{f} : A \to A$ to be
$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes 1} B \otimes A \xrightarrow{! \otimes 1} I \otimes A \xrightarrow{u^L_{\otimes}} A.$$
This is a restriction.

7. An important example of a copy category is provided by the Kleisli category for the exception monad $(\_ + 1)$ on a distributive category. In the induced restriction structure, if $f : A \to B + 1$ then $\bar{f}$ is given by

$$A \xrightarrow{\Delta} A \times A \xrightarrow{1 \times f} A \times (B + 1) \to (A \times B) + (A \times 1) \xrightarrow{p_0 + p_1} A + 1.$$

The total maps are those which factorize through the unit $B \to B + 1$.

8. Inverse monoids [13] provide a source of restriction monoids. An inverse monoid has for each m a unique (partial) inverse n such that $mnm = m$ and $nmn = n$. Letting m° denote the unique partial inverse of m then $\bar{m} = m^\circ m$.
The total maps are the monic or left cancellable elements. Since every monic is a section these are also the sections. See Section 2.3.2 for the details.
9. The monoid $\mathbb{N}$ of natural numbers under the operation *sup* has two restriction structures with the same total maps: in the first $\bar{n} = n$, while in the second $\bar{n}$ is n if n is odd or zero, and $n - 1$ otherwise.
10. Sometimes something looks very much like a restriction but is not! If **X** is a regular category then $\mathsf{Span}(\mathbf{X})$ has a "restriction" structure. The restriction idempotents are spans of identical monics. The "restriction" of an arbitrary span is obtained by factorizing the left leg into a regular epi followed by a mono, and then using this mono to form the restriction idempotent. Conditions (R.1)–(R.3) are all true, but (R.4) fails.
The "total" maps are those spans whose left leg is already regular epimorphic. We may think of these spans as non-deterministic maps.
11. For completeness, we include examples to establish the independence of the other axioms. In any pointed category, we can define $\bar{f} = 0$, and this will satisfy (R.2) –(R.4) but not (R.1).
On a distributive category, there is a monad $(\_ + C)$ for each object C, constructed similarly to the exception monad described in Example 7 above. Given an arrow $f : A \to B + C$ in the Kleisli category, we define $\bar{f} : A \to A + C$ by

$$A \xrightarrow{\Delta} A \times A \xrightarrow{1 \times f} A \times (B + C) \to (A \times B) + (A \times C) \xrightarrow{p_0 + p_1} A + C$$

and this satisfies (R.1), (R.3), and (R.4), but not (R.2) unless C is a subobject of the terminal object.
Consider the commutative monoid $\mathbb{N}$ of the natural numbers under addition. Adjoin an element ε satisfying $\varepsilon + 0 = 0 + \varepsilon = \varepsilon$, $\varepsilon + \varepsilon = 0$, and $\varepsilon + n = n + \varepsilon = n$, for all $n > 0$. Now define $\bar{0} = \bar{\varepsilon} = 0$, and $\bar{n} = \varepsilon$ for all $n > 0$. This satisfies (R.1), (R.2), and (R.4), but not (R.3).
12. There is also the dual notion of *corestriction* category. Consider the category $\mathsf{stabLat}$, whose objects are (meet) semi-lattices (that is posets with a top and binary meets) and whose maps are stable homomorphisms (maps which preserve the binary meets, but not necessarily the top). Given $f : L_1 \to L_2$ the corestriction $\bar{f} : L_2 \to L_2$ is defined by

$$\bar{f}(y) = y \wedge f(\top).$$

The maps f with $\bar{f}=1$ are the top-preserving maps; they form a subcategory of stabLat called sLat.

13. In general, for any restriction category $\mathbf{X}$, the slice category $\mathbf{X}/A$ is a restriction category for any object A of $\mathbf{X}$, with restrictions being formed as in $\mathbf{X}$.
14. The category Proj of projective spaces over the reals, and "homogeneous polynomial" morphisms is another example of a restriction category.[4] This is a non-full subcategory of the category Par of sets and partial functions, with the usual restriction structure.

    Recall that $\mathbb{R}\mathsf{P}^n$ is the quotient of $\mathbb{R}^{n+1}$ by the relation $(x_0,\dots,x_n)\sim(\lambda x_0,\dots,\lambda x_n)$. The objects of Proj are the sets of the form $\mathbb{R}\mathsf{P}^n$ for some $n\in\mathbb{N}$, and a partial function from $\mathbb{R}\mathsf{P}^n$ to $\mathbb{R}\mathsf{P}^m$ lies in Proj if and only it is induced by some (necessarily homogeneous) polynomial function $\mathbb{R}^{n+1}\to\mathbb{R}^{m+1}$. We leave to the reader the verification that if a partial function from $\mathbb{R}\mathsf{P}^n$ to $\mathbb{R}\mathsf{P}^m$ lies in Proj then so too does its restriction; as an example, the map $\mathbb{R}\mathsf{P}^1\to\mathbb{R}\mathsf{P}^1$ induced by the polynomial function $\mathbb{R}^2\to\mathbb{R}^2:(x,y)\mapsto(p(x,y),q(x,y))$ has restriction induced by the polynomial function $\mathbb{R}^2\to\mathbb{R}^2:(x,y)\to(r(x,y)x,r(x,y)y)$ where $r(x,y)=p(x,y)^2+q(x,y)^2$. If we think of projective space as comprising lines through the origin in Euclidean space, then these maps send lines to lines, and are regarded as being equal if they send the same lines to the same lines. The partiality arises as a given map can take a line constantly to the origin; we then regard the map as being partial on that line.

2.1.4. The 2-category associated to a restriction category

If $\mathbf{X}$ is a category with a restriction structure, then there is a natural notion of 2-cell which makes $\mathbf{X}$ into a (locally ordered) 2-category: given arrows f and g from C to D, we define $f\leqslant g$ if $f=g\bar{f}$. All the axioms for a 2-category are easily seen to hold.

In fact, this point of view seems to be less useful than one might expect, but the 2-cell structure will arise from time to time. Especially important is the case of two restriction idempotents e, e' on the same object; then $e\leqslant e'$ if and only if $e=e'e$.

2.2. The 2-category of restriction categories

Having introduced the restriction categories in the previous sections, we now come to the morphisms of restriction categories. Here it is natural to take those functors which preserve the restriction. We shall also, in Section 2.2.2 consider a certain class of natural transformations between these functors, giving a 2-category rCat of restriction categories. First, however, we treat only its underlying category rCat_0 and show that it is monadic over the category Cat_0 of categories. We shall see in Remark 2.14 that this does not extend to a 2-monadicity result for rCat.

We establish the monadicity of rCat_0 over Cat_0 by noticing that restriction categories can be finitely algebraically presented over categories. However, we also provide an explicit description of the adjoint. In particular, this means that the monadicity can be

[4] We thank Terry Bisson for drawing our attention to this example.

established directly by checking the Beck condition (i.e. that the underlying functor creates coequalizers of split coequalizers): and this may be a useful exercise for the reader. Having an explicit description of the adjoint also, of course, allows us to construct the free restriction category associated to a category and so to obtain some further examples of restriction categories.

2.2.1. The category rCat_0

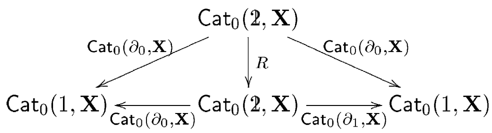
First we define the (ordinary) category of restriction categories. A functor $F:\mathbf{X}\to\mathbf{X}'$ between restriction categories is said to be a *restriction functor* if $F(\bar{g})=\overline{F(g)}$ for all arrows g in $\mathbf{X}$. Restriction categories and restriction functors form a category which we call rCat_0; it has an evident forgetful functor $\mathscr{U}:\mathsf{rCat}_0\to\mathsf{Cat}_0$. We shall prove that this functor $\mathscr{U}$ is monadic, and then give an explicit description of the monad.

In proving the monadicity of rCat_0, we observe that Cat_0 is a locally finitely presentable category, and then give a presentation, in the sense of [16], for a finitary monad on Cat_0, whose category of algebras is rCat_0. These presentations involve *operations* and *equations*, each having an arity which is a finitely presentable category. We shall also give an explicit description of the adjunction.

Theorem 2.3. $\mathscr{U}:\mathsf{rCat}_0\to\mathsf{Cat}_0$ *is monadic via a finitary monad.*

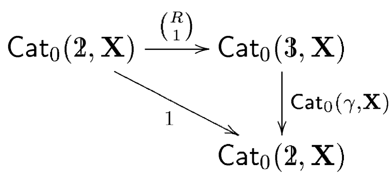
Proof. Write $\mathbb{2}$ for the "arrow category" $(0\to 1)$, and $\mathbb{3}$ for the category $(0\to 1\to 2)$. Write ∂_0 and ∂_1 for the functors from the terminal category 1 to $\mathbb{2}$, and write $\gamma:\mathbb{2}\to\mathbb{3}$ for the functor taking 0 to 0 and 1 to 2.

To provide a category $\mathbf{X}$ with the structure of a restriction category, one first gives a function $R:\mathsf{Cat}_0(\mathbb{2},\mathbf{X})\to\mathsf{Cat}_0(\mathbb{2},\mathbf{X})$, assigning an arrow $\bar{f}$ to each arrow f in $\mathbf{X}$; this is the only operation. We now impose two equations of arity 1



which together express the condition that $\bar{f}$ be an endomorphism of the domain of f.

The functions $R:\mathsf{Cat}_0(\mathbb{2},\mathbf{X})\to\mathsf{Cat}_0(\mathbb{2},\mathbf{X})$ and $1:\mathsf{Cat}_0(\mathbb{2},\mathbf{X})\to\mathsf{Cat}_0(\mathbb{2},\mathbf{X})$ induce a function $\binom{R}{1}:\mathsf{Cat}_0(\mathbb{2},\mathbf{X})\to\mathsf{Cat}_0(\mathbb{3},\mathbf{X})$, and we now impose the equation



which says that the assignment $f\mapsto\bar{f}$ satisfies axiom (R.1).

In a similar fashion one can express axioms (R.2)–(R.4) using equations of arity $\bullet \leftarrow \bullet \rightarrow \bullet$, $\bullet \leftarrow \bullet \rightarrow \bullet$, and $\bullet \rightarrow \bullet \rightarrow \bullet$, respectively. □

Corollary 2.4. rCat_0 *is locally finitely presentable, and so complete and cocomplete.*

Proof. rCat_0 is the category of algebras for a finitary monad on the locally finitely presentable category Cat_0. It follows, for example by the final remark of Chap. 2 in [1], that rCat_0 is locally finitely presentable, and so in particular complete and cocomplete. □

Remark 2.5. Since $\mathscr{U}$ is monadic, limits in rCat_0 are constructed as in Cat_0. For general colimits in rCat_0, however, we have no explicit description, although it is easy to see that coproducts are formed in rCat_0 as in Cat_0, whence we conclude that rCat_0 is *extensive*, in the sense of [5, 6].

The rest of this section is devoted to giving an explicit description of the left adjoint $\mathscr{F}$ to $\mathscr{U} : \mathsf{rCat}_0 \to \mathsf{Cat}_0$. To motivate the construction, we first recall from Lemma 2.1 that if $f = ug$ then $\bar{g}\bar{f} = \bar{f}$; we shall use this repeatedly in what follows.

For each object $X \in \mathbf{X}$ we write $\Phi(X)$ for the set of all sets of arrows with domain X satisfying

- If $M \in \Phi(X)$ then $1_X \in M$;
- If $M \in \Phi(X)$ and $ug \in M$ then $g \in M$ – that is, M is *right-factor closed*;
- for each $M \in \Phi(X)$ there are $f_1, \ldots, f_n \in M$ such that for every $g \in M$ there is an $i \in \{1, \ldots, n\}$ and an arrow u in $\mathbf{X}$ with $ug = f_i$ – that is, M is *finitely generated.*

If $K = \{f_i : X \to Z_i \mid i \in I\}$ is a set of arrows with domain X, and $g : Y \to X$ is any arrow, then we write Kg for the set $\{f_i g \mid i \in I\}$, and $\Downarrow K$ for the set $\{f : X \to Z \mid uf = f_i$ for some $i \in I$ and some $u : Z \to Z_i\}$. Clearly $\Downarrow K$ is right-factor closed, and so if K is finite then $\Downarrow K$ will be in $\Phi(X)$.

Lemma 2.6. (i) $\Downarrow\{1_X\}$ *is the smallest set in* $\Phi(X)$;
(ii) *if* K_1, K_2 *are sets of arrows with domain* X *then* $\Downarrow(K_1 \cup K_2) = \Downarrow K_1 \cup \Downarrow K_2$;
(iii) *if* $M_1, M_2 \in \Phi(X)$ *then* $M_1 \cup M_2 \in \Phi(X)$;
(iv) *if* $f : X \to Y$ *and* $M \in \Phi(Y)$ *then* $Mf \in \Phi(X)$;
(v) *if* $M_1, M_2 \in \Phi(Y)$, *and* $f : X \to Y$ *then* $(M_2 \cup M_2)f = (M_1 f) \cup (M_2 f)$;
(vi) *if* $f : X \to Y$ *and* $g : Y \to Z$ *and* $M \in \Phi(Z)$ *then* $(Mg)f = M(gf)$.

Proof. We prove only three of the claims.
(i) Recall that 1_X must be in each set of $\Phi(X)$.
(iii) The union of two right-factor closed subsets is right-factor closed. The preservation of finite generation follows from (ii).
(iv) If $M = \Downarrow\{g_1, \ldots, g_n\}$ then $Mf = \Downarrow\{g_1 f, \ldots, g_n f\}$. □

This amounts to the observation that these data give a fibred join-semilattice. Later we shall see that fibred meet-semilattices arise naturally in the context of restrictions so it is natural to consider the complements of these right-factor closed sets. These are sieves with the property that the arrows *not* contained in the sieve are finitely generated: hence we may call them *cofinite sieves*. Thus, it would be possible to present the construction below using these sieves. [5]

Remark 2.7. The preceding lemma includes the crucial fact that these cofinite sieves are closed under binary intersection and universal quantification along an arrow, allowing us to define a functor $\Gamma_{\mathbf{X}}: \mathbf{X}^{\mathrm{op}} \to \mathsf{stabLat}$ taking an object X to the meet-semilattice of cofinite sieves on X. We shall return to this in Section 4.

We now form a new category $\mathscr{F}\mathbf{X}$ with the same objects as $\mathbf{X}$, and with an arrow from X to Y being a pair (f,M), where $M \in \Phi(X)$, and $f \in M$, with the codomain of f being Y. The identity on X is $(1_X, \Downarrow 1_X)$, and composition is given by $(g,M')(f,M) = (gf, M \cup M'f)$. It is straightforward to show that this is a category: notice that

$$(1, \Downarrow \{1\})(f,M) = (f, M \cup \Downarrow \{1\}f) = (f, M \cup \Downarrow \{f\}) = (f,M),$$

where the last step uses the fact that $f \in M$.

Furthermore, $\mathscr{F}\mathbf{X}$ becomes a restriction category when we define $\overline{(f,M)} = (1,M)$:

Proposition 2.8. *$\mathscr{F}\mathbf{X}$ is a restriction category.*

Proof. We must check the restriction identities:

(R.1) $(f,M)\overline{(f,M)} = (f,M)(1,M) = (f,M \cup M) = (f,M)$.

(R.2) $\overline{(g,N)}\,\overline{(f,M)} = (1,N)(1,M) = (1,M \cup N) = (1,M)(1,N) = \overline{(f,M)}\,\overline{(g,N)}$.

(R.3) $\overline{\overline{(g,N)}(f,M)} = \overline{(g,N)(1,M)} = \overline{(g,M \cup N)} = (1,M \cup N) = (1,N)(1,M) = \overline{(g,N)}\,\overline{(f,M)}$.

(R.4) $\overline{(g,N)}(f,M) = (1,N)(f,M) = (f,M \cup Nf) = (f,M)(1,M \cup Nf) = (f,M)\overline{(g,N)(f,M)}$. □

There is a functor $N: \mathbf{X} \to \mathscr{U}\mathscr{F}\mathbf{X}$ which is the identity on objects, and takes an arrow $f: X \to X'$ in $\mathbf{X}$ to $(f, \Downarrow\{f\})$. Preservation of identities is trivial, and

$$\begin{aligned} N(g)N(f) &= (g, \Downarrow \{g\})(f, \Downarrow \{f\}) = (gf, \Downarrow \{f\} \cup (\Downarrow \{g\})f) \\ &= (gf, \Downarrow \{gf\}) = N(gf). \end{aligned}$$

This functor will turn out to be the unit of an adjunction between $\mathscr{F}$ and $\mathscr{U}$.

Remark 2.9. In fact N is interesting from another point of view: the only monics it preserves are the sections. This is because the set $\Downarrow\{f\}$ for a non-section f is strictly bigger than $\Downarrow\{1\}$ and thus $\overline{N(f)}$ is a non-trivial idempotent, hence cannot be monic.

[5] The second author did, in fact, present the construction in this manner to the Australian Category Seminar in July 1998).

In this way the embedding $\mathbf{X} \to \mathscr{U}\mathscr{F}\mathbf{X}$ demonstrates the fact that the only absolute monics are the sections.

Example 2.10. Here are some examples of $\mathscr{F}\mathbf{X}$ for particular $\mathbf{X}$:

(i) If $\mathbf{X}$ is a groupoid then the M in each map (f, M) in $\mathscr{F}\mathbf{X}$ must be the set of *all* maps with the same domain as f; thus $M = \Downarrow 1_X$ and so $\mathscr{F}\mathbf{X} = \mathbf{X}$. More generally, if X is an object for which every arrow with domain X is a section, then the only map in $\mathscr{F}\mathbf{X}$ of the form (f, M) is $(f, \Downarrow 1_X)$. Conversely, if $\mathscr{F}\mathbf{X} = \mathbf{X}$ and $f: X \to Y$ is an arrow in $\mathbf{X}$ which is not a section, then $(1_X, \Downarrow f)$ is not of the form $(1_X, \Downarrow 1_X)$, and so $\mathscr{F}\mathbf{X} \neq \mathbf{X}$. Thus $\mathscr{F}\mathbf{X} = \mathbf{X}$ if and only if every arrow is a section; that is, if and only if $\mathbf{X}$ is a groupoid. In this case the restriction structure on $\mathscr{F}\mathbf{X}$ is the trivial one.

(ii) If $\mathbf{X}$ is a preorder and $X \leqslant Y$ a non-invertible map, then the arrow $(1_x, \Downarrow \{X \leqslant Y\}) : X \to X$ in $\mathscr{F}\mathbf{X}$ is not an identity, and so $\mathscr{F}\mathbf{X}$ is not a preorder. In fact, a map in $\mathscr{F}\mathbf{X}$ from X to Y can exist only if $X \leqslant Y$, in which case it consists of a set M of objects of $\mathbf{X}$ satisfying four conditions: $Y \in M$, if $Z \in M$ then $X \leqslant Z$, if $X \leqslant W \leqslant Z$ and $Z \in M$ then $W \in M$, and there exists a finite set $\{Y_i\}_{i \in I}$ of objects of $\mathbf{X}$ with the property that an object Z of $\mathbf{X}$ satisfying $X \leqslant Z$ lies in M if and only if $Z \leqslant Y_i$ for some $i \in I$. An alternative description of the maps from X to Y (with $X \leqslant Y$) is as the subsets of $\mathbf{X}$ which can be written as a finite union of intervals $\bigcup_{i \in I} [X, Y_i]$ where at least one Y_i has $Y \leqslant Y_i$.

(iii) In the special case that the preorder $\mathbf{X}$ is a totally ordered set, then we may choose the maximum Y' of the Y_i; in this case a map in $\mathscr{F}\mathbf{X}$ from X to Y is an object Y' for which $X \leqslant Y \leqslant Y'$.

(iv) We now describe $\mathscr{F}\mathsf{Set}$. First we consider arrows $X \to Y$ with X non-empty. Such an arrow has the form $(f, \Downarrow \{f_1, \ldots, f_n\})$; but now we can factorize each f_i as

$$X \overset{e_i}{\twoheadrightarrow} Z_i \overset{m_i}{\rightarrowtail} Y_i$$

and now each Z_i is non-empty since X is so, and thus each m_i splits. It follows that $\Downarrow \{f_1, \ldots, f_n\} = \Downarrow \{e_1, \ldots, e_n\}$. Thus a map from X to Y is a pair (f, M), where $f: X \to Y$, and M is a set of quotients of X which is right-factor closed, finitely generated, and contains the image of f. Alternatively, we can replace quotients of X by the corresponding equivalence relations on X.

Now consider the maps from X to Y where $X = \emptyset$. Such a map consists of an arrow $f: X \to Y$ in $\mathbf{X}$, and a set M of maps with domain X satisfying certain conditions. There is exactly one arrow in $\mathbf{X}$ from X to Y, and to give the set of maps with domain X is just to give the set of objects which are their codomains. Thus, a map in $\mathscr{F}\mathbf{X}$ from $\emptyset$ to Y consists simply of a set M of objects of $\mathbf{X}$ satisfying certain conditions. These conditions reduce to: M contains both $\emptyset$ and Y, and if M contains any non-empty set then it contains all non-empty sets. Thus there is exactly one arrow from $\emptyset$ to any non-empty set, and there are exactly two endomorphisms of $\emptyset$.

Remark 2.11. The restriction idempotents in $\mathscr{F}\mathsf{Set}$ may be viewed as measuring "squashability". The equivalence relations of the restriction idempotents are then seen as limiting the degree to which a map with that restriction idempotent can quotient its domain. Thus, we should regard the maps in $\mathscr{F}\mathsf{Set}$ as being equipped with generalized apartness relations. It is interesting to note that in [21] a very similar idea (there called the "category of worlds") was used to provide a semantics for modeling *non-interference* in a programming language.

Suppose now that we have a functor $H : \mathbf{X} \to \mathscr{U}\mathbf{Y}$ where $\mathbf{Y}$ is a restriction category. We define $\hat{H}X = HX$ for an object X of $\mathbf{X}$, and

$$\hat{H}(f, \Downarrow\{g_1, \ldots, g_n\}) = Hf\overline{Hg_1}\ldots\overline{Hg_n}$$

for an arrow $(f, \Downarrow\{g_1, \ldots, g_n\})$ of $\mathbf{X}$. This is well-defined, by:

Lemma 2.12. *If* $\Downarrow\{g_1, \ldots, g_n\} = \Downarrow\{g'_1, \ldots, g'_m\}$ *then* $\overline{Hg_1}\ldots\overline{Hg_n} = \overline{Hg'_1}\ldots\overline{Hg'_m}$.

Proof. We show that

$$\overline{Hg_1}\ldots\overline{Hg_n} = \overline{Hg_1}\ldots\overline{Hg_n}\,\overline{Hg'_1}\ldots\overline{Hg'_m}$$

from which the result follows by symmetry. For each j there exists an i such that $g_i = yg'_j$ for some y, and so $\overline{g_i} = \overline{g_i}\,\overline{g'_j}$, whence the desired equality. □

In proving functoriality of $\hat{H}$ it is useful to observe first that $\hat{H}$ preserves the restriction. Let $(f, \Downarrow\{g_1, \ldots, g_n\})$ be an arrow in $\mathscr{F}\mathbf{X}$ from A to B. By definition of $\mathscr{F}\mathbf{X}$, we can write $g_j = yf$ for some j, and so $\bar{f}\,\overline{g_j} = \bar{f}\,\overline{yf} = \overline{yf} = \overline{g_j}$. Now

$$\begin{aligned}
\overline{\hat{H}(f, \Downarrow\{g_1, \ldots, g_n\})} &= \overline{Hf\overline{Hg_1}\ldots\overline{Hg_n}}\\
&= \overline{Hf}\,\overline{Hg_1}\ldots\overline{Hg_n}\\
&= \overline{Hg_1}\ldots\overline{Hg_n}\\
&= \hat{H}(1, \Downarrow\{g_1, \ldots, g_n\})\\
&= \hat{H}(\overline{(f, \Downarrow\{g_1, \ldots, g_n\})})
\end{aligned}$$

and so $\hat{H}$ does indeed preserve the restriction. Also

$$\begin{aligned}
&\hat{H}(f', \Downarrow\{g'_1, \ldots, g'_n\})\hat{H}(f, \Downarrow\{g_1, \ldots, g_m\})\\
&\quad = Hf'\overline{Hg'_1}\ldots\overline{Hg'_n}Hf\overline{Hg_1}\ldots\overline{Hg_m}\\
&\quad = Hf'.Hf.\overline{Hg'_1.Hf}\ldots\overline{Hg'_n.Hf}\ \overline{Hg_1}\ldots\overline{Hg_m}\\
&\quad = H(f'f)\overline{H(g'_1f)}\ldots\overline{H(g'_nf)}\ \overline{Hg_1}\ldots\overline{Hg_m}\\
&\quad = \hat{H}(f'f, (\Downarrow\{g'_1, \ldots, g'_n\})f \cup (\Downarrow\{g_1, \ldots, g_m\}))
\end{aligned}$$

which completes the proof that $\hat{H}$ is a restriction functor. By construction, $\hat{H}N = H$, and moreover $\hat{H}$ is clearly the unique restriction functor with this property, whence we conclude that $\mathscr{F}$ is left adjoint to $\mathscr{U}$.

Although we now set about adding 2-cells to rCat_0, the monad described in this section cannot be enriched to become a 2-monad. It does admit enrichment over *invertible* 2-cells (for the importance of such monads see [2]) but we shall not make any use of this fact.

2.2.2. The 2-category rCat

A natural transformation between restriction functors is called a *restriction transformation* if all of its components are total. The restriction categories, restriction functors, and restriction transformations form a 2-category called rCat, whose underlying ordinary category is of course rCat_0. Notice the choice of 2-cells; the reason for this choice will become apparent in Section 3.

Proposition 2.13. *The* 2*-category* rCat *is complete.*

Proof. We know already that the underlying category rCat_0 of rCat is complete. We can deduce the existence of all conical limits in rCat if we know that rCat admits tensors with the arrow category $\mathbb{2}$. In fact it is easy to see that the tensor with $\mathbb{2}$ of a restriction category $\mathbf{X}$ is given by the product in rCat of $\mathbb{2}$ (equipped with the trivial restriction) and $\mathbf{X}$.

To show that rCat is complete, it remains to show that it admits *cotensors* with $\mathbb{2}$. That is, for each restriction category $\mathbf{X}$, we must construct a restriction category $\mathbf{X}^{\mathbb{2}}$, restriction functors $\partial_0, \partial_1 : \mathbf{X}^{\mathbb{2}} \to \mathbf{X}$, and a restriction transformation $\lambda : \partial_0 \to \partial_1$; and these data must be universal. This universality says that for any restriction category $\mathbf{Y}$ with restriction functors $f_0, f_1 : \mathbf{Y} \to \mathbf{X}$ and a restriction transformation $\phi : f_0 \to f_1$, there is a unique restriction functor $\Phi : \mathbf{Y} \to \mathbf{X}^{\mathbb{2}}$ for which $\partial_0 \Phi = f_0$, $\partial_1 \Phi = f_1$, and $\lambda\Phi = \phi$; furthermore, given also $g_0, g_1 : \mathbf{Y} \to \mathbf{X}$ and $\psi : g_0 \to g_1$ inducing $\Psi : \mathbf{Y} \to \mathbf{X}^{\mathbb{2}}$, and $\alpha_0 : f_0 \to g_0$ and $\alpha_1 : f_1 \to g_1$ satisfying $\psi\alpha_0 = \alpha_1\phi$ there is a unique $A : \Phi \to \Psi$ satisfying $\partial_0 A = \alpha_0$ and $\partial_1 A = \alpha_1$.

An object of $\mathbf{X}^{\mathbb{2}}$ is a total map in $\mathbf{X}$, and an arrow in $\mathbf{X}^{\mathbb{2}}$ from $t : X \to Y$ to $t' : X' \to Y'$ is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

in $\mathbf{X}$. The restriction structure is given by $\overline{(f,g)} = (\bar{f}, \bar{g})$; note that $\bar{g}t = t\overline{gt} = t\overline{t'f} = t\bar{f}$, since t' is total, and so $(\bar{f}, \bar{g})$ is indeed an endomorphism of t in $\mathbf{X}^{\mathbb{2}}$. The verification that $\mathbf{X}^{\mathbb{2}}$ satisfies the universal property of cotensor products is straightforward. □

Remark 2.14. From the construction of cotensors in rCat, we see that the forgetful 2-functor rCat $\to$ Cat does not preserve cotensors, thus cannot have a left 2-adjoint, let alone be 2-monadic.

Proposition 2.15. *The* 2*-category* rCat *admits coproducts, and tensor products with arbitrary small categories.*

Proof. We have already seen that rCat_0 admits coproducts; by the previous result rCat has cotensor products with $\mathbb{2}$, and so we conclude that rCat has coproducts. As for the tensor product of a restriction category $\mathbf{X}$ with a category C, it is given by the product in rCat of $\mathbf{X}$ with the category C equipped with the trivial restriction. □

Corollary 2.16. *The* 2*-category* rCat *is cocomplete.*

Proof. By Proposition 2.4, we know that the underlying category rCat_0 has all conical colimits, and since rCat has cotensor products, it follows that rCat has all conical colimits. Since by the previous proposition rCat has tensor products, we conclude that it is cocomplete. □

The completeness and cocompleteness of rCat allow us to perform a large number of constructions that will be useful in future papers [8, 9]. Completeness allows us to form the Eilenberg–Moore object and the Kleisli object of a monad; and also comma objects and inserters: the latter two in particular are of interest in connection with datatypes. Cocompleteness allows us to construct free objects with various structures or properties.

We should also note that there is an inclusion of the 2-category of all categories into rCat as the trivial restriction categories. A functor from a trivial restriction category always factorizes (uniquely) through the total category of the codomain. This immediately gives a 2-adjunction:

$$\mathsf{Cat} \underset{\mathsf{Total}}{\overset{\mathsf{Triv}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{rCat}$$

making categories with a trivial restriction a full coreflective sub-2-category.

A category with a restriction has a natural poset enrichment using the definition $f \leqslant g \Leftrightarrow f = g\bar{f}$ which allows one to regard it as a 2-category/bicategory of partial maps (see [4, 15]). We do not pursue this here beyond the observation that it allows us to construct a left 2-adjoint to the inclusion $\mathsf{Triv} : \mathsf{Cat} \to \mathsf{rCat}$. For as each $\mathbf{X}$ in rCat can be regarded as a 2-category, we can use the 2-cell structure to provide a congruence on $\mathbf{X}$. Factoring out by this congruence gives the universal trivial restriction category associated with $\mathbf{X}$. Thus, it provides the left 2-adjoint to the inclusion of Cat into rCat, and we have:

Proposition 2.17. *The inclusion* $\mathsf{Triv} : \mathsf{Cat} \to \mathsf{rCat}$ *is a fully faithful* 2*-functor with both adjoints.*

2.3. Properties of maps

In the presence of a restriction structure it is possible to introduce a weakening of the usual notion of isomorphism. A *restricted isomorphism* is a map f with a *restricted* *inverse* g in the sense that $\bar{f} = gf$ and $\bar{g} = fg$. It turns out that in a restriction category, every arrow is a restricted isomorphism if and only if for every arrow f there is a unique arrow g such that $fgf = f$ and $gfg = g$. One-object categories with this last property are called *inverse monoids* [13] by semigroup theorists, and we shall adopt this terminology for the more general situation of categories.

The section ends by discussing idempotents. The idea of freely splitting certain idempotents is absolutely fundamental to the whole development in which we shall repeatedly desire to split the restriction idempotents.

2.3.1. Restricted monics, sections, and isomorphisms

A map f in a restriction category is said to be *restricted monic* if $fh = fk$ implies that $\bar{f}h = \bar{f}k$, a *restricted section* if there is an h with $hf = \bar{f}$ and a *restricted isomorphism* if there is an h with $hf = \bar{f}$ and $fh = \bar{h}$.

Lemma 2.18. *In a restriction category*:

(i) *If f and g are restricted monic then gf is restricted monic.*
(ii) *If gf is restricted monic and g is total then f is restricted monic.*
(iii) *If f is a restricted section then f is restricted monic.*
(iv) *If f and g are restricted sections then gf is a restricted section.*
(v) *If gf is a restricted section and g is total then f is a restricted section.*
(vi) *If f and g are restricted isomorphisms then gf is a restricted isomorphism.*
(vii) *If f is a restricted isomorphism then its restricted inverse is unique.*

Proof. (i) Suppose $gfx = gfx'$ then as g is restricted monic we have $\bar{g}fx = \bar{g}fx'$ so that $f\overline{gf}x = f\overline{gf}x'$; now as f is restricted monic we have $\bar{f}\,\overline{gf}x = \bar{f}\,\overline{gf}x'$ which gives the desired equality since $\bar{f}\,\overline{gf} = \overline{gf}$.

(ii) If $fx = fx'$ then certainly $gfx = gfx'$, so that $\overline{gf}x = \overline{gf}x'$. But using the fact that g is total, $\overline{gf} = \overline{\bar{g}f} = \bar{f}$ which gives the result.

(iii) If $fx = fx'$ then $\bar{f}x = hfx = hfx' = \bar{f}x'$.

(iv) If $hf = \bar{f}$ and $kg = \bar{g}$ then

$$hkgf = h\bar{g}f = hf\overline{gf} = \bar{f}\overline{gf} = \overline{gf}$$

showing that gf is a restricted section.

(v) This uses the fact that when g is total, $\overline{gf} = \bar{f}$, so that if $hgf = \overline{gf}$ then hg provides a partial retraction for f.

(vi) This follows directly from the fact that restricted sections compose.

(vii) Suppose $gf = \bar{f}$, $fg = \bar{g}$, $g'f = \bar{f}$, and $fg' = \bar{g}'$. Then we must show that $g = g'$. This is obtained by:

$$g = g\bar{g} = gfg = g'fg = g'fg'fg = g'fgfg' = g'fg' = g'. \qquad \square$$

It is important to realize that functors which preserve restrictions also preserve all the various classes of maps considered in this section, with the exception of restricted monics; the easy proof of the following lemma is omitted.

Lemma 2.19. *If $F:\mathbf{X}\to\mathbf{Y}$ is a restriction functor then F preserves*:
(i) *total maps*,
(ii) *restriction idempotents*,
(iii) *restricted sections*,
(iv) *restricted isomorphisms.*

2.3.2. Inverse categories

A special, but important, class of categories with restriction are those for which every map is a restricted isomorphism. We call these *inverse categories*. The main result of this section establishes two rather different presentations of the structure of inverse category, which show that one-object inverse categories are precisely inverse monoids.

Notice that the total maps in an inverse category have $\bar{f}=1$ thus $f^\circ f=1$ (where f° denotes the restricted inverse) and so the total maps in an inverse category f are precisely the sections.

Theorem 2.20. *For a category* $\mathbf{X}$, *the following are equivalent*:
(i) $\mathbf{X}$ *is an inverse category*, *that is*, *it has a restriction structure for which every map is a restricted isomorphism*,
(ii) *every morphism $f:A\to B$ has a unique $g:B\to A$ with $fgf=f$ and $gfg=g$*,
(iii) *there is a functor $(\_)^\circ:\mathbf{X}\to\mathbf{X}^{op}$ which is the identity on objects and satisfies*

$$(f^\circ)^\circ=f,$$
$$ff^\circ f=f,$$
$$ff^\circ gg^\circ=gg^\circ ff^\circ.$$

Moreover, *the structures in* (i) *and* (iii) *are unique.*

Proof. (i) ⇒ (ii): Suppose in an inverse category we have $fgf=f$ and $gfg=g$ then we must establish that g is the restricted inverse f° of f.

First note that $\bar{g}f=f\overline{gf}=fgf\overline{gf}=fgf=f$, and so $\overline{gf}=\overline{\bar{g}f}=\bar{f}$.

For any idempotent e we have $e=e\bar{e}=e\overline{e\bar{e}}=\bar{e}e=e^\circ ee=e^\circ e=\bar{e}$ and so all idempotents are restriction idempotents. Thus $gf=\overline{gf}=\bar{f}$ and, similarly, $fg=\bar{g}$. Now using the fact that restriction idempotents commute, we have $g=gfg=gff^\circ fg=gfgff^\circ =gff^\circ ff^\circ=f^\circ fgff^\circ=f^\circ ff^\circ=f^\circ$.

(ii) ⇒ (iii): Define f° to be the unique arrow satisfying $ff^\circ f=f$ and $f^\circ ff^\circ=f^\circ$. The only non-trivial conditions to check are $ff^\circ gg^\circ=gg^\circ ff^\circ$ and functoriality.

We shall prove that all idempotents commute, from which it follows that $ff^\circ gg^\circ=gg^\circ ff^\circ$. Let e and e' be idempotents, and let $x=(e'e)^\circ$; then $e'exe'e=e'e$ and $xe'ex=x$. Now

$$(exe')^2=exe'exe'=exe'$$

and so exe' is idempotent, giving $(exe')^\circ=exe'$. Also

$$(e'e)(exe')(e'e)=e'exe'e=e'e,$$
$$(exe')(e'e)(exe')=exe'exe'=exe'$$

and so $e'e = (exe')^\circ = exe'$, and so $(e'e)^\circ = e'e$. Finally

$$(e'e)(ee')(e'e) = e'ee'e = e'e$$

and similarly

$$(ee')(e'e)(ee') = ee'$$

so that $e'e = (e'e)^\circ = ee'$.

Functoriality is now easy: $gff^\circ g^\circ gf = gg^\circ gff^\circ f = gf$ and $f^\circ g^\circ gff^\circ g^\circ = f^\circ ff^\circ g^\circ gg^\circ = f^\circ g^\circ$, whence $(gf)^\circ = f^\circ g^\circ$, and similarly $1^\circ = 1$.

(iii) $\Rightarrow$ (i): We set $\bar{x} = x^\circ x$, then (R.1) and (R.2) are immediate. Furthermore under this definition each arrow is a restricted isomorphism. For (R.3) we have

$$\overline{g\bar{f}} = (gf^\circ f)^\circ gf^\circ f = f^\circ fg^\circ gf^\circ f = g^\circ gf^\circ f = \bar{g}\bar{f}.$$

Finally for (R.4) we have

$$\bar{g}f = g^\circ gf = g^\circ gff^\circ f = ff^\circ g^\circ gf = f(gf)^\circ gf = f\overline{gf}. \qquad \square$$

The theorem gives immediately:

Corollary 2.21. *A one-object inverse category is precisely an inverse monoid.*

As idempotents are their own restricted inverses we record the following interesting fact about inverse categories:

Corollary 2.22. *In an inverse category all idempotents are restriction idempotents.*

There is a full sub-2-category InvCat of rCat consisting of the inverse categories. Observe that any functor between inverse categories is a restriction functor, since the structure of a restricted inverse is algebraic. Observe also that the total maps in an inverse category are precisely the split monomorphisms.

Remark 2.23. A standard example of an inverse category is the category of partial injective functions between sets (these are partial functions which are injective over their domain). More generally, let $\mathscr{C}$ be a category with pullbacks in which every arrow is a monomorphism. Then the category of partial maps in $\mathscr{C}$ is an inverse category. In fact if a category of partial maps **X** is an inverse category, then it must be of this form. Categories of partial maps will be discussed in Section 3.

Proposition 2.24. InvCat *is a full coreflective sub-2-category of* rCat *which is complete and extensive.*

Proof. The right adjoint sends a restriction category to the subcategory of restricted isomorphisms. $\square$

An inverse category with a trivial restriction is a groupoid. Thus, InvCat is to the 2-category Gpd of groupoids as rCat is to Cat. In particular, if one factors out by the 2-cell structure induced on an inverse category the result is a groupoid.

2.3.3. Split restrictions

We say that a restriction structure on a category is *split* if all the restriction idempotents split. These idempotents – the arrows f with $\bar{f}=f$ – have the unusual property of being determined once one knows either the section or the retraction:

Lemma 2.25. *In any restriction category*:
(i) *If* $rm=1$ *and* $sm=1$ *with* $mr=\bar{r}$ *and* $ms=\bar{s}$ *then* $r=s$.
(ii) *If* $rm=1$ *and* $rn=1$ *with* $mr=\bar{r}$ or $nr=\bar{r}$ *then* $n=m$.

Proof. (i) We use the fact that the idempotents mr and ms commute in

$$r = rmsmr = rmrms = s.$$

(ii) If we have, say, $mr=\bar{r}$, then:

$$mr = mrnr = \bar{r}nr = nr\overline{rnr} = nr\bar{r} = nr.$$

Now as r is epimorphic we have $n=m$. □

The monic part of the splitting of a restriction idempotent we call a *restriction monic*. The first part of the lemma informs us that each restriction monic splits a unique restriction idempotent. Since restriction monics are monic, they are certainly restricted monic; on the other hand, every restriction idempotent is a restricted monic, but cannot be a restriction monic unless it is an identity.

We shall often wish to formally split certain idempotents (typically the restriction idempotents themselves) in a restriction category. For any category $\mathbf{X}$ we may pick any set of idempotents in the category and formally split them: if E is the set of idempotents we denote this category $K_E(\mathbf{X})$. An object of $K_E(\mathbf{X})$ is an element of E, while an arrow from e_1 to e_2 is an arrow f of $\mathbf{X}$ satisfying $e_2fe_1=f$.

In the case when $\mathbf{X}$ has a restriction we define the restriction of each $f=e_2fe_1$ to be $\bar{f}e_1$.

Proposition 2.26. *If* $\mathbf{X}$ *is a category with a restriciton and E is any set of idempotents of* $\mathbf{X}$ *then* $K_E(\mathbf{X})$ *inherits a restriction as above. Furthermore, if* $\mathbf{X}$ *is an inverse category then* $K_E(\mathbf{X})$ *is an inverse category.*

If E contains all the identities then there is a restriction-preserving inclusion $\mathbf{X}\to K_E(\mathbf{X})$.

Proof. We must verify that $K_E(\mathbf{X})$ is a restriction category and for this we first need to check that $\bar{f}e_1 : e_1\to e_1$. This is the case as $\bar{f}e_1 = e_1\overline{fe_1} = e_1\bar{f}$. Next, we need to

check the four restriction axioms:
(R.1) $f\bar{f}e_1 = fe_1 = f$;
(R.2) $\underline{\bar{f}e_1\bar{g}}e_1 = \underline{e_1\bar{f}\bar{g}e_1} = e_1\underline{\bar{g}\bar{f}}e_1 = \bar{g}e_1\bar{f}e_1$;
(R.3) $g\bar{f}e_1e_1 = ge_1\overline{\bar{f}e_1}e_1 = g\bar{f}e_1 = \bar{g}\bar{f}e_1 = \bar{g}e_1\bar{f}e_1$;
(R.4) $\bar{g}e_2f = e_2f\overline{ge_2f} = fe_1\overline{gf} = f\overline{gf}e_1$.
Thus if E contains the identities, $K_E(\mathbf{X})$ is a restriction category containing $\mathbf{X}$ as a full sub-restriction-category.

If $\mathbf{X}$ is an inverse category then $f : e_1 \to e_2$ in $K_E(\mathbf{X})$ has $e_1 f^\circ e_2$ as a partial inverse because

$$e_1 f^\circ e_2 f = e_1 f^\circ f = e_1 \bar{f} = \bar{f} e_1$$

and

$$f e_1 f^\circ e_2 = f f^\circ e_2 = \overline{f^\circ} e_2. \qquad \square$$

If $\mathbf{X}$ is a restriction category and E is the set of restriction idempotents, then we write $K_r(\mathbf{X})$ for $K_E(\mathbf{X})$.

The 2-category rCat has an important full sub-2-category, comprising those objects with a split restriction, which we call rCat_s. In fact, the inclusion has a left biadjoint which takes a restriction category $\mathbf{X}$ to $K_r(\mathbf{X})$. Since $K_r(\mathbf{X})$ is an inverse category if $\mathbf{X}$ is one, this also provides a left biadjoint to the inclusion of the full sub-2-category InvCat_s of InvCat given by the split inverse categories.

It is useful, however to know that these inclusion 2-functors actually have 2-adjoints rather than biadjoints. Although we do not give explicit constructions for these 2-adjoints, we can prove their existence by general means. Of course 2-adjoints are biadjoints, and biadjoints are determined up to equivalence, and so the 2-adjoints will be naturally equivalent to the biadjoints described above. The proof is fairly technical and rather terse, and the reader whose tastes do not incline towards such questions should feel free to skip over it.

Proposition 2.27. rCat_s *is contained in* rCat *as a full reflexive sub*-2-*category*; *similarly* InvCat_s *is a full reflexive sub*-2-*category of* InvCat. *In each case the left adjoint is equivalent to the* 2-*functor taking* $\mathbf{X}$ *to* $K_r(\mathbf{X})$ (*where* $\mathbf{X}$ *is either a restriction category or an inverse category*).

Proof. We saw in Corollary 2.4 that rCat_0 is locally finitely presentable, being the category of algebras for a finitary monad on the locally finitely presentable category Cat_0. It will now follow by Adamek and Rosicky [1, Corollary 2.49] that $(\mathsf{rCat}_s)_0$ is locally finitely presentable, and reflective in rCat_0, if we can prove that $(\mathsf{rCat}_s)_0$ is closed in rCat_0 under limits and filtered colimits. But this is easy to see since limits and filtered colimits are formed in rCat_0 as they are in Cat_0. It will now follow from [2, Proposition 3.1] that the inclusion $\mathsf{rCat}_s \to \mathsf{rCat}$ has a left (2-)adjoint if we can show that rCat_s is closed in rCat under cotensor products with $\mathbf{2}$. This is obvious from the description of such cotensor products in the proof of Proposition 2.13.

This completes the proof that the inclusion $J : \mathsf{rCat}_s \to \mathsf{rCat}$ has a left adjoint; we call it L. Since K_r is left biadjoint to J, we conclude that K_r is naturally equivalent to L. Now if $\mathbf{X}$ is an inverse category then $K_r(\mathbf{X})$ is an inverse category and $L(\mathbf{X}) \simeq K_r(\mathbf{X})$, and so $L(\mathbf{X})$ is also an inverse category. Thus L restricts to a left adjoint L_{inv} to the inclusion $J_{\text{inv}} : \mathsf{InvCat}_s \to \mathsf{InvCat}$. □

It follows that both InvCat_s and rCat_s are closed in rCat under arbitrary limits, and it is easy to see that they are also closed under arbitrary coproducts.

If we write $\mathscr{V} : (\mathsf{rCat}_s)_0 \to \mathsf{Cat}_0$ for the evident forgetful functor, combining Theorem 2.3 and Proposition 2.27 we see that $\mathscr{V}$ has a left adjoint, and so it is reasonable to consider whether $\mathscr{V}$ is monadic.

Theorem 2.28. *$\mathscr{V} : (\mathsf{rCat}_s)_0 \to \mathsf{Cat}_0$ is monadic.*

Proof. Certainly $\mathscr{V}$ is conservative and has a left adjoint, and so it will suffice to check the Beck condition. Given the monadicity of $\mathscr{U}$, this amounts to showing that if

$$\mathbf{X} \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{Y} \overset{Q}{\to} \mathbf{Z}$$

is a coequalizer in rCat_0 which is $\mathscr{U}$-split, then $\mathbf{Z}$ lies in $(\mathsf{rCat}_s)_0$ if $\mathbf{X}$ and $\mathbf{Y}$ do so. But any restriction idempotent e in $\mathbf{Z}$ can be written Qd for some d in $\mathbf{Y}$, and now $e = \bar{e} = \overline{Qd} = Q\bar{d}$, and so e is the image of the restriction idempotent $\bar{d}$ in $\mathbf{Y}$. Since $\bar{d}$ splits in $\mathbf{Y}$, applying Q gives a splitting of e in $\mathbf{Z}$. □

2.4. Restriction semifunctors and rCat_{sf}

In a future paper [9] in which we study (one-object) categories of partial recursive functions it will be useful to consider morphisms of categories, which are not required to preserve the identity. We shall call them semifunctors, and they will facilitate the discussion of the "functorial" structure of these categories in the absence of an explicit-type system. It turns out that these semifunctors provide us with an alternative description of the 2-category rCat_s.

If $\mathbf{X}$ and $\mathbf{Y}$ are categories, define a *semifunctor* from $\mathbf{X}$ to $\mathbf{Y}$ to be a function F from the objects of $\mathbf{X}$ to the objects of $\mathbf{Y}$, along with functions $F : \mathbf{X}(X, X') \to \mathbf{Y}(FX, FX')$ satisfying $F(gf) = Fg.Ff$, but not being required to preserve identities. Given semifunctors $F, G : \mathbf{X} \to \mathbf{Y}$, a *natural transformation* from F to G comprises an arrow $\alpha_X : FX \to GX$ in $\mathbf{Y}$ for each object of X, such that for each arrow $f : X \to X'$ in $\mathbf{X}$, we have $Gf.\alpha_X = \alpha_{X'}.Ff$. If $\mathbf{X}$ and $\mathbf{Y}$ are restriction categories, then a semifunctor $F : \mathbf{X} \to \mathbf{Y}$ is said to be a *restriction semifunctor* if it satisfies $F\bar{f} = \overline{Ff}$ for every arrow f. Given restriction semifunctors $F, G : \mathbf{X} \to \mathbf{Y}$, a natural transformation $\alpha : F \to G$ is said to be a *restriction transformation* if $\overline{\alpha_X} = F(1_X)$ for all X. Clearly this definition is consistent with our earlier definition of restriction transformations between restriction functors. There is an evident 2-category rCat_{sf} of restriction categories, restriction semifunctors, and restriction transformations, and an inclusion 2-functor $\mathscr{H} : \mathsf{rCat}_s \to \mathsf{rCat}_{sf}$;

we remark that the identity 2-cell id_F in rCat_{sf} on a restriction semifunctor $F:\mathbf{X}\to\mathbf{Y}$ has component $F(1_X)$ at X.

Proposition 2.29. *The* 2*-functor* $\mathscr{H}:\mathsf{rCat}_s\to\mathsf{rCat}_{sf}$ *is a biequivalence of* 2*-categories.*

Proof. Clearly $\mathscr{H}$ is locally fully faithful; that is, if $\mathbf{X}$ and $\mathbf{Y}$ are split restriction categories, then $\mathscr{H}:\mathsf{rCat}_s(\mathbf{X},\mathbf{Y})\to\mathsf{rCat}_{sf}(\mathscr{H}\mathbf{X},\mathscr{H}\mathbf{Y})$ is fully faithful. Suppose now that $K:\mathbf{X}\to\mathbf{Y}$ is a restriction semifunctor. For each object X of $\mathbf{X}$, we know that $\overline{K(1_X)}=K(\overline{1_X})=K(1_X)$, and so $K(1_X)$ is a restriction idempotent. Let $m_X:K'X\to KX$ and $r_X:KX\to K'X$ be a splitting of $K(1_X)$. Given an arrow $f:X\to X'$ in $\mathbf{X}$, there is a unique arrow $K'f$ in $\mathbf{Y}$ rendering commutative

$$\begin{array}{ccccc} KX & \xrightarrow{r_X} & K'X & \xrightarrow{m_X} & KX \\ {\scriptstyle Kf}\downarrow & & \vdots\,{\scriptstyle K'f} & & \downarrow{\scriptstyle Kf} \\ KX' & \xrightarrow[r_{X'}]{} & K'X' & \xrightarrow[m_{X'}]{} & KX'. \end{array}$$

Now given $f:X\to X'$ and $g:X'\to X''$, we have $m_{X''}.K'g.K'f.r_X=Kg.m_{X'}.K'f.r_X=Kg.Kf.m_X.r_X=K(gf).m_X.r_X=m_{X''}.K'(gf).r_X$, and so $K'g.K'f=K'(gf)$ since $m_{X''}$ is mono and r_X is epi. Similarly, we have $m_X.K'(1_X).r_X=K(1_X).m_X.r_X=m_X.r_X.m_X.r_X=m_X.r_X$, and so $K'(1_X)=1$. Thus K' is a functor; furthermore we have $m_X.K'(\bar{f}).r_X=K(\bar{f}).m_X.r_X=\overline{Kf}.m_X.r_X=m_X.\overline{Kf.m_X}.r_X=m_X.\overline{m_{X'}.K'f}.r_X=m_X.\overline{K'f}.r_X$, and so $K'(\bar{f})=\overline{K'f}$. Thus K' is a restriction functor. We have natural transformations $m:K'\to K$ and $r:K\to K'$, and for each X we have $\overline{m_X}=1=K'(1_X)$, and $\overline{r_X}=m_X.r_X=K(1_X)$, and so m and r are 2-cells in rCat_{sf}. Clearly $r.m=1$, but also $(m.r)_X=m_X.r_X=K(1_X)=(id_K)_X$, and so r is inverse to m in rCat_{sf}.

We have now proved that $\mathscr{H}:\mathsf{rCat}_s\to\mathsf{rCat}_{sf}$ is locally an equivalence. It remains only to show that $\mathscr{H}$ is biessentially surjective on objects; that is, for every restriction category $\mathbf{X}$, we must find a split restriction category which is equivalent *in the* 2*-category* rCat_{sf} to $\mathbf{X}$. We take the split restriction category $K_r(\mathbf{X})$. There is a fully faithful restriction functor $j:\mathbf{X}\to K_r(\mathbf{X})$. There is also a restriction semifunctor $r:K_r(\mathbf{X})\to\mathbf{X}$ which takes an object (X,e) to X, and an arrow $f:(X,e)\to(X',e')$ to $f:X\to X'$. Clearly $rj=1$; we shall show that $jr\cong 1$ in rCat_{sf}. Given an object (X,e) of $K_r(\mathbf{X})$, we have $jr(X,e)=(X,1)$, and we write $\alpha_{(X,e)}$ for e, seen as an arrow from $(X,1)$ to (X,e). Now if $f:(X,e)\to(X',e')$ is an arrow in $K_r(\mathbf{X})$, then we have $e'f=fe$, and so the $\alpha_{(X,e)}$ form the components of a natural transformation $jr\to 1$; also $\overline{\alpha_X}=\bar{e}=e=jr(e)=jr(1_{(X,e)})$, and so $\alpha:jr\to 1$ is a 2-cell in rCat_{sf}. Similarly, we write $\beta_{(X,e)}$ for e, seen as an arrow from (X,e) to $(X,1)$. Once again these are natural, and $\overline{\beta_X}=\bar{e}=e=1_{(X,e)}$, and so we have a 2-cell $\beta:1\to jr$ in rCat_{sf}. Finally $(\beta\alpha)_{(X,e)}=\beta_{(X,e)}\alpha_{(X,e)}=e.e=e=(id_{jr})_{(X,e)}$, and so $\beta\alpha=1$; and $(\alpha\beta)_{(X,e)}=\alpha_{(X,e)}\beta_{(X,e)}=e.e=e=1_{(X,e)}$, and so $\alpha\beta=1$, which completes the proof.

Concretely, a biequivalence inverse to $\mathscr{H}$ is given by $K_r:\mathsf{rCat}_{sf}\to\mathsf{rCat}_s$; where $K_r(\mathbf{X})$ for a restriction category $\mathbf{X}$ is defined as above, $K_r(F)$ for a restriction semifunctor $F : \mathbf{X} \to \mathbf{Y}$ is defined by $K_r(F)(X, e) = (FX, Fe)$ and $K_r(F)(f) = Ff$, and $K_r(\alpha)$ for a restriction transformation $\alpha : F \to G$ is defined by $K_r(\alpha)_{(X,e)} = \alpha_X.Fe$. $\square$

3. Categories of partial maps

It is really in this section that one discovers what having a restriction is all about! For, at last, we make quite explicit the sense in which it expresses the partiality of a map. To do this we have to start by introducing the categorical setting in which one can discuss partial maps sensibly: namely categories with a specified stable system of monics. These form the objects of a 2-category $\mathcal{M}$Cat. A stable system of monics is essentially what was called a *dominion* in [24], an *admissible system of subobjects* in [22], a *notion of partial* in [23], and a *domain structure* in [18].

The main goal of this section is to construct an equivalence of 2-categories between rCat$_s$ and $\mathcal{M}$Cat, underlying our claim that restriction categories model categories of partial maps.

3.1. Stable systems of monics

In any category a collection of monics, $\mathcal{M}$, which includes all isomorphisms and is closed under composition is called a *system of monics*. Such a system of monics $\mathcal{M}$ is said to be *stable* if for any $m \in \mathcal{M}$ and any $f : A \to B$ the pullback

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[f]{} & B \end{array}$$

exists and has $m' \in \mathcal{M}$. We call such a pullback an $\mathcal{M}$-pullback.

A stable system of monics actually has a further property: if $m'a = m$ with $m \in \mathcal{M}$ and m' an arbitrary monomorphism, then $a \in \mathcal{M}$. To see this note that

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle m'} \\ A & \xrightarrow[m]{} & B \end{array}$$

is a pullback.

We give some examples of stable systems of monics:

- In any category, the isomorphisms form a stable system of monics, giving the trivial $\mathcal{M}$-category structure on the category.
- In a category with pullbacks, the monics form a stable system of monics.
- In an extensive category [5, 6], the coproduct injections form a stable system of monics.
- In the category of posets with a bottom element, and bottom-preserving homomorphisms, there is a stable system of monics given by the “downward-closed” subsets (the ideals).

- In the category of the previous example there is another stable system of monics given by the full inclusions; in fact the isomorphisms and the monics give two further examples on this category, and so it has at least four stable systems of monics.
- If $\mathscr{M}$ and $\mathscr{M}'$ are stable systems of monics then their intersection $\mathscr{M} \cap \mathscr{M}'$ is a stable system.
- We may also form the "join" of two stable systems of monics by considering

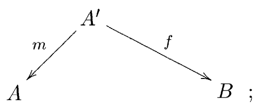
$$\mathscr{M} \vee \mathscr{M}' = \{m'_k m_k \dots m'_1 m_1 \mid m_i \in \mathscr{M}, m'_i \in \mathscr{M}'\}$$

and this too is a stable system of monics.

An *$\mathscr{M}$-category* is a pair $(\mathscr{C}, \mathscr{M})$, where $\mathscr{C}$ is a category, and $\mathscr{M}$ is a stable system of monics in $\mathscr{C}$. Given such an $\mathscr{M}$-category we may form its category of partial maps $\mathsf{Par}(\mathscr{C}, \mathscr{M})$:

Objects: $A \in \mathscr{C}$

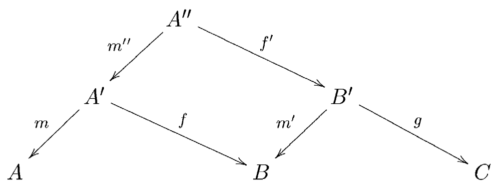
Arrows (from A to B): Pairs (m, f) where $m: A' \to A$ is in $\mathscr{M}$ and $f: A' \to B$ is arbitrary:



factored out by the equivalence relation $(m, f) \sim (m', f')$ whenever there is an isomorphism α with $m'\alpha = m$ and $f'\alpha = f$.

Identity: $(1_A, 1_A): A \to A$.

Composition: By pullback



That this is a category is well known: the fact that pulling back is only determined up to isomorphism is compensated for by the equivalence relation on the maps. Only a little less obvious is the fact that this category has a canonical split restriction. The original maps of $\mathscr{C}$ embed into $\mathsf{Par}(\mathscr{C}, \mathscr{M})$ by $f \mapsto (1, f)$ and are called the total partial maps.

Proposition 3.1. $\mathsf{Par}(\mathscr{C}, \mathscr{M})$ *has a split restriction given by* $\overline{(m, f)} = (m, m)$. *Furthermore, a map is total in* $\mathsf{Par}(\mathscr{C}, \mathscr{M})$ *with respect to this restriction if and only if it is total as a partial map.*

Proof. We must check the restriction category axioms:

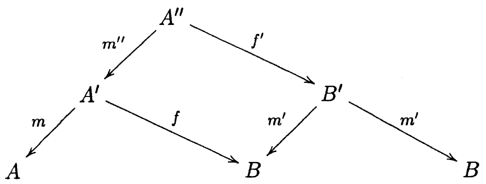
(R.1) For this we must check $(m, f)(m, m) = (m, f)$, which is immediate as the pullback of m over itself is the identity.

(R.2) We must check $(m',m')(m,m) = (m,m)(m',m')$ but this is clear as both composites give the restriction idempotent $(m \cap m', m \cap m')$.

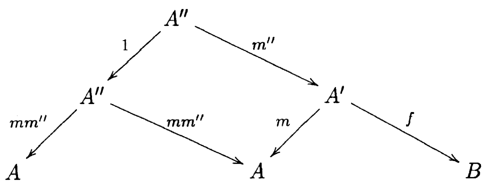
(R.3) This follows most easily by observing

$$\begin{aligned}\overline{(m',f')\overline{(m,f)}} &= \overline{(m',f')(m,m)} = (m \cap m', m \cap m') \\ &= (m',m')(m,m) = \overline{(m',f')}(m,f).\end{aligned}$$

(R.4) We must check that the following two composites



and



are the same. This follows from the form of the composites given above and the fact that $m'f' = fm''$.

Notice that $(m,1)(1,m) = (1,1)$ and $(1,m)(m,1) = (m,m)$, so that we have a splitting of each restriction idempotent.

If $\overline{(m,f)} = (1,1)$ then (m,m) is equivalent to $(1,1)$, and so m must be an isomorphism. Thus, (m,f) is equivalent to $(1, fm^{-1})$, and the total maps in the restriction category are indeed the total partial maps. □

3.2. *The 2-category $\mathscr{M}$Cat*

We should now like to define a 2-category $\mathscr{M}$Cat in such a way that Par becomes a 2-functor from $\mathscr{M}$Cat to rCat. We define an *$\mathscr{M}$-functor* between $\mathscr{M}$-categories $(\mathscr{C},\mathscr{M})$ and $(\mathscr{D},\mathscr{N})$ to be a functor $F:\mathscr{C}\to\mathscr{D}$ taking elements of $\mathscr{M}$ to elements of $\mathscr{N}$ and preserving $\mathscr{M}$-pullbacks. A natural transformation $\alpha: F\to G$ between $\mathscr{M}$-functors $F,G:(\mathscr{C},\mathscr{M})\to(\mathscr{D},\mathscr{N})$ is said to be *$\mathscr{M}$-cartesian* if for every $m:A\to B$ in $\mathscr{M}$, the naturality square

$$\begin{array}{ccc} FA & \xrightarrow{Fm} & FB \\ {\scriptstyle \alpha A}\downarrow & & \downarrow{\scriptstyle \alpha B} \\ FB & \xrightarrow[Gm]{} & GB \end{array}$$

is a pullback. The $\mathscr{M}$-categories, $\mathscr{M}$-functors, and $\mathscr{M}$-cartesian natural transformations form a 2-category $\mathscr{M}\mathsf{Cat}$.

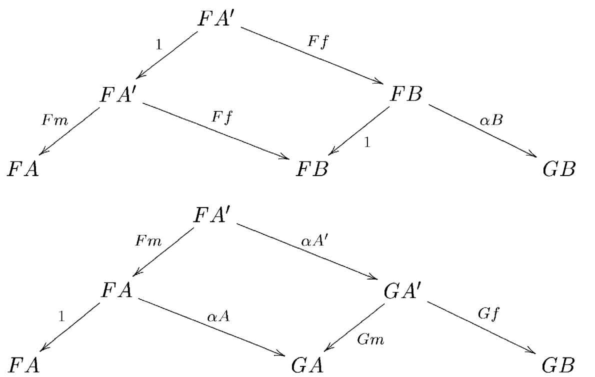
Proposition 3.2. *There is a* 2*-functor* $\mathsf{Par}:\mathscr{M}\mathsf{Cat}\to\mathsf{rCat}$ *taking* $(\mathscr{C},\mathscr{M})$ *to* $\mathsf{Par}(\mathscr{C},\mathscr{M})$.

Proof. Given an arrow $F:(\mathscr{C},\mathscr{M})\to(\mathscr{D},\mathscr{N})$, we define $\mathsf{Par}(F)(A)=FA$ for each object A of $\mathscr{C}$, and $\mathsf{Par}(F)(m,f)=(Fm,Ff)$ for each arrow. This is well-defined since $Fm\in\mathscr{N}$, and functorial since F preserves pullbacks along $\mathscr{M}$-maps. Clearly it preserves the restriction structure, making $\mathsf{Par}(F)$ into a restriction functor $\mathsf{Par}(\mathscr{C},\mathscr{M})\to\mathsf{Par}(\mathscr{D},\mathscr{N})$.

Given $F,G:(\mathscr{C},\mathscr{M})\to(\mathscr{D},\mathscr{N})$ and an $\mathscr{M}$-cartesian natural transformation $\alpha:F\to G$, we define $\mathsf{Par}(\alpha)A:\mathsf{Par}(F)(A)\to\mathsf{Par}(G)(A)$ to be the total partial map $(1_A,\alpha A)$ from FA to GA. The naturality condition we must check is

$$\begin{array}{ccc} FA & \xrightarrow{(Fm,Ff)} & FB \\ {\scriptstyle(1,\alpha A)}\downarrow & & \downarrow{\scriptstyle(1,\alpha A)} \\ GA & \xrightarrow[(Gm,Gf)]{} & GB \end{array}$$

and this amounts to checking that the following composites of partial maps are the same:



where in the second diagram we have used $\mathscr{M}$-cartesianness of α. The equality of these two partial maps follows by the naturality condition $\alpha B.Ff=Gf.\alpha A$.

We have now given all the data for a 2-functor $\mathsf{Par}:\mathscr{M}\mathsf{Cat}\to\mathsf{rCat}$; verifying the various functoriality conditions is straightforward. □

3.3. *$\mathscr{M}\mathsf{Cat}$ is 2-equivalent to rCat_s*

We have observed that the image of Par lies within the 2-category rCat_s of split restriction categories. We now show that this is a 2-equivalence. To do this we shall provide a 2-functor in the other direction which constructs an $\mathscr{M}$-category from a split restriction category. We shall call this 2-functor $\mathscr{M}\mathsf{Total}$.

Suppose then that $\mathbf{X}$ is a split restriction category and define $\mathscr{M}\mathsf{Total}(\mathbf{X}) = (\mathsf{Total}(\mathbf{X}), \mathscr{M}_{\mathbf{X}})$, where $\mathscr{M}_{\mathbf{X}}$ comprises the restriction monics in $\mathbf{X}$.

Proposition 3.3. *$\mathscr{M}\mathsf{Total}(\mathbf{X})$ is an $\mathscr{M}$-category.*

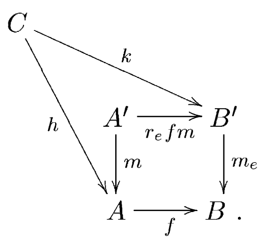
Proof. We must show that $\mathscr{M}_{\mathbf{X}}$ is a stable system of monics. Every arrow in $\mathscr{M}_{\mathbf{X}}$ is a section and so certainly is monic. Every isomorphism splits an identity, and identities are restriction idempotents, thus $\mathscr{M}_{\mathbf{X}}$ contains the isomorphisms. The class $\mathscr{M}_{\mathbf{X}}$ is also closed under composition, since, if $rm = 1$, $r'm' = 1$, $mr = \overline{mr}$, and $m'r' = \overline{m'r'}$; then $m'mrr' = m'\overline{mr}r' = m'r'\overline{mrr'} = \overline{m'r'}\,\overline{mrr'}$ and so $m'mrr'$ is a restriction idempotent, whence $m'mrr' = \overline{m'mrr'}$, and also $r'rmm' = r'm' = 1$.

It remains to show that $\mathscr{M}_{\mathbf{X}}$ is stable. If $f : A \to B$ is a total map, and $e : B \to B$ is a restriction idempotent split by $m_e : B_e \to B$ and $r_e : B_e \to B$, then let the splitting of $\overline{ef}$ be given by $m : A' \to A$ and $r : A \to A'$. The square

$$\begin{array}{ccc} A' & \xrightarrow{r_e f m} & B' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m_e} \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes since $m_e r_e f m = efm = f\overline{ef}m = fm$. The maps f, m, and m_e are all total and so fm is total, and so $mr_e fm$ is total, and so finally $r_e fm$ is total. Thus the above square is a commutative diagram in $\mathsf{Total}(\mathbf{X})$. We claim that it is a pullback.

Suppose, then, that we have another commutative square of total maps:



Since m_e and m are monic, it will suffice to find a factorization of h through m. But this amounts to showing that $\overline{ef}h = h$, which follows by

$$\overline{ef}h = h\overline{efh} = h\overline{em_e k} = h\overline{m_e k} = h. \qquad \square$$

In fact, it is easy to see that $\mathscr{M}\mathsf{Total}$ can be defined on arrows and 2-cells, as follows. If $F : \mathbf{X} \to \mathbf{Y}$ is a restriction functor between categories with split restrictions, then F restricts to a functor $\mathsf{Total}(F) : \mathsf{Total}(\mathbf{X}) \to \mathsf{Total}(\mathbf{Y})$ which takes restriction monics to restriction monics, since F is a restriction functor. Moreover, $\mathsf{Total}(F)$ preserves pullbacks along restriction monics since these are constructed using only composition and splittings of restriction idempotents. Thus, $\mathsf{Total}(F)$ is an $\mathscr{M}$-functor from $\mathscr{M}\mathsf{Total}(\mathbf{X})$ to $\mathscr{M}\mathsf{Total}(\mathbf{Y})$ which we call $\mathscr{M}\mathsf{Total}(F)$. If $F, G : \mathbf{X} \to \mathbf{Y}$ are restriction functors, and $\alpha : F \to G$ is a natural transformation whose components are total, then the components of α form a natural transformation $\mathsf{Total}(\alpha) : \mathsf{Total}(F) \to \mathsf{Total}(G)$. A typical naturality square for $\mathsf{Total}(\alpha)$ is given by

$$\begin{array}{ccc} FA & \xrightarrow{Fm} & FB \\ {\scriptstyle \alpha A}\downarrow & & \downarrow{\scriptstyle \alpha B} \\ GA & \xrightarrow[Gm]{} & GB \end{array}$$

where $rm = 1$ and $mr = \overline{mr}$. Now

$$Fm\,.\,Fr = F(mr) = F\overline{mr} = \overline{F(mr)} = \overline{\alpha B.F(mr)} = \overline{G(mr).\alpha B}$$

and so Fm splits the restriction idempotent $\overline{G(mr).\alpha B}$. Also $\alpha A = G(rm).\alpha A = Gr.\alpha B.Fm$, and so by the construction of pullbacks in $\mathsf{Total}(\mathbf{X})$, the above naturality square is a pullback, and $\mathsf{Total}(\alpha)$ is $\mathscr{M}_{\mathbf{X}}$-cartesian, hence is a 2-cell $\mathscr{M}\mathsf{Total}(\alpha) : \mathscr{M}\mathsf{Total}(F) \to \mathscr{M}\mathsf{Total}(G)$. These data now form a 2-functor $\mathscr{M}\mathsf{Total} : \mathsf{rCat} \to \mathscr{M}\mathsf{Cat}$.

Theorem 3.4. *The* 2*-functors* $\mathscr{M}\mathsf{Total}$ *and* Par *are part of an equivalence of* 2*-categories between* $\mathscr{M}\mathsf{Cat}$ *and* rCat_s.

Proof. Define $\Phi : \mathbf{X} \to \mathsf{Par}(\mathsf{Total}(\mathbf{X}), \mathscr{M})$ to be the identity on objects and to take an arrow $f : A \to B$ in $\mathbf{X}$ to

$$\Phi(f) = \quad A \xleftarrow{\;m\;} A' \xrightarrow{\;fm\;} B$$

where m is the restriction monic of the restriction idempotent $\bar{f}$. Observe that $\overline{fm} = \overline{\bar{f}m} = \bar{m} = 1$, so that fm is total. Clearly Φ preserves identities. Given $f : A \to B$ and $g : B \to C$, where m and r split $\bar{f}$, and n and s split $\bar{g}$, we have

$$\Phi(g)\Phi(f) = \quad \begin{array}{ccccccccc} & & & & B'' & & & & \\ & & & {\scriptstyle n'}\swarrow & & \searrow{\scriptstyle f'} & & & \\ & & A' & & & & B' & & \\ & {\scriptstyle m}\swarrow & & \searrow{\scriptstyle fm} & & {\scriptstyle n}\swarrow & & \searrow{\scriptstyle gn} & \\ A & & & & B & & & & C \end{array}$$

where n' splits $\overline{gfm}$, and $f' = sfmn'$. Since n' splits $\overline{gfm}$, and m splits $\bar{f}$, the composite mn' splits $m\overline{gfm}r = \overline{gf}mr = \overline{gf}\bar{f} = \overline{gf}$. Also $gnf' = gnsfmn' = g\bar{g}fmn' = gfmn'$, and so it follows that $\Phi(g)\Phi(f)$ is equal to

$$\Phi(gf) = \quad A \xleftarrow{\;mn'\;} A'' \xrightarrow{\;gfmn'\;} C$$

and so Φ is a functor.

Since Φ is the identity on objects, it will be invertible if we can show that it is fully faithful. Suppose then that (m, f) is an arrow in $\mathsf{Par}(\mathsf{Total}(\mathbf{X}), \mathscr{M})$ from A to B. By

Lemma 2.25 there is a unique r with $rm = 1$ and $mr = \overline{mr}$. Then $\Phi(fr) = (m, frm) = (m, f)$ and so Φ is full. On the other hand, if $\Phi(g) = (m, f)$ then $gm = f$ and $mr = \bar{g}$, and so $g = g\bar{g} = gmr = fr$, giving faithfulness of Φ.

It is clear that this construction of Φ is natural in $\mathbf{X}$, giving an isomorphism between $\mathsf{Par} \circ \mathscr{M}\mathsf{Total}$ and the identity 2-functor on rCat.

On the other hand, given an $\mathscr{M}$-category $(\mathscr{C}, \mathscr{M})$, we have already seen that the category of total maps in $\mathsf{Par}(\mathscr{C}, \mathscr{M})$ is just $\mathscr{C}$, while the restriction monics are just the monics in $\mathscr{M}$, and so we deduce an isomorphism between $\mathscr{M}\mathsf{Total} \circ \mathsf{Par}$ and the identity 2-functor on $\mathscr{M}\mathsf{Cat}$. □

This equivalence restricts to inverse categories as follows: on the one side one has the 2-category InvCat_s of inverse categories whose idempotents split, on the other side one has the 2-category of those $\mathscr{M}$-categories all of whose morphisms are $\mathscr{M}$-maps. In the one-object case this amounts to the equivalence between inverse monoids and *division categories* described in [12, Section VII.8].

Corollary 3.5. *The* 2-*functor* $\mathsf{Par} : \mathscr{M}\mathsf{Cat} \to \mathsf{rCat}$ *is fully faithful with image given by the split restriction categories; moreover* Par *has a left adjoint given by formally splitting the restriction idempotents and then applying* $\mathscr{M}\mathsf{Total}$.

Proof. Combine Theorem 3.4 and Proposition 2.27. □

As a further corollary we have:

Corollary 3.6. *The category* $\mathscr{M}\mathsf{Cat}_0$ *of* $\mathscr{M}$*-categories and functors preserving this structure is monadic over* Cat_0, *with the forgetful functor taking an* $\mathscr{M}$*-category* $(\mathscr{C}, \mathscr{M})$ *to* $\mathsf{Par}(\mathscr{C}, \mathscr{M})$.

Proof. Combine Theorems 2.28 and 3.4. □

3.4. Representation theorems

The Vagner–Preston Representation theorem [13, Theorem 5.1.7] for inverse monoids says:

Theorem 3.7 (Vagner–Preston). *If S is an inverse monoid then there exists a set X and a monoid monomorphism from S to the monoid I_X of partial injective maps from X to X.*

We wish to give a categorical explanation of this theorem. If we have a restriction category $\mathbf{X}$, in particular if we have an inverse category, then we can split the restriction idempotents and extract the total maps to obtain $\mathsf{Total}(K_r(\mathbf{X}))$. This is a category, and if $\mathbf{X}$ is an inverse category, it will be a category with pullbacks, all of whose arrows are monic. We may now represent $\mathsf{Total}(K_r(\mathbf{X}))$ via the Yoneda embedding:

$$Y : \mathsf{Total}(K_r(\mathbf{X})) \to [\mathsf{Total}(K_r(\mathbf{X}))^{\mathrm{op}}, \mathsf{Set}]$$

which is fully faithful and preserves any limits which exist.

The total category is really an $\mathscr{M}$-category and we can use the $\mathscr{M}$-maps to induce (following [18]) a system of monics on the presheaf category. We shall say that an arrow $m : F \to G$ in $[\mathsf{Total}(K_r(\mathbf{X})^{\mathrm{op}}, \mathsf{Set}]$ is an $\hat{\mathscr{M}}$-map if the pullback of m, along any arrow $YC \to G$ with domain a representable functor, is in $\mathscr{M}$. It is now straightforward to check that $\hat{\mathscr{M}}$ is a stable system of monics. Observe, furthermore, that the poset of $\hat{\mathscr{M}}$-subobjects of a representable functor is isomorphic to the poset of $\mathscr{M}$-subobjects of the representing object.

This means that $K_r(\mathbf{X})$ can be fully and faithfully represented as the $\hat{\mathscr{M}}$-partial maps in the presheaf category. The original category $\mathbf{X}$ as a full subcategory of $K_r(\mathbf{X})$ will therefore also admit a fully faithful representation as $\hat{\mathscr{M}}$-partial maps. This is already a much stronger representation result than the Vagner–Preston theorem!

Theorem 3.8. *If* $\mathbf{X}$ *is a restriction category then there is a fully faithful restriction-preserving representation of* $\mathbf{X}$ *as the* $\hat{\mathscr{M}}$*-partial maps of the presheaf category* $[\mathsf{Total}(K_r(\mathbf{X}))^{\mathrm{op}}, \mathsf{Set}]$.

If $\mathbf{X}$ is an inverse category, we now have a fully faithful representation of $\mathbf{X}$ as $\hat{\mathscr{M}}$-partial maps between representable objects in the presheaf category; notice that these partial maps will necessarily be restricted isomorphisms. If $\mathbf{X}$ is an inverse monoid, then the representation will be an isomorphism to the $\hat{\mathscr{M}}$-partial endomorphisms of $Y(\star)$ in the presheaf category.

To get a representation back in Set itself we may use the functor

$$\sum : [\mathsf{Total}(K_r(\mathbf{X})), \mathsf{Set}] \to \mathsf{Set}$$

which takes a functor F to the coproduct $\sum_{X \in \mathbf{X}} FX$. This functor is faithful and preserves pullbacks, so it gives the desired representation in Set for a one-object restriction category:

Theorem 3.9. *If* S *is any restriction monoid then there is a set* X *and a restriction-preserving monoid monomorphism* $r : S \to \mathsf{Par}(X)$ *where* $\mathsf{Par}(X)$ *is the set of partial endofunctions of* X.

Notice that, in the Vagner–Preston theorem at the start of the section, we do not have to mention that the restriction is preserved: having a restriction inverse is a property of inverse monoids and is preserved by any homomorphism. However, this is not the case for a restriction monoid so it is a crucial ingredient of the representation theorem that the representation preserve this structure.

4. Fibrations and subobjects

Often when one wishes to verify a program written in a programming language which supports some simple types it is useful to introduce a finer-type system which allows one to express properties such as the preconditions that an input should satisfy or the postconditions an output should satisfy.

One may model this situation by a category $\mathbf{X}$ with, sitting above each object, a formal set of subobjects. One must then be able to formally pullback these subobjects along the maps of the base category. This structure amounts to a functor from $\mathbf{X}^{\mathrm{op}}$ to the category Poset of posets and order-preserving maps, or in other words a fibred poset. The question we now address concerns the problem of realizing such formal subobjects as actual subobjects. We shall present the construction by using a series of adjunctions: first the 2-equivalence $\mathsf{rCat} \simeq \mathscr{M}\mathsf{Cat}_s$ developed in Theorem 3.4, secondly the adjunction of Proposition 2.27 between rCat and rCat_s, and thirdly the adjunction we shall establish in Theorem 4.2 below between $\mathsf{sLatFib}$ and rCat. This gives a conceptual development of the universal way in which a formally given system of subobjects can be added to a category.

4.1. *The restriction fibration*

If $\mathbf{X}$ is a restriction category we define a functor

$$\mathbf{X}^{\mathrm{op}} \overset{\mathrm{RId}_{\mathbf{X}}}{\longrightarrow} \mathsf{Poset}$$

as follows. For an object X of $\mathbf{X}$, we define $\mathrm{RId}_{\mathbf{X}}(X)$ to be the poset of restriction idempotents on X, ordered as usual by $e \leqslant e'$ if and only if $e = e'e$. Given an arrow $f: X \to Y$ in $\mathbf{X}$, we define $\mathrm{RId}_{\mathbf{X}}(f): \mathrm{RId}_{\mathbf{X}}(Y) \to \mathrm{RId}_{\mathbf{X}}(X)$ to take a restriction idempotent e on Y to the restriction idempotent $\overline{ef}$ on X. If $e \leqslant e'$ then $e = e'e = ee'$, and so $\overline{e'f}\,\overline{ef} = \overline{e'f}\,\overline{ee'f} = \overline{ee'f} = \overline{ef}$, and so $\overline{ef} \leqslant \overline{e'f}$, whence $\mathrm{RId}_{\mathbf{X}}(f)$ is indeed a poset homomorphism. Also $\mathrm{RId}_{\mathbf{X}}(f)\mathrm{RId}_{\mathbf{X}}(g)(e) = \mathrm{RId}_{\mathbf{X}}(f)\overline{eg} = \overline{\overline{eg}f} = \overline{f\,\overline{egf}} = \bar{f}\,\overline{egf} = \overline{egf} = \mathrm{RId}_{\mathbf{X}}(gf)(e)$ and $\mathrm{RId}_{\mathbf{X}}(1)(e) = \overline{e1} = e$, and so $\mathrm{RId}_{\mathbf{X}}$ is a functor.

Applying the Grothendieck construction to $\mathrm{RId}_{\mathbf{X}}: \mathbf{X}^{\mathrm{op}} \to \mathsf{Poset}$, we obtain a fibration $\partial: \mathscr{R}(\mathbf{X}) \to \mathbf{X}$. Explicitly, an object of $\mathscr{R}(\mathbf{X})$ is an object X of $\mathbf{X}$, equipped with a restriction idempotent e on X. An arrow in $\mathscr{R}(\mathbf{X})$ from (X, e) to (X', e') is an arrow $f: X \to X'$ in $\mathbf{X}$ satisfying $e \leqslant \overline{e'f}$, that is, $e = \overline{e'f}e$. Composition and identities are formed as in $\mathbf{X}$, and the fibration $\partial: \mathscr{R}(\mathbf{X}) \to \mathbf{X}$ merely forgets the restriction idempotent on an object. Given an object (X, e) of $\mathscr{R}(\mathbf{X})$, and an arrow $j: W \to X$ of $\mathbf{X}$, the cartesian lifting of j is the arrow $j: (W, \overline{ej}) \to (X, e)$.

In fact, $\mathscr{R}(\mathbf{X})$ is also a restriction category and $\partial: \mathscr{R}(\mathbf{X}) \to \mathbf{X}$ is a restriction functor; the restriction of an arrow $f: (X, e) \to (X', e')$ is given by $\bar{f}: (X, e) \to (X, e)$, where $\bar{f}$ is the restriction of f seen as an arrow of $\mathbf{X}$. We must check that $\bar{f}$ is indeed an arrow from (X, e) to (X, e), but this follows since $e = \overline{e'f}e = \overline{fe'f}e = \bar{f}e$, and so $e \leqslant \bar{f}$. The four restriction axioms on $\mathscr{R}(\mathbf{X})$ follow immediately from the corresponding axioms on $\mathbf{X}$, since $\partial: \mathscr{R}(\mathbf{X}) \to \mathbf{X}$ is faithful. We call $\partial: \mathscr{R}(\mathbf{X}) \to \mathbf{X}$ the *restriction fibration* associated to the restriction category $\mathbf{X}$. It turns out that $\partial: \mathscr{R}(\mathbf{X}) \to \mathbf{X}$ is actually a *fibration in the bicategory* rCat in the sense of [25], but we shall not pursue this point of view here.

The posets $\mathrm{RId}_{\mathbf{X}}(X)$ are actually meet-semilattices, with binary meets given by $e_1 \wedge e_2 = e_1 e_2$, and the identity 1_X as top element. Also we have $\mathrm{RId}_{\mathbf{X}}(f)(e_1 \wedge e_2) =$

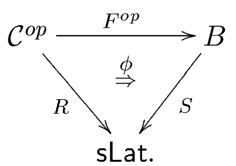
$\overline{e_1e_2f} = \overline{e_1f\overline{e_2f}} = \overline{e_1f}\,\overline{e_2f} = \mathrm{RId}_{\mathbf{X}}(f)(e_1) \wedge \mathrm{RId}_{\mathbf{X}}(f)(e_2)$, and so $\mathrm{RId}_{\mathbf{X}}(f)$ preserves binary meets; on the other hand $\mathrm{RId}_{\mathbf{X}}(f)(1) = \bar{f}$, and so $\mathrm{RId}_{\mathbf{X}}(f)$ preserves the top element if and only if f is total. We have now shown that $\mathrm{RId}_{\mathbf{X}}: \mathbf{X}^{\mathrm{op}} \to \mathsf{Poset}$ lifts to a functor (of the same name) $\mathrm{RId}_{\mathbf{X}}: \mathbf{X}^{\mathrm{op}} \to \mathsf{stabLat}$; moreover, the restriction of this functor to $\mathsf{Total}(\mathbf{X})^{\mathrm{op}}$ lands in sLat. If we apply the Grothendieck construction to this new functor $\mathrm{RId}^t_{\mathbf{X}}: \mathsf{Total}(\mathbf{X})^{\mathrm{op}} \to \mathsf{sLat}$ we obtain a fibration $\partial^t: \mathscr{R}^t(\mathbf{X}) \to \mathsf{Total}(\mathbf{X})$, and we have the following pullback in Cat:

$$\begin{array}{ccc} \mathscr{R}^t(\mathbf{X}) & \longrightarrow & \mathscr{R}(\mathbf{X}) \\ {\scriptstyle \partial^t}\big\downarrow & & \big\downarrow{\scriptstyle \partial} \\ \mathsf{Total}(\mathbf{X}) & \longrightarrow & \mathbf{X}. \end{array}$$

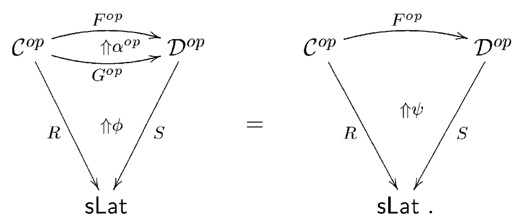
4.2. *The 2-category* sLatFib

To give a functor $\mathscr{C}^{\mathrm{op}} \to \mathsf{sLat}$ is equivalent to giving a fibration over $\mathscr{C}$ whose fibres are meet-semilattices, and whose inverse image functors are meet-semilattice homomorphisms; either such structure we call a fibred meet-semilattice. We shall now define a 2-category $\mathsf{sLatFib}$ of fibred meet-semilattices, providing a domain for the study of the functor $\mathrm{RId}^t_{\mathbf{X}}: \mathsf{Total}(\mathbf{X})^{\mathrm{op}} \to \mathsf{sLat}$ of the previous section.

An object of $\mathsf{sLatFib}$ is a pair $(\mathscr{C}, R)$, where $\mathscr{C}$ is a category and $R: \mathscr{C}^{\mathrm{op}} \to \mathsf{sLat}$ a functor, while an arrow from $(\mathscr{C}, R)$ to $(\mathscr{D}, S)$ comprises a functor $F: \mathscr{C} \to \mathscr{D}$ and a natural transformation

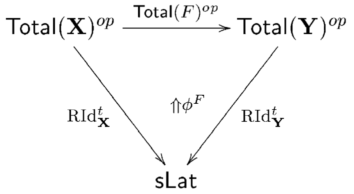


Finally a 2-cell from (F, ϕ) to (G, ψ) is a natural transformation $\alpha: F \to G$ such that



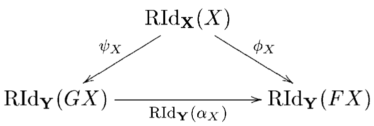
The purpose of this section is to provide a "forgetful" 2-functor $\mathscr{R}_t: \mathsf{rCat} \to \mathsf{sLatFib}$, which takes a restriction category $\mathbf{X}$ to the fibred meet-semilattice $\mathrm{RId}^t_{\mathbf{X}}: \mathsf{Total}(\mathbf{X})^{\mathrm{op}} \to \mathsf{sLat}$, defined in the previous section, which equips $\mathsf{Total}(\mathbf{X})$ with the system of subobjects comprising the restriction monics.

If $F : \mathbf{X} \to \mathbf{Y}$ is a restriction functor, we define $\mathscr{R}_t(F) : \mathscr{R}_t(\mathbf{X}) \to \mathscr{R}_t(\mathbf{Y})$ to be the pair $(\mathsf{Total}(F), \phi^F)$, where ϕ^F is the natural transformation



whose component at an object X of $\mathbf{X}$ is the map $\phi^F_X : \mathrm{RId}_{\mathbf{X}}(X) \to \mathrm{RId}_{\mathbf{Y}}(FX)$ defined by $\phi^F_X(e) = Fe$. If $e \leqslant e'$ then $e = e'e$, and so $Fe = F(e'e) = Fe'.Fe$, giving $Fe \leqslant Fe'$, and F is order-preserving; also $F(e_1 \wedge e_2) = F(e_1 e_2) = Fe_1.\ Fe_2 = Fe_1 \wedge Fe_2$ and so F is a semilattice homomorphism. As for naturality, if $f : X \to Y$ is an arrow in $\mathsf{Total}(\mathbf{X})^{\mathrm{op}}$, and e is a restriction idempotent on Y, then $\phi^F_X(\mathrm{RId}_{\mathbf{X}}(f)(e)) = \phi^F_X(\overline{ef}) = F(\overline{ef}) = \overline{F(ef)} = \overline{Fe.Ff} = \overline{\phi^F_Y(e).\ Ff} = \mathrm{RId}_{\mathbf{Y}}(Ff)(\phi^F_Y(e))$ and so ϕ^F is natural, and $(\mathsf{Total}(F), \phi^F)$ does indeed form an arrow $\mathscr{R}_t(F) : \mathscr{R}_t(\mathbf{X}) \to \mathscr{R}_t(\mathbf{Y})$ in $\mathsf{sLatFib}$.

If $F, G : \mathbf{X} \to \mathbf{Y}$ are restriction functors, and $\alpha : F \to G$ is a restriction transformation, then we have a natural transformation $\mathsf{Total}(\alpha) : \mathsf{Total}(F) \to \mathsf{Total}(G)$ having the same components as α. We claim that $\mathsf{Total}(\alpha)$ is a 2-cell in $\mathsf{sLatFib}$ from $\mathscr{R}_t(F) = (\mathsf{Total}(F), \phi)$ to $\mathscr{R}_t(G) = (\mathsf{Total}(G), \psi)$. We need only check that the diagram



commutes, and for a restriction idempotent e on X, we have $\mathrm{RId}_{\mathbf{Y}}(\alpha_X)(\psi_X(e)) = \mathrm{RId}_{\mathbf{Y}}(\alpha_X)(Ge) = \overline{Ge.\alpha_X} = \overline{\alpha_X.Fe} = \overline{Fe} = Fe = \phi_X(e)$. Thus we can define $\mathscr{R}_t(\alpha) = \mathsf{Total}(\alpha)$. We have now given all the data for a 2-functor $\mathscr{R}_t : \mathsf{rCat} \to \mathsf{sLatFib}$, and the verification of the functoriality conditions is straightforward, giving:

Proposition 4.1. $\mathscr{R}_t : \mathsf{rCat} \to \mathsf{sLatFib}$ *is a 2-functor.*

4.3. *The 2-reflection* sLatFib *to* rCat

Our aim now is to provide a 2-adjoint for $\mathscr{R}_t$. Thus given a fibred meet-semilattice $R : \mathscr{C}^{\mathrm{op}} \to \mathsf{sLat}$ we wish to show how to build an associated restriction category $\mathscr{S}(\mathscr{C}, R)$. The objects of $\mathscr{S}(\mathscr{C}, R)$ are just the objects of $\mathscr{C}$, while an arrow in $\mathscr{S}(\mathscr{C}, R)$ from B to C is a pair (f, e) where $f : B \to C$ is an arrow of $\mathscr{C}$, and $e \in RB$. The composite of arrows $(f, e) : B \to C$ and $(g, e') : C \to D$ is the pair $(gf, e \wedge (Rf)e')$ and the identity on B is $(1_B, \top)$. This is easily shown to be a category. The restriction structure is given by $\overline{(f, e)} = (1, e)$, and the restriction axioms are satisfied because

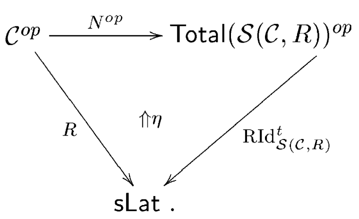
(R.1) $(f, e)\overline{(f, e)} = (f, e)(1, e) = (f, e \wedge e) = (f, e)$;

(R.2) $\overline{(g, e')}\,\overline{(f, e)} = (1, e')(1, e) = (1, e \wedge e') = (1, e' \wedge e) = (1, e)(1, e') = \overline{(f, e)}\,\overline{(g, e')}$;

(R.3) $\overline{(g, e')\overline{(f, e)}} = \overline{(g, e')(1, e)} = (1, e \wedge e') = (1, e')(1, e) = \overline{(g, e')}\,\overline{(f, e)}$;

(R.4) $\overline{\overline{(g,e')}(f,e)} = \overline{(1,e')(f,e)} = \overline{(f,e \wedge (Rf)e')} = (1,e \wedge (Rf)e') = (1,e)(1,(Rf)e') = \overline{(f,e)}\,\overline{(g,e')(f,e)}$.

Observe that the total maps are precisely those (f,e) with $e=\top$, while the restriction idempotents are those (f,e) with $f=1$. There is an evident isomorphism of categories $N:\mathscr{C}\to\mathsf{Total}(\mathscr{S}(\mathscr{C},R))$ which is the identity on objects and takes an arrow $f:C\to D$ in $\mathscr{C}$ to $(f,\top)$. For each object C of $\mathscr{C}$, there is an isomorphism of semilattices $\eta_C:RC\to \mathrm{RId}_{\mathscr{S}(\mathscr{C},R)}(NC)$ taking $e\in RC$ to $(1_C,e)$. If $f:C\to D$ is in $\mathscr{C}$, and $e\in RD$, we have $\mathrm{RId}_{\mathscr{S}(\mathscr{C},R)}(Nf)(\eta_D(e))=\mathrm{RId}_{\mathscr{S}(\mathscr{C},R)}(f,\top)(1_D,e)=\overline{(1_D,e)(f,\top)}=\overline{(f,(Rf)e)}=(1_C,(Rf)e)=\eta_C((R_f)e)$, and so we have a natural isomorphism



It follows that $(N,\eta):(\mathscr{C},R)\to\mathscr{R}_t\mathscr{S}(\mathscr{C},R)$ is an isomorphism in $\mathsf{sLatFib}$.

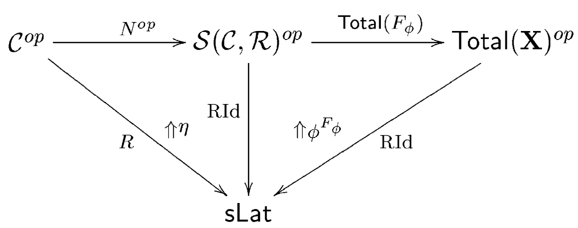
We now prove that $(N,\eta):(\mathscr{C},R)\to\mathscr{R}_t\mathscr{S}(\mathscr{C},R)$ has the universal property of the unit of an adjunction $\mathscr{S}\dashv\mathscr{R}_t$.

Suppose then that $\mathbf{X}$ is a restriction category, and $(F,\phi):(\mathscr{C},R)\to\mathscr{R}_t(\mathbf{X})$ is an arrow in $\mathsf{sLatFib}$. We shall define a functor $F_\phi:\mathscr{S}(\mathscr{C},R)\to\mathbf{X}$. On objects we define $F_\phi C=FC$, and given an arrow $(f,e):B\to C$ in $\mathscr{S}(\mathscr{C},R)$ we define $F_\phi(f,e)$ to be the composite

$$FB\overset{\phi_B(e)}{\to}FB\overset{Ff}{\to}FC.$$

Now $F_\phi((g,e')(f,e))=F_\phi(gf,e\wedge(Rf)e')=f(gf)\phi_B(e\wedge(Rf)e')=Fg.\,Ff.\phi_B((Rf)e').\phi_B(e)=Fg.\phi_B(e').Ff.\phi_B(e)=F_\phi(g,e')F_\phi(f,e)$, and $F_\phi(1,\top)=F1.\phi_B(\top)=1$, and so F_ϕ is a functor. Also $F_\phi\overline{(f,e)}=F_\phi(1,e)=F1.\phi_B(e)=\phi_B(e)=\overline{\phi_B(e)}=\overline{Ff.\phi_B(e)}=\overline{F_\phi(f,e)}$, where in the penultimate step we have used the fact that Ff is total; we conclude that F_ϕ is a restriction functor.

Now if C is an object of $\mathscr{C}$ then $\mathsf{Total}(F_\phi)(N(C))=\mathsf{Total}(F_\phi)(C)=FC$, while if $f:B\to C$ is an arrow of $\mathscr{C}$, we have $\mathsf{Total}(F_\phi)(N(f))=\mathsf{Total}(F_\phi)(f,\top)=F_\phi(f,\top)=Ff$, and so $\mathsf{Total}(F_\phi)N=F$. Further, if $e\in RC$, then $\phi_C^{F_\phi}(\eta_C(e))=\phi_C^{F_\phi}(1_C,e)=F_\phi(1_C,e)=\phi_C(e)$ and so the composite



is equal to ϕ, whence $\mathscr{R}_t(F_\phi).(N,\eta)=(F,\phi)$.

Moreover, suppose that $H:\mathscr{S}(\mathscr{C},R)\to\mathbf{X}$ is a restriction functor satisfying $\mathscr{R}_t(H).(N,\eta)=(F,\phi)$. The equality $\mathsf{Total}(H).N^{\mathrm{op}}=F$ proves that H agrees with F_ϕ on objects and on total morphisms, while the equality of the comparison natural transformations ensures that H and F_ϕ agree on restriction idempotents. Since every arrow of $\mathscr{S}(\mathscr{C},R)$ can be written as the composite of a total arrow and a restriction idempotent, it follows that $H=F_\phi$. Thus we have proved the one-dimensional aspect of the universal property.

As for the two-dimensional part, suppose that (F,ϕ) and (G,ψ) are arrows from $(\mathscr{C},R)$ to $\mathscr{R}_t(\mathbf{X})$, and that $\alpha:(F,\phi)\to(G,\psi)$ is a 2-cell. We must produce a unique 2-cell $\beta:F_\phi\to G_\psi$ such that $\mathscr{R}_t(\beta).(N,\eta)=\alpha$. Let the component β_C at C of β be just α_C. Observe that α is a natural transformation between functors with codomain $\mathsf{Total}(\mathbf{X})$, and so certainly each β_C will be natural. The only thing to check is that β as defined is a natural transformation from $F_\phi\to G_\psi$. Note that the condition that the natural transformation $\alpha:F\to G$ be a 2-cell from (F,ϕ) to (G,ψ) amounts to saying that for each object C of $\mathscr{C}$, and each $e\in RC$, we have $\psi_C(e)=\mathrm{RId}(\alpha_C).\phi_C(e)=\overline{\phi_C(e).\alpha_C}$. If $(f,e):B\to C$ is an arrow of $\mathscr{S}(\mathscr{C},R)$, then $\beta_C.\,F_\phi(f,e)=\alpha_C.\,Ff.\phi_B(e)=Gf.\alpha_B.\phi_B(e)$ $=Gf.\alpha_B.\overline{\psi_B(e)\alpha_B}=Gf.\overline{\psi_B(e)}.\alpha_B=Gf.\psi_B(e).\alpha_B=G_\psi(f,e).\beta_B$ and so β is natural.

Thus we have proved:

Theorem 4.2. *There is an adjunction*:

$$\mathsf{sLatFib}\ \underset{\mathcal{R}_t}{\overset{\mathcal{S}}{\rightleftarrows}}\ \mathsf{rCat}\qquad (\mathcal{S}\dashv\mathcal{R}_t)$$

with invertible unit.

We can now combine several of the adjunctions already constructed to obtain an adjunction between $\mathsf{sLatFib}$ and $\mathscr{M}\mathsf{Cat}$, and so provide as promised a way of universally realizing formal subobjects.

Given an $\mathscr{M}$-category $(\mathbf{X},\mathscr{M})$ we can form the fibred semilattice $(\mathbf{X},Sub_{\mathscr{M}})$, where $Sub_{\mathscr{M}}:\mathbf{X}^{\mathrm{op}}\to\mathsf{sLat}$ takes an object X to the meet-semilattice of $\mathscr{M}$-subobjects of X. This is the object part of a fully faithful 2-functor

$$\mathscr{I}:\mathscr{M}\mathsf{Cat}\to\mathsf{sLatFib}$$

which is the composite of $\mathsf{Par}:\mathscr{M}\mathsf{Cat}\to\mathsf{rCat}$ and $\mathscr{R}_t:\mathsf{rCat}\to\mathsf{sLatFib}$. We now have:

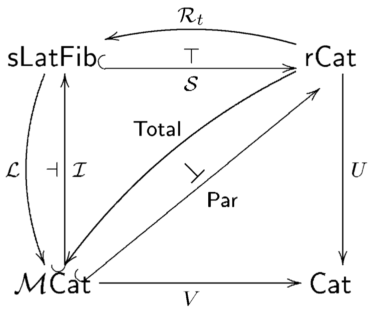
Theorem 4.3. *The fully faithful* 2*-functor* $\mathscr{I}:\mathscr{M}\mathsf{Cat}\to\mathsf{sLatFib}$ *has a left adjoint* $\mathscr{L}$.

Proof. Combine Theorem 3.4, Proposition 2.27, and Theorem 4.2. □

Remark 4.4. The left adjoint $\mathscr{F}:\mathsf{Cat}_0\to\mathsf{rCat}_0$ to the forgetful functor $\mathscr{U}:\mathsf{Cat}_0\to\mathsf{rCat}_0$ has certain similarities to $\mathscr{S}:\mathsf{sLatFib}\to\mathsf{rCat}$. We saw in Remark 2.7 how to associate a functor $\Gamma:\mathscr{C}^{\mathrm{op}}\to\mathsf{stabLat}$ to a category $\mathscr{C}$; the functor took an object C to the semilattice of cofinite sieves on C. The passage from $(\mathscr{C},\Gamma)$ to the free restriction category $\mathscr{F}\mathscr{C}$ is now formally identical to the passage from $(\mathscr{C},R)$ to $\mathscr{S}(\mathscr{C},R)$. The analogy can be made precise: both constructions can be described in terms of a certain weighted limit, but we shall not do so here.

5. Conclusion

The main relationships established in this paper can be summarized by the following diagram of 2-categories and 2-functors:



where $U \circ \mathsf{Par} \cong V$, $\mathscr{R}_t \circ \mathsf{Par} \cong \mathscr{I}$, and $\mathsf{Total} \circ \mathscr{S} \cong \mathscr{L}$; and where moreover the underlying functors $\mathscr{U}$ of U and $\mathscr{V}$ of V are monadic. The image of $\mathsf{Par} : \mathscr{M}\mathsf{Cat} \to \mathsf{rCat}$ is equivalent to rCat_s, which in turn is biequivalent to rCat_{sf}.

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