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MISCELLANEOUS ERROR BOUNDS FOR MULTIQUADRIC AND RELATED INTERPOLATORS

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Abstract—We establish several types of *a priori* error bounds for multiquadric and related interpolators. The results are stated and proven in the general multivariate case. These estimates show, for example, that in many cases such interpolators converge very quickly and can be used in the recovery of band limited functions from discrete data. We also include numerical experiments which illustrate the theoretical results.

1. INTRODUCTION

This paper is concerned with error bounds for a certain class of interpolation problems. For the most part, we will restrict our attention to the case of interpolators of the form

$$s_\gamma(x) = c_0 + \sum_{j=1}^{N} c_j\, h_\gamma(x - x_j), \tag{1}$$

where

$$h_\gamma(x) = -\sqrt{\gamma^2 + |x|^2}\,, \tag{2}$$

$X = \{x_1, \dots, x_N\}$ is a collection of points in $\mathbf{R}^n$, and γ is a fixed positive constant. The function h_γ which is conditionally positive definite of order one, is often referred to as a *multiquadric* and the interpolators (1) are called *multiquadrics*. For a detailed background with historical remarks see Hardy's survey paper [1].

There is a rapidly growing body of literature concerned with multiquadric interpolation; in addition to [1] and the pertinent references therein, [2–7] are representative of current work. The present volume is another example.

Most of the cited work is not concerned with the role of the parameter γ which is usually normalized to be one. However, recently several authors [7,8] have noticed that this parameter can meaningfully affect the interpolator; in certain cases, the deviation can be significantly decreased by increasing γ. In Subsections 2.1–2.3, we record several observations concerning this matter which may partially explain this phenomenon; in particular, we include a local error bound in terms of the parameter γ in Subsection 2.2 and study the behavior of the interpolator as γ tends to infinity in Subsection 2.3.

These interpolators can be used for discretizing many of the problems involving differential and integral equations in preparation for numerical implementations. In many such applications the derivatives of the interpolators play a significant role. In Subsection 2.4, we record an error bound involving such derivatives.

Section 3 is devoted to various remarks concerning the material in Section 2. For example, in Subsection 3.2, we indicate that these results are valid in a much more general setting; in Subsection 3.3, we include comments on other types of error bounds.

The results of certain numerical calculations are summarized in Section 4. These computations concretely illustrate the behavior of the interpolators considered here. However, perhaps more importantly, they provide credence to some natural questions raised earlier.

The error bounds we are talking about arise as follows: Suppose that the constants $c_0, \ldots, c_N$ defining the interpolator (1) are chosen so that

$$s_\gamma(x_i) = f(x_i), \qquad i = 1, \ldots, N, \tag{3}$$

where f is a continuous function some of whose properties are known. The error bounds are *a priori* estimates of the deviation $|f(x) - s_\gamma(x)|$ in terms of certain parameters associated with f, h_γ, and $X = \{x_1, \ldots, x_N\}$.

2. *A PRIORI* ERROR BOUNDS

2.1. *Some Heuristics and Background*

Suppose Ω is a cube in $\mathbf{R}^n$ and

$$\delta = \sup_{y \in \Omega} \left(\inf_{x \in X} (|x - y|) \right).$$

If s_γ satisfies (1)–(3) for x in Ω, we wish to estimate the error $|f(x) - s_\gamma(x)|$ in terms of δ and parameters which depend on f and h_γ. In particular, we are interested in the role played by the parameter γ when h_γ is the function defined by (2). To facilitate the discussion, we adopt the notation $s_\gamma(x) = s_\gamma(f, x)$ to explicitly indicate the dependence of s_γ on f in this case.

Consider the extreme case $\gamma = 0$. In this instance, $h_0(x) = |x|$ and it is not difficult to see that, no matter how smooth the function f,

$$|f(x) - s_0(x)| = O(\delta^{n+1})$$

is the best estimate one can generally expect in terms of δ. This is particularly clear in the case $n = 1$ since $s_0(x)$ is simply the piecewise linear interpolator of f. In the general multivariate case, this is most easily seen by restricting ones attention to the case when $\Omega = \mathbf{R}^n$ and $X = \delta \mathbf{Z}^n$.

To get an idea of what we should expect for large values of γ, recall that Theorem 1 in [5] roughly says that in this case, if f is sufficiently smooth then the estimate

$$|f(x) - s_\gamma(x)| \leq C_1\, E_k(h_\gamma, C_2\delta)^{1/2}$$

is possible for every $k = 1, 2, \ldots$, where C_1 and C_2 are constants independent of δ and

$$E_k(h_\gamma, \epsilon) = \inf_{p \in \mathcal{P}_k} \left\{ \sup_{|x| < \epsilon} |h_\gamma(x) - p(x)| \right\}.$$

$\mathcal{P}_k$ is the class of polynomials of degree $\leq k$. Since

$$h_\gamma(x) = \gamma\sqrt{1 + |\gamma^{-1}x|^2} = \gamma \left\{ \sum_{j=0}^{\infty} a_j |\gamma^{-1}x|^{2j} \right\},$$

it is clear that $E_k(h_\gamma, \epsilon) = O(\epsilon^{2m}/\gamma^{2m+1})$ whenever $k = 2m - 2$ or $2m - 1$. Hence, by taking k to be a positive even integer, we have

$$|f(x) - s_\gamma(x)| \leq C_1\, C_2^k\, \gamma^{-k-1} \delta^k$$

which, asymptotically as δ tends to 0, favors large values of γ if the constants C_1 and C_2, which may depend on γ and k, can be controlled.

A precise result is given in the next subsection. Before doing so, however, we review some notation and concepts used in the discussion.

The Fourier transform $\hat{f}$ of an integrable function f is defined by

$$\hat{f}(x) = \int_{\mathbf{R}^n} f(x)\, e^{-i\langle x,\xi\rangle}\, dx.$$

All Fourier transforms are to be interpreted in the distributional sense. In particular, the Fourier transform $\hat{h}_\gamma$ of h_γ is a distribution which is equal to a C^∞ function outside every neighborhood of the origin; thus, for ξ in $\mathbf{R}^n\backslash\{0\}$ we identify $\hat{h}_\gamma$ with this function. An explicit formula for $\hat{h}_\gamma(\xi)$, $\xi \in \mathbf{R}^n\backslash\{0\}$, can be found in [7,9] but it will not be used here. Only the behavior of $\hat{h}_\gamma(\xi)$ in neighborhoods of infinity plays a significant role in what follows.

Another important ingredient in deriving our estimates is the following: if the constants c_j in (1) are chosen so that $\sum_{j=1}^N c_j = 0$, the interpolator s_γ which satisfies (3) is a solution to the variational problem

$$\min\left\{\|g\|_{C_{h_\gamma}} : g(x_i) = f(x_i), \qquad i = 1,\dots,N\right\},$$

where C_{h_γ} is a certain semi-Hilbert space and $\|\cdot\|_{C_{h_\gamma}}$ is the corresponding semi-norm. In other words, $\|\cdot\|^2_{C_{h_\gamma}}$ is a certain semi-definite quadratic form determined by h_γ; the interpolator s_γ minimizes this form over all interpolators g. For more details see [5].

We close this subsection by warning the reader of some potentially ambiguous notation: Most of the time, the symbol x_j denotes the j^{th} point in a collection of points that are a subset of $\mathbf{R}^n$, for example, as in equations (1) and (3) in the introduction. On a few occasions, however, we find it convenient to denote the i^{th} coordinate of the point x by x_i. Since the meaning should be clear from the context, we hope no confusion arises from this usage.

2.2. Estimates for Band Limited and Related Functions

To obtain a precise conclusion consider the following setup:

(i) f is a continuous function on $\mathbf{R}^n$.
(ii) $X = \{x_1,\dots,x_N\}$ is a finite subset of $\mathbf{R}^n$.
(iii) Ω_a is a cube with sides of length a defined by

$$\Omega_a = \{x = (x_1,\dots,x_n) : 0 \le x_i \le a\},$$

where x_i, $i = 1,\dots,n$, is the i^{th} coordinate of x.
(iv) $\delta = \sup_{y\in\Omega_a}(\inf_{x\in X}|x-y|)$.
(v) $s_\gamma(f,x) = c_0 + \sum_{j=1}^N c_j\sqrt{\gamma^2 + |x-x_j|^2}$ where

$$c_0 + \sum_{j=1}^N c_j\sqrt{\gamma^2 + |x_k - x_j|^2} = f(x_k), \qquad k = 1,\dots,N \text{ and } \sum_{j=1}^N c_j = 0.$$

Observe that

$$\sup_{x\in\Omega_a}|f(x) - s_\gamma(f,x)| = \sup_{x\in\Omega_{a/\gamma}}|f(\gamma x) - s_\gamma(f,\gamma x)|, \tag{4}$$

and

$$s_\gamma(f,\gamma x) = c_0 + \sum_{j=1}^N c_j\,\gamma\sqrt{1 + |x - \gamma^{-1}x_j|^2}.$$

So, if we let $g(x) = f(\gamma x)$ and $s_1(g,x) = s_\gamma(f,\gamma x)$, then $s_1(g,x)$ is the minimum C_{h_1} norm interpolator of g on $\gamma^{-1}X = \{\gamma^{-1}x_1,\dots,\gamma^{-1}x_N\}$, where h_1 is the function defined by $h_1(x) = \sqrt{1+|x|^2}$. Hence, by virtue of Theorem 2 in [5], if δ is sufficiently small and if $1 \le \gamma \le a$, there is a constant λ which is independent of f, a, γ, and δ such that

$$\sup_{x\in\Omega_{a/\gamma}}|g(x) - s_1(g,x)| \le \lambda^{\gamma/\delta}\|g\|_{C_{h_1}}. \tag{5}$$

In order that the last inequality lead to a useful error bound, f must be chosen so that the norm $\|g\|_{C_{h_1}}$ is finite. First, recall that in this particular case C_{h_1} can be characterised as the class of those distributions u such that each component of the Fourier transform of $\operatorname{grad} u$ is in $L^2(\mathbf{R}^n, (|\xi|^2 \hat{h}_1(\xi))^{-1}\, d\xi)$; moreover,

$$\|u\|_{C_{h_1}}^2 = \sum_{i=1}^{n} \int_{\mathbf{R}^n} |\xi_i\, \hat{u}(\xi)|^2 \,(|\xi|^2 \hat{h}_1(\xi))^{-1}\, d\xi. \tag{6}$$

In view of this characterization, it is not difficult to obtain simple conditions on functions f for which (5) leads to meaningful error bounds. We give two examples.

Let B_σ, for a positive parameter σ, be the class of band limited functions f in $L^2(\mathbf{R}^n)$ defined by

$$B_\sigma = \left\{ f \in L^2(\mathbf{R}^n) : \hat{f}(\xi) = 0 \text{ if } |\xi| > \sigma \right\}. \tag{7}$$

Clearly, any function f in B_σ is also in C_{h_1}. Furthermore, if ϵ_h and C_h are positive constants such that

$$(\hat{h}_1(\xi))^{-1} \le C_h\, e^{\epsilon_h |\xi|}; \tag{8}$$

then, when $g(x) = f(\gamma x)$, we have

$$\begin{aligned}\|g\|_{C_{h_1}}^2 &\le C \int_{\mathbf{R}^n} |\gamma^{-n} \hat{f}(\gamma^{-1}\xi)|^2\, e^{2\epsilon_h |\xi|}\, d\xi \\ &\le C \sup_{|\xi| \le \gamma\sigma} \left\{ e^{2\epsilon_h |\xi|} \right\} \cdot \gamma^{-n} \cdot \int_{|\xi|<\sigma} |\hat{f}(\xi)|^2\, d\xi \\ &\le C\, e^{2\epsilon_h \sigma\gamma}\, \|f\|_{L^2(\mathbf{R}^n)}^2,\end{aligned}$$

whenever $\gamma \ge 1$ and f is in B_σ.

If σ is a positive number, let E_σ be the class of those functions f in $L^2(\mathbf{R}^n)$ defined by

$$E_\sigma = \left\{ f \in L^2(\mathbf{R}^n) : \|f\|_{E_\sigma} < \infty \right\}, \tag{9}$$

where

$$\|f\|_{E_\sigma}^2 = \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2\, e^{|\xi|^2/\sigma}\, d\xi.$$

Again, any function f in E_σ is also in C_{h_1} and when $g(x) = f(\gamma x)$ and $\gamma \ge 1$, we have

$$\begin{aligned}\|g\|_{C_{h_1}}^2 &\le C \sup_{\xi \in \mathbf{R}^n} \left\{ e^{2\epsilon_h |\xi| \gamma - |\xi|^2/\sigma} \right\} \cdot \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2\, e^{|\xi|^2/\sigma} d\xi \\ &\le C\, e^{\epsilon_h^2 \gamma^2 \sigma}\, \|f\|_{E_\sigma}^2.\end{aligned}$$

We summarize these observations as follows:

THEOREM 1. *Suppose we have the setup described by items (i)–(v) listed above and $1 \le \gamma \le a$. Then there is a constant λ which satisfies $0 < \lambda < 1$ and which is independent of f, a, γ, and δ such that the following holds:*

- *If f is in B_σ, then for sufficiently small δ*

$$\sup_{x \in \Omega_a} |f(\xi) - s_\gamma(f, x)| \le C\, e^{\epsilon_h \sigma\gamma}\, \lambda^{\gamma/\delta}\, \|f\|_{L^2(\mathbf{R}^n)}. \tag{10}$$

- *If f is in E_σ, then for sufficiently small δ*

$$\sup_{x \in \Omega_a} |f(x) - s_\gamma(f, x)| \le C\, e^{\epsilon_h^2 \gamma^2 \sigma/2}\, \lambda^{\gamma/\delta}\, \|f\|_{E_\sigma}. \tag{11}$$

- *In both cases, ϵ_h is the constant in the exponent in inequality (8) and C is a constant independent of f, a, γ, δ, λ, and σ.*
- *In both cases,*

$$|f(x) - s_\gamma(f, x)| = O(\lambda^{\gamma/\delta}) \qquad \text{as } \delta \to 0, \tag{12}$$

whenever x is in Ω_a.

Before moving on to the next subsection, we note that if

$$f(x) = P(x)\,e^{-c|x-y|^2},$$

where $P(x)$ is a polynomial, c is a positive constant and y is any point in $\mathbf{R}^n$, then f is in E_σ whenever $\sigma > 2c$. Such f's were used in the numerical experiments reported on in [8].

We also note that for band limited functions f in B_σ (10) gives an upper bound on the pointwise error which is proportional to $(e^{\epsilon_h\sigma}\lambda^{1/\delta})^\gamma$. Hence, for example, if δ and/or σ are sufficiently small so that $e^{\epsilon_h\sigma}\lambda^{1/\delta} < 1/100$, then this error bound can be decreased by a factor of 100 by simply increasing the value of γ by one.

2.3. Asymptotic Estimates as $\gamma \to \infty$

Suppose X is a closed subset of $\mathbf{R}^n$ and Ω is a cube which is invariant under dilation. Such a cube must necessarily have side $a = \infty$. For example, $\Omega = \mathbf{R}^n$ or $\Omega = \{x = (x_1, \ldots, x_n) : x_i \geq 0,\ i = 1, \ldots, n\}$. In this case, if

$$\delta = \sup_{y\in\Omega}\left(\inf_{x\in X}|x-y|\right)$$

is finite, then X cannot be a finite subset of $\mathbf{R}^n$ and s_γ cannot be defined via a finite number of coefficients which are solutions to a finite number of linear equations. Nevertheless, if f is in C_{h_γ}, with $h_\gamma(x) = -\sqrt{\gamma^2 + |x|^2}$, then $s_\gamma(x)$ can be uniquely defined as the element of minimal C_{h_γ} norm which coincides with f on X. If X is discrete, then s_γ is still of form (1) but where the sum is taken over the infinite set X. See [5] for more details. Note that the conclusions of Theorem 1 are valid in this context with $1 \leq \gamma < \infty$.

Now suppose f is in B_σ and consider estimate (10) in this context. If

$$\kappa = -\frac{\log\lambda}{\delta},$$

then $\kappa > 0$ and (10) can be re-expressed as

$$|f(x) - s_\sigma(f, x)| \leq C(e^{\epsilon_h\sigma-\kappa})^\gamma\,\|f\|_{L^2(\mathbf{R}^n)}, \tag{13}$$

whenever x is in Ω. Hence, it should be clear that if $\sigma < \kappa/\epsilon_h$, then the deviation $|f(x) - s_\gamma(f, x)|$ tends to zero as γ tends to infinity.

These considerations allow us to conclude that s_γ, $\gamma \to \infty$, can be used as a summability method in the recovery of a certain class of band limited functions which are sampled on the set X. This class of band limited functions contains B_σ whenever $\sigma < \kappa/\epsilon_h$ but estimate (13) is too crude to identify this class any more closely.

We summarize these results in the theorem below. First we remind the reader of the basic setup.

(i) Ω is a cube which is invariant under dilation, e.g., $\Omega = \mathbf{R}^n$ or $\Omega = \{x = (x_1, \ldots, x_n) : x_i > 0,\ i = 1, \ldots, n\}$.
(ii) X is a closed subset of $\mathbf{R}^n$.
(iii) $\delta = \sup_{y\in\Omega}(\inf_{x\in X}|x-y|)$ is finite.
(iv) If f is in C_{h_γ} with $h_\gamma(x) = -\sqrt{\gamma^2 + |x|^2}$, then s_γ is the minimum C_{h_γ} norm interpolator of f on X. For example, if X is discrete, namely $X = \{x_j\}_{j=1}^{\infty}$, then s_γ is of form (1) where the sum is taken over x_j in X and $s_\gamma(f, x_j) = f(x_j)$ for all x_j in X. Any function f in $L^2(\mathbf{R}^n)$ which is band limited is also in C_{h_γ}.

THEOREM 2. *Suppose we have the setup described by items (i)–(iv) listed above. Then there is a class of band limited functions f such that*

$$\lim_{\gamma\to\infty} s_\gamma(x) = f(x)$$

uniformly on Ω. If

$$0 < \sigma < -\frac{\log \lambda}{\epsilon_h \delta},$$

where ϵ_h and λ are the constants in inequality (10), then this class contains B_σ; indeed, there are positive constants γ_0 and ρ, $0 < \rho < 1$, which are independent of f such that for all x in Ω

$$|f(x) - s_\gamma(x)| \le \rho^\gamma \, \|f\|_{L^2(\mathbf{R}^n)},$$

whenever f is in B_σ and $\gamma > \gamma_0$.

2.3.1. The Special Case When X is a Lattice

If X is a lattice in $\mathbf{R}^n$, then by mimicking the development in [10] one can easily arrive at a convenient formula for $s_\gamma(f, x)$. For example, if $X = a\mathbf{Z}^n$, where $a > 0$ and $\mathbf{Z}^n$ is the integer lattice, the Fourier transform of s_γ is given by

$$\hat{s}_\gamma(f, \xi) = \frac{\hat{f}_a(\xi)}{\hat{h}_{\gamma,a}(\xi)} \, \hat{h}_\gamma(\xi), \tag{14}$$

where $h_\gamma(x) = -\sqrt{\gamma^2 - |x|^2}$, $\hat{f}$ and $\hat{h}_\gamma$ are the Fourier transforms of f and h_γ respectively, and $\hat{f}_a$ and $\hat{h}_{\gamma,a}$ are defined by

$$\hat{f}_a(\xi) = \sum_{j \in \mathbf{Z}^n} \hat{f}\left(\xi - \frac{2\pi j}{a}\right)$$

and

$$\hat{h}_{\gamma,a}(\xi) = \sum_{j \in \mathbf{Z}^n} \hat{h}_\gamma\left(\xi - \frac{2\pi j}{a}\right).$$

If

$$\frac{\hat{f}_a(\xi)}{\hat{h}_{\gamma,a}(\xi)} = \sum_{j \in \mathbf{Z}^n} c_j \, e^{-ia(j,\xi)},$$

where the coefficients c_j decay sufficiently rapidly, then

$$s_\gamma(f, x) = \sum_{j \in \mathbf{Z}^n} c_j \sqrt{\gamma^2 + |x - aj|^2}.$$

We remind the reader that the Fourier transform of $-\sqrt{\gamma^2 + |x|^2}$ is a distribution which is equal to a positive infinitely differentiable function on $\mathbf{R}^n \setminus \{0\}$, decays exponentially at infinity, and behaves like $O(|\xi|^{-n-1})$ at the origin. Thus, if we take

$$\frac{\hat{h}_\gamma(2\pi j/a)}{\hat{h}_{\gamma,a}(2\pi j/a)} = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{otherwise,} \end{cases}$$

then s_γ is well defined via (14) whenever $\hat{f}$ is a compactly supported function in $L^2(\mathbf{R}^n)$. Indeed in this case, we may write

$$s_\gamma(f, x) = \sum_{j \in \mathbf{Z}^n} f(aj) \, L_{\gamma,a}(x - aj), \tag{15}$$

where

$$\hat{L}_{\gamma,a}(\xi) = \frac{\hat{h}_\gamma(\xi)}{\hat{h}_{\gamma,a}(\xi)}.$$

That the $s_\gamma(f, x)$ given by (15) is the minimum C_{h_γ} norm interpolator of f on aZ^n follows from arguments which are analogous to those used in [11].

If we re-express the Fourier transform of $L_{\gamma,a}$ as

$$\hat{L}_{\gamma,a}(\xi) = \left\{ 1 + \sum_{j\neq 0} \frac{\hat{h}_1(\gamma(\xi - 2\pi j/a))}{\hat{h}_1(\gamma\xi)} \right\}^{-1},$$

then, as in [10], from the behavior of the of the ratios in the above identity we may conclude that

$$\lim_{\gamma\to\infty} \hat{L}_{\gamma,a}(\xi) = \chi_a(\xi) \tag{16}$$

in $L^p(\mathbf{R}^n)$ where $1 \le p < \infty$ and $\chi_a(\xi)$ is the characteristic function of the cube

$$Q_{\pi/a} = \{\xi = (\xi_1, \ldots, \xi_n) \, : \, |\xi_i| \le \frac{\pi}{a} \qquad \text{for } i = 1, \ldots, n\}.$$

We summarize these observations as follows:

THEOREM 3. *Suppose f is in $L^2(\mathbf{R}^n)$, $\hat{f}$ is supported in $Q_{\pi/a}$, and $X = aZ^n$, $a > 0$. Then, $s_\gamma(f,x)$ is given by (15) and*

$$\lim_{\gamma\to\infty} s_\gamma(f,x) = f(x) \tag{17}$$

in $L^2(\mathbf{R}^n)$ and uniformly on R^n.

It should be mentioned that relation (17) holds uniformly on compacta for a somewhat more general class of distributions f whose Fourier transforms are supported in $Q_{\pi/a}$. Unfortunately, the decay properties of $L_{\gamma,a}$ at infinity do not permit the level of generality which is valid for similar recovery formulas in terms of polyharmonic splines, see [12]. Details of this and related material will appear elsewhere.

Theorems 2 and 3 seem to hint that the interpolator $s_\gamma(f,x)$ may be useful in prediction and sampling applications. See [13] for a survey of this material.

2.4. Estimates on Derivatives

In this subsection, we record a result which indicates that the derivatives of multiquadric interpolators approximate the corresponding derivatives of the interpolatees. We will use the following setup:

(i) γ is a fixed positive number and f is any element of $\mathcal{C}_{h_\gamma}$.

(ii) X is a closed subset of $\mathbf{R}^n$ such that

$$\delta = \sup_{y\in\mathbf{R}^n} \left(\inf_{x\in X} |x - y| \right)$$

is finite. A typical example of such an X is a scaled integer lattice, $X = a\mathbf{Z}^n$ where $a = \delta/\sqrt{2}$.

(iii) s_γ is the minimum $\mathcal{C}_{h_\gamma}$ norm interpolator of f on X.

In other words, s_γ is the unique element in $\mathcal{C}_{h_\gamma}$ which satisfies

$$s_\gamma(x) = f(x) \qquad \text{for all } x \text{ in } X,$$

and

$$\|s_\gamma\|_{\mathcal{C}_{h_\gamma}} = \min\left\{ \|g\|_{\mathcal{C}_{h_\gamma}} : g = f \text{ on } X \right\}.$$

If X is a lattice, then there is a convenient formula for s_γ, see the previous subsection.

THEOREM 4. *Suppose we have the setup described by items (i)–(iii) listed above. Then for any multi-index α and parameters $r > r_0$ and $2 \le p \le \infty$,*

$$\|D^\alpha f - D^\alpha s_\gamma\|_{L^p(\mathbf{R}^n)} \le C\,\delta^r\, \|f\|_{\mathcal{C}_{h_\gamma}},$$

where C is a constant independent of δ and f. The parameter r_0 depends only on α.

Before going into details we remind the reader that for a continuous function g on $\mathbb{R}^n$

$$\|g\|_{L^p(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} |g(x)|^p \, dx \right\}^{1/p},$$

if $0 < p < \infty$ and if $p = \infty$

$$\|g\|_{L^\infty(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |g(x)|.$$

Also, recall that if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and if g is a smooth function, then

$$D^\alpha g(x) = \frac{\partial^{\alpha_1} \cdots \partial^{\alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} g(x).$$

We also bring attention to the fact that the lower bound r_0 on r is needed only in the case $\delta > 1$. If $\delta < 1$ then, for obvious reasons, no lower bound is needed on r.

PROOF. The theorem is an easy consequence of the following four elementary facts:

(a) If g is a smooth function and $|\alpha| < k$, then for any positive ϵ and any p, $1 \le p \le \infty$,

$$\|D^\alpha g\|_{L^p(\mathbb{R}^n)} \le C \left\{ \epsilon^{-|\alpha|} \|g\|_{L^p(\mathbb{R}^n)} + \epsilon^{k-|\alpha|} \sum_{|\beta|=k} \|D^\beta g\|_{L^p(\mathbb{R}^n)} \right\},$$

where C is a constant independent of ϵ and g.

(b) If $2 \le p \le \infty$ and the integer k satisfies $k > n(\frac{1}{2} - \frac{1}{p})$, then

$$\|D^\alpha g\|_{L^p(\mathbb{R}^n)} \le C \left\{ \|D^\alpha g\|_{L^2(\mathbb{R}^n)} + \sum_{|\beta|=k} \|D^{\alpha+\beta} g\|_{L^2(\mathbb{R}^n)} \right\},$$

where C is a constant independent of g and α.

(c) If g is in C_{h_γ} and the multi-index α satisfies $2|\alpha| > n + 1$, then

$$\|D^\alpha g\|_{L^2(\mathbb{R}^n)} \le C \|g\|_{C_{h_\gamma}},$$

where C is a constant independent of g.

(d) If k is an integer which satisfies $k > n/p$ and g is a smooth function such that $g(x) = 0$ for all x in X, then

$$\|g\|_{L^p(\mathbb{R}^n)} \le C \delta^k \sum_{|\alpha|=k} \|D^\alpha g\|_{L^p(\mathbb{R}^n)},$$

where C is a constant independent of g and δ.

To see how these items imply the desired result, apply items (d), (b), and (c), in that order, to $g = f - s_\gamma$ to get

$$\|f - s_\gamma\|_{L^p(\mathbb{R}^n)} \le C \delta^m \|f - s_\gamma\|_{C_{h_\gamma}}.$$

Since s_γ is the minimum C_h norm interpolator of f,

$$\|f - s_\gamma\|_{C_{h_\gamma}} \le \|f\|_{C_{h_\gamma}}. \tag{18}$$

The last two inequalities imply that for any integer m, $m \ge m_0$,

$$\|f - s_\gamma\|_{L^p(\mathbb{R}^n)} \le C\delta^m \|f\|_{C_{h_\gamma}}, \tag{19}$$

where C is a constant independent of f and δ. Now, by virtue of item (a), for sufficiently large k, we may write

$$\|D^\alpha f - D^\alpha s_\gamma\|_{L^p(\mathbb{R}^n)} \le C\left\{\epsilon^{-|\alpha|}\|f - s_\gamma\|_{L^p(\mathbb{R}^n)} + \epsilon^{k-|\alpha|}\sum_{|\beta|=k}\|D^\beta(f - s_\gamma)\|_{L^p(\mathbb{R}^n)}\right\}.$$

Use inequality (18) and items (b) and (c) to get the estimate

$$\|D^\alpha(f - s_\gamma)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{C_{k,\gamma}}.$$

The last two inequalities together with (19) imply that

$$\|D^\alpha f - D^\alpha s_\gamma\| \le C\left\{\epsilon^{-|\alpha|}\delta^m + \epsilon^{k-|\alpha|}\right\}\|f\|_{C_k}.$$

Choosing $m = k$ and $\epsilon = \delta$ gives us the desired result.

To complete the proof, we must verify items (a)–(d). Items (a) and (b) are essentially folklore; variants may be found [14]. Item (c) is an easy consequence of (6) and the fact that $|\xi|^{2|\alpha|}\hat{h}(\xi)$ is a bounded function of ξ whenever $2|\alpha| \ge n+1$. Item (d) is essentially the contents of Theorem 1 in [15]. ∎

3. GENERAL REMARKS

3.1. Derivatives

Error bounds analogous to those given by Theorems 1 and 2 are also valid for derivatives, $D^\alpha f(x) - D^\alpha s_\gamma(f, x)$. They can be established by using the full generality of inequality (4.15) in [9] to obtain a derivative variant of inequality (5) in Subsection 2.2.

Note that the bounds given in Theorem 4 are global but require a stronger hypothesis. Although the asymptotic estimate is not quite as good as that given in Theorem 1 the proof, however, is considerably simpler.

Elementary results such as that given by Theorem 4 and its proof were essentially the motivation for the more involved estimates given in [5,16,17].

3.2. Generalizations

In order to maintain clarity and simplicity, the results in Section 2 were stated and explicitly proven for multiquadric interpolators. However, the mindful reader should recognize that appropriate variants of these results hold for a much wider subclass of interpolators considered in [5].

For example, suppose we have the following setup:

(i) The function h is continuous and conditionally positive definite of order m, $m \ge 0$. In addition to this, suppose h satisfies
  - $\hat{h}$ coincides with a continuous positive function on $\mathbb{R}^n\backslash\{0\}$;
  - for all sufficiently large k,

$$\int_{\mathbb{R}^n} |\xi|^k\,\hat{h}(\xi)\,d\xi \le \rho^k k^k, \tag{20}$$

  where ρ is a fixed positive constant;
  - for some positive constants ϵ_h and C,

$$\left(\hat{h}(\xi)\right)^{-1} \le C\,e^{2\epsilon_h|\xi|}. \tag{21}$$

(ii) For $\gamma > 0$ the function h_γ is defined via

$$h_\gamma(x) = h(\gamma^{-1}x).$$

Observe that h_γ is also continuous and conditionally positive definite of order m.

(iii) X is a closed subset of $\mathbf{R}^n$ and Ω_a and δ are as in (iii) and (iv) of Subsection 2.2.
(iv) If f is in C_{h_γ} then $s_\gamma(f,x)$ is the minimal C_{h_γ} norm interpolator of f on X.
(v) Note that the classes of functions B_σ and E_σ, $\sigma > 0$, defined via (7) and (9) are contained in each C_{h_γ}, $\gamma > 0$.

THEOREM 5. *The conclusions of Theorem 1 are valid if we have the setup described by items (i)–(v) listed above. If Ω_a is taken to be a cube which is invariant under dilation, then the conclusions of Theorem 2 are also valid with this setup.*

Examples of functions h which enjoy the properties listed in item (i) include

$$h(x) = \frac{\Gamma(a/2)}{(1+|x|^2)^{a/2}},$$

where a is any real number $\neq 0, -2, -4, \dots$ and Γ denotes the classical Gamma function. Note that for positive a, the constant factor $\Gamma(a/2)$ can essentially be ignored but for negative a it adjusts the sign of h so that it is conditionally positive definite of order $m > -a/2$. The celebrated multiquadric is the case $a = -1$. Particularly appealing is the case $a = n+1$, since in this instance $\hat{h}$ is a constant multiple of $e^{-|\xi|}$.

It should be clear that different variants of Theorem 5 can be proven by merely changing the nature of the bounds (20) and (21). For example, the important case

$$h(x) = e^{-|x|^2}$$

fails to satisfy (21); indeed, if $h_\gamma(x) = h(x/\gamma)$ then E_σ fails to be contained in C_{h_γ} whenever $\gamma^2 > 4/\sigma$. Nevertheless, B_σ is contained in C_{h_γ} for all $\gamma > 0$ and appropriate variants of estimate (10) and Theorem 2 hold in this case also.

The point is this: *Exponential error bounds are valid for a wide class of interpolators and various classes of interpolatees. The precise nature of these bounds is determined by conditions which are variants of (20) and (21) and the nature of the function being interpolated.* Under these conditions various analogs of the theorems in Section 2 can be verified by using appropriate mutations of the corresponding arguments found there. We also stress that although the examples of h considered here are all radial this restriction is not part of the hypothesis of Theorem 5.

In the case when X is a lattice the class of band limited functions in $L^2(\mathbf{R}^n)$ for which analogs of Theorem 3 hold can be readily identified more precisely. Indeed if h is a radial function, such that its Fourier transform is a rapidly decreasing function of the radius, then the class consists of exactly those functions f in $L^2(\mathbf{R}^n)$ whose Fourier transform vanishes outside a cube of side length π/a centered at the origin. Indeed, arguments similar to those used here and in [10] can be used to show that the Lagrange function $L_{\gamma,a}(x)$ converges to $\mathrm{sinc}(x/a)$, see also [12]. Other types of h's will reconstruct other classes of band limited functions. We also mention that the lattice need not be a scaled version of the integer lattice, appropriate analogs hold for any lattice in $\mathbf{R}^n$. Details of this and related material will appear elsewhere.

3.3. Error Bounds for Continuous Functions

The hypotheses in the theorems of Section 2 essentially require f to be a real analytic function. What happens if we simply use the natural assumption that f is merely continuous? Certainly if X is discrete then, under certain mild assumptions on f, the interpolator $s(f,x)$, in terms of translates of h and, if necessary, low degree polynomials, of f on X can be computed by solving the appropriate system of linear equations. In view of classical approximation theory, it is reasonable to suspect estimates of the form

$$|f(x) - s(f,x)| \le C\delta^\alpha,$$

whenever f satisfies a Lipschitz (Hölder) condition of order α, $\alpha > 0$.

Unfortunately, in the general case such a bound is not easy to establish. Without the assumption that f is in C_h, it is difficult to obtain an estimate of the interpolator, $|s(f,x)|$, in terms of the data, $f(y)$, $y \in X$.

On the other hand, when X is a lattice there is an explicit formula for $s(f,x)$ in terms of the data, for example see formulas (14) and (15) in Subsection 2.3. In this case it is not difficult to establish the suspected estimates. We are currently preparing the details of this and related material for publication.

4. NUMERICAL COMPUTATIONS

4.1. *Comparison of h's*

Interpolators which are essentially linear combinations of translates of one function h are very natural and computationally attractive. The well known univariate piecewise polynomial splines are a classical example. As indicated in the introduction, the example $h_\gamma(x) = -\sqrt{\gamma^2 + |x|^2}$, known also as the multiquadric, is currently quite popular. On the other hand, the material presented here does not distinguish the multiquadric from other members of the class of h's considered in Subsection 3.2, such as for example $h_\gamma(x) = (\gamma^2 + |x|^2)^{-(n+1)/2}$. I am not aware of any work which does. Indeed, comparing the behavior of these two examples at infinity, the second seems more appealing.

One objective of the numerical experiments reported on below was to make some sort of quantitative comparison between several choices of h. These choices were the following:

(i) the multiquadric or MQ

$$h_\gamma(x) = -\sqrt{\gamma^2 + |x|^2},$$

(ii) the reciprocal multiquadric or RMQ

$$h_\gamma(x) = \frac{1}{\sqrt{\gamma^2 + |x|^2}},$$

(iii) the Poissonian or P

$$h_\gamma(x) = \frac{1}{(\gamma^2 + |x|^2)^{(n+1)/2}},$$

(iv) the Gaussian or G

$$h_\gamma(x) = e^{-|x|^2/\gamma^2}.$$

A more detailed description of these experiments can be found in Subsection 4.3.

4.2. *Dependence on γ and δ*

Another objective of these numerical experiments was to quantitatively display the behavior of these interpolators as functions of the parameters γ and δ. In particular, Theorems 1 and 5 predict that a small increase in the parameter γ should result in a dramatic improvement in the error for sufficiently small δ. I wanted to see specific numerical examples of this.

This brings us to one unpleasant feature of these interpolators which must be mentioned. The functions h_γ essentially tend to a constant as the parameter γ gets large. As a result, the matrices which must be inverted in order to evaluate the coefficients c_j's, see (1), become very poorly conditioned for large values of γ. Theoretically the matrices are invertible for all positive values of γ but numerically for large γ they are essentially singular. In the case of scattered data interpolation, I am not aware of any efficient schemes which overcome this unpleasantness. See the remarks concerning this matter in Section 5.

We do remind the reader that in the cases where X is a lattice, formula (15), together with the fast Fourier transform, leads to efficient evaluation of the interpolators for any positive value of γ.

4.3. Calculations I

The function f defined by

$$f(x) = \frac{\sin \pi x}{\pi x} \tag{22}$$

is the canonical example of a member of B_π in the case $n = 1$. The tables displayed below summarize the results of numerical experiments involving this interpolatee. The point of these experiments was to study the behavior of the interpolators mentioned in Subsection 4.1 versus each other and as functions of γ. For simplicity they were done in the univariate case, $n = 1$. The setup was as follows:

Five sets of knots labeled $\delta = 2, 1, 0.5, 0.25$, and 0.125 were generated. The j^{th} member of the set labeled δ was randomly chosen in the subinterval $[(j-1)\delta, j\delta]$, $j = 1, \dots, N$ where $N = 8/\delta$, of the interval [0,8]. More specifically the knots labeled δ were given by

$$x_j = (j-1)\delta + \delta X_j, \qquad j = 1, \dots, N,$$

where the X_j's were independent pseudo-random variables uniformly distributed on [0,1]. The X_j's were generated using the canned MATLAB subroutine **rand** starting with the **seed**=0. Namely, starting with **seed**=0 the values $X_1, \dots, X_4$ were generated for the case $\delta = 2$ via the MATLAB command **rand (1,4)**, then using the current **seed** the values $X_1, \dots, X_8$ were generated for the case $\delta = 1$ via the MATLAB command **rand (1,8)**, etc. These knots were then used to generate five sets of data via (22). Namely, the data was given by

$$y_j = \frac{\sin \pi x_j}{\pi x_j}, \qquad j = 1, \dots, N.$$

The corresponding interpolators generated by the h_γ's designated as MQ, RMQ, P, and G in Subsection 4.1 were computed by solving the appropriate system of equations for the coefficients c_j. More specifically, in the case of MQ the system of equations

$$c_0 + \sum_{j=1}^{N} c_j \sqrt{\gamma^2 + |x_i - x_j|^2} = y_i, \qquad i = 1, \dots, N$$

$$c_1 + \dots + c_N = 0$$

was solved. In the other three cases, the system

$$\sum_{j=1}^{N} c_j \, h_\gamma(x_i - x_j) = y_i, \qquad i = 1, \dots, N$$

was solved with the appropriate h_γ. The four choices $\gamma = 1$, 2, 3, and 4 were used. The canned MATLAB linear equation solver was used to solve these systems; this algorithm is essentially a variant of Gaussian elimination.

The interpolators were evaluated at the 801 points $x = 0, 0.01, 0.02, \dots, 8.00$. We will refer to this set as Q below and use $|Q| = 801$. In the case of MQ, the interpolator was given by the formula

$$s_\gamma(x) = c_0 + \sum_{j=1}^{N} c_j \sqrt{\gamma^2 + |x - x_j|^2},$$

in the cases RMQ, P, and G it was given by

$$s_\gamma(x) = \sum_{j=1}^{N} c_j \, h_\gamma(x),$$

where h_γ was the appropriate function described in Subsection 4.1.

Several types of meaningful errors were computed. Specifically computed were

$$\max_{x\in Q}\{|s_\gamma(x)-f(x)|\},$$

and the average error

$$\frac{1}{|Q|}\sum_{x\in Q}|s_\gamma(x)-f(x)|,$$

which we call the L^∞ and L^1 deviations respectively. For completeness, the root-mean-square error

$$\sqrt{\frac{1}{|Q|}\sum_{x\in Q}|s_\gamma(x)-f(x)|^2}$$

was also computed; it is called the L^2 deviation.

The results are summarized in Tables 1, 2, and 3. The numbers labeled with an asterisk (*) are somewhat unreliable since the matrices corresponding to the linear systems for c_j's had a condition number greater than 10^{17}. The numbers labeled with a double asterisk (**) were obviously grossly distorted because of this phenomenon. Figures 1 through 5 contain selected plots of these interpolators with the interpolatee in the "background" and the set of knots clearly displayed.

Table 1. L^∞ deviations.

		$\delta=2$	$\delta=1$	$\delta=0.5$	$\delta=0.25$	$\delta=0.125$
	$\gamma=1$	0.5613	0.4238	0.0494	0.0064	$0.174\cdot10^{-3}$
MQ	$\gamma=2$	0.5489	0.2814	0.0194	$0.728\cdot10^{-4}$	$0.472\cdot10^{-5}$*
	$\gamma=3$	0.5374	1.1931	0.0106	$0.479\cdot10^{-4}$*	$0.402\cdot10^{-5}$*
	$\gamma=4$	0.5276	2.3103	0.0106	$0.87\cdot10^{-4}$*	$0.785\cdot^{-5}$ **
	$\gamma=1$	0.5766	0.7718	0.0287	0.0054	$0.315\cdot10^{-3}$
RMQ	$\gamma=2$	0.5846	0.2747	0.0276	$0.596\cdot10^{-3}$	$0.103\cdot10^{-5}$*
	$\gamma=3$	0.5695	0.4715	0.0115	$0.816\cdot10^{-4}$*	$0.159\cdot10^{-5}$*
	$\gamma=4$	0.5547	1.3991	0.0078	$0.545\cdot10^{-4}$*	$0.125\cdot10^{-4}$ **
	$\gamma=1$	0.5962	0.8553	0.0184	0.0025	$0.273\cdot10^{-3}$
P	$\gamma=2$	0.5961	0.4107	0.0283	$0.906\cdot10^{-3}$	$0.139\cdot^{-5}$*
	$\gamma=3$	0.5828	0.2316	0.0110	$0.759\cdot10^{-4}$	$0.116\cdot10^{-5}$*
	$\gamma=4$	0.5670	1.0757	0.0058	$0.170\cdot10^{-4}$*	$0.101\cdot10^{-5}$*
	$\gamma=1$	0.5551	0.7717	0.0102	$0.659\cdot10^{-4}$	$0.165\cdot10^{-6}$*
G	$\gamma=2$	0.6126	0.8793	0.0039	$0.115\cdot10^{-5}$*	$0.947\cdot10^{-7}$*
	$\gamma=3$	0.5778	4.0237	0.0186*	$0.148\cdot10^{-3}$*	$0.820\cdot10^{-4}$ **
	$\gamma=4$	0.5469	7.0282	0.2989*	0.0452**	0.0251**

We bring the reader's attention to the following:

- With the particular normalizations described and used here the matrices corresponding to G were the "first" to become seriously ill-conditioned. Those corresponding to P were "last."
- Although the values labeled by the asterisks are unreliable, if anything, they are larger than the error would be if the interpolators were calculated accurately. Looking at Table 2 with this in mind, it is interesting to compare the numbers in the rectangles (δ, MQ) with the corresponding numbers in the rectangles (δ, G) particularly in the range $\delta = 0.5$, 0.25, and 0.125.

We leave it to the reader to draw his own impressions and possible conclusions from these numerics.

Table 2. L^1 deviations.

		$\delta=2$	$\delta=1$	$\delta=0.5$	$\delta=0.25$	$\delta=0.125$
MQ	$\gamma=1$	0.1252	0.0540	0.0035	0.0002	$0.001\cdot10^{-3}$
	$\gamma=2$	0.1326	0.0369	0.0017	$0.263\cdot10^{-5}$	$0.037\cdot10^{-6}*$
	$\gamma=3$	0.1325	0.0857	0.0004	$0.231\cdot10^{-5}*$	$0.058\cdot10^{-6}*$
	$\gamma=4$	0.1299	0.1509	0.0007	$0.494\cdot10^{-5}*$	$0.754\cdot10^{-5}**$
RMQ	$\gamma=1$	0.1215	0.0752	0.0022	0.0002	$0.015\cdot10^{-4}$
	$\gamma=2$	0.1327	0.0473	0.0021	$0.220\cdot10^{-4}$	$0.007\cdot10^{-6}*$
	$\gamma=3$	0.1407	0.0505	0.0006	$0.246\cdot10^{-5}*$	$0.035\cdot10^{-6}*$
	$\gamma=4$	0.1401	0.1030	0.0005	$0.240\cdot10^{-5}*$	$0.124\cdot10^{-5}**$
P	$\gamma=1$	0.1183	0.0803	0.0019	0.0001	$0.015\cdot10^{-4}$
	$\gamma=2$	0.1302	0.0540	0.0021	$0.319\cdot10^{-4}$	$0.010\cdot^{-6}*$
	$\gamma=3$	0.1423	0.0412	0.0009	$0.213\cdot10^{-5}$	$0.015\cdot10^{-6}*$
	$\gamma=4$	0.1440	0.0866	0.0004	$0.067\cdot10^{-5}*$	$0.047\cdot10^{-6}*$
G	$\gamma=1$	0.1186	0.0705	0.0008	$0.221\cdot10^{-5}$	$0.015\cdot10^{-7}*$
	$\gamma=2$	0.1509	0.0920	0.0001	$0.056\cdot10^{-6}*$	$0.158\cdot10^{-7}*$
	$\gamma=3$	0.1548	0.2669	0.0003*	$0.385\cdot10^{-4}*$	$0.748\cdot10^{-5}**$
	$\gamma=4$	0.1421	0.4150	0.0052*	0.0085**	0.0047**

Table 3. L^2 deviations.

		$\delta=2$	$\delta=1$	$\delta=0.5$	$\delta=0.25$	$\delta=0.125$
MQ	$\gamma=1$	0.1915	0.1067	0.0100	0.0010	$0.087\cdot10^{-4}$
	$\gamma=2$	0.1926	0.0511	0.0041	$0.108\cdot10^{-4}$	$0.243\cdot10^{-6}*$
	$\gamma=3$	0.1904	0.1876	0.0013	$0.109\cdot10^{-4}*$	$0.233\cdot10^{-6}*$
	$\gamma=4$	0.1869	0.3645	0.0021	$0.136\cdot10^{-4}*$	$0.102\cdot10^{-5}**$
RMQ	$\gamma=1$	0.1951	0.1719	0.0060	0.0008	$0.0153\cdot10^{-4}$
	$\gamma=2$	0.2011	0.0763	0.0056	$0.887\cdot10^{-4}$	$0.059\cdot10^{-6}*$
	$\gamma=3$	0.2021	0.0813	0.0015	$0.114\cdot10^{-4}*$	$0.088\cdot10^{-6}*$
	$\gamma=4$	0.1991	0.2258	0.0015	$0.841\cdot10^{-5}*$	$0.166\cdot10^{-5}**$
P	$\gamma=1$	0.1890	0.1891	0.0039	0.0004	$0.035\cdot10^{-4}$
	$\gamma=2$	0.2034	0.1013	0.0058	$0.134\cdot10^{-3}$	$0.077\cdot^{-6}*$
	$\gamma=3$	0.2059	0.0553	0.0021	$0.106\cdot10^{-4}$	$0.073\cdot10^{-6}*$
	$\gamma=4$	0.2041	0.1766	0.0012	$0.259\cdot10^{-5}*$	$0.078\cdot10^{-6}*$
G	$\gamma=1$	0.1910	0.1656	0.0021	$0.944\cdot10^{-5}$	$0.090\cdot10^{-7}*$
	$\gamma=2$	0.2178	0.1649	0.0004	$0.165\cdot10^{-6}*$	$0.204\cdot10^{-7}*$
	$\gamma=3$	0.2151	0.6444	0.0017*	$0.484\cdot10^{-4}*$	$0.102\cdot10^{-4}**$
	$\gamma=4$	0.1999	1.0826	0.0267*	0.0111**	0.0061**

4.4. *Calculations II*

To illustrate the ease with which these interpolants can be computed in the case when the set of knots X is a lattice, we calculated the Lagrange function $L_{\gamma,1}$ for the Poissonian P when X is the integer lattice. Recall that in the n variate case the Poissonian is $h_\gamma(x) = (\gamma^2+|x|^2)^{-(n+1)/2}$ whose Fourier transform is a constant multiple of $\exp(-\gamma|\xi|)$. Thus, we evaluated $L_{\gamma,1}(x)$ by computing the inverse Fourier transform of

$$\hat{L}_{\gamma,1}(\xi) = \exp(-\gamma|\xi|)\left\{\sum_{j\in\mathbb{Z}^n}\exp(-\gamma|\xi-2\pi j|)\right\}^{-1}$$

via the fast Fourier transform algorithm. Figure 6 contains the graphs of $y = L_{\gamma,1}(x)$ in the case

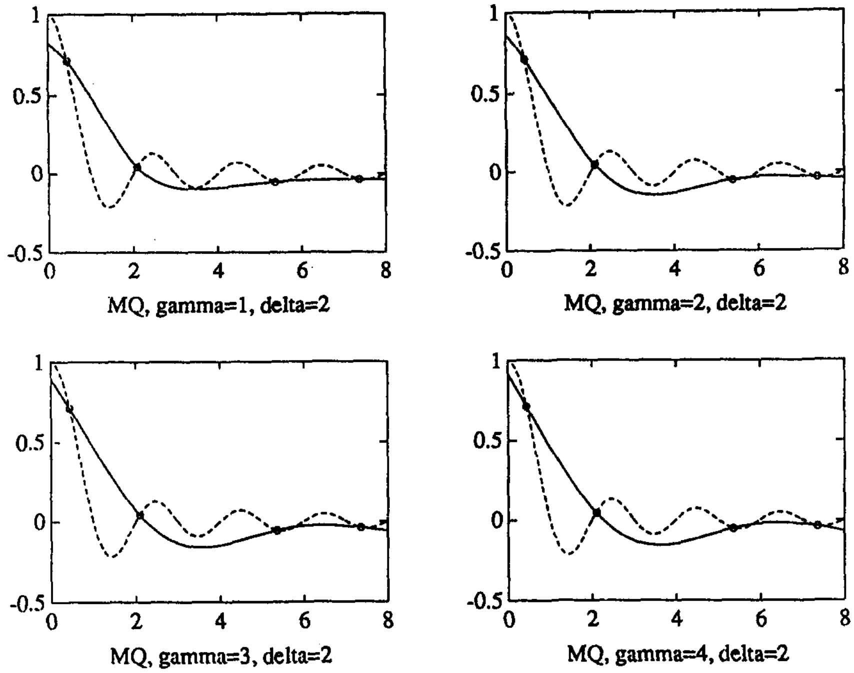

Figure 1. Multiquadric interpolators with knots labeled $\delta = 2$ and parameters $\gamma = 1, 2, 3$ and 4.

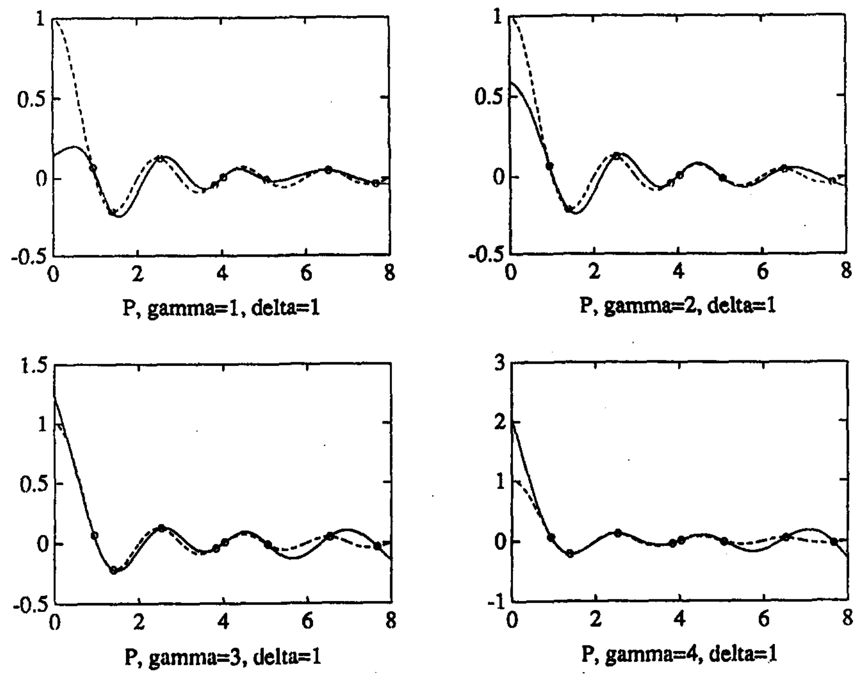

Figure 2. Poissonian interpolators with knots labeled $\delta = 1$ and parameters $\gamma = 1, 2, 3$ and 4. Note the auto-scaling on the y axis.

$n = 1$ and $\gamma = 1, 4, 16$, and 64. Figure 7 contains the surface $z = L_{\gamma,1}(x)$ in the case $n = 2$ and $\gamma = 2$.

5. CONCLUDING REMARKS

One goal of this paper was to present and prove *a priori* error bounds for a class of interpolation problems associated with conditionally positive definite functions. Another objective, perhaps more important than the first, was to stimulate further study of these methods.

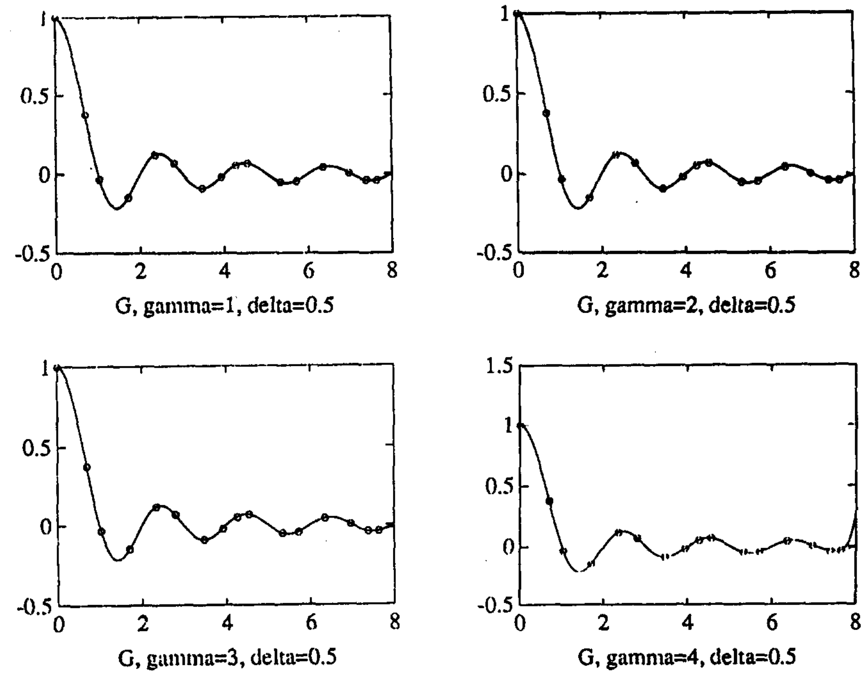

Figure 3. Gaussian interpolators with knots labeled $\delta = 0.5$ and parameters $\gamma = 1$, 2, 3 and 4. Note the auto-scaling on the y axis.

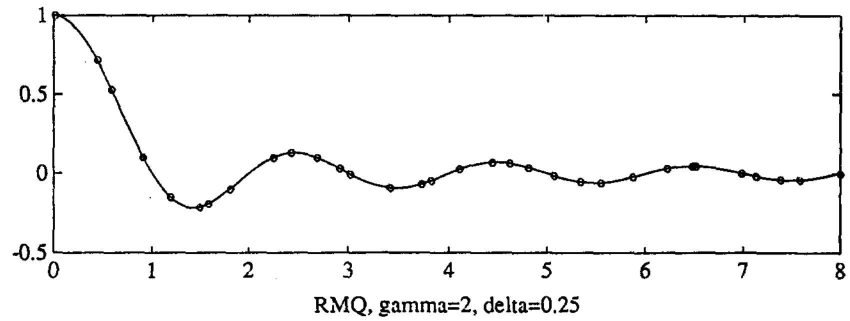

Figure 4. Reciprocal multiquadric interpolator with knots labeled $\delta = 0.25$ and parameter $\gamma = 2$.

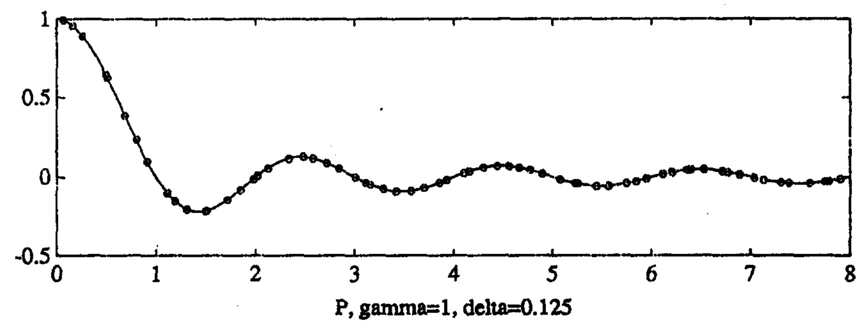

Figure 5. Poissonian interpolator with knots labeled $\delta = 0.125$ and $\gamma = 1$.

For example, the error bounds show that these methods have excellent approximation theoretic properties. On the other hand, the ill-conditionedness of the interpolation matrix, $(a_{ij}) = (h(x_i - x_j))$, poses a significant obstacle to certain potential scattered data applications. Motivated by the case when the set of knots, X, is a lattice it seems reasonable to suspect that a solution to

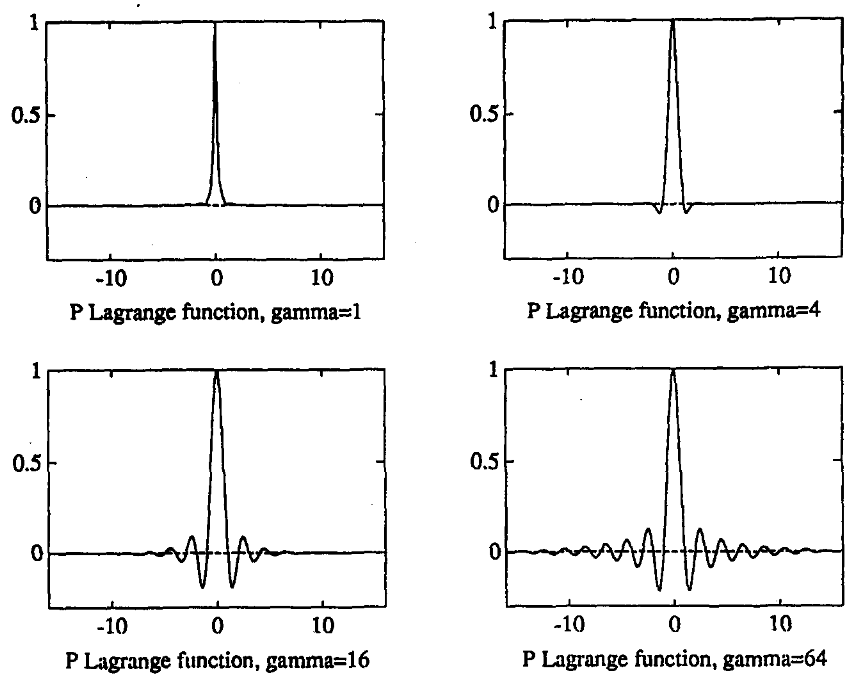

Figure 6. Graphs of $y = L_{\gamma,1}(x, y)$ for the Poissonian in the cases $n = 1$ and $\gamma = 1, 4, 16, 64$.

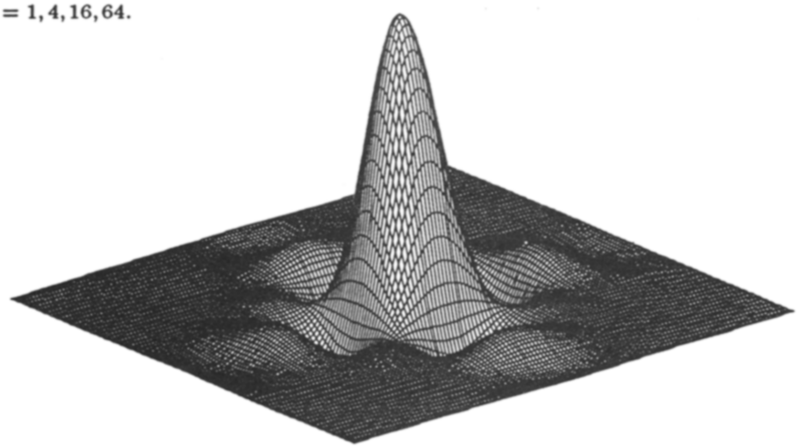

Figure 7. Plot of surface $z = L_{\gamma,1}(x, y)$ for the Poissonian in the case $n = 2$ and $\gamma = 16$. Here $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$.

this difficulty may lay in considering bases for the space of interpolators which are of the form

$$B_k(x) = \sum_{j=1}^{N} b_{kj}\, h(x - x_j), \qquad k = 1, \ldots, N,$$

or, if necessary,

$$B_k(x) = p_k(x) + \sum_{j=1}^{N} b_{kj}\, h(x - x_j), \qquad k = 1, \ldots, N,$$

where the p_k's are appropriately chosen polynomials and $\{x_1, \ldots, x_N\}$ is the set of knots. The basis functions B_k should be easy to evaluate and the interpolation matrix, $(a_{ij}) = (B_i(x_j))$ should be numerically easy to invert, the ideal case being of course the identity matrix. We point out that direct application of difference schemes may not be appropriate in cases where h is relatively 'flat.' Also, simply requiring the B_i's to decay at infinity may not be sufficient as can be seen from the numerical examples involving the Gaussian and Poissonian. Of course, it may be that some completely new ideas are needed to deal with this.

I gratefully acknowledge the generosity and warm hospitality of Mike Powell and his group, mainly Martin Buhmann and Brad Baxter, during a visit to Cambridge in January 1991. They brought [7] and similar work involving the parameter γ to my attention during numerous friendly discussions which were conducted at that time, indeed, Theorem 3 presented here was conceived during these conversations.

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