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Note

A closure concept in factor-critical graphs

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Abstract

A graph G is called n-factor-critical if the removal of every set of n vertices results in a graph with a 1-factor. We prove the following theorem: Let G be a graph and let x be a locally n-connected vertex. Let $\{u,v\}$ be a pair of vertices in $V(G)-\{x\}$ such that $uv \notin E(G)$, $x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G(u) \cup N_G(v) \cup \{u,v\}$. Then G is n-factor-critical if and only if $G+uv$ is n-factor-critical.

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1. Introduction

We consider only finite simple graphs and follow Chartrand and Lesniak [5] for general terminology and notation. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a subset A of $V(G)$, $G[A]$ denotes the subgraph of G induced by A and $G-A$ is the subgraph of G induced by $V(G)-A$. We often identify $G[A]$ with A. Further, if F is a subgraph of G, we may write simply $G[F]$ instead of $G[V(F)]$ and $G-F$ instead of $G-V(F)$.

For a vertex $v \in V(G)$, $N_G(v)$ denotes the *neighbourhood* of v in G and let $\deg_G(v) = |N_G(v)|$ denote the *degree* of v. Further, let $N_G[v]$ denote $N_G(v) \cup \{v\}$. If $G[N_G(v)]$ is k-connected, then v is called *locally k-connected*. A locally connected vertex v is said to be *eligible* if $N_G(v)$ induces a noncomplete graph. The *local completion* of G at v is the operation of replacing the induced subgraph $G[N_G(v)]$ by the complete graph $K_{|N_G(v)|}$. A graph G is said to be *claw-free* if G contains no induced subgraph isomorphic to $K_{1,3}$. *Ryjáček n-closure* $C^n_{\mathrm{R}}(G)$ is a graph obtained from a claw-free graph G by

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iteratively performing local completion at eligible locally n-connected vertices until no more edges can be added.

A graph G of order p is *k-factor-critical* (*k-fc* in brief), where k is an integer of the same parity as p with $0 \leqslant k \leqslant p$, if $G - X$ has a perfect matching for any set X of k vertices of G. In particular, G is 0-factor-critical if and only if G has a perfect matching.

Ryjáček 1-closure (or simply Ryjáček closure) was introduced in the study of the existence of hamiltonian cycles in claw-free graphs.

Theorem A (Ryjáček [8]). *Let G be a claw-free graph. Then G is hamiltonian if and only if $C_{\mathrm{R}}^{1}(G)$ is hamiltonian.*

The concept of Ryjáček closure is a condition on the neighbourhood structure of a subgraph of G. Similarly, Broersma [2], and Broersma and Schiermeyer [3] (and [4], etc.) gave other closure concepts in terms of neighbourhood conditions on four vertices.

Theorem B (Broersma [2]). *Let $\{u, v, x, y\}$ be a subset of four vertices of a graph G such that $uv \notin E(G), xy \in E(G)$, and $\{x, y\} \subset N_G(u) \cap N_G(v)$. If $N_G(x) \cup N_G(y) \subset N_G[u] \cup N_G[v]$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Theorem C (Broersma and Schiermeyer [3]). *Let $\{u, v, x, y\}$ be a subset of four vertices of a graph G such that $uv \notin E(G), x \in N_G(u) \cap N_G(v)$, and $y \in N_G(u)$. If $N_G(y) \subset N_G(v) \cup \{u\}$ and $N_G(x) \subset N_G[y] \cup \{v\}$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Graphs satisfying the conditions of Theorem B contain claw-free graphs. Because if $\{u, v, x, y\}$ satisfies $uv \notin E(G), xy \in E(G)$, and $\{x, y\} \subset N_G(u) \cap N_G(v)$, and if there exists a vertex $w \in N_G(x) \cup N_G(y) - (N_G[u] \cup N_G[v])$, then either $\{x, u, v, w\}$ or $\{y, u, v, w\}$ must induce a claw.

On the other hand, Plummer and Saito [7] proved the following theorems by using local completion.

Theorem D. *Let G be a claw-free graph and let x be a locally n-connected eligible vertex. Let G' be the graph obtained from G by local completion at x in G. Then G is n-factor-critical if and only if G' is n-factor-critical.*

Corollary E. *Let G be a claw-free graph. Then G is n-factor-critical if and only if $C_{\mathrm{R}}^{n}(G)$ is n-factor-critical.*

Our purpose in this note is to extend Theorem D by using a neighbourhood condition. We will prove the following theorem.

Theorem 1. *Let G be a graph and let x be a locally n-connected vertex. Let $\{u, v\}$ be a pair of vertices in $V(G) - \{x\}$ such that $uv \notin E(G)$, $x \in N_G(u) \cap N_G(v)$, and*

$N_G(x) \subset N_G[u] \cup N_G[v]$. Then G is n-factor-critical if and only if $G + uv$ is n-factor-critical.

By the observation after Theorem C, the hypothesis of Theorem 1 is clearly weaker than that of Theorem D. That is, if $\{u, v, x\}$ satisfies $uv \notin E(G)$ and $x \in N_G(u) \cap N_G(v)$, and if there exists a vertex $w \in N_G(x) - (N_G[u] \cup N_G[v])$, then $\{x, u, v, w\}$ induces a claw. Further, let G be a claw-free graph and x an eligible vertex. Also suppose $\{u, v\} \subset N_G(x)$ with $uv \notin E(G)$. Then the induced subgraph $H = G[N_G[x]]$ is clearly claw free. Therefore, by the observation as in the above, we have $N_H(x) \subset N_H[u] \cup N_H[v]$. Even when we add the edge uv to H, this procedure does not have an influence upon this inclusion relation. This implies that the local completion G' of G at a vertex x can be obtained by iteratively joining a pair $\{u, v\} \subset N_G(x)$ satisfying the conditions of Theorem 1.

Based on Theorem 1, if a graph H can be obtained from a graph G by iteratively joining all pairs $\{u, v\}$ satisfying the conditions $uv \notin E(G)$, $x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$ for some vertex (resp. locally n-connected vertex) x, and if H contains no such pair, then H is called a closure (resp. an n-closure) of G and denoted by $\mathrm{cl}(G)$ (resp. $\mathrm{cl}_n(G)$). Note that $\mathrm{cl}(G)$ and $\mathrm{cl}_n(G)$ can be different for each positive integer n.

From Theorem 1, we have the following.

Corollary 2. *Let G be a graph. Then G is n-factor-critical if and only if $\mathrm{cl}_n(G)$ is n-factor-critical.*

Ryjáček [8] and Bollobás et al. [1] proved that if a graph G is claw-free, then $C_{\mathrm{R}}^k(G)$ is uniquely determined for each integer k. However, in general, our closure is not determined uniquely. For our closure, there exist graphs G which have different closures. We present such a graph here. Let $W_6 = K_1 \oplus C_6$ be a wheel, where '$\oplus$' denotes the join, $K_1 = \{u\}$ is a complete graph, and $C_6 = v_1 v_2 \ldots v_6$ is a cycle. We set $G = (W_6 - uv_4) + v_3 v_5$. Then we can recursively join the pair $\{v_2 v_6\}$, $\{v_1 v_3\}$, $\{v_1 v_5\}$, $\{v_1 v_4\}$, $\{v_2 v_5\}$, $\{v_3 v_6\}$, $\{v_4 u\}$, $\{v_2 v_4\}$, and $\{v_4 v_6\}$. Then we have $\mathrm{cl}_1(G) = K_7$. On the other hand, we can recursively join the pair $\{v_1 v_3\}$, $\{v_1 v_5\}$, $\{v_1 v_4\}$, $\{v_2 v_5\}$, $\{v_3 v_6\}$, and $\{v_4 u\}$. Now we cannot join further pairs. Then we have $K_7 - \{v_2 v_6, v_2 v_4, v_4 v_6\}$ as a different closure $\mathrm{cl}_1(G)$.

As some variations of Theorem 1, we have the following theorems that are similar to Theorems B and C.

Theorem 3. *Let $\{u, v, x, y\}$ be a subset of four vertices of a graph G such that x is locally n-connected, $uv \notin E(G)$, and $\{x, y\} \subset N_G(u) \cap N_G(v)$. If $N_G(x) \subset N_G[u] \cup N_G[v]$ or $N_G(y) \subset N_G[u] \cup N_G[v]$, then G is n-factor-critical if and only if $G + uv$ is n-factor-critical.*

Theorem 4. *Let $\{u, v, x, y\}$ be a subset of four vertices of a graph G such that $x \in N_G(u) \cap N_G(v)$ is locally n-connected, $uv \notin E(G)$, and $y \in N_G(u) \cup N_G(v)$. If $N_G(y) \subset N_G[u] \cup N_G[v]$ and $N_G(x) \subset N_G[y] \cup \{u, v\}$, then G is n-factor-critical if and only if $G + uv$ is n-factor-critical.*

2. Proofs of theorems

We use the following two lemmas in the proofs of theorems.

Lemma F (Favaron [6]). *A graph G is n-factor-critical if and only if* $\mathrm{o}(G-B) \leqslant |B|-n$ *for every* $B \subset V(G)$ *with* $|B| \geqslant n$, *where* $\mathrm{o}(G)$ *denotes the number of odd components of* G.

Lemma G (Plummer and Saito [7]). *Let G be a graph and H, a spanning subgraph of G. If H is n-factor-critical, then G is n-factor-critical.*

Proof of Theorem 1. Let G, x, u, and v be as in the statement of the theorem. By Lemma G, necessity is obvious. We prove sufficiency by contradiction.

Suppose $G+uv$ is n-fc but G is not n-fc. Then, by Lemma F, there exists a subset $B \subset V(G)$ with $|B| \geqslant n$ such that $\mathrm{o}(G-B) > |B| - n \geqslant \mathrm{o}[(G+uv)-B]$. Notice that $|V(G)| \equiv n \pmod 2$ since $G+uv$ is n-fc. Since $\mathrm{o}(G-B)+|B| \equiv |V(G)| \pmod 2$ and $\mathrm{o}[(G+uv)-B] \geqslant \mathrm{o}(G-B)-2$, we have $\mathrm{o}(G-B)-2 = \mathrm{o}[(G+uv)-B] = |B|-n$. Therefore, we may assume $u \in C_1$ and $v \in C_2$, where C_1 and C_2 are odd components of $G-B$. Now let $C_3, \ldots, C_{|B|-n+2}$ be the other odd components of $G-B$. These components are also odd components of $(G+uv)-B$. Now since $E_G(C_1, C_2) = \emptyset$ and $x \in N_G(u) \cap N_G(v)$, we may assume $x \in B$.

Case 1: $|B| = n$. In this case, two vertices u and $v \in N_G(x)$ are separated by $N_G(x) \cap B$ in $G[N_G(x)]$. Since $|N_G(x) \cap B| < n$, this contradicts the assumption that x is locally n-connected.

Case 2: $|B| > n$. We can take a vertex subset $S \subset B - \{x\}$ with $|S| = n$. Since $G+uv$ is n-fc, $(G+uv)-S$ has a perfect matching so that every vertex of $B-S$ is matched with a vertex of distinct components $C_3, \ldots, C_{|B|-n+2}$. In particular, we may assume x is matched with a vertex w of C_3. However, since $N_G(x) \subset N_G[u] \cup N_G[v]$, w is adjacent to u or v in G. This is impossible since $E_G(C_1 \cup C_2, C_3) = \emptyset$, which completes the proof. □

The condition of being locally n-connected cannot be deleted from the hypotheses of Theorem 1. As in [7], we let $G = K_n \oplus (C_1 \cup C_2)$, where '$\oplus$' denotes the join and $C_1 = C_2 = K_{2l+1}$. Then since $G - K_n$ consists of two odd components C_1 and C_2, G is not n-fc. Suppose $x \in K_n$, $u \in C_1$, and $v \in C_2$. Then x is locally $(n-1)$-connected and $\{x, u, v\}$ satisfies the conditions of Theorem 1; that is, $uv \notin E(G)$, $x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$. Now it is easy to check that $G+uv$ is n-fc.

Since our proofs of Theorems 3 and 4 are the almost same as that of Theorem 1, we only present an outline.

Proof of Theorems 3 and 4. Let x, y, u, v be vertices satisfying the conditions of Theorem 3 (or 4). By Lemma G, it suffices to prove that if $G+uv$ is n-fc, then G is n-fc. Suppose that $G+uv$ is n-fc, but G is not n-fc. Then, there exists a vertex subset B with $|B| \geqslant n$ such that $\mathrm{o}(G-B) = \mathrm{o}((G+uv)-B)+2 = |B|-n+2$. Let $C_1, \ldots, C_{|B|-n+2}$ be the odd components of $G-B$. Without loss of generality, $u \in C_1$ and $v \in C_2$. Further, since $x \in N_G(u) \cap N_G(v)$, we may assume $x \in B$. Note that $C_3, \ldots, C_{|B|-n+2}$ are also odd components of $(G+uv)-B$. By the same argument as in the proof of Theorem 1, we may assume that $|B|>n$.

In Theorem 3, if $N_G(x) \subset N_G[u] \cup N_G[v]$, then we are done by Theorem 1. Therefore, we may assume that $N_G(y) \subset N_G[u] \cup N_G[v]$ and $N_G(x) \not\subset N_G[u] \cup N_G[v]$. Since $y \in N_G(u) \cap N_G(v)$, y is in B. Since $|B|>n$, we can take a subset $S \subset B - \{y\}$ with $|S|=n$. Then y must be matched with a vertex of $\bigcup_{i=3}^{|B|-n+2} C_i$. Because $(G+uv)-S$ has a 1-factor. This contradicts the assumption $N_G(y) \subset N_G[u] \cup N_G[v]$.

In Theorem 4, since $y \in N_G(u) \cup N_G(v)$, y is in B or $C_1 \cup C_2$. If y is in B, then by the argument similar to that in last half of the previous proof, we have a contradiction. Therefore, y is in $C_1 \cup C_2$. However, since $|B|>n$, we can take a subset $S \subset B - \{x\}$ with $|S|=n$. Then since $(G+uv)-S$ has a 1-factor, x must be matched with a vertex of $\bigcup_{i=3}^{|B|-n+2} C_i$, which contradicts the assumption $N_G(x) \subset N_G[y] \cup \{u,v\}$. □

One might conjecture that the result like Theorem 1 holds for the *factor-extendability*, that is, the following statement holds.

Let G be a graph and let $\{u,v,x\}$ be a subset of $V(G)$ such that $x \in N_G(u) \cap N_G(v)$ is locally $2n$-connected, $uv \notin E(G)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$. Then G is n-extendable if and only if $G+uv$ is n-extendable.

Here, G is said to be *k-extendable* (*k-ext* in brief) if every matching of size k in G can be extended to a perfect matching.

The factor-criticality and the extendability actually have many similar results. For example, in [7], Plummer and Saito also proved the following theorem on extendability that is similar to Theorem D.

Theorem H. *Let G be a claw-free graph and let x be a locally $2n$-connected eligible vertex. Let G' be the graph obtained from G by local completion at x. If G' is n-extendable, then G is n-extendable.*

As a concluding remark, we show that there exists a non-n-ext graph G such that G satisfies the conditions of the statement, but $G+uv$ is n-ext.

Example. Let w,x,y,z be four vertices. We set $X=(n-1)K_2 \cup \{y,z\}$ and $Y=K_p \cup K_q \cup \{w\}$, where p,q are odd integers greater than n. And let $G=(\{x\} \oplus (X \oplus Y)) - xw$. Further, let u (resp. v) be a vertex of K_p (resp. K_q). Then G satisfies that $x \in N_G(u) \cap N_G(v)$ is locally $2n$-connected, $uv \notin E(G)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$. And we can check that $G+uv$ is n-ext. On the other hand, if we can take a matching $M=\{xz\} \cup E((n-1)K_2)$, then we have $o((G-V(M))-\{y\})=3>1=|\{y\}|$, where $V(M)$ denotes the set of endvertices of edges in M. Therefore, G is not n-ext.

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