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Existence results for some fourth-order nonlinear elliptic problems of local superlinearity and sublinearity ☆

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Abstract

In this paper we study the existence of positive solutions for the problem

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u) & \text{in } \Omega, \\ u \geqslant 0, \quad u \not\equiv 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $c < \lambda_1(\Omega)$ and $f(x,u)$ satisfies the local superlinearity and sublinearity condition.

Keywords: Variational method; Biharmonic operator; Superlinear; Sublinear; Positive solutions; Suspension bridge

1. Introduction

This paper is concerned with the study of existence of positive solutions for the problem

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u) & \text{in } \Omega, \\ u \geqslant 0, \quad u \not\equiv 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Δ^2 denotes the biharmonic operator, $c \in R$, Ω is a bounded domain in R^N and $f: \Omega \times R^+ \to R$ is a Carathéodory function.

These fourth-order problems have been studied by many authors. In [1] there is a survey of results obtained in this direction. At the same time, in [1] it has been point out that this type of nonlinearity can furnish a model to study travelling waves in suspension bridges.

There are many results about (1.1) when $c < \lambda_1(\Omega)$ (where $(\lambda_k(\Omega))_{k \geqslant 1}$ is the sequence of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$) and $f(x,u) = b[(u+1)^+ - 1]$. In [2] Tarantello found a negative solution when $b \geqslant \lambda_1(\Omega) * (\lambda_1(\Omega) - c)$ by a degree argument. Lazer and McKenna in [3] proved the existence of $2k-1$ solutions when $\Omega \subset R$ is an interval and $b > \lambda_k(\Omega) * (\lambda_k(\Omega) - c)$ by the global bifurcation method. Micheletti and Pistoia in [4] proved that there exist two solutions when $b > \lambda_1(\Omega) * (\lambda_1(\Omega) - c)$ and three solutions when b is close to $\lambda_k(\Omega) * (\lambda_k(\Omega) - c)$ for a more general nonlinearity g by variational method.

In this paper we use the following terminologies. Problem (1.1) is said to be sublinear (superlinear) at 0 if there exist $\alpha > \lambda_1(\Omega) * (\lambda_1(\Omega) - c)$ and $s_0 > 0$ such that

$$f(x,s) \geqslant (\leqslant)\, \alpha s \quad \text{for a.e. } x \in \Omega \text{ and all } 0 \leqslant s \leqslant s_0; \tag{1.2}$$

here $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$. Problem (1.1) is said to be superlinear (or sublinear) at ∞ if there exist $\beta > \lambda_1(\Omega) * (\lambda_1(\Omega) - c)$ and $s_1 > 0$ such that

$$f(x,s) \geqslant (\leqslant)\, \beta s \quad \text{for a.e. } x \in \Omega \text{ and all } s \geqslant s_1. \tag{1.3}$$

Zhang in [5] proved the existence of weak solutions when $f(x,u)$ is sublinear at ∞ by variational method.

It is the purpose to study the problem (1.1) when f satisfies the local superlinearity and sublinearity by variational methods. Our method is similar to De Figueiredo's method in [6].

A good example to which our results apply is

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda a(x) u^q + b(x) u^p & \text{in } \Omega, \\ u \geqslant 0, \quad u \not\equiv 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a parameter and the exponents p and q satisfy $0 \leqslant q < 1 < p$ with $p \leqslant 2^* - 1$ if $N \geqslant 3$, $p < +\infty$ if $N = 1$ or 2. Here $2^* = 2N/(N-2)$.

2. Main results

We will always assume $N \geqslant 3$, $c < \lambda_1(\Omega)$, denote by σ' the Hölder conjugate of σ, by $\lambda_1(\Omega)$ the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ and by $\lambda_1(\Omega_1)$ the first eigenvalue of $-\Delta$ on $H_0^1(\Omega_1)$.

Given a bounded domain $\Omega \subset R^N$, let V denote the Hilbert space $H^2(\Omega) \cap H_0^1(\Omega)$. Denote the $L^p(\Omega)$ norm by $\|u\|_p$, the norm on $H_0^1(\Omega)$ is given by $\|u\|_{0,1} = (\int_\Omega |\nabla u|^2\, dx)^{1/2}$, the norm on V is given by

$$\|u\|_V = \left(\int_\Omega \left[(\Delta u)^2 - c|\nabla u|^2 \right] dx \right)^{1/2}.$$

Note that $\|u\|_V^2 \geqslant (\lambda_1(\Omega) - c)\|u\|_{0,1}^2$ for all $u \in V$, that is, the norm $\|u\|_V$ is stronger than the norm $\|u\|_{0,1}$.

Let $f : \Omega \times R^+ \to R$ be a Carathéodory function and satisfy the following conditions:

(H1) $f(x,0) \geqslant 0$ for a.e. $x \in \Omega$;

(H2) There exist $1 \leqslant \sigma < 2^*$, $d_1 \in L^{\sigma'}(\Omega)$, $d_2 > 0$ such that

$$\left|f(x,s)\right| \leqslant d_1(x) + d_2|s|^{\sigma-1}$$

for a.e. $x \in \Omega$ and all $s > 0$;

(H3) There exist $\Theta > 2$, $1 \leqslant r < 2$, $d \in L^{(2^*/r)'}(\Omega)$, with $d \geqslant 0$ a.e. in Ω, $s_0 \geqslant 0$, such that

$$\Theta F(x,s) \leqslant s f(x,s) + d(x)s^r$$

for a.e. $x \in \Omega$ and all $s > s_0$, where $F(x,s) = \int_0^s f(x,t)\,dt$;

(H4) There exist $0 \leqslant q < 1 < p < 2^* - 1$, $a_0 \in L^{\sigma_q}(\Omega)$, with $\sigma_q = (2^*/(q+1))'$ and $a_0 \geqslant 0$ a.e. in Ω, $b_0 \in L^{\sigma_p}(\Omega)$, with $\sigma_p = (2^*/(p+1))'$ and $b_0 \geqslant 0$ a.e. in Ω, such that

$$f(x,s) \leqslant a_0 s^q + b_0 s^p$$

for a.e. $x \in \Omega$ and all $s \geqslant 0$;

(H5) There exist a nonempty subdomain $\Omega_1 \subset \Omega$, $\Theta_1 > \lambda_1(\Omega_1) * (\lambda_1(\Omega_1) - c)$, $s_1 > 0$, such that

$$F(x,s) \geqslant \Theta_1 \frac{s^2}{2}$$

for a.e. $x \in \Omega_1$ and all $0 \leqslant s \leqslant s_1$;

(H6) There exist a nonempty open subset $\Omega_2 \subset \Omega$, $\Theta_2 > 0$, $s_2 \geqslant 0$, such that

$$F(x,s) \geqslant \Theta_2 s^2$$

for a.e. $x \in \Omega_2$ and all $s \geqslant s_2$, with the additional requirement that the function $d(x)$ appearing in (H3) is bounded on Ω_2.

There are some comments on the conditions (H2), (H3), (H4), (H5), (H6) in [6], so we omit them. Here we comment the hypothesis (H1) again. When assuming (H1), we will always understand that $f(x,s)$ has been extended for $s < 0$ by putting $f(x,s) = f(x,0)$ for a.e. $x \in \Omega$ and all $s < 0$. The extension technology had been used in [8,9] too. Doing so, any solution $u \in V$ of

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

is automatically $\geqslant 0$ in Ω.

Indeed, if u is the solution of Eq. (2.1), then $-u^-$ (where $u^- = \max\{-u, 0\}$) satisfies the system

$$\begin{cases} \Delta u^- + cu^- = -v & \text{in } \Omega, \\ \Delta v = f(x, -u^-) & \text{in } \Omega, \\ u^- = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Note that the function $v = 0$ on $\partial\Omega$. Hence, the weak maximum principle applied to the second equation of (2.2) show that

$$v(x) \leqslant 0 \quad \text{for } x \in \Omega.$$

Thus, the right-hand side of the first equation of (2.2) is positive in Ω. Because $u^- = 0$ on $\partial\Omega$ and $c < \lambda_1(\Omega)$, the weak maximum principle applied to the first equation of (2.2) in turn implies that $u^-(x) \leqslant 0$ for all $x \in \Omega$, but $u^-(x) \geqslant 0$ for all $x \in \Omega$, so $u^-(x) = 0$ for all $x \in \Omega$.

Solving (1.1) reduces to look for nontrivial solutions of (2.1) where f has been extended as above.

In addition (H5) and (H6) imply $a_0 \not\equiv 0$ and $b_0 \not\equiv 0$ in (H4).

The main results of this paper are the following theorems.

Theorem 2.1. *Under conditions* (H1)–(H6) *there exists* $\eta = \eta(p, q, N) > 0$ *such that problem* (1.1) *has at least two solutions for* a_0, b_0:

$$\|a_0\|_{\sigma_q}^{p-1} \|b_0\|_{\sigma_p}^{1-q} < \eta.$$

One of them, call it v*, satisfies* $\Phi(v) > 0$*, while the other, call it* w*, satisfies* $\Phi(w) < 0$*, where* Φ *denotes the associated energy*:

$$\Phi(u) := \frac{1}{2} \int_\Omega \left[(\Delta u)^2 - c|\nabla u|^2\right] dx - \int_\Omega F(x, u)\, dx.$$

In addition, if f *varies in such a way that the coefficients in* (H4) *satisfy*

$$a_0 \to 0 \quad \text{in } L^{\sigma_q}(\Omega) \quad \text{and} \quad b_0 \text{ is bounded in } L^{\sigma_p}(\Omega),$$

then the solution $w = w_f$ *can be constructed such that* $w_f \to 0$ *in* V.

Applying this theorem to Eq. (1.4), we have

Corollary 2.1. *Assume in* (1.4) $\lambda > 0$, $0 \leqslant q < 1 < p < 2^* - 1$, $a \in L^{\tau_q}(\Omega)$ *with* $\tau_q < \sigma_q$, $b \in L^{\tau_p}(\Omega)$ *with* $\tau_p < \sigma_p$*, and, in addition,* $a(x) \geqslant 0$ *a.e. in* Ω *in case* $q = 0$*. Suppose*

(I1) *there exists a nonempty open subset* $\Omega_1 \subset \Omega$ *such that, on* Ω_1, $a(x) \geqslant \epsilon_1$ *for some* $\epsilon_1 > 0$ *and* $b(x)$ *is bounded from below;*

(I2) *there exists a nonempty open subset* $\Omega_2 \subset \Omega$ *such that, on* Ω_2, $b(x) \geqslant \epsilon_2$ *for some* $\epsilon_2 > 0$ *and* $a(x)$ *is bounded from above and from below.*

Then there exists $\tilde{\eta} = \tilde{\eta}(p, q, N) > 0$ *such that if*

$$\lambda < \frac{\tilde{\eta}}{\|a^+\|_{\sigma_q} \|b^+\|_{\sigma_p}^{(1-q)(p-1)}},$$

then problem (1.4) *has at least two solutions* v *and* w *such that* $\Psi(v) > 0$ *and* $\Psi(w) < 0$*, where* Ψ *denotes the energy functional associated to* (1.4)*. Moreover, if* $\lambda \to 0$*, the solution* $w = w_f$ *can be constructed such that* $w_f \to 0$ *in* V.

3. Proofs

Lemma 3.1. *Assume* (H1)–(H3). *Then Φ satisfies the* (PS) *condition on V.*

Proof. As already observed in connection with (H1), f is extended on all $\Omega \times R$ by putting $f(x,s) = f(x,0)$ for a.e. $x \in \Omega$ and all $s < 0$. It is then clear that (H2) implies that Φ is a C^1 functional on V.

To see that Φ satisfies (PS) condition, first note that by (H2) the map $s \to f(\cdot,s)$ takes bounded set in $L^\sigma(\Omega)$ into bounded sets in $L^{\sigma/(\sigma-1)}(\Omega) \subset H^{-1}(\Omega)$. So, and since for $\sigma < 2^*$, by Rellich's theorem the space $H_0^1(\Omega)$ embeds into $L^\sigma(\Omega)$ compactly, the map $K: H_0^1(\Omega) \to H^{-1}(\Omega)$, given by $K(s) = f(\cdot,s)$, is compact. Now $D\Phi(s) = \Delta^2 s + c\Delta s - f(\cdot,s)$, and hence by Proposition 2.2.2 of [7] it suffices to show that any (PS)-sequence (u_n) for Φ is bounded in V.

Let (u_n) be a (PS) sequence, i.e., $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$. So, for Θ as in (H3) and for some $\varepsilon_n \to 0$ and some constant C,

$$\Theta\Phi(u_n) - \Phi'(u_n)u_n \leqslant C + \varepsilon_n \|u_n\|_V,$$

where $\|\cdot\|_V$ denotes the norm on V. So we have

$$\left(\frac{\Theta}{2} - 1\right)\|u_n\|_V^2 - \int_\Omega \big(\Theta F(x,u_n) - u_n f(x,u_n)\big) \leqslant C + \varepsilon_n \|u_n\|_V.$$

By (H3),

$$\left(\frac{\Theta}{2} - 1\right)\|u_n\|_V^2 \leqslant c' + \int_\Omega d(x)\big(u_n^+\big)^r\,dx + \varepsilon_n \|u_n\|_V,$$

where c' is another constant. Since $r < 2$, we deduces that $\|u_n\|_V$ remains bounded. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let* $0 \leqslant q < 1 < p$, $A > 0$, $B > 0$, *and consider the function*

$$\Psi_{A,B}(t) = t^2 - At^{q+1} - Bt^{p+1}$$

for $t \geqslant 0$. *Then* $\max\{\Psi_{A,B}(t):\ t \geqslant 0\} > 0$ *if and only if*

$$A^{p-1}B^{q-1} < \frac{(p-1)^{p-1}(1-q)^{1-q}}{(p-q)^{p-q}} := \eta_1(p,q).$$

Moreover, for

$$t = t_B := \left[\frac{1-q}{B(p-q)}\right]^{1/(p-1)},$$

one has

$$\Psi_{A,B}(t_B) = t_B^2\left[\frac{p-1}{p-q} - AB^{(1-q)/(p-1)}\left(\frac{p-q}{1-q}\right)^{(1-q)/(p-1)}\right]. \tag{3.1}$$

For the proof see Lemma 3.2 of [6].

Proof of Theorem 2.1. First, we will get the first solution by the classical mountain pass theorem. We observe that $\Phi(0)=0$. By (H4), Hölder inequality, and Sobolev inequality, we have

$$\begin{aligned}\Phi(u) &\geqslant \frac{1}{2}\|u\|_V^2 - \int_\Omega \left[\frac{a_0(u^+)^{q+1}}{q+1} + \frac{b_0(u^+)^{p+1}}{p+1}\right] dx \\ &\geqslant \frac{1}{2}\|u\|_V^2 - m_1\|a_0\|_{\sigma_q}\|u\|_{0,1}^{q+1} - m_2\|b_0\|_{\sigma_p}\|u\|_{0,1}^{p+1} \\ &\geqslant \frac{1}{2}\|u\|_V^2 - c_1\|a_0\|_{\sigma_q}\|u\|_V^{q+1} - c_2\|b_0\|_{\sigma_p}\|u\|_V^{p+1} \end{aligned}\tag{3.2}$$

for all $u \in V$, where $m_1 = (q+1)^{-1}S^{-(q+1)/2}$, $m_2 = (p+1)^{-1}S^{-(p+1)/2}$, $c_1 = m_1 \times (\lambda_1(\Omega)-c)^{-(q+1)/2}$, $c_2 = m_2(\lambda_1(\Omega)-c)^{-(p+1)/2}$, and

$$S := \inf\left\{\int_\Omega |\nabla u|^2\,dx\colon\ u \in H_0^1(\Omega) \text{ and } \int_\Omega |u|^{2^*}\,dx = 1\right\}.$$

We now apply to (3.2) Lemma 3.2 with $A = 2c_1\|a_0\|_{\sigma_q}$ and $B = 2c_2\|b_0\|_{\sigma_p}$. This gives that for all $u \in V$ with $\|u\|_V = t_B$, we have

$$\Phi(u) \geqslant \frac{1}{2}\Psi_{A,B}(t_B) > 0$$

provided that $A^{p-1}B^{q-1} < \eta_1(p,q)$, i.e., provided that

$$\|a_0\|_{\sigma_q}^{p-1}\|b_0\|_{\sigma_p}^{1-q} < \frac{\eta_1(p,q)}{(2c_1)^{p-1}(2c_2)^{1-q}} := \eta(p,q,N). \tag{3.3}$$

So we have obtained, under (3.3), a "range of mountains" around 0.

Now we will look for some $u_2 \in V$ such that $\Phi(tu_2) \to -\infty$ as $t \to +\infty$ by condition (H6). Let us choose s_3 sufficiently large so that $s_3 > s_0$ (from (H3)) and, for some $\Theta_3 > 0$, $F(x,s) \geqslant \Theta_3 s^2 + 1$ for a.e. $x \in \Omega_2$ and all $s \geqslant s_3$, which is clearly possible by (H6). For $x \in \Omega_2$ and $s \geqslant s_3$, we then divide the inequality of (H3) by $sF(x,s)$, integrate from s_3 to s and take the exponential to get

$$F(x,s) \geqslant F(x,s_3)\left(\frac{s}{s_3}\right)^\Theta \exp\left(-d(x)\int_{s_3}^{s} \frac{t^{r-1}}{F(x,t)}\,dt\right).$$

So, by (H6),

$$F(x,s) \geqslant cs^\Theta \tag{3.4}$$

for a.e. $x \in \Omega_2$ and all $s \geqslant s_3$, where $c > 0$ is a constant. We now take a function $u_2 \in V \cap H^\Theta(\Omega)$ with support in Ω_2 and $u_2 \geqslant 0$, $u_2 \not\equiv 0$, and consider tu_2 with $t \geqslant t_2$, where t_2 is such that measure of $\{x \in \Omega_2\colon\ tu_2(x) \geqslant s_3\}$ is > 0. We have

$$\Phi(tu_2) = \frac{t^2}{2}\|u_2\|_V^2 - \int_{\Omega_2} F(x,tu_2)\,dx$$

and, consequently, splitting the integral over Ω_2 into an integral over $\{x \in \Omega_2\colon\ tu_2(x) < s_3\}$ and an integral over $\{x \in \Omega_2\colon\ tu_2(x) \geqslant s_3\}$, and applying (3.4) to the latter, we get, for some constants c', c'' with $c'' > 0$,

$$\Phi(tu_2) \leqslant \frac{t^2}{2}\|u_2\|_V^2 + c' - ct^\Theta \int\limits_{\{x\in\Omega_2\colon\ tu_2(x)\geqslant s_3\}} (u_2)^\Theta\,dx \leqslant \frac{t^2}{2}\|u_2\|_V^2 + c' - c''t^\Theta.$$

Since $\Theta > 2$, this latter relation implies that $\Phi(tu_2) \to -\infty$ as $t \to +\infty$.

Under (3.3), we apply the mountain pass theorem, which yields a critical point v of Φ with

$$\Phi(v) \geqslant \frac{1}{2}\Psi_{A,B}(t_B) > 0. \tag{3.5}$$

This function v is a nontrivial solution of (2.2), so we get the first solution v.

Next, we will get the second solution by local minimization. We will show that there exists $u_1 \in V$ such that

$$\Phi(tu_1) < 0 \tag{3.6}$$

for all $t > 0$ sufficiently small by condition (H5). Take for u_1 the positive eigenfunction associated to the principle eigenvalue of $-\Delta$ on $H_0^1(\Omega_1)$. It is known that $u_1 \in L^\infty(\Omega_1)$. By (H5), we have, for $t > 0$ sufficiently small (in fact $t \leqslant s_1/\|u_1\|_\infty$, so $0 \leqslant tu_1 \leqslant s_1$),

$$\Phi(tu_1) = \frac{t^2}{2}\|u_1\|_V^2 - \int\limits_{\Omega_1} F(x,tu_1)\,dx \leqslant \frac{t^2}{2}\left(\|u_1\|_V^2 - \Theta_1\int\limits_{\Omega_1}(u_1)^2\,dx\right).$$

The term in parentheses is < 0 since $\Theta_1 > \lambda_1(\Omega_1) * (\lambda_1(\Omega_1) - c)$. So we have (3.6).

It follows from (3.6) that the minimum of the (weakly lower semicontinuous) functional Φ on the closed ball in V with center 0 and radius t_B is achieved in the corresponding open ball and thus yields a nontrivial solution w of (2.2) with

$$\Phi(w) < 0 \quad \text{and} \quad \|w\|_V < t_B. \tag{3.7}$$

So, we get the second solution.

This completes the proof of the existence of at least two solutions in Theorem 2.1.

We now turn to the study of the asymptotic behavior of one of these two solutions. When f varies in such a way that $a_0 \to 0$ in $L^{\sigma_q}(\Omega)$ and b_0 remains bounded in $L^{\sigma_p}(\Omega)$, we fix $\alpha \in]0, 1/(1-q)[$ and observe that for $t_B = \|a_0\|_{\sigma_q}^\alpha$ and for all u with $\|u\|_V = t_B$, $t_B \to 0$. By (3.2), we have

$$\begin{aligned}\Phi(u) &\geqslant \frac{1}{2}\|a_0\|_{\sigma_q}^{2\alpha} - c_1\|a_0\|_{\sigma_q}^{1+\alpha(q+1)} - c_2\|b_0\|_{\sigma_p}\|a_0\|_{\sigma_q}^{\alpha(p+1)} \\ &= \|a_0\|_{\sigma_q}^{2\alpha}\left(\frac{1}{2} - c_1\|a_0\|_{\sigma_q}^{1-\alpha(1-q)} - c_2\|b_0\|_{\sigma_p}\|a_0\|_{\sigma_q}^{\alpha(p-1)}\right).\end{aligned}$$

Since $1 - \alpha(1-q) > 0$, the expression is positive for $\|a_0\|_{\sigma_q}$ sufficiently small, so we have

$$\Phi(u) \geqslant \frac{1}{2}\Psi_{A,B}(t_B) > 0 \quad \text{for all } u \text{ with } \|u\|_V = \|a_0\|_{\sigma_q}^\alpha = t_B.$$

By (3.7), the corresponding solution w will converge to 0 in V. This complete the proof of Theorem 2.1. □

Proof of Corollary 2.1. It is easy to verify that for each $\lambda > 0$, conditions (H1)–(H6) of Theorem 2.1 hold. □

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