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Aluffi torsion-free ideals

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ABSTRACT

A special class of algebras which are intermediate between the symmetric and the Rees algebras of an ideal was introduced by P. Aluffi in 2004 to define characteristic cycle of a hypersurface parallel to conormal cycle in intersection theory. These algebras are recently investigated by A. Nasrollah Nejad and A. Simis who named them Aluffi algebras. For a pair of ideals $J \subseteq I$ of a commutative ring R, the Aluffi algebra of I/J is called Aluffi torsion-free if it is isomorphic to the Rees algebra of I/J. In this paper, ideals generated by 2-minors of a $2 \times n$ matrix of linear forms and also edge ideals of graphs are considered and some conditions are presented which are equivalent to Aluffi torsion-free property of them.

Introduction

In the remarkable paper [1], Paolo Aluffi introduced an intermediate graded algebra between a symmetric algebra and the Rees algebra which he called quasi-symmetric algebra. His purpose was to describe the characteristic cycle of a hypersurface, parallel to well-known conormal cycle in intersection theory. A. Nasrollah Nejad and A. Simis in [14] and then in [15] called such an algebra the *Aluffi algebra*. Given a commutative ring R and ideals $J \subset I \subset R$, the Aluffi algebra of I/J is defined by

$$\mathcal{A}_{R/J}(I/J) := \mathcal{S}_{R/J}(I/J) \otimes_{\mathcal{S}_R(I)} \mathcal{R}_R(I).$$

The Aluffi algebra is squeezed as $\mathcal{S}_{R/J}(I/J) \twoheadrightarrow \mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ and is moreover a residue ring of the ambient Rees algebra $\mathcal{R}_R(I)$. The kernel of the right-hand surjection is called the module of Valabrega–Valla as defined in [17] which is the torsion of the Aluffi algebra [15]. Thus, provided

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that I has a regular element modulo J, the Rees algebra of I/J is the Aluffi algebra modulo its torsion. The question which motivated this paper is: when is the surjection $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ an isomorphism. For importance of this question in commutative algebra and intersection theory, we call a pair of ideals $J \subset I$, Aluffi torsion-free if the surjection $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is injective.

Some important examples of Aluffi torsion-free pairs have been appeared explicitly in the following two results. The first one is due to Huneke [12] who assumes that I is an ideal whose extension $(I+J)/J$ on the quotient ring R/J is generated by a d-sequence. The second one is due to Herzog, Simis and Vasconcelos and is what they called “Artin–Rees lemma on the nose” [11]. They have considered that, both ideals I and I/J are of linear type over R and R/J, respectively. By the structure of the Aluffi algebra, it is shown in [15] that the assumption in the second result to the effect that I be of linear type over R does not intervene the result. Nasrollah Nejad and Simis in [15] give necessary and sufficient conditions for these algebras to be isomorphic in terms of I-standard basis of J and also relates this isomorphism with the relation type number of I/J over R/J and the Artin–Rees number of J relative to I.

In geometric settings, let $X \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} Z$ be closed embeddings of schemes with $J \subset I \subset R$ the ideal sheaves of Y and X in Z, respectively. Let $\widetilde{Z} = \mathrm{Proj}(\mathcal{R}_R(I)) \xrightarrow{\pi} Z$ be the blowup of Z along X and $\widetilde{Y} = \mathrm{Proj}(\mathcal{R}_{R/J}(I/J))$ be the blowup of Y along X. Note that $\widetilde{Y}$ embeds in $\widetilde{Z}$ as the strict transform of Y under $\widetilde{Z} \xrightarrow{\pi} Z$. Let $E = \pi^{-1}(X)$ be the exceptional divisor of the blowup. Then, E is a subscheme of $\pi^{-1}(Y)$. Let $\mathfrak{R} = \mathfrak{R}(E, \pi^{-1}(Y))$ be the residual scheme of E in $\pi^{-1}(Y)$. Here “residual” is taken in the sense of [7, Definition 9.2.1]. In terms of the ideal sheaves, $\mathfrak{R}$ is characterized by the equation $\mathcal{I}_{\mathfrak{R}}.\mathcal{I}_E = \mathcal{I}_{\pi^{-1}(Y)}$, where $\mathcal{I}_E$, $\mathcal{I}_{\pi^{-1}(Y)}$ are respectively the ideals of E and $\pi^{-1}(Y)$ in $\widetilde{Z}$. Aluffi in [1, Theorem 2.12] proved that $\mathrm{Proj}(\mathcal{A}_{R/J}(I/J)) = \mathfrak{R}(E, \pi^{-1}(Y))$. Fulton in [7, B. 6.10] shows that if i and j are regular embeddings, then $\mathfrak{R} = \widetilde{Y}$ which is equivalent to say that $J \cap I^n = JI^{n-1}$ for all sufficiently large n. S. Keel in [13, Theorem 1] shows that this result holds as long as $X \hookrightarrow Y$ is a linear embedding and $Y \hookrightarrow Z$ is a regular embedding. The goal of the present work is to find some examples of Aluffi torsion-free pairs which are in the main streams of research in commutative algebra and algebraic geometry. To this goal, two major examples are worked out. First we classify all Aluffi torsion-free ideals generated by 2-minors of a $2 \times n$ matrix of linear forms. Then, we characterize all Aluffi torsion-free edge ideals of simple graphs.

In Section 2, we consider J as an ideal generated by 2-minors of a $2 \times n$ matrix of linear forms and I stands for the Jacobian ideal of J. We prove that the pair $J \subseteq I$ is Aluffi torsion-free if and only if in the Kronecker–Weierstrass normal form of the matrix, there is no any Jordan block. More precisely, Theorem 2.3 asserts that these conditions are equivalent to say that $I_r(\Theta) = \mathfrak{m}^r$, where r is codimension of J, Θ stands for the Jacobian matrix of J and $\mathfrak{m}$ is the homogeneous maximal ideal of $k[\mathbf{X}] = k[x_1, \ldots, x_n]$. This motivates us to conjecture that, if $J \subset k[\mathbf{X}]$ is an ideal of codimension $r \geqslant 2$, generated by 2-forms, and if I denotes the ideal generated by r-minors of the Jacobian matrix Θ of J, then I is $\mathfrak{m}$-primary if and only if $I = \mathfrak{m}^r$ (Conjecture 2.6).

Section 3 is devoted to find conditions for edge ideal of a graph and its Jacobian ideal to be Aluffi torsion-free pair. In this regard, we give some necessary and sufficient conditions for graphs equivalent to the Aluffi torsion-free property. Finally, we present several examples of graphs which are Aluffi torsion-free or not.

Some of the results of this paper have been conjectured after explicit computations performed by the computer algebra systems Singular [9] and CoCoA [6].

1. Torsion-free Aluffi algebras

Let R be a commutative ring and I an ideal of R. The two most common and important commutative algebras related to the ideal I are the symmetric algebra $\mathcal{S}_R(I)$ and the Rees algebra $\mathcal{R}_R(I)$. Recall that these algebras are defined as

$$\mathcal{R}_R(I) := \bigoplus_{t \geqslant 0} I^t u^t \simeq R[Iu] \subset R[u], \qquad \mathcal{S}_R(I) := \bigoplus_{t \geqslant 0} \mathcal{S}_R^t(I),$$

where $\mathcal{S}_R^t(I) = T_R^t(I)/((x \otimes y - y \otimes x) \cap T_R^t(I))$ and $T_R^t(I)$ is the tensor algebra of order t. The definition of $\mathcal{R}_R(I)$ immediately implies that, it is torsion-free over the base ring R. A natural surjection of standard R-graded algebras arises from the definition:

$$\mathcal{S}_R(I) \twoheadrightarrow \mathcal{R}_R(I). \tag{1}$$

This map is injective locally on the primes $p \in \operatorname{spec}(R)$ such that $I \nsubseteq p$. It follows from the general arguments that, provided that I has some regular elements, the kernel is the R-torsion submodule (ideal) of the symmetric algebra. If the map in (1) is injective, one says that the ideal I is of linear type, a rather non-negligible notion in parts of syzygy theory of ideals.

Definition 1.1. (See [15].) Let R be Noetherian and $J \subset I$ be ideals of R. The Aluffi algebra of I/J is

$$\mathcal{A}_{R/J}(I/J) := \mathcal{S}_{R/J}(I/J) \otimes_{\mathcal{S}_R(I)} \mathcal{R}_R(I).$$

We have the following surjections:

$$\mathcal{S}_{R/J}(I/J) \twoheadrightarrow \mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J).$$

The kernel of the second surjection is the so-called module of Valabrega–Valla (see [17], also [18, 5.1]) which is

$$\mathcal{W}_{J\subset I} = \bigoplus_{t\geqslant 2} \frac{J \cap I^t}{JI^{t-1}}. \tag{2}$$

Of course, as an ideal, this kernel is generated by finitely many homogeneous elements, but as a graded R/J-module, it is conceivable that it may fail this property. By [15, Proposition 2.5] the Valabrega–Valla module gives the torsion of the Aluffi algebra.

Definition 1.2. A pair of ideals $J \subset I$ of a ring R is said to be Aluffi torsion-free if the map $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is injective.

Note that by [7, B. 6.10] and (2), a pair of ideals $J \subset I$ is Aluffi torsion-free if and only if $J \cap I^n = JI^{n-1}$ for all positive integers n.

Example 1.3. Let $a_1, \dots, a_r$ be a regular sequence in a Noetherian ring R and let $I = \langle a_1, \dots, a_r\rangle$. Then, for each $i = 1, \dots, r$, the pair $J = (a_1^n, \dots, a_i^n) \subset I^n$ is Aluffi torsion-free.

Lemma 1.4. *Let $R = k[\mathbf{X}]$ and $J \subset R$ be an ideal generated by forms of the same degree $d \geqslant 1$. Then, $J \cap \mathfrak{m}^{rt} \subset J\mathfrak{m}^{r(t-1)}$ for every $t \geqslant 0$ and $r \geqslant d$.*

Proof. Let $f_1, \dots, f_m$ be generators of J and let F be a form on f_i's such that $F \in \mathfrak{m}^{rt}$. Then $F = \sum_{i=1}^m g_i f_i$, where $g_i = \sum a_\alpha \mathbf{X}^\alpha \in R_{rt-d+\delta}$ for $\delta \geqslant 0$. Since $R_{rt-d+\delta} = R_{r-d+\delta}.R_{rt-r}$, we can rewrite g_i as

$$g_i = \sum_{\substack{|\alpha|=r-d+\delta \\ |\beta|=rt-r}} a_{\alpha,\beta} \mathbf{X}^{\alpha+\beta}, \quad \text{hence} \quad F = \sum_{|\alpha|=r-d+\delta} \mathbf{X}^\alpha \left(\sum_{\substack{i=1 \\ |\beta|=rt-r}}^{s} (\mathbf{X}^\beta) f_i \right).$$

Therefore, $F \in J\mathfrak{m}^{rt-r}$, as required. □

Let $R = k[\mathbf{X}]$ be the $\mathbb{N}$-graded polynomial ring over a field k, $J \subset R$ be a homogeneous ideal and $I \subset R$ be the Jacobian ideal of J, by which we always mean the ideal $(J, I_r(\Theta))$ where $r = \mathrm{ht}(J)$ and Θ stands for the Jacobian matrix of a set of generators of J. More precisely, if $J = (f_1, \ldots, f_s)$, then

$$\Theta = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_s}{\partial x_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_s}{\partial x_n} \end{bmatrix}.$$

Corollary 1.5. *With the above assumptions and notations, if $I_r(\Theta) = \mathfrak{m}^r$, then the pair $J \subseteq I$ is Aluffi torsion-free.*

Proof. Let t be a positive integer. Then, we have

$$\begin{aligned} J \cap I^t &= J \cap \big(J, I_r(\Theta)\big)^t = J \cap \big(J, \mathfrak{m}^r\big)^t \\ &= J \cap \big(J^t, J^{t-1}\mathfrak{m}^r, \ldots, J\mathfrak{m}^{r(t-1)}\big) + J \cap \mathfrak{m}^{rt} \\ &= J\big(J, \mathfrak{m}^r\big)^{t-1} + J \cap \mathfrak{m}^{rt} \subseteq JI^{t-1} + J \cap \mathfrak{m}^{rt}. \end{aligned}$$

Lemma 1.4 implies that $J \cap \mathfrak{m}^{rt} \subseteq J\mathfrak{m}^{r(t-1)} \subseteq JI^{t-1}$. □

2. Ideal of 2-minors of a 2 × *n* matrix of linear forms

We recall the Kronecker–Weierstrass normal form of a $2 \times n$ matrix of linear forms [8]. Assume that k is an algebraically closed field. Let S be the polynomial ring in variables x_{ij}, y_{ij}, z_{ij} over k. Let M be a $2 \times n$ matrix of linear forms of S. Then, M is conjugate to a matrix obtained by concatenation of certain blocks such as

$$[D_1 \mid \cdots \mid D_k \mid J_1 \mid \cdots \mid J_s \mid B_1 \mid \cdots \mid B_t], \tag{3}$$

where D_i is a "nilpotent block" of length $n_i + 1$:

$$D_i = \begin{bmatrix} x_{i1} & x_{i2} & \cdots & x_{in_i} & 0 \\ 0 & x_{i1} & \cdots & x_{i,n_i-1} & x_{in_i} \end{bmatrix},$$

J_i is a "Jordan block" of length m_i with eigenvalue $\lambda_i \in k$:

$$J_i = \begin{bmatrix} y_{i1} & y_{i2} & \cdots & y_{im_i} \\ \lambda_i y_{i1} & y_{i1} + \lambda_i y_{i2} & \cdots & y_{i,m_i-1} + \lambda_i y_{im_i} \end{bmatrix},$$

and B_i is a "scroll block" of length l_i:

$$B_i = \begin{bmatrix} z_{i1} & z_{i2} & \cdots & z_{i,l_i-1} & z_{il_i} \\ z_{i0} & z_{i1} & \cdots & z_{i,l_i-2} & z_{i,l_i-1} \end{bmatrix}.$$

Let $I_2(M)$ be the ideal generated by 2-minors of M. Since this ideal does not change under conjugation of the matrix, we will assume that M is in the form of Kronecker–Weierstrass normal form.

Lemma 2.1. *(See [19].) Let M be a $2 \times n$ matrix of linear forms in the Kronecker–Weierstrass normal form*

$$[D_1 \mid \cdots \mid D_k \mid J_{11} \mid \cdots \mid J_{1l_1} \mid \cdots \mid J_{s_1} \mid \cdots \mid J_{sl_s} \mid B_1 \mid \cdots \mid B_t],$$

where each J_{ij} is a Jordan block with length p_{ij} and eigenvalue λ_i. Suppose that, there is at least one Jordan block with eigenvalue zero and

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_j \\ 0 & y_1 & \cdots & y_{j-1} \end{bmatrix}$$

be the Jordan block with smallest length. Then, the ideal $(I_2(M) : y_1)$ is generated by all indeterminates appearing in the second row of M.

Proof. Let M' be the matrix obtained by deleting the column $\left[\begin{smallmatrix} y_1 \\ 0 \end{smallmatrix}\right]$ and substituting y_1 with 0 in the matrix M. Denote by J the ideal generated by indeterminates in the second row of M. Then we have the following sequence:

$$0 \to \frac{S}{J}(-1) \xrightarrow{y_1} \frac{S}{I_2(M)} \to \frac{S}{(I_2(M'), y_1)} \to 0. \tag{4}$$

We claim that this sequence is exact. To prove it, we compare Hilbert series of them. By [5, (2.2.3), (2.5.5)], the Hilbert series of $S/I_2(M)$ is

$$\frac{1}{(1-\nu)^t}\left(\frac{1+A\nu}{1-\nu} + \sum_{i=1}^{s}\sum_{j=1}^{l_i} \frac{p_{ij}}{(1-\nu)^{l_i-j+1}}\right) + G(\nu),$$

where $A = \sum_{i=1}^{t} m_i - 1$, and $G(\nu)$ is a polynomial which is the Hilbert series of a matrix consisting of all nilpotent blocks of M. In the other hand,

$$HS_{S/(I_2(M'),y_1)}(\nu) = HS_{S'/I_2(M')}(\nu),$$

where S' is the ring S without y_1. Since M' has one column less than M, then

$$HS_{S'/I_2(M')}(\nu) = \frac{1}{(1-\nu)^t}\left(\frac{1+A\nu}{1-\nu} + \sum_{i=1}^{s}\sum_{j=1}^{l_i} \frac{p_{ij}}{(1-\nu)^{l_i-j+1}} - \frac{\nu}{(1-\nu)^{l_1}}\right) + G(\nu).$$

Hence

$$HS_{S/I_2(M)}(\nu) - HS_{S/(I_2(M'),y_1)}(\nu) = \frac{\nu}{(1-\nu)^{t+l_1}},$$

where l_1 is the number of Jordan blocks with eigenvalue zero. Note that the number of indeterminates which does not appear in the second column of M is $t + l_1$. Therefore, S/J is isomorphic with a polynomial ring with $t + l_1$ indeterminates. Thus, the sequence (4) is exact and $J = (I_2(M) : y_1)$. □

Note that in the above lemma, assuming that y_1 is in the Jordan block with the smallest length is necessary. For example, in the matrix

$$\begin{bmatrix} y_1 & y_2 & w_1 & w_2 & w_3 \\ 0 & y_1 & 0 & w_1 & w_2 \end{bmatrix},$$

we have $y_2 w_1 \in I_2(M)$ but y_2 is not in the second row.

Proposition 2.2. *Let M be a $2 \times m$ matrix of linear forms in the Kronecker–Weierstrass normal form* (3)*. Then, the height of $I_2(M)$ is given by the following formulas.*

(i) *If M consists of only $k \geqslant 1$ nilpotent blocks, then*

$$\operatorname{ht}\bigl(I_2(M)\bigr) = \sum_{i=1}^{k} n_i.$$

(ii) *If M consists of $t \geqslant 1$ scroll and $k \geqslant 0$ nilpotent blocks, then*

$$\operatorname{ht}\bigl(I_2(M)\bigr) = \sum_{i=1}^{k} n_i + \sum_{i=1}^{t} l_i - 1.$$

(iii) *If M consists of $k \geqslant 0$ nilpotent, $t \geqslant 0$ scroll and $s \geqslant 1$ Jordan blocks, then*

$$\operatorname{ht}\bigl(I_2(M)\bigr) = \sum_{i=1}^{k} n_i + \sum_{i=1}^{t} l_i + \sum_{i=1}^{s} m_i - \gamma,$$

where γ is the maximum number of Jordan blocks with the same eigenvalue.

Proof. (i) Let M be of the form

$$M = \left[\begin{array}{ccccc|c|ccccc} x_{1,1} & x_{1,2} & \cdots & x_{1,n_1} & 0 & \cdots & x_{k,1} & x_{k,2} & \cdots & x_{k,n_k} & 0 \\ 0 & x_{1,1} & \cdots & x_{1,n_1-1} & x_{1,n_1} & & 0 & x_{k,1} & \cdots & x_{k,n_k-1} & x_{k,n_k} \end{array}\right].$$

By [3, p. 15],

$$I_2(M) = \langle x_{1,1}, x_{1,2}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, x_{k,2}, \ldots, x_{k,n_k} \rangle^2.$$

Therefore, (i) is clear.

(ii) If M consists of only t scroll blocks, then by [5], the Hilbert series of $S/I_2(M)$ is equal to

$$\frac{1 + (m-1)\nu}{(1-\nu)^{t+1}}.$$

This proves the assertion in case (ii) when we have only scroll blocks.

Suppose that M consists of $t \geqslant 1$ scroll and $k \geqslant 1$ nilpotent blocks. In this case, proof is by induction on number of columns of M. Let x_{11} be the first indeterminate in the first nilpotent block. We have the following short exact sequence:

$$0 \to \frac{S}{I_2(M) : x_{11}} \xrightarrow{x_{11}} \frac{S}{I_2(M)} \to \frac{S}{(I_2(M), x_{11})} \to 0.$$

Note that

$$\frac{S}{(I_2(M), x_{11})} \simeq \frac{S'}{I_2(M')},$$

where M' is the matrix obtained by deleting the first column of M and replacing 0 instead of x_{11} in the second column of M, and S' is the polynomial ring S without x_{11}. By induction hypothesis, there is $h'(\nu) \in \mathbb{Z}[\nu]$ such that, the Hilbert series of $S'/I_2(M')$ is of the form

$$\frac{h'(\nu)}{(1-\nu)^{c-1-(\delta-2)}} = \frac{h'(\nu)}{(1-\nu)^{c-(\delta-1)}},$$

where $\delta = \sum_{i=1}^{k} n_i + \sum_{i=1}^{t} l_i$, and c is number of all indeterminates appearing in M.

If $l_i \geqslant 3$, for $i = 1, \dots, t$, then, the ideal $I_2(M) : x_{11}$ is generated by all indeterminates. If for some $1 \leqslant i \leqslant t$, $1 \leqslant l_i \leqslant 2$, then, $z_{i,l_i}^u \in I_2(M) : x_{11}$, for some positive integer u. Since the ideal $(I_2(M) : x_{11})$ is zero-dimensional, therefore, the Hilbert series of $S/(I_2(M) : x_{11})(-1)$ is simply $\nu h(\nu)$ for some $h(\nu) \in \mathbb{Z}[\nu]$. By using the above short exact sequence and additive property of Hilbert series, we obtain the Hilbert series of $S/I_2(M)$:

$$H_{S/I_2(M)}(\nu) = \frac{\nu h(\nu)(1-\nu)^{c-(\delta-1)} + h'(\nu)}{(1-\nu)^{c-(\delta-1)}}.$$

In this fraction, the numerator is not divisible by $(1-\nu)$. Therefore, dimension of $S/I_2(M)$ is $c-(\delta-1)$ and height of $I_2(M)$ is $\delta - 1$. This completes the proof of case (ii).

(iii) Suppose that M has $s \geqslant 1$ Jordan blocks. Also in this case, the proof is by induction on number of columns of M. Let γ be the maximum number of Jordan blocks with the same eigenvalues λ. After some suitable elementary column and row operations, we obtain a matrix conjugate to M such that lengths and types of all blocks are preserved and the blocks with eigenvalue λ have become to blocks with eigenvalue zero (for details, see the proof of the main theorem in [4]). Let y_{11} be the first indeterminate in the smallest Jordan block with eigenvalue zero. The above short exact sequence is valid if we substitute x_{11} by y_{11}. In this case,

$$\frac{S}{(I_2(M), y_{11})} \simeq \frac{S'}{I_2(M')},$$

where M' is the matrix obtained by M deleting first column and replacing 0 instead of y_{11}, and S' is the polynomial ring S without y_{11}. By induction hypothesis, there is $h'(\nu) \in \mathbb{Z}[\nu]$ such that, the Hilbert series of $S'/I_2(M')$ is of the form

$$\frac{h'(\nu)}{(1-\nu)^{c-1-(\delta-\gamma-1)}} = \frac{h'(\nu)}{(1-\nu)^{c-(\delta-\gamma)}},$$

where $\delta = \sum_{i=1}^{k} n_i + \sum_{i=1}^{t} l_i + \sum_{i=1}^{s} m_i$. Note that, the ideal $(I_2(M) : y_{11})$ is generated by all indeterminates appearing in the second row of M [21]. The number of indeterminates appearing in the second row is $\delta - \gamma$. Therefore, the Hilbert series of $S/(I_2(M) : y_{11})(-1)$ is

$$\frac{\nu}{(1-\nu)^{c-(\delta-\gamma)}}.$$

The Hilbert series of $S/I_2(M)$ is

$$H_{S/I_2(M)}(\nu) = \frac{\nu + h'(\nu)}{(1-\nu)^{c-(\delta-\gamma)}}.$$

Therefore, dimension of $S/I_2(M)$ is $c - (\delta - \gamma)$ and height of $I_2(M)$ is $\delta - \gamma$. $\square$

Theorem 2.3. *Let M be a $2 \times n$ matrix of linear forms in a polynomial ring S over an algebraically closed field k. Suppose that $I_2(M)$ has codimension $r > 1$. Denote by Θ the Jacobian matrix of $I_2(M)$. Then, the following conditions are equivalent*:

(a) $I_r(\Theta) = \mathfrak{m}^r$;
(b) *the Kronecker–Weierstrass normal form of M does not have any Jordan block, or it consists of only some nilpotent blocks and some Jordan blocks of length* 1;
(c) *the pair $I_2(M) \subseteq (I_2(M), I_r(\Theta))$ is Aluffi torsion-free,*

where $\mathfrak{m}$ is the irrelevant maximal ideal of S and $I_r(\Theta)$ is the ideal generated by r-minors of Θ.

Proof. (a) $\Rightarrow$ (b) Let M be a matrix which has at least one Jordan block. Suppose that γ is the maximum number of Jordan blocks with the same eigenvalue λ. As stated in the proof of Proposition 2.2, we may assume that λ is zero. Let the block J_1 be one of the Jordan blocks with length greater than 1 and eigenvalue zero. It is in the form:

$$\begin{bmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m_1} \\ 0 & y_{1,1} & \cdots & y_{1,m_1-1} \end{bmatrix}.$$

By Proposition 2.2, height of $I_2(M)$ is

$$r = \sum_1^k n_i + \sum_1^s m_i + \sum_1^t l_i - \gamma.$$

But, the variable y_{1,m_1} appears only in $r-1$ quadratic forms in generators of $I_2(M)$ and therefore, it appears only in $r-1$ rows in the Jacobian matrix of $I_2(M)$. This is enough to know that y_{1,m_1}^r is not in the ideal of r-minors of the Jacobian matrix of $I_2(M)$ and then, $I_r(\Theta) \neq \mathfrak{m}^r$.

If there is no any Jordan block of length greater than 1, and there is at least one scroll block and some at least one Jordan block of length 1, then the variable z_{1l_1} appears in $r-1$ quadratic forms in $I_2(M)$ and the same argument as above shows that $z_{1l_1}^r \notin I_r(\Theta)$.

(b) $\Rightarrow$ (a) If the Kronecker–Weierstrass normal form of M does not have any Jordan block, then, M falls within one of the following cases.

(i) M has only scroll blocks.
(ii) M has only nilpotent blocks.
(iii) M has nilpotent and scroll blocks.
(iv) M has nilpotent and Jordan blocks of length 1.

In each case, we show that $I_r(\Theta) = \mathfrak{m}^r$.

Case (i). First assume that there is only one scroll block:

$$M = \begin{bmatrix} z_1 & z_2 & \cdots & z_{m-1} & z_m \\ z_0 & z_1 & \cdots & z_{m-2} & z_{m-1} \end{bmatrix}.$$

We prove that for each monomial of degree $m-1$, there is an $(m-1)$-minor of Θ such that the monomial is initial term of the minor with lexicographic order. By

$$A = \left[z_{i_1}\ z_{i_2} \ldots z_{i_r}\ \middle|\ (c_{11}, c_{12})\ (c_{21}, c_{22}) \ldots (c_{r1}, c_{r2})\right]$$

we mean the r-minor of Θ such that the entry $[A]_{kl} = \partial f_{(c_{l1}, c_{l2})} / \partial z_k$, where $f_{(c_{l1}, c_{l2})}$ is the 2-minor of M obtained by columns c_{l1} and c_{l2}. The following equations are clear:

$$
\begin{aligned}
z_0^{m-1} &= \bigl[z_2\, z_3 \dots z_m \bigm| (1,2)\,(1,3)\dots(1,m)\bigr],\\
z_1^{m-1} &= \bigl[z_1\, z_3 \dots z_m \bigm| (1,2)\,(2,3)\dots(2,m)\bigr],\\
&\vdots\\
z_i^{m-1} &= \bigl[z_i\, z_{i-2} \dots z_0\, z_{i+2} \dots z_m \bigm| (i,i+1)\,(i-1,i)\dots(1,i)\,(i+1,i+2)\dots(i+1,m)\bigr],\\
&\vdots\\
z_{m-1}^{m-1} &= \bigl[z_{m-1}\, z_{m-3} \dots z_0 \bigm| (m-1,m)\,(m-2,m-1)\dots(1,m-1)\bigr],\\
z_m^{m-1} &= \bigl[z_{m-2}\, z_{m-3} \dots z_0 \bigm| (m-1,m)\,(m-2,m)\dots(1,m)\bigr].
\end{aligned}
$$

All above minors are upper triangular.

Let $z_{j_1}^{a_1} z_{j_2}^{a_2} \cdots z_{j_s}^{a_s}$ be a given monomial of degree $r = m-1$. Take the minor

$$\bigl[z_{d_1} \dots z_{d_r} \bigm| (h_{11}, h_{12})\,(h_{21}, h_{22}) \dots (h_{r1}, h_{r2})\bigr]$$

such that a_1 of z_i's are first a_1 entries of the minor of $z_{j_1}^{m-1}$. Then, for the succeeding a_2 of z_i's choose first a_2 entries of the minor of $z_{j_2}^{m-1}$, which they are not appeared in the previous chooses and also the columns are not repeated. Continuing this process, we get a minor which its main diagonal is the given monomial $z_{j_1}^{a_1} z_{j_2}^{a_2} \cdots z_{j_s}^{a_s}$ and this monomial is initial of the minor. To show the last statement, note that entries below the main diagonal do not effect the initialness of the main diagonal. Example 2.4 illustrates concretely this argument.

Now let M be of the form

$$M = \left[\begin{array}{ccccc|c|ccccc} z_{1,1} & z_{1,2} & \cdots & z_{1,l_1-1} & z_{1,l_1} & \cdots & z_{c,1} & z_{c,2} & \cdots & z_{c,l_c-1} & z_{c,l_c} \\ z_{1,0} & z_{1,1} & \cdots & z_{1,l_1-2} & z_{1,l_1-1} & & z_{c,0} & z_{c,1} & \cdots & z_{c,l_c-2} & z_{c,l_c-1} \end{array}\right].$$

First consider the lexicographic order on terms of S with respect to $z_{1,0} > z_{1,1} > \cdots > z_{c,l_c}$ and write the generators of $I_2(M)$ with this order:

$$I_2(M) = (f_1, \dots, f_t, f_{t+1}, \dots, f_k),$$

where $z_{1,0}$ appears in $f_1, \dots, f_t$ and does not appear in $f_{t+1}, \dots, f_k$. Then, the Jacobian matrix of $I_2(M)$ is of the form:

$$\Theta = \left[\begin{array}{ccccccccc|ccccc} -z_{1,2} & -z_{1,3} & \cdots & -z_{1,l_1} & \cdots & -z_{c,1} & \cdots & -z_{c,l_c} & & 0 & 0 & \cdots & 0 & 0 \\ \hline & & & & * & & & & & & & \Theta' & & \end{array}\right]. \tag{5}$$

In this matrix, the block Θ' is Jacobian matrix of $I_2(M')$ where M' is a matrix obtained by deleting first column of M. By induction on number of columns of M, we have $I_{r-1}(\Theta') = \mathfrak{m}'^{r-1}$, where $\mathfrak{m}'$ is the ideal $\mathfrak{m}$ without $z_{1,0}$. By the form of Θ, it is clear that

$$z_{i,j} \langle z_{1,1}, z_{1,2}, \dots, z_{c,l_c} \rangle^{r-1} \subseteq I_r(\Theta), \quad 1 \leqslant i \leqslant c,\ 1 \leqslant j \leqslant l_i,\ (i,j) \neq (1,1).$$

Therefore,

$$\langle z_{1,2}, \dots, z_{1,l_1}, \dots, z_{c,1}, \dots, z_{c,l_c-1}, z_{c,l_c} \rangle^r \subseteq I_r(\Theta).$$

In other hand, if we assume the degree reverse lexicographic order with respect to $z_{1,0} > z_{1,1} > \cdots > z_{c,l_c}$, the Jacobian matrix of $I_2(M)$ is of the form:

$$\left[\begin{array}{ccccc|ccccccc} & & \Theta'' & & & & & & * & & & \\ \hline 0 & 0 & \cdots & 0 & 0 & -z_{1,0} & -z_{1,1} & \cdots & -z_{1,l_1-1} & \cdots & -z_{c,0} & \cdots & -z_{c,l_c-2} \end{array}\right],$$

where the block Θ'' is Jacobian matrix of $I_2(M'')$ where M'' is a matrix obtained by deleting the last column of M. Note that the latter matrix is obtained by some changes of columns of the matrix Θ. Again by induction on number of columns of the matrix M, we have $I_{r-1}(\Theta'') = \mathfrak{m}''^{r-1}$, where $\mathfrak{m}''$ is the ideal $\mathfrak{m}$ without z_{c,l_c}. Then, it is clear that

$$z_{i,j}\langle z_{1,0}, z_{1,2}, \ldots, z_{c,l_c-1}\rangle^{r-1} \subseteq I_r(\Theta), \quad 1 \leqslant i \leqslant c,\ 0 \leqslant j \leqslant l_i - 1,\ (i,j) \neq (c, l_c - 1).$$

Therefore,

$$\langle z_{1,0}, \ldots, z_{1,l_1-1}, \ldots, z_{c,0}, \ldots, z_{c,l_c-1}, z_{c,l_c-2}\rangle^r \subseteq I_r(\Theta).$$

Changing the first and last blocks of M and repeating the above argument, completes the proof in this case.

Case (ii). If the matrix M consists of only nilpotent blocks, then by Proposition 2.2, $I_2(M) = \mathfrak{m}^2$ and clearly $I_r(\Theta) = \mathfrak{m}^r$.

Case (iii). Let M be a matrix obtained by concatenation of some scroll blocks and some nilpotent blocks:

$$M = [D_1 \mid \cdots \mid D_r \mid B_1 \mid \cdots \mid B_t].$$

Let x_{11} be the first entry of the first nilpotent block D_1. Then, $x_{11}^2 \in I_2(M)$ and with the same method of case (i), the Jacobian matrix of $I_2(M)$ will be in the form of (5) with all indeterminates appearing in the top-left block. Using induction on number of columns of M proves the theorem in this case.

Case (iv). Let M be a matrix consisting of $k \geqslant 0$ nilpotent blocks and s Jordan blocks:

$$M = \left[\begin{array}{ccc|c|ccc|c|c|c|c|c|c} x_{1,1} & \cdots & 0 & \cdots & x_{k,1} & \cdots & 0 & y_1 & \cdots & y_\gamma & y_{\gamma+1} & \cdots & y_s \\ 0 & \cdots & x_{1,n_1} & \cdots & 0 & \cdots & x_{k,n_k} & 0 & \cdots & 0 & \lambda_1 y_{\gamma+1} & \cdots & \lambda_s y_s \end{array}\right].$$

If $k > 0$, then the same argument as case (iii) concludes case (iv). If there is no any nilpotent block, take y_1, y_2, y_s and use the induction argument as in case (iii).

(a) $\Rightarrow$ (c) It follows from Corollary 1.5.

(c) $\Rightarrow$ (b) Let M has Jordan blocks of length greater than 1. In this case, it is clear that $f = (y_{1,1} y_{1,m_1}^{r-1}) \in I_r(\Theta) \setminus I_2(M)$ but, $f^2 \in I_2(M)$. Therefore, $f^2 \in I_2(M) \cap I_r(\Theta)^2$ but, $f^2 \notin I_2(M) I_r(\Theta)$. Let M has $t > 0$ scroll blocks and $s > 0$ Jordan blocks of length 1. Then, $f_1 = z_{1l_1-1} z_{1l_1}^{r-1}$ and $f_2 = z_{1l_1}^{r-1} y_1$ are in $I_r(\Theta)$, but they are not in $I_2(M)$. In other hand, $f_1 f_2 \in I_2(M) \cap I_r(\Theta)^2$ but, $f_1 f_2 \notin I_2(M) I_r(\Theta)$. $\square$

Example 2.4. Let M be the matrix

$$\begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 \\ z_0 & z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{bmatrix}.$$

Following is illustration of some monomials as initials of minors.

$$z_0^6=\big[z_2\ z_3\ z_4\ z_5\ z_6\ z_7 \mid (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)\ (1,7)\big],$$
$$z_1^6=\big[z_1\ z_3\ z_4\ z_5\ z_6\ z_7 \mid (1,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)\ (2,7)\big],$$
$$z_2^6=\big[z_2\ z_0\ z_4\ z_5\ z_6\ z_7 \mid (2,3)\ (1,2)\ (3,4)\ (3,5)\ (3,6)\ (3,7)\big],$$
$$z_3^6=\big[z_3\ z_1\ z_0\ z_5\ z_6\ z_7 \mid (3,4)\ (2,3)\ (1,3)\ (4,5)\ (4,6)\ (4,7)\big],$$
$$z_4^6=\big[z_4\ z_2\ z_1\ z_0\ z_6\ z_7 \mid (4,5)\ (3,4)\ (2,4)\ (1,4)\ (5,6)\ (5,7)\big],$$
$$z_5^6=\big[z_5\ z_3\ z_2\ z_1\ z_0\ z_7 \mid (5,6)\ (4,5)\ (3,5)\ (2,5)\ (1,6)\ (6,7)\big],$$
$$z_6^6=\big[z_6\ z_4\ z_3\ z_2\ z_1\ z_0 \mid (6,7)\ (5,6)\ (4,6)\ (3,6)\ (2,6)\ (1,6)\big],$$
$$z_7^6=\big[z_5\ z_4\ z_3\ z_2\ z_1\ z_0 \mid (6,7)\ (5,7)\ (4,7)\ (3,7)\ (2,7)\ (1,7)\big],$$
$$z_0z_1^2z_4z_7^2=\operatorname{In}\big(\big[z_2\ z_3\ z_4\ z_0\ z_5\ z_1 \mid (1,2)\ (2,3)\ (2,4)\ (1,4)\ (6,7)\ (2,7)\big]\big).$$

The sub-matrix corresponding to the last monomial is

$$\begin{bmatrix} -z_0 & 2z_2 & z_3 & 0 & 0 & z_6 \\ 0 & -z_1 & z_2 & z_1 & 0 & 0 \\ 0 & 0 & -z_1 & -z_0 & 0 & 0 \\ -z_2 & 0 & 0 & -z_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -z_7 & 0 \\ 2z_1 & -z_3 & z_4 & z_3 & 0 & -z_7 \end{bmatrix}.$$

Remark 2.5. Let $X \subset \mathbb{P}_k^n$ be a projective algebraic set of dimension d with defining ideal $I_2(M)$ where M is a matrix of linear forms in $k[x_0,\dots,x_n]$ and k is algebraically closed. Theorem 2.3 gives a criterion to check nonsingularity of X, that is, the Kronecker–Weierstrass normal form of M does not have any Jordan block, or it consists of only some nilpotent blocks and some Jordan blocks of length 1 if and only if X is nonsingular.

Note that by proof of the above theorem, in case that M does not have any Jordan block, then the ideal $I_r(\Theta)$ is $\mathfrak{m}$-primary but, in the case that M has Jordan blocks, it is not $\mathfrak{m}$-primary. This means that the following conditions are equivalent:

(a) $I_r(\Theta)=\mathfrak{m}^r$;
(b) $I_r(\Theta)$ is $\mathfrak{m}$-primary.

This remark initiates the following conjecture.

Conjecture 2.6. *Let J denote the ideal generated by quadrics in a polynomial ring S, such that $r=\operatorname{ht}(J)\geqslant 2$. Then, the following conditions are equivalent*:

(a) $I_r(\Theta)=\mathfrak{m}^r$;
(b) $I_r(\Theta)$ *is $\mathfrak{m}$-primary,*

where Θ is the Jacobian matrix of J and $\mathfrak{m}$ is the irrelevant maximal ideal of S.

Corollary 2.7. *Let $J \subset R=k[x_1,\dots,x_n]$ denote a codimension 2 ideal generated by 3 quadrics with the following free resolution*:

$$0\to R^2\to R^3\to J\to 0.$$

Let $I_2(\Theta)$ denote the ideal generated by the 2-minors of the Jacobian matrix Θ of the generators of J. If $I_2(\Theta)$ is $\mathfrak{m}=(x_1,\dots,x_n)$-primary, then the pair $J \subset (J, I_2(\Theta))$ is Aluffi torsion-free. In particular $V(J)\subseteq \mathbb{P}^{n-1}$ is nonsingular.

Proof. By the Hilbert–Burch theorem, J is generated by 2-minors of the syzygy matrix M of J. By assumption, the transpose of M is a 2×3 matrix of linear forms in R. Since $I_2(\Theta)$ is $\mathfrak{m}$-primary, Theorem 2.3 implies that the Kronecker–Weierstrass normal form of M does not have Jordan block and $I_2(\Theta) = \mathfrak{m}^2$. Then by Corollary 1.5, the pair $J \subset (J, I_2(\Theta))$ is Aluffi torsion-free. Since the Jacobian ideal has codimension n, then the additional assertion at the end of the statement is clear. □

Recall that an $n \times n$ (generic) Hankel matrix is of the form

$$H = \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_2 & x_3 & \cdots & x_n & x_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{n-1} & x_n & \cdots & x_{2n-3} & x_{2n-2} \\ x_n & x_{n+1} & \cdots & x_{2n-2} & x_{2n-1} \end{bmatrix},$$

and a generalized Hankel matrix is concatenation of some Hankel matrices (with different indeterminates).

Corollary 2.8. *Let J be the ideal of* 2*-minors of a generalized Hankel matrix. Then, the pair $J \subseteq (J, I_r(\Theta))$ is Aluffi torsion-free.*

Proof. By [20, Theorem 2.2], J is generated by 2-minors of a $2 \times m$ matrix which has only scroll blocks. Now, use Theorem 2.3 to complete the proof. □

Examples 2.9.

(i) The rational normal scroll in $\mathbb{P}^d_k$, could be realized as the variety of the ideal J generated by 2-minors of a matrix consisting only scroll blocks [10]. If I is the Jacobian ideal of J, then by Theorem 2.3, the pair $J \subseteq I$ is Aluffi torsion-free.

(ii) Consider the rational map $F : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^4_k$ given by

$$F(y_0 : y_1 : y_2) = \left(y_0^2 : y_1^2 : y_0 y_1 : y_0 y_2 : y_1 y_2\right).$$

The image of this map is given by the ideal

$$J = \langle x_2^2 - x_0 x_1, x_2 x_3 - x_0 x_4, x_2 x_4 - x_1 x_3 \rangle.$$

Note that J is generated by 2-minors of the matrix

$$\begin{bmatrix} x_2 & x_1 & x_4 \\ x_0 & x_2 & x_3 \end{bmatrix},$$

which consists of two scroll blocks. Therefore, the pair $J \subseteq I$ is Aluffi torsion-free.

3. Edge ideal of a graph

Let I be a monomial ideal in the polynomial ring $k[x_1, \ldots, x_n]$. It is known that the ideal of r-minors of the Jacobian matrix of I is again a monomial ideal (see [16] and [2]). We provide another simple proof for this fact in Lemma 3.1.

Let M be an $m \times n$ matrix and $1 \leqslant r \leqslant \min\{m, n\}$ be an integer. A transversal of length r in M or an r-transversal of M is a product of r entries of M with different rows and columns. In other words, an r-transversal of M is product of entries of the main diagonal of an $r \times r$ sub-matrix of M after suitable changes of columns and rows.

Lemma 3.1. *Let I be an ideal of $k[x_1,\dots,x_n]$ generated by monomials $m_1,\dots,m_s$. Let Θ be the Jacobian matrix of I and $1 \leqslant r \leqslant \min\{n,s\}$. Then, any r-minor of Θ is a monomial.*

Proof. Let $f = [a_1,\dots,a_r \mid b_1,\dots,b_r]$ represent an r-minor of Θ. That is, $1 \leqslant a_1 < a_2 < \cdots < a_r \leqslant n$ are rows and $1 \leqslant b_1 < b_2 < \cdots < b_r \leqslant s$ are columns of the matrix Θ appearing in the chosen r-minor. The corresponding sub-matrix is

$$\begin{bmatrix} \frac{\partial m_{a_1}}{\partial x_{b_1}} & \frac{\partial m_{a_2}}{\partial x_{b_1}} & \cdots & \frac{\partial m_{a_r}}{\partial x_{b_1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial m_{a_1}}{\partial x_{b_r}} & \frac{\partial m_{a_2}}{\partial x_{b_r}} & \cdots & \frac{\partial m_{a_r}}{\partial x_{b_r}} \end{bmatrix}.$$

Note that, any term of f is an r-transversal. This term is nonzero if in any factor $\frac{\partial m_{a_i}}{\partial x_{b_j}}$ of it, m_{a_i} is divisible by x_{b_j} and in this case, $\frac{\partial m_{a_i}}{\partial x_{b_j}} = \gamma \frac{m_{a_i}}{x_{b_j}}$, where the integer γ is the highest power of x_{b_j} appearing in m_{a_i}. Therefore, any nonzero term of f is of the form:

$$\beta \frac{m_{a_1} \cdots m_{a_r}}{x_{b_1} \cdots x_{b_r}},$$

where β is an integer. The minor f is sum of the same monomials with possibly different coefficients and therefore, it is a monomial. □

Let G be a finite simple graph on a vertex set $V(G) = \{v_1,\dots,v_n\}$. Recall that the edge ideal $I(G)$ of G is the ideal in the ring $k[x_1,\dots,x_n]$ generated by $x_i x_j$ provided that $\{v_i, v_j\}$ is an edge in G. Let v be a vertex in G. Degree of v is number of all vertices adjacent to v. For a subset A of $V(G)$, the set of all vertices adjacent to some vertices in A is called neighborhood of A and denoted by $N(A)$. A subset B of vertices of G is called an independent set if there is no any edge between each two vertices of B. A matching in G is a subset of edges of G such that there is no any common vertex between any two of them. In this section, we identify any edge v_i with the corresponding indeterminate x_i.

Lemma 3.2. *Let G be a graph with n vertices, $I(G)$ edge ideal of G and Θ the Jacobian matrix of $I(G)$. Let $g \in k[x_1,\dots,x_n]$ be a monomial and r a positive integer. The following conditions are equivalent.*

(i) *g is an r-transversal of Θ.*
(ii) *There are r different edges $e_1 = \{x_{1_1}, x_{1_2}\},\dots,e_r = \{x_{r_1}, x_{r_2}\}$ such that vertices $x_{1_1},\dots,x_{r_1}$ are different and $g = x_{1_2} \cdots x_{r_2}$.*

Moreover, let the set $\{x_{i_1},\dots,x_{i_s}\}$ is independent. Then there is an r-transversal of the form $g = x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s}$ with $0 \leqslant \alpha_j \leqslant \deg(x_{i_j})$ for $1 \leqslant j \leqslant s$ and $\sum \alpha_j = r$, if and only if $|N(\{x_{i_1},\dots,x_{i_s}\})| \geqslant r$.

Proof. Generators of the ideal $I(G)$ are of the form $x_i x_j$ where $\{x_i, x_j\}$ is an edge in G and each entry of the Jacobian matrix Θ is zero or of the form x_i where x_i is belonging to an edge in G. Equivalence of (i) and (ii) is clear by definition of r-transversal of Θ.

By Lemma 3.1, any r-transversal of Θ is a monomial of degree r. Let $g = x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s}$ be an r-transversal of Θ. It means that there is an $r \times r$ sub-matrix of Θ, which admits b_1 times x_{i_1}, …, and b_s times x_{i_s} in different rows and columns. In the matrix Θ, the entry x_{i_j} appears exactly $\deg(v_{i_j})$ times. Therefore $\alpha_j \leqslant \deg(x_{i_j})$ for each $1 \leqslant j \leqslant s$. Moreover, if $A = \{x_{i_1},\dots,x_{i_s}\}$ is an independent set of vertices, then the set $N(A)$ contains vertices which are adjacent to some vertices in A and there are $|N(A)|$ different edges between A and $N(A)$ with different ends in B. Now, it is clear that there is an r-transversal of the form $g = x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s}$ with $0 \leqslant \alpha_j \leqslant \deg(x_{i_j})$ for $1 \leqslant j \leqslant s$ and $\sum \alpha_j = r$, if and only if $|N(A)| \geqslant r$. $\square$

We say that a graph G is Aluffi torsion-free if the pair $I(G) \subseteq (I(G), I_r(\Theta))$ is Aluffi torsion-free, where r is height of $I(G)$ and Θ is Jacobian matrix of $I(G)$.

Theorem 3.3. *Let G be a graph and $\mathrm{ht}(I(G)) = r > 1$. Then G is not Aluffi torsion-free if and only if there are adjacent vertices x_1, x_2 and other vertices $x_{i_1}, \dots, x_{i_s}$ for some integer $s \geqslant 1$, such that*

(i) *the sets $\{x_1, x_{i_1}, \dots, x_{i_s}\}$ and $\{x_2, x_{i_1}, \dots, x_{i_s}\}$ both are independent, and*
(ii) $|N(\{x_{i_1}, \dots, x_{i_s}\})| = r - 1$.

Proof. Let G be not Aluffi torsion-free. Then, there is an integer $t \geqslant 2$ such that

$$I(G) \cap \big(I(G), I_r(\Theta)\big)^t \neq I(G)\big(I(G), I_r(\Theta)\big)^{t-1}. \tag{6}$$

Note that the right-hand side is always a subset of the left-hand side and it is enough to check the reverse inclusion. Let g be a monomial in left-hand side which is not in right-hand side of (6). Then $g = g_1 \cdots g_t$ such that $g_i \in (I(G), I_r(\Theta))$. If for some $1 \leqslant i \leqslant t$, $g_i \in I(G)$, then $g = g_i(g_1 \cdots g_{i-1} g_{i+1} \cdots g_t) \in I(G)(I(G), I_r(\Theta))^{t-1}$, which is a contradiction. Note that an r-transversal g_i belongs to $I(G)$ if and only if the set of vertices appearing in g_i is not independent.

The monomial g is in $I(G)$ then there are adjacent vertices x_k, x_l such that $x_k x_l \mid g$, but $x_k x_l \nmid g_i$ for each $i = 1, \dots, t$. Without loss of generality, let $x_k \mid g_1$ and $x_l \mid g_2$. In this situation, $g_1 g_2 \in I(G) \cap (I(G), I_r(\Theta))^2$. If $g_1 g_2 \in I(G)(I(G), I_r(\Theta))$, then $g_3 g_4 \cdots g_t \in (I(G), I_r(\Theta))^{t-2}$ and $g \in I(G)(I(G), I_r(\Theta))^{t-1}$ which is again a contradiction. Therefore, we may assume that $g = g_1 g_2 \in I(G) \cap (I(G), I_r(\Theta))^2 \setminus I(G)(I(G), I_r(\Theta))$ and $g_i \in I_r(\Theta) \setminus I(G)$ for $i = 1, 2$. Moreover, $x_1 \mid g_1$, $x_2 \mid g_2$ and x_1 is adjacent to x_2.

Assume that $g_1 = x_1 x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s}$ and $g_2 = x_2 x_{j_1}^{\beta_1} \cdots x_{j_t}^{\beta_t}$, such that $\sum \alpha_i = \sum \beta_j = r - 1$ and both sets $A = \{x_1, x_{i_1}, \dots, x_{i_s}\}$ and $B = \{x_2, x_{j_1}, \dots, x_{j_t}\}$ are independent. If the set $\{x_{i_1}, \dots, x_{i_s}, x_{j_1}, \dots, x_{j_t}\}$ is dependent, then $g_1 g_2 \in (I(G))^2 \subseteq I(G)(I(G), I_r(\Theta))$, a contradiction. By the same argument, it is not possible that x_1 is adjacent to some vertex in $B \setminus \{x_2\}$ and simultaneously x_2 is adjacent to some vertex in $A \setminus \{x_1\}$. Assume that x_2 is not adjacent to any vertex in $A \setminus \{x_1\}$. We claim that the vertices x_1, x_2 and $x_{i_1}, \dots, x_{i_s}$ satisfy conditions (i) and (ii).

By the procedure of the above argument, the vertices x_1, x_2 and $x_{i_1}, \dots, x_{i_s}$ clearly satisfy conditions (i). In other hand, $x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s}$ is an $(r-1)$-transversal of Θ and by Lemma 3.2, $|N(\{x_{i_1}, \dots, x_{i_s}\})| \geqslant r - 1$. We know that $x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s} x_{j_1}^{\beta_1} \cdots x_{j_t}^{\beta_t}$ is not in $I_r(\Theta)$ and thus there is no any r-transversal of Θ dividing it. This means that for any subset C of $\{x_{i_1}, \dots, x_{i_s}, x_{j_1}, \dots, x_{j_t}\}$, $|N(C)| < r$. Therefore $|N(\{x_{i_1}, \dots, x_{i_s}\})| = r - 1$, as required.

Conversely, let there are vertices x_1, x_2 and $x_{i_1}, \dots, x_{i_s}$ satisfying conditions (i) and (ii). Let $g_3 = x_2 x_{i_1} \cdots x_{i_s}$. Then g_3 is an r-transversal of Θ and $g_1 g_3 \in I(G) \cap (I(G), I_r(\Theta))^2$. By Lemma 3.2, condition (ii) guarantees that $g_1 g_3 / x_1 x_2 \notin (I(G), I_r(\Theta))$. Therefore, G is not Aluffi torsion-free. $\square$

Examples 3.4.

(i) A complete graph K_n for $n > 2$ is Aluffi torsion-free. Because all vertices are adjacent to each other and there is no any vertex satisfying condition (i) of the above theorem.
(ii) A complete r-partite graph is Aluffi torsion-free. In contrary if it is not Aluffi torsion-free, then, there are two adjacent vertices v_1, v_2 and at least one another vertex w which is adjacent to none of v_1 and v_2. In this case, v_1 and w belongs to the same part and also v_2 and w belongs to the same part. Therefore v_1 and v_2 are in the same part which is a contradiction.
(iii) A complete graph minus edges in a matching is Aluffi torsion-free, where by a graph G minus an edge e, we mean a graph resulting from G which the edge e is deleted and the vertices at the

ends of e are remaining. Note that, if G is a complete graph minus a matching, then any vertex can be independent to at most only one other vertex. Therefore, item (i) of Theorem 3.3 is not valid.

(iv) The cycles C_3 and C_4 are Aluffi torsion-free, but, for each $n \geqslant 5$, the cycle C_n is not Aluffi torsion-free. Let n be even and $\{v_1, \dots, v_n\}$ be the set of vertices of G such that $v_i \sim v_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $v_n \sim v_1$. Take v_1 and v_2 which are adjacent and $v_4, v_6, \dots, v_{n-2}$ which are independent. Clearly condition (i) of Theorem 3.3 is satisfied. Note that, $\mathrm{ht}(I(G)) = \frac{n}{2}$ and degree of each vertex is 2. Moreover, $N(\{v_4, v_6, \dots, v_{n-2}\}) = \{v_3, v_5, \dots, v_{n-1}\}$ which has cardinality $\frac{n}{2} - 1$. This is condition (ii) of Theorem 3.3. If n is odd, then, the vertices v_1, v_2 and $v_4, v_6, \dots, v_{n-1}$ by the same argument as above, satisfy conditions of Theorem 3.3.

(v) Any path P_n is not Aluffi torsion-free. It follows by the same argument as item (iv) taking the same vertices.

(vi) A star graph is not Aluffi torsion-free. Recall that a graph G is called star if there is a vertex v, such that all other vertices are adjacent to v and there is no any other edge.

Remark 3.5. Let G be a finite simple graph. Then, for $J = I(G)$, the edge ideal of G, Conjecture 2.6 holds.

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