Bases for Coordinate Rings of Conjugacy Classes of Nilpotent Matrices

Mark Shimozono

Department of Mathematics, Virginia Tech, Blacksburg, Virginia 24061-0123
E-mail: mshimo@math.vt.edu

and

Jerzy Weyman

Department of Mathematics, Northeastern University, Boston, Massachusetts
E-mail: weyman@neu.edu

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A general conjecture is given for an explicit basis of the coordinate ring of the closure of the conjugacy class of a nilpotent matrix. This conjecture is proven when the partition given by the transpose Jordan type of the nilpotent matrix is a hook or has two parts.

1. INTRODUCTION

Let $X_{\mu}$ be the closure of the nilpotent conjugacy class in the $n \times n$ matrices over an algebraically closed field $k$, whose Jordan form has blocks of sizes given by the transpose $\mu'$ of the partition $\mu$ of $n$, where $\mu'_i = \#(i \mid \mu_i \geq j)$. We give a construction that conjecturally produces a basis for the coordinate ring $k[X_{\mu}]$ of $X_{\mu}$ (see Conjecture 40) valid in arbitrary characteristic. This conjecture is proven when the partition $\mu$ is a hook or has two parts.

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The varieties $X_\mu$ have been extensively studied. Their equations are known in characteristic zero [38, 40]. In arbitrary characteristic they are normal and have a resolution of singularities $R: Z_\mu \to X_\mu$ such that $R'^i q_*(\alpha_{Z_\mu}) = 0$ for $i > 0$ [8, 20, 28]. For any fixed $\mu$, the coordinate rings $k[X_\mu]$ have a universal basis over any algebraically closed field $k$; that is, there is a $Z$-form $A_{\mu, Z}$ of $C[X_\mu]$ such that $A_{\mu, Z} \otimes k \cong k[X_\mu]$ for every algebraically closed field $k$ [28].

Our approach to the construction of a basis exploits the structure of the ring $k[X_\mu]$ as a rational graded $G = GL(n, k)$-module. Matrix conjugation induces a graded action of $G$ on $k[\mathfrak{g}]$ and $k[X_\mu]$, and restriction of functions gives a canonical ring- and $G$-module epimorphism $k[\mathfrak{g}] \to k[X_\mu]$. Moreover, each of the coordinate rings $k[\mathfrak{g}]$ and $k[X_\mu]$ has a good filtration [8, 28], that is, a filtration by $G$-modules whose filter quotients are isomorphic to direct sums of Schur modules.

We first construct a basis $B$ of highest weight vectors for $C[\mathfrak{g}]$, that is, a basis of the $U$-invariants $C[\mathfrak{g}]^U$ in $C[\mathfrak{g}]$, where $U$ is the unipotent subgroup of upper unitriangular matrices in $\mathfrak{g}$. The dimensions of the subspaces of $U$-invariants in $C[\mathfrak{g}]$ of a given dominant weight in a given degree are encoded in the graded character of $C[\mathfrak{g}]$. It is decomposed explicitly as a sum of irreducible characters using a Cauchy formula and the Littlewood–Richardson rule. This Cauchy formula arises from a filtration of $C[\mathfrak{g}]$ given by bitableaux, a construction which in this case was known to Young and was later generalized by Doubilet, Rota, and Stein [9]. Given a combinatorial datum from the graded character decomposition, we construct a corresponding $U$-invariant in $C[\mathfrak{g}]$ expressed as an explicit $Z$-linear combination of bitableaux. These $U$-invariants comprise the desired basis $B$ of $C[\mathfrak{g}]^U$.

For a given partition $\mu$, one would like to choose a distinguished subset $B_\mu$ of $B$ that forms a basis of $C[X_\mu]^U$. Let $h_{\mu, a, d}$ be the multiplicity of the irreducible $G$-module $S_a$ of highest weight $a$ in the $d$th degree component $C[X_\mu]^U$. These multiplicities are encoded in the Poincaré polynomial

$$P_{\mu, \mu}(q) = \sum_{d \geq 0} q^d h_{\mu, a, d}. \tag{1.1}$$

Much is known in the case $\mu = (1^n)$, where $X_{(1^n)}$ is the cone of all nilpotent matrices in $\mathfrak{g}$. In a pioneering work, Kostant [19] showed that $X_{(1^n)}$ is a complete intersection cut out by the fundamental $G$-invariants in $C[\mathfrak{g}]$. Using this fact, Hesselink [15] and Peterson [29] independently gave a formula for $P_{\mu, \mu}(q)$, which was observed by Gupta [12] to coincide with a Kostka–Foulkes polynomial [27, III.6]. But the latter polynomial has a combinatorial description due to Lascoux and Schützenberger [24]. This gives a second formula for the graded character of $C[\mathfrak{g}]$. It is easy to guess which terms in this formula should survive in passing to the factor $C[X_{(1^n)}]$. 
Using a bijection between the two indexing sets given by the two formulas for the graded character of \( \mathbb{C}[g] \), a distinguished subset \( \mathcal{B}_{(1^r)} \) of \( \mathcal{B} \) is obtained. Indeed, \( \mathcal{B}_{(1^r)} \) is shown to be a basis of \( \mathbb{C}[X_{(1^r)}]^U \). Linear independence is achieved using the straightening formula for bitableaux [9].

For general \( \mu \), we give a conjectural description of the coefficients \( b_{\mu, a, d} \) that generalizes a formula for the Kostka–Foulkes polynomials due to Lascoux [21]. Let \( a \) be a dominant integral weight whose associated highest weight module occurs in \( \mathbb{C}[X_{\mu}] \). For such weights one may write \( k = -a \), and \( \lambda = a + (k^n) \). To each column strict tableau of content \( (k^n) \) and shape \( \lambda \), we associate a partition of \( n \) called its catabolism type (see Section 8 for the definition). We conjecture that \( b_{\mu, a, d} \) is the number of column strict tableaux \( S \) of shape \( \lambda \), content \( (k^n) \), and charge \( d \), whose catabolism type dominates \( \mu \). We also associate to each such tableau \( S \) a \( G \)-module homomorphism from the Schur module \( S_\lambda \) to a factor of \( \mathbb{C}[X_{\mu}]_\mu \), and conjecture that these maps realize a good filtration of \( \mathbb{C}[X_{\mu}]_\mu \).

These conjectures are proven when \( \mu \) is a hook or two-row partition.

Let \( R_{\mu} \) be the coordinate ring of the schematic intersection of \( X_{\mu} \) with the diagonal matrices. Our problem is closely related to the construction of bases for \( R_{\mu} \) that reflect its structure as a graded module for the symmetric group \( \Sigma_n \) [2, 13]. More precisely, let \( a \) be a dominant integral weight occurring in \( k[X_{\mu}] \) with \( a_\mu = -1 \), and let \( \lambda \) be the partition of \( n \) given by \( \lambda_i = a + 1 \). The multiplicity of the irreducible \( \Sigma_n \)-module of the transpose shape \( \lambda' \) in the \( d \)th degree component \( R_{\mu, d} \) is \( b_{\mu, a, d} \) [40].

The paper is organized as follows. Section 2 gives combinatorial definitions. Section 3 recounts properties of Schur modules. In Section 4 combinatorial formulas are derived for the decomposition of the graded characters of \( \mathbb{C}[g] \) and \( \mathbb{C}[X_{\tau}] \) into irreducibles. Section 5 reviews the notion of a bitableau, which is then used to construct many \( U \)-invariants in \( \mathbb{C}[g] \). From among these, bases for \( \mathbb{C}[g]^U \) and \( \mathbb{C}[X_{\tau}]^U \) are chosen in Sections 6 and 7, respectively, with the help of the graded character formulas obtained in Section 4. In Section 8 the notion of the catabolism type of a tableau is introduced, and a basis for \( \mathbb{C}[X_{\mu}]^U \) is given when \( \mu \) is a hook or two-part partition.

In Section 9 it is shown that universal bases exist for \( k[X_{\mu}] \). In Section 9.2 the notion of a good filtration is used to extend the aforementioned bases of \( \mathbb{C}[X_{\mu}]^U \) to bases of \( k[X_{\mu}] \) for special \( \mu \) in arbitrary characteristic. The Appendix contains proofs of some technical combinatorial results.

2. COMBINATORIAL PRELIMINARIES

Young tableaux are used in this paper to index bases for various \( G \)-modules and to give various multiplicities, including the constants \( b_{\mu, a, d} \).
mentioned in the introduction. With this in mind, this section starts with definitions for tableaux. This is followed by a review of several operations on tableaux that are defined in terms of the Knuth or plactic equivalence. The section concludes with a detailed discussion of several concepts related to the Littlewood–Richardson (LR) rule, a key component of the graded character formulas in Section 4.

An alphabet is a totally ordered set. Its elements are called letters. The two alphabets we use most often are the positive integers and the interval of integers $[1, n] = \{1 < 2 < \cdots < n\}$, which will be abbreviated to $[n]$. A subalphabet is a subset of an alphabet with the inherited order. A word is a finite sequence of letters. A subword is a subsequence. If $w$ is a word in the alphabet $A$ and $B \subseteq A$, the restriction $w|_B$ of $w$ to $B$ is the subword of $w$ consisting of the letters of $w$ that are in $B$. Let $\text{rev}(w) = \cdots w_2 w_1$, denote the reverse of the word $w = w_1 w_2 \cdots$, where the $w_i$ are letters. The content of a word $w$ is the sequence $\text{content}(w) = (m_1(w), m_2(w), \ldots)$, where $m_i(w)$ denotes the multiplicity of the letter $i$ in $w$. A standard word of length $n$ is one of content $(1^n)$.

A composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a sequence of nonnegative integers, almost all zero. We write $|\alpha|$ for the sum $\alpha_1 + \alpha_2 + \cdots$. A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ is a composition whose entries weakly decrease. Let $l(\lambda)$ denote the length (number of nonzero entries) of the partition $\lambda$. In this paper it is assumed that partitions have length at most $n$. The dominance partial order on partitions $\lambda \trianglerighteq \mu$ is defined by $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$, for all $i$.

A diagram is a finite set of pairs of positive integers. The Ferrers diagram of the partition $\lambda$ is defined by

$$D(\lambda) = \{(i, j) | 1 \leq j \leq \lambda_i\}.$$ 

If $\lambda$ and $\mu$ are partitions, define the skew diagram or skew shape $\lambda/\mu$ by the set difference $\lambda/\mu = D(\lambda) - D(\mu)$, provided that $D(\mu) \subseteq D(\lambda)$.

Let $\tilde{\nu}$ denote the partition whose diagram is given by the 180-degree rotation of the skew diagram $(\nu^\circ)/\nu$. Explicitly, $\tilde{\nu}_i = \nu_1 - \nu_{n+1-i}$ for $1 \leq i \leq n$.

A tableau of the skew shape $D$ in the alphabet $A$ is a map $T : D \rightarrow A$. We use the English convention for depicting tableaux, in which the tableau $T$ is viewed as a partial matrix in which the entry in the $i$th row and $j$th column contains the letter $T(i, j)$ for $(i, j) \in D$. The tableau $T$ is column strict if the entries of $T$ weakly increase in each row from left to right and strictly increase in each column from top to bottom. If $T$ is a column strict tableau in the alphabet $A$ and $B$ is a subinterval of $A$, let $T|_B$ be the column strict tableau consisting of the letters of $T$ that are in $B$. The row-reading word of $T$ is defined by $\text{word}(T) = \cdots u^2 u^1$, where $u^i$ is
the word obtained by reading the $i$th row of $T$ from left to right. The column-reading word of $T$ is the word $\text{colword}(T) = v^1 v^2 \cdots$, where $v^j$ is the word obtained by reading the $j$th column of $T$ from bottom to top. The content of a tableau is the content of its word. A standard tableau is a column strict tableau whose word is standard. For each composition $\alpha$, let the key tableau $K(\alpha)$ be the unique column strict tableau of shape $\lambda$ and content $\alpha$, where $\lambda$ is the partition obtained by sorting the parts of $\alpha$ into weakly decreasing order. Explicitly, the $j$th column of $K(\alpha)$ consists of the numbers $i$ such that $\alpha_i \geq j$. For a partition $\lambda$, the $i$th row of the key tableau $K(\lambda)$ consists of $\lambda_i$ copies of the letter $i$.

The Knuth equivalence (also called the plactic equivalence) relation $\sim_K$ on words is the transitive symmetric closure of relations of the form

\[
uxyw \sim_K uzxy \text{ if } x \leq y < z
\]
\[
uyxz \sim_K uyzx \text{ if } x < y \leq z,
\]

where $x$, $y$, and $z$ are letters and $u$ and $v$ are words [18]. The Knuth equivalence satisfies the following properties:

(K1) Knuth equivalent words have the same content.

(K2) The row- and column-reading words of a (skew) column strict tableau are Knuth equivalent.

(K3) For every word $w$ there is a unique column strict tableau $P(w)$ of partition shape whose row-reading word (or equivalently, whose column-reading word) is Knuth equivalent to $w$.

(K4) Knuth equivalence is preserved under restriction to subalphabets that are subintervals.

Let $w = w_1 w_2 \cdots w_N$, where $w_i$ is a letter. Denote by $Q(w)$ the recording tableau of the Schensted row insertion of the word $w$ [34], that is, the unique standard tableau in the alphabet $[N]$ such that

\[
\text{shape}(Q(w)_{[1,k]}) = \text{shape}(P(w_1 w_2 \cdots w_k))
\]

for all $1 \leq k \leq N$.

By abuse of language we often refer to the Knuth equivalence class of a (skew) column strict tableau instead of that of its row-reading (or column-reading) word.

Let $T$ be a column strict tableau of partition shape in the alphabet $[n]$. The evacuation $\text{ev}(T)$ of $T$ (with respect to the alphabet $[n]$) is the unique column strict tableau of partition shape in the alphabet $[n]$ such that

\[
\text{shape}(\text{ev}(T)_{[1,r]}) = \text{shape}(P(T|_{[n+1-r,n]}))
\]

for all $1 \leq r \leq n$. 

\[
(2.1)
\]
Example 1. Let \( n = 5 \). A column strict tableau \( S \) and the intermediate tableaux \( P(S_{[k,n]}) \) involved in the computation of \( ev(S) \) are given below.

\[
\begin{align*}
S &= P(S_{[1,5]}) = \begin{array}{cccc}
1 & 1 & 2 & 5 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 5 \\
4 & 5 & 5 & 5
\end{array} \\
P(S_{[2,5]}) &= \begin{array}{cccc}
2 & 2 & 4 & 5 \\
3 & 3 & 4 & 5 \\
4 & 4 & 5 & 5 \\
5 & 5 & 5 & 5
\end{array} \\
P(S_{[3,5]}) &= \begin{array}{cccc}
3 & 4 & 5 & 5 \\
4 & 5 & 5 & 5 \\
5 & 5 & 5 & 5
\end{array} \\
P(S_{[4,5]}) &= \begin{array}{cccc}
4 & 4 & 5 & 5 \\
5 & 5 & 5 & 5
\end{array} \\
P(S_{[5,5]}) &= \begin{array}{cccc}
5 & 5 & 5 & 5 \\
6 & 5 & 5 & 5
\end{array}
\end{align*}
\]

Suppose \( w \) is a word in the alphabet \([n]\). Denote by \( w^\# \) the reverse of the word obtained from \( w \) by replacing each letter \( i \) by the complementary letter \( n + 1 - i \).

Theorem 2 [25]. Let \( w \) be a word in the alphabet \([n]\).

1. \( w \) is the row- (resp. column-) reading word of a column strict tableau of the skew shape \( D \) if and only if \( w^\# \) is the row- (resp. column-) reading word of a column strict tableau of shape given by the 180-degree rotation of \( D \).
2. \( P(w^\#) = ev(P(w)) \).
3. \( Q(w^\#) = ev(Q(w)) \).

The symmetric group on the alphabet \( \mathcal{A} \) has an action on words in the alphabet \( \mathcal{A} \) that factors through Knuth equivalence [25], which we recount here. Let \( \rho \) be a permutation. Consider the following operator on words (also denoted \( \rho \)) called the automorphism of conjugation or plactic permutation operator corresponding to \( \rho \). Suppose first that \( \rho \) is the transposition \((s, s)\) of the letters \( r \) and \( r + 1 \). Let \( w \) be a word. Viewing each letter \( r \) (resp. \( r + 1 \)) in \( w \) as a right (resp. left) parenthesis, perform the usual matching of parentheses, ignoring other letters. Say that an occurrence of the letter \( r \) or \( r + 1 \) in \( w \) is \( r \)-paired (resp. \( r \)-unpaired) if it corresponds to a matched (resp. unmatched) parenthesis. The subword of \( r \)-unpaired letters must have the form \( r^n(r + 1)^n \). Define \( s_\rho w \) to be the word obtained by replacing the \( r \)-unpaired subword \( r^n(r + 1)^n \) of \( w \) by \( r^n(r + 1)^n \).

Example 3. Let \( r = 2 \). A word \( w \), the \( r \)-parenthesization of \( w \), and the word \( s_\rho w \) are given below. The \( r \)-unpaired subwords are underlined.

\[
\begin{align*}
w &= 2 3 3 2 1 3 2 2 4 2 1 3 2 2 3 2 2 3 3 2 3 \\
s_2w &= 2 3 3 2 1 3 2 2 4 2 1 3 2 3 2 3 3 3 2 3
\end{align*}
\]
Note that if \( w \) contains the same number of \( r \)'s and \((r + 1)\)'s, then \( s_rw = w \). For an arbitrary permutation \( \rho \) and factorization \( \rho = s_{i_1} s_{i_2} \cdots s_{i_p} \) of \( \rho \) into adjacent transpositions, define \( \rho w = s_{i_1} s_{i_2} \cdots s_{i_p} w \). The proof of the following theorem appears in the Appendix.

**Theorem 4.25.** (1) The plactic permutation operators \( s_r \) satisfy the Moore–Coxeter relations for the symmetric group, and hence define an action of the symmetric group on words.

2. content(\( \rho w \)) = \( \rho(\text{content}(w)) \).

3. Let \( T \) be a column strict tableau. Then there is a unique column strict tableau \( \rho T \) of the same shape as \( T \) such that \( \text{word}(\rho T) = \rho \text{word}(T) \).

4. \( P(\rho w) = \rho P(w) \).

5. \( Q(\rho w) = Q(w) \).

6. Let \( B \) be the minimal interval such that \( \rho(i) = i \) for all \( i \in B \). Then \( (\rho w)|_B = \rho(w|_B) \) and \( (\rho w)|_{B^c} = w|_{B^c} \) where \( B^c \) is the complement of \( B \), and the subwords \( \rho w|_B \) and \( w|_B \) (resp. \( \rho w|_{B^c} \) and \( w|_{B^c} \)) occupy the same positions in the words \( \rho w \) and \( w \).

7. Let \( w = uu' \) and \( \rho w = u'u' \) where \( u \) and \( u' \) have the same length. Then \( \rho uu' = u'uu' \).

From [24] we recall the definition of the charge, a \( \mathbb{Z}_+ \)-valued function on words. Suppose first that \( w \) is a standard word of length \( n \). Affix an index \( c_i \) to the letter \( i \) in \( w \) according to the rule that \( c_1 = 0 \) and \( c_i = c_{i-1} + 1 \) if \( i \) appears to the left of \( i - 1 \) in \( w \) and \( c_i = c_{i-1} + 1 \) if \( i \) appears to the right of \( i - 1 \) in \( w \). Let

\[
\text{charge}(w) = c_1 + c_2 + \cdots + c_n
\]

Next, if \( w \) has partition content \( \mu \), then define

\[
\text{charge}(w) = \text{charge}(w^1) + \text{charge}(w^2) + \cdots
\]

where \( w \) is partitioned into disjoint standard subwords \( w^j \) of length \( \mu^j \), using the following left circular reading. To compute \( w^1 \), start from the right end of \( w \) and scan to the left. Choose the first 1 encountered, then the first 2 that occurs to the left of the selected letter 1, etc. If at any point there is no \( i + 1 \) to the left of the selected letter \( i \), circle around to the right end of \( w \) and continue scanning to the left. This process selects the subword \( w^1 \) of \( w \). Erase the letters of \( w^1 \) from \( w \) and repeat this process, obtaining the subword \( w^2 \). Continue until all the letters of \( w \) have been exhausted.
Example 5. Let \( w \) be the row-reading word of the tableau \( S \) in Example 1. The words \( w \) and its subwords \( w^1 \) and \( w^2 \) appear below.

\[
\begin{align*}
  w & = 4 \quad 3 \quad 2 \quad 3 \quad 4 \quad 1 \quad 1 \quad 2 \quad 5 \quad 5 \\
  w^1 & = 4 \quad 3 \quad 2 \quad 1 \quad 5 \\
  w^2 & = 3 \quad 4 \quad 1 \quad 2 \quad 5 
\end{align*}
\]

The charges of the subwords \( w^1 \) and \( w^2 \) are calculated below. Each index \( c_i \) is written below the letter \( i \).

\[
\begin{array}{cccccccc}
  4 & 3 & 2 & 1 & 5 & 3 & 4 & 1 & 2 & 5 \\
  0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 3 \\
\end{array}
\]

So \( \text{charge}(w) = \text{charge}(w^1) + \text{charge}(w^2) = 1 + 7 = 8 \).

Finally, for a word \( w \) of arbitrary content, let \( \rho \) be a permutation such that \( \rho w \) has partition content. Then define \( \text{charge}(w) = \text{charge}(\rho w) \).

Define the charge of a (skew) column strict tableau to be the charge of its row-reading word. This makes sense since the charge is constant on Knuth equivalence classes [26].

We require a special case of a remarkable formula for the charge due to Lascoux, Leclerc, and Thibon.

Theorem 6 [22]. Let \( S \) be a column strict tableau of partition shape in the alphabet \( [n] \), \( \delta_i(S) \) the minimum of the number of \( r \)-unpaired letters \( r \) and the number of \( r \)-unpaired letters \( r + 1 \) in \( S \), and \( \Sigma_n \) the symmetric group on the set \( [n] \). Then

\[
\text{charge(} \text{ev}(S)) = \frac{1}{n!} \sum_{\rho \in \Sigma_n} \sum_{r=1}^{n-1} r \delta_r(\rho S) \quad (2.2)
\]

where \( \rho \) acts as a plactic permutation operator.

Let \( u_r(w) \) denote the number of \( r \)-unpaired letters \( r + 1 \) in \( w \).

Remark 7. When \( S \) has content \( (k^n) \), \( \delta_r(S) = u_r(S) \) for \( 1 \leq r \leq n - 1 \) and \( \rho S = S \) for all \( \rho \in \Sigma_n \). In this special case,

\[
\text{charge(} \text{ev}(S)) = \sum_{r=1}^{n-1} r u_r(S) \quad (2.3)
\]

We now discuss concepts related to the Littlewood–Richardson rule (cf. Theorem 17 and Corollary 18). Consider the following predicate on words depending on a partition \( \tau \) called the \( \tau \)-lattice property. The word \( w \) is said to be \( \tau \)-lattice if content(\( u \)) + \( \tau \) is a partition for every final subword \( u \) of \( w \). The word \( w \) is said to be lattice if it is \( \tau \)-lattice where \( \tau \) is the empty
A column strict tableau is said to be $\tau$-lattice if its row-reading word is.

Let CST$(D, \gamma)$ be the set of column strict tableaux of shape $D$ and content $\gamma$, and let LRT$_\tau(D, \gamma)$ be the subset of $\tau$-lattice tableaux in CST$(D, \gamma)$. If $\tau$ is empty, the subscript $\tau$ is suppressed.

**Remark 8.** Let $\lambda$ be a partition. The set LRT($\lambda, \lambda$) consists of a single tableau, namely, the key tableau $K(\lambda)$ of content $\lambda$. If $D$ is the skew diagram given by the 180-degree rotation of the Ferrers diagram of $\lambda$, then the set LRT$(D, \lambda)$ consists of a single tableau, whose columns are given by those of $K(\lambda)$ but in reverse order.

**Example 9.** Let $n = 5$, $\lambda = (8,6,4,4,3)$, $\mu = (5,4,3,2,0)$, and $\sigma = (5,3,2,1,0)$. A tableau $T \in$ LRT($\lambda/\mu, \sigma$) is given below.

$$
\begin{array}{cccccc}
\boxempty & \boxempty & \boxempty & \boxempty & \boxempty & 1 & 1 \\
\boxempty & \boxempty & \boxempty & \boxempty & 1 & 2 \\
\boxempty & \boxempty & 2 & 3 \\
2 & 3 & 4 \\
\end{array}
$$

**Example 10.** Let $D = (5,3,1,1,0)$, $\tau = (5,4,3,2,0)$ and $\gamma = (2,2,2,2,2)$. We have $S \in$ LRT$_\tau(D, \gamma)$ where $S$ is the tableau in Example 1.

We require various criteria for $\tau$-latticeness.

**Lemma 11.** The following are equivalent.

1. The word $w$ is $\tau$-lattice.
2. $u_r(w) \leq \tau_r - \tau_{r+1}$ for every $r \geq 1$.
3. Some word Knuth equivalent to $w$ is $\tau$-lattice.
4. Every word Knuth equivalent to $w$ is $\tau$-lattice.

**Proof.** The lemma follows immediately from two observations. First, the number of $r$-paired letters is preserved under Knuth equivalence; this can be verified directly from the definitions. Second, $u_r(w)$ has the following formula:

$$u_r(w) = \max\{m_{r+1}(v) - m_r(v) \mid v \text{ is a final subword of } w\} \quad (2.4)$$

To see this, without loss of generality, assume $w$ is a word in the two letter alphabet $[r, r + 1]$. If $w$ has no $r$-paired letters $(2.4)$ clearly holds. Otherwise $w$ can be written in the form $v'(r + 1)v''$, where $v'$ and $v''$ are words. Both sides of $(2.4)$ are unchanged in passing from $w$ to $v'v''$, so $(2.4)$ holds by induction on the length of $w$. $\blacksquare$
For tableaux of rectangular content \((k^n)\), the lattice properties transform nicely under evacuation.

**Proposition 12.** Let \(T\) be a tableau of partition shape and content \((k^n)\). Then

1. \(u_r(\text{ev}(T)) = u_{n-r}(T)\) for \(1 \leq r \leq n - 1\).
2. \(T\) is \(\tau\)-lattice if and only if \(\text{ev}(T)\) is \(\tau\)-lattice.

**Proof.** (2) follows immediately from (1) and Lemma 11. To show (1), we have

\[
u_r(\text{ev}(T)) = u_r(P(\text{word}(T)\#)) = u_r(\text{word}(T)\#) = u_{n-r}(T)
\]

Applying Theorem 2 (2) to \(\text{word}(T)\) proves the first equality. The second equality follows from Lemma 11. The last equality holds since the \(r\)-paired letters in \(w\) correspond to the \((n - r)\)-paired letters in \(w\#\).

### 3. Schur Modules

In this section we review a construction for Schur modules given in [1] and define some maps involving Schur modules for later use. In characteristic zero the Schur modules give representatives for each of the isomorphism classes of irreducible finite dimensional \(G\)-modules. In this section \(k\) is assumed to be a field.

An integral weight \(a = (a_1, a_2, \ldots, a_n)\) is a sequence of integers of length \(n\). The dominance partial order on partitions can be extended to integral weights using the same definition: \(a \succ b\) if \(a_1 + \cdots + a_i \geq b_1 + \cdots + b_i\) for all \(i\). The weight \(a\) is dominant if \(a_1 \geq a_2 \geq \cdots \geq a_n\). Let \(E \equiv k^n\) be the defining representation of \(G \equiv GL(n, k)\). If characteristic of \(k\) is zero, the inequivalent irreducible \(G\)-modules are given by the Schur modules \(S_a(E)\) (defined below) as \(a\) runs over the dominant integral weights.

Let \(D\) be a diagram with \(m\) columns and \(F\) a \(G\)-module. Let \(\wedge^D\) be the functor

\[
\wedge^D(F) = \bigotimes_{j=1}^m \wedge^{D_j}(F),
\]

where \(D_j\) is the \(j\)th column of \(D\) and \(\wedge^{D_j}(F)\) is a copy of \(\wedge^{[D_j]}(F)\) whose tensor positions are indexed by the cells of \(D_j\). Suppose for a moment that...
Define the $G$-equivariant map $\eta_\ell$ by

\[
\Lambda^{[D_{\ell}]}(F) \otimes \Lambda^{[D_{\ell}]}(F) \xrightarrow{\Delta \otimes \text{id}} \\
\Lambda^{[D_{\ell}]}(F) \otimes \Lambda^*(F) \otimes \Lambda^{[D_{\ell}]}(F) \xrightarrow{\text{id} \otimes m} \\
\Lambda^{[D_{\ell}]}(F) \otimes \Lambda^{[D_{\ell}]}(F)
\]

where $\Delta$ and $m$ are the exterior diagonalization and multiplication maps.

Now let $D$ be an arbitrary skew diagram $\lambda/\mu$ with $m$ columns. The skew Schur functor $S_D$ is defined by

\[
S_D(F) = (\Lambda^D(F))/R_D
\]

where $R_D$ is the submodule of $\Lambda^D(F)$ generated by the images of the $G$-equivariant maps $\eta_{j,r}$ for $1 \leq j \leq m$ and $\mu_j - \mu_{j+1} < r \leq \lambda_{j+1} - \mu_{j+1}$, given by

\[
\Lambda^{[D_{\ell}]}(F) \otimes \cdots \otimes \Lambda^{[D_{\ell}]}(F) \otimes \Lambda^{[D_{\ell}]}(F) \otimes \cdots \Lambda^{[D_{\ell}]}(F) \\
\eta_{j,r} \downarrow \\
\Lambda^D(F)
\]

where $\eta_{j,r}$ is the map $\eta_j$ in the $j$th and $(j+1)$st tensor factors and the identity in the others. Let $d_D$ denote the natural $G$-equivariant projection

\[
d_D : \Lambda^D(F) \rightarrow S_D(F)
\]

Recall that $E$ denotes the defining representation of $G$. For a partition $\lambda$, write $S_\lambda(E) = S_{D(\lambda)}(E)$. For the dominant integral weight $\alpha$ with $-k = a_\alpha < 0$, let $\lambda$ be the partition defined by $\lambda_i = a_i + k$. Define the Schur module $S_\alpha(E)$ by

\[
S_\alpha(E) = S_\alpha(E) \otimes (\Lambda^\alpha(E^*) \otimes k)
\]

Example 13. $S_{(p, 0^{p-1})}(E)$ is the $p$th symmetric power of $E$ for any nonnegative integer $p$, and $S_{(1, 0^{p-1})}(E)$ is the $p$th exterior power $\Lambda^p(E)$ of $E$ for $0 \leq p \leq n$. In particular, $S_{(0^p)}(E) \cong k$ is the trivial $G$-module, $S_{(1,0^{p-1})}(E) \cong E$ is the defining representation of $G$, $S_{(1^p)}(E)$ is the determinant module for $G$, and $S_{(0^{p-1})}(E)$ is the dual module $E^*$. 
Remark 14. Let us fix an ordered basis \( \{e_i \mid 1 \leq i \leq n\} \) for the defining \( G \)-module \( E \). Given a word \( u = u_1 u_2 \cdots u_p \) in the alphabet \([n]\), let \( e_u \in \Lambda^p(E) \) be defined by \( e_u = e_{u_1} \wedge e_{u_2} \wedge \cdots \wedge e_{u_p} \). For each tableau \( T \) of skew shape \( D \) in the alphabet \([n]\), let \( e_T \in \Lambda^p(E) \) denote the element

\[
e_T = \bigotimes_j e_{u^j}
\]

where \( u^j \) is the word obtained by reading the \( j \)th column of \( T \) from top to bottom. A basis for \( SE \) is given by \( de \) where \( T \) ranges over the set of column strict tableaux of shape \( D \) in the alphabet \([n]\).

Next we review isomorphisms involving dual modules. There is a \( G \)-module isomorphism \( \Lambda^p(E^*) \cong \Lambda^p(E^*)^* \) sending \( f_1 \wedge f_2 \wedge \cdots \wedge f_p \in \Lambda^p(E^*) \) to the map \( v_1 \wedge v_2 \wedge \cdots \wedge v_p \mapsto \det(f_i(v_j))_{1 \leq i, j \leq p} \). Exterior multiplication \( \Lambda^p(E) \otimes \Lambda^{n-p}(E) \rightarrow \Lambda^n(E) \) induces a \( G \)-module isomorphism

\[
\Lambda^p(E^*) \cong \Lambda^{n-p}(E) \otimes \Lambda^n(E^*).
\]

(3.7)

Let \( \nu \) be a partition. Recall that \( \tilde{\nu} \) is the partition defined by \( \tilde{\nu}_i = \nu_i - \nu_{i+1-j} \) for \( 1 \leq i \leq n \). Since the lengths of the \( j \)th column of \( D(\nu) \) and the \( (\nu_1 + 1 - j) \)-th column of \( D(\tilde{\nu}) \) sum to \( n \), tensor powers of the map (3.7) give the isomorphism

\[
\Lambda^{D(\nu)}(E^*) \cong \Lambda^{D(\tilde{\nu})}(E) \otimes (\Lambda^n(E^*))^{\otimes \nu_1}.
\]

(3.8)

This induces the isomorphism

\[
S_\nu(E^*) \cong S_\nu(E) \otimes (\Lambda^n(E^*))^{\otimes \nu_1}.
\]

(3.9)

The map (3.9) can be described explicitly using the tableau basis in Remark 14. Let \( \{e^*_i \mid 1 \leq i \leq n\} \) be the basis of \( E^* \) dual to the basis \( \{e_i \mid 1 \leq i \leq n\} \) of \( E \). Consider the alphabet \([n]^* = \{1^* > 2^* > \cdots > n^*\} \) (note the reverse order), where \( i^* \) stands for the basis vector \( e^*_i \). Let \( Q \) be a tableau of shape \( \nu \) in the alphabet \([n]^* \), whose columns are strictly increasing from top to bottom. Let \( e_Q^* \in \Lambda^{D(\nu)}(E^*) \) be defined in the obvious way. Define \( \tilde{Q} \) to be the tableau of shape \( \tilde{\nu} \) with strictly increasing columns in the alphabet \([n]\) such that the \( j \)th column of \( \tilde{Q} \) (with stars removed) and the \( (\nu_1 + 1 - j) \)-th column of \( \tilde{Q} \) are complementary subsets in \([n]\). Note that \( \tilde{Q} \) is column strict if and only if \( Q \) is. Then up to sign, the map (3.9) is given by

\[
d_\nu(e_Q^*) \mapsto d_\nu \left( e_{\tilde{Q}}^* \otimes (e^*_n \wedge e^*_n \wedge \cdots \wedge e^*_1)^{\otimes \nu_1} \right).
\]

(3.10)
For the integral weight \(a\), let \(a^* = (-a_n, -a_{n-1}, \ldots, -a_1)\). Clearly \(a^*\) is dominant if and only if \(a\) is. The map (3.9) induces an isomorphism \(S_a(E^*) \cong S_{a^*}(E)\). Let \(\alpha\) and \(\beta\) be the unique pair of partitions such that \(l(\alpha) + l(\beta) \leq n\) and \(a = \alpha + \beta^*\); that is, \(\alpha\) and \(\beta\) consist of the positive and negative parts of \(a\), respectively. Assume now that \(|a| = 0\) and write \(\lambda = a + (k^n)\) where \(k = -a_n = \beta_1\). Define \(d_a\) to be the composite map

\[
\Lambda^{D(\alpha)}(E) \otimes \Lambda^{D(\beta)}(E^*) \\
\Lambda^{D(\alpha)}(E) \otimes \Lambda^{D(\beta)}(E) \otimes (\Lambda^n(E^*))^{\otimes \beta_1} \\
\Lambda^{D(\lambda)}(E) \otimes (\Lambda^n(E^*))^{\otimes \beta_1} \\
d_a \circ \text{id} \\
S_\lambda(E) \otimes (\Lambda^n(E^*))^{\otimes \beta_1} = S_{\Lambda}(E).
\]

Let \(D\) and \(D'\) be skew shapes such that their side-by-side juxtaposition \(DD'\) is also a skew shape. The following commutative diagram defines an epimorphism of \(G\)-modules \(S_D(E) \otimes S_D(E) \to S_{DD'}(E)\).

\[
\Lambda^D(E) \otimes \Lambda^{D'}(E) \to \Lambda^{DD'}(E) \\
d_{D_D} \circ \text{id} \\
S_D(E) \otimes S_D(E) \to S_{DD'}(E)
\]

**Remark 15.** By Remark 14, a basis for \(S_{DD'}(E)\) is given by the images of vectors of the form \(e_T \otimes e_{T'}\), where the tableau \(TT'\), defined by the columnwise juxtaposition of \(T\) and \(T'\), is column strict of shape \(DD'\) in the alphabet \([n]\).

**Remark 16.** Consider the following special cases of the map (3.12).

(1) Let \(\alpha\) and \(\beta\) be partitions with \(l(\alpha) + l(\beta) \leq n\), \(D' = D(\alpha), a = \alpha + \beta^*, k = \beta_1,\) and \(a = \lambda - (k^n)\). Let \(D\) be the diagram \(D(\beta)\), so that \(DD'\) is the diagram of the partition \(\lambda\). Say that the pair of tableaux \((P, Q)\) is a *complementary pair of type \((\alpha, \beta^*)* if \(P\) is column strict of shape \(\alpha\) in the alphabet \([n]\), \(Q\) is column strict of shape \(\beta\) in the alphabet \([n]^\ast\), and \(QP\) is a column strict tableau of shape \(\lambda\). The map (3.12) gives a \(G\)-module epimorphism \(S_\alpha(E) \otimes S_\beta(E^*) \to S_{\lambda}(E)\). By Remark 15 a basis for \(S_\alpha(E)\) is given by the vectors \(d_a(e_p \otimes e_{Q}^n)\) for the complementary pairs \((P, Q)\) of type \((\alpha, \beta^*)\).
(2) Let $D$ be the 180-degree rotation $((\beta^\circ_n)'/\tilde{\beta})$ of the Ferrers diagram $D(\beta)$. In this case $DD'$ is the skew shape $((\beta^\circ_n) + \alpha)/\tilde{\beta}$. The map

$$S_a(E) \otimes S_{\tilde{\beta}}(E) \to S_a(E) \otimes S((\beta^\circ_n) + \alpha)/\tilde{\beta}(E) \to S((\beta^\circ_n) + \alpha)/\tilde{\beta}(E)$$

(3.13)

is an isomorphism. To see this, in light of Remarks 14 and 15, it is enough to show that if $T$ and $T'$ are column strict tableaux of shape $(\beta^\circ_n)/\tilde{\beta}$ and $\alpha$, respectively, then the tableau $TT'$ is column strict. Any letter in the $r$th row of $T$ is at most $r$, since $T$ is column strict of shape $(\beta^\circ_n)/\tilde{\beta}$, which is the 180-degree rotation of a partition diagram. Any letter in the $r$th row of $T'$ is at least $r$, since $T'$ is column strict of the partition shape $\alpha$. Therefore $TT'$ is column strict.

Now assume that $k$ is a field of characteristic zero. We conclude this section with the Littlewood–Richardson rule, a combinatorial description of the multiplicity of an irreducible $G$-module in a skew Schur module or in a tensor product of Schur modules.

Let $M$ and $M'$ be finite dimensional $G$-modules. Define their intertwining coefficient by

$$\langle M, M' \rangle = \dim \text{Hom}_G(M, M'), \quad (3.14)$$

the dimension of the space of $G$-module homomorphisms $M \to M'$.

For the dominant integral weights $a, b$, and $d$, define the coefficients $LR_{ab}^d$ by

$$LR_{ab}^d = \langle S_a(E), S_a(E) \otimes S_b(E) \rangle.$$ 

Note that $LR_{ab}^d = 0$ unless $|d| = |a| + |b|$, where $|a| = a_1 + a_2 + \cdots + a_n$. It follows from the definitions that

$$LR_{ab}^d = LR_{a+(p+q)\lambda}, \quad (3.15)$$

for any integers $p$ and $q$. By choosing $p$ and $q$ to be sufficiently large, the calculation of the coefficient $LR_{ab}^d$ reduces to the case when all three weights are partitions, where the following classical formula applies.

**Theorem 17** (Littlewood–Richardson (LR) rule [23]). *For the partitions $\lambda, \mu, \nu$, the coefficient $LR_{\lambda/\mu}^\lambda$ is given by the cardinality of the set of tableaux $LRT(\lambda/\mu, \nu)$ (cf. Section 2).*

It is well known (cf. [27] 1.5.3)) that the multiplicity of an irreducible module in a skew Schur module is also given by an LR coefficient:

$$\langle S_{\lambda/\mu}(E), S_{\nu}(E) \rangle = LR_{\lambda/\mu, \nu}^\lambda.$$ 

(3.16)
The intertwining coefficient of two skew Schur modules $S_{\lambda/\mu}(E)$ and $S_{\sigma/\tau}(E)$ can be computed from the LR rule. For any diagrams $D$ and $D'$ let $D \otimes D' = (D + (r, 0)) \cup (D' + (0, c))$, where $D + (r, 0)$ is the downward translation of the diagram $D$ by the number $r$ of rows of $D'$ and $D' + (0, c)$ is the rightward translation of the diagram $D'$ by the number $c$ of columns of $D$. If $D$ and $D'$ are skew shapes then so is $D \otimes D'$, and $S_{D \otimes D'}(E) \equiv S_D(E) \otimes S_{D'}(E)$. Applying (3.16) and the LR rule to the skew shape $\lambda/\mu \otimes \tau$ and the partition $\sigma$, one immediately obtains the following formula, which has appeared in the literature in various forms (e.g., [42]).

**Corollary 18** (cf. [30]). Let $\lambda/\mu$ and $\sigma/\tau$ be two skew shapes. Then the coefficient

$$\langle S_{\lambda/\mu}(E), S_{\sigma/\tau}(E) \rangle$$

is the cardinality of the set of tableaux $\text{LRT}_c(\lambda/\mu, \sigma - \tau)$ (cf. Section 2 and the Appendix).

## 4. Graded Characters of Coordinate Rings

In this section the graded characters of $\mathbb{C}[G]$ and $\mathbb{C}[X_{(\nu)]}$ are expressed as sums of irreducible characters. The calculations rest on the work of Kostant [19]. These formulas yield index sets for bases of $\mathbb{C}[G]^{\nu}$ and $\mathbb{C}[X_{(\nu)]}^{\nu}$.

The character $\text{ch}(M)$ of the finite dimensional $G$-module $M$ is the Laurent polynomial in the variables $x_1, x_2, \ldots, x_n$ defined by

$$\text{ch}(M) = \text{tr}(x|M)$$

where $\text{tr}(x|M)$ is the trace of the action on $M$ by the diagonal matrix $x$ with eigenvalues $x_1, x_2, \ldots, x_n$. If $M$ is a graded $G$-module whose homogeneous components $M_d$ are $G$-stable and finite dimensional then the graded character $\text{ch}_q(M)$ of $M$ is the formal power series in $q$ defined by

$$\text{ch}_q(M) = \sum_{d \geq 0} q^d \text{ch}(M_d) \tag{4.1}$$

It is convenient to view the ring $k[G]$ in the following way. Let $E$ be the defining $G$-module with fixed basis $\{e_i | 1 \leq i \leq n\}$ and let $E^*$ be its dual vector space with dual basis $\{e^*_j | 1 \leq j \leq n\}$. $G$ acts on $E^*$ by $(g f)(v) = f(g^{-1} v)$ for $f \in E^*, g \in G, v \in E$. $G$ acts diagonally on the tensor product $E \otimes E^*$ by $g(v \otimes f) = g v \otimes f \circ g^{-1}$. The $n \times n$ matrices $g$ can be identified with $E \otimes E^*$ via $E_{ij} \mapsto e_i \otimes e^*_j$, where $E_{ij}$ is the matrix with 1 in the
$(i, j)$-th entry and zeros elsewhere. The action of $G$ on $E \otimes E^*$ corresponds to conjugation on $gl$. The coordinate ring $k[gl]$ of $gl$ is viewed as the symmetric algebra on the dual of $E \otimes E^*$. But $E \otimes E^*$ is self-dual as a vector space via the isomorphism induced by taking transposes in $gl$.

Therefore $k[gl] \cong \text{Sym}(E \otimes E^*)$ where $t_{ij} \mapsto e_i \otimes e_j^*$. This given, we expand $\text{ch}_q(C[gl])$ using the Cauchy formula and Littlewood–Richardson rule.

\[
\text{ch}_q(C[gl]) = \sum_{d \geq 0} q^d \text{ch}(S_d(E \otimes E^*))
= \sum_{d \geq 0} q^d \sum_{|\nu| = d} \text{ch}(S_{\nu}(E) \otimes S_{\nu^*}(E^*))
= \sum q^{\nu} \sum_{a} LR_{\nu a} \text{ch}(S_{a}(E)).
\] (4.2)

The sum runs over all partitions $\nu$. Note that the weights $a$ that appear in the sum satisfy $|a| = |\nu| + |\nu^*| = 0$. Let us rewrite (4.2) using partitions instead of dominant weights. Let $P_k$ be the set of partitions $\lambda$ such that $|\lambda| = kn$ and $f(\lambda) < n$ (that is, $\lambda_n = 0$). Each dominant integral weight $a$ with $|a| = 0$ can be written uniquely as $a = \lambda - (k^n)$ where $k = -a_n$ and $\lambda \in P_k$. Let $\lambda^+ = \lambda + ((\nu_1 - k)\nu)$. Using (3.14) we have

\[
LR_{\nu a} = LR_{\lambda^+ a}.
\] (4.3)

From (4.2) we have

\[
\text{ch}_q(A) = \sum_{k \geq 0} (x_1 \cdots x_n)^{-k} \sum_{\lambda \in P_k} S_{\lambda}(x) \sum_{\nu} q^{\nu} LR_{\nu a} \text{ch}_{S_{\lambda}(E)}(S_{a}(E)).
\] (4.4)

where $S_{\lambda}(x) = \text{ch}(S_{\lambda}(E))$ is the Schur polynomial. This expresses $\text{ch}_q(A)$ in terms of the irreducible characters $\text{ch}(S_{\lambda}(E)) = S_{a}(x) = (x_1 \cdots x_n)^{-k} S_{\lambda}(x)$.

Remark 19. Let $I$ be the set of triples $(\lambda, \nu, T)$ where $\lambda \in P_k$ for some $k$, $\nu$ is a partition, and $T \in \text{LRT}(\lambda^+ / \nu, \nu)$. Formula (4.4) shows that the set $I$ indexes a basis for $C[gl]^U$ such that the triple $(\lambda, \nu, T)$ indexes a $U$-invariant polynomial in $C[gl]$ of degree $|\nu|$ and weight $\lambda - (k^n)$.

Example 20. Let $n = 5$. An example of such a triple $(\lambda, \nu, T) \in I$ is given by $\lambda = (5, 3, 1, 1, 0)$, $\nu = (5, 3, 2, 1, 0)$ and $T$ as in Example 9. Here $k = 2$, $\lambda^+ = (8, 6, 4, 4, 3)$, and $\nu = (5, 4, 3, 2, 0)$.
To obtain another formula for \( \chi_q(A) \) we rewrite the LR coefficient in (4.4):

\[
\text{LR}^\nu_{\nu'} = \left< S_{\lambda^+}(E), S_p(E) \otimes S_{q}(E) \right> = \left< S_{\lambda^+}(E), S_{p+(q \cdot \nu')}(E) \right> \\
= \left< S_{\lambda^+}(E) \otimes S_p(E), S_{p+(q \cdot \nu')}(E) \right> = \left| \text{LRT}_p(\lambda^+, (\nu^\nu)) \right| \\
= \left| \text{LRT}_p(\lambda, (k^n)) \right| .
\]

(4.5)

The second equality uses the isomorphism \( S_p(E) \otimes S_{q}(E) \rightarrow S_{p+(q \cdot \nu')}(E) \) given by (3.13) with \( \alpha = \nu \) and \( \beta = \nu' \). The third equality follows from (3.16). The fourth equality is given by Corollary 18. The last equality is given by the bijection

\[
\text{LRT}_p(\lambda, (k^n)) \rightarrow \text{LRT}_p(\lambda^+, (\nu^\nu))
\]

that adjoins the rectangular key tableau \( K((\nu_1 - k)^n) \) to the left of a tableau in \( \text{LRT}_p(\lambda, (k^n)) \) (cf. the Appendix).

From (4.4) and (4.5) we have

\[
\chi_q(A) = \sum_{k \geq 0} (x_1 \cdots x_n)^{-k} \sum_{\lambda \in \mathcal{P}_k} S_{\lambda}(x) \sum_{\nu} \left| \text{LRT}_p(\lambda, (k^n)) \right|
\]

\[
= \sum_{k \geq 0} (x_1 \cdots x_n)^{-k} \sum_{\lambda \in \mathcal{P}_k} S_{\lambda}(x) \sum_{S \in \text{CST}(\lambda, (k^n))} \sum_{S \text{ is } \nu\text{-lattice}} q^{\nu_1 m} \tag{4.6}
\]

Lemma 11 gives the following parametrization of the set of partitions \( \nu \) such that \( S \) is \( \nu \)-lattice, by \( n \)-tuples of nonnegative integers. Let

\[
m_r = \nu_r - \nu_{r+1} - u_r(S) \tag{4.7}
\]

for all \( 1 \leq r \leq n \), with the conventions \( \nu_{n+1} = 0 \) and \( u_n(S) = 0 \). From (4.6) and (4.7) we have

\[
\chi_q(A) = \sum_{k \geq 0} (x_1 \cdots x_n)^{-k} \sum_{\lambda \in \mathcal{P}_k} S_{\lambda}(x) \sum_{S \in \text{CST}(\lambda, (k^n))} \sum_{m \in \mathbb{Z}_+^n} q^{\sum_{r=1}^{n-1} m_r u_r(S)}
\]

\[
= \prod_{r=1}^{n} (1 - q^r)^{-1} \sum_{k \geq 0} (x_1 \cdots x_n)^{-k} \sum_{\lambda \in \mathcal{P}_k} S_{\lambda}(x) \sum_{S \in \text{CST}(\lambda, (k^n))} q^{\sum_{r=1}^{n-2} m_r u_r(S)} . \tag{4.8}
\]
At this point we require the following theorem of Kostant [19].

**Theorem 21.** (1) $\mathbb{C}[\mathfrak{g}]^G$ is the polynomial ring $\mathbb{C}[t_{ij}]_{1 \leq i, j \leq n}$, where $t_{ij}$ is the coefficient of $(-\lambda)^{n-r}$ in the characteristic polynomial $\det(T - \lambda I)$ of the matrix $T = (t_{ij})_{1 \leq i, j \leq n}$ (Chevalley).

(2) $\mathbb{C}[\mathfrak{g}] \cong \mathbb{C}[X_{1^{\nu}}] \otimes \mathbb{C}[\mathfrak{g}]^G$ as graded $G$-modules.

(3) $X_{1^{\nu}}$ is irreducible and is a complete intersection in the $n \times n$ matrices, cut out by the invariants $t_1, t_2, \ldots, t_n$.

(4) Let $\mathcal{B}_0$ be a set of homogeneous polynomials in $\mathbb{C}[\mathfrak{g}]$ that form a basis of $\mathbb{C}[X_{1^{\nu}}]^G$. Then a basis for $\mathbb{C}[\mathfrak{g}]^G$ is given by the polynomials of the form $\zeta \prod_{i=1}^{n} t_i^{m_i}$ where $\zeta \in \mathcal{B}_0$ and $m_i \in \mathbb{Z}_+$.

It follows that

\[ \text{ch}_q(\mathbb{C}[\mathfrak{g}]) = \text{ch}_q(\mathbb{C}[\mathfrak{g}]^G) \text{ch}_q(\mathbb{C}[X_{1^{\nu}}]) = \prod_{r=1}^{n} (1 - q^r)^{-1} \text{ch}_q(\mathbb{C}[X_{1^{\nu}}]). \]

Combined with (4.8) we have

\[ \text{ch}_q(\mathbb{C}[X_{1^{\nu}}]) = \sum_{k \geq 0} (-1)^k \sum_{\lambda \in \mathcal{P}_k} S_\lambda(x) \sum_{S \in \text{CST}(\lambda, (k^n))} q^{\sum_{i=1}^{n} l_{\lambda_i}(S)} \]

This formula is equivalent to one given in Hesselink [15]. Let $\mu$ be a partition of $n$. Define the formal power series $P_{\alpha\mu}(q) \in \mathbb{Z}_+[q]$ by

\[ \text{ch}_q(\mathbb{C}[X_{\mu}]) = \sum_{a} \text{ch}(S_a(E)) P_{\alpha\mu}(q) \quad (4.10) \]

where $a$ runs over the dominant integral weights. The power series $P_{\alpha\mu}(q)$ is in fact a polynomial. This follows from the inequality

\[ P_{\alpha\mu}(1) = \langle \mathbb{C}[X_{\mu}], S_a(E) \rangle \leq \langle \mathbb{C}[X_{1^{\nu}}], S_a(E) \rangle = |\text{CST}(\lambda, (k^n))| < \infty \]

where $a = \lambda - (k^n)$ with $\lambda \in \mathcal{P}_k$ as usual. This inequality holds due to the canonical epimorphism of graded $G$-modules $\mathbb{C}[X_{1^{\nu}}] \to \mathbb{C}[X_{\mu}]$ given by restriction of functions from $X_{1^{\nu}}$ to the subvariety of $X_{\mu}$.

Now (4.9) can be rewritten as

\[ P_{\alpha(1^{\nu})}(q) = \sum_{S \in \text{CST}(\lambda, k^n)} q^{\sum_{i=1}^{n} l_{\lambda_i}(S)}. \quad (4.11) \]
As observed by Gupta [12], Hesselink's formula [15] for \( \text{ch}_\nu(\mathbb{C}[X_{(\nu)}]) \) is equivalent to

\[
P_{\nu(k^n)}(q) = K_{\lambda(k^n)}(q) \tag{4.12}
\]

where \( K_{\lambda}(q) \) is the Kostka–Foulkes polynomial (cf. [27, Section III.6]). This polynomial has the combinatorial description [24]

\[
K_{\lambda}(q) = \sum_{S \in \text{CST}(\lambda, \tau)} q^{\text{charge}(S)}. \tag{4.13}
\]

The equivalence of the formulas (4.11) and (4.13) with \( \tau = (k^n) \) follows from (2.3).

Remark 22. Let \( I_0 \) be the set of all pairs \((\lambda, S)\) where \( \lambda \in \mathcal{P}_k \) for some \( k \) and \( S \in \text{CST}(\lambda, (k^n)) \). The set \( I_0 \) indexes a basis for \( \mathbb{C}[X_{(\nu)}]^{U(k)} \) such that the pair \((\lambda, S)\) indexes a \( U \)-invariant polynomial in \( \mathbb{C}[X_{(\nu)}] \) of weight \( \lambda - (k^n) \) and degree \( \text{charge}(S) \), by (4.11), (4.13), and (2.3). Let \( I' = I_0 \times \mathbb{Z}_+^n \) be the set of all triples \((\lambda, S, m)\) where \((\lambda, S) \in I_0 \) and \( m \in \mathbb{Z}_+^n \). Theorem 21 shows that \( I' \) indexes a basis for \( \mathbb{C}[\mathbb{Z}]^{T} \) such that the triple \((\lambda, S, m)\) indexes a \( U \)-invariant polynomial of weight \( \lambda - (k^n) \) and degree \( \text{charge}(S) + \sum_{i=1}^n m_i \).

5. BITABLEAUX

This section reviews the bitableau construction and the straightening formula for bitableaux [9] and gives a construction for linear combination of bitableaux which are \( U \)-invariants of prescribed weight and degree.

Let \( E \) and \( F \) be two alphabets. By abuse of notation the alphabet \( E \) will be regarded as an ordered basis for a \( k \)-vector space also denoted by \( E \). The vector space \( E \otimes F \) has a basis given by the symbols \( (a|b) \) for the letters \( a \in E \) and \( b \in F \). Let \( P \) and \( Q \) be tableaux of the partition shape \( \nu \) in the alphabets \( E \) and \( F \), respectively. Define the bitableau \((P|Q) \in \text{Sym}(E \otimes F)\) by

\[
(P|Q) = \prod_s \det((P(i, s)|Q(j, s)))_{1 \leq i, j \leq \nu'} \tag{5.1}
\]

which is the product of determinants coming from the columns of the tableaux. The bitableau \((P|Q)\) is said to have shape \( \nu \). Since permuting the entries in a column of \( P \) or \( Q \) only changes the bitableau \((P|Q)\) by the sign of the permutation, it is clear that up to sign, every bitableau is zero or equal to a bitableau whose columns are strictly increasing. A bitableau is called \textit{standard} if \( P \) and \( Q \) are column strict tableaux in their respective
alphabets. Consider the following partial order on pairs of tableaux \((P, Q)\) of the same partition shape. Let \((P, Q) < (P', Q')\) be defined by the following set of tiebreakers:

1. \(\text{shape}(P) < \text{shape}(P')\).
2. \(\text{colword}(P) <_{\text{lex}} \text{colword}(P')\).
3. \(\text{colword}(Q) <_{\text{lex}} \text{colword}(Q')\).

By abuse of notation we write \((P|Q) < (P'|Q')\) when we mean \((P, Q) < (P', Q')\).

**Theorem 23** (straightening formula). [9]. The standard bitableaux in the alphabets \(E\) and \(F\) form a basis for \(\text{Sym}(E \otimes F)\). Moreover, suppose that \((P|Q)\) is a bitableau that is not standard, where \(P\) and \(Q\) have strictly increasing columns. Then there are unique constants \(c_i \in \mathbb{Z}\) and standard bitableaux \((P_i|Q_i)\) such that

\[
(P|Q) = \sum_i c_i (P_i|Q_i)
\]

where \(\text{content}(P) = \text{content}(P_i)\), \(\text{content}(Q) = \text{content}(Q_i)\), and \((P_i|Q_i) < (P|Q)\) for all \(i\).

Using only the dominance order on the shape of bitableaux, one obtains a filtration of \(\text{Sym}(E \otimes F)\) whose composition factors give the summands of the Cauchy formula. Let \(\nu\) be a partition of the nonnegative integer \(d\). Define the submodules \(M_{\vartriangleleft \nu}\) (resp. \(M_{\triangleleft \nu}\)) of \(\text{Sym}(E \otimes F)\) to be the span of all bitableaux whose shape is a partition of \(d\) and is less than or equal to \(\nu\) (resp. strictly less than \(\nu\)) in the dominance order.

**Corollary 24.** \(\text{Sym}(E \otimes F)\) has a filtration by the \(\text{GL}(E) \times \text{GL}(F)\)-modules \(M_{\vartriangleleft \nu}\) such that

\[
M_{\vartriangleleft \nu}/M_{\triangleleft \nu} \cong S_\nu(E) \otimes S_\nu(F).
\]

**Remark 25.** The following specialization of the bitableau construction yields a construction of Specht for the Schur module \(S_\nu(E)\). Consider the \(\text{GL}(E)\)-module \(S_\nu(E)\) given by the subspace of \(\text{Sym}(E \otimes F)\) spanned by the bitableaux of shape \(\nu\) of the form \((\nu|K(\nu))\), where \(P\) is a tableau in the alphabet \(E\). There is an isomorphism of \(\text{GL}(E)\)-modules \(S_\nu(E) \rightarrow S_\nu(E)\) given by \(d_\nu(e_P) \mapsto (P|K(\nu))\). A similar result holds when the roles of \(E\) and \(F\) are switched.

The rest of the section is devoted to the construction of linear combinations of bitableaux that are \(U\)-invariants in \(k[\mathfrak{g}] \cong \text{Sym}(E \otimes E^*)\).
Let \( \nu \) be a partition of \( d \). Let \( \phi_\nu \) be the \( GL(E) \times GL(F) \)-module homomorphism
\[
\phi_\nu : \Lambda^{D(\nu)}(E) \otimes \Lambda^{D(\nu)}(F) \to S_d(E \otimes F)
\]
given by
\[
\phi_\nu(e_P \otimes f_Q) = (P|Q)
\]
where \( P \) and \( Q \) are tableaux of shape \( \nu \) in the alphabets \( E \) and \( F \), respectively, and \( e_P \) and \( f_Q \) are defined as in (3.6).

Given a bijection of diagrams \( b : D \to D' \), we construct a \( GL(E) \)-module map \( \psi_b : \Lambda^{D}(E) \to \Lambda^{D'}(E) \) as follows. Let \( D_{ij} = D_i \cap b^{-1}(D'_j) \) be the set of cells in the \( i \)th column \( D_i \) of \( D \) whose images under \( b \) lie in the \( j \)th column \( D'_j \) of \( D' \). The collection of sets \( D_{ij} \) for fixed \( i \) and varying \( j \) is a set partition of \( D_i \). Similarly the collection \( b(D_{ij}) \) for fixed \( j \) and varying \( i \) is a set partition of \( D'_j \). Define the \( GL(E) \)-module map \( \psi_b : \Lambda^{D}(E) \to \Lambda^{D'}(E) \) by

\[
\Lambda^{D}(E) = \bigotimes_i \Lambda^{D_i}(E) \\
\bigotimes_i \Lambda^{D'_j}(E) \\
\bigotimes_j \Lambda^{D'_j}(E) = \Lambda^{D'}(E).
\]
The second map identifies tensor positions in \( D \) with those in \( D' \) via the bijection \( b \).

Recall that \( k[q] \) can be identified with \( \text{Sym}(E \otimes E^*) \) by the \( G \)-equivariant isomorphism given by \( t_{ij} \mapsto e_i \otimes e_j^* \). For \( 0 \leq r \leq n \) define the \( G \)-equivariant map \( \theta_r : k \to \Lambda^r(E) \otimes \Lambda^r(E^*) \) by
\[
1 \mapsto \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} (e_{i_1} \wedge \cdots \wedge e_{i_r}) \otimes (e_{i_r}^* \wedge \cdots \wedge e_{i_1}^*) \quad (5.4)
\]
The composition \( \phi_{(\nu)} \circ \theta_r : k \to S_r(E \otimes E^*) \) sends 1 to the basic \( G \)-invariant \( t_r \) for \( 1 \leq r \leq n \) (cf. Theorem 21).

We now give a construction that produces many \( U \)-invariants in \( k[q] \). Let \( \lambda \in \mathcal{P}_k \) for some \( k \) and \( a = \lambda - (k^n) \). Write \( a = \alpha + \beta^* \) where \( \alpha \) and \( \beta \) are partitions with \( k(\alpha) + k(\beta) \leq n \). Let \( \nu \) be a partition (of the
integer $d$, say) that contains $\alpha$ and $\beta$. Let $j_\alpha: D(\alpha) \to D(\nu)$ and $j_\beta: D(\beta) \to D(\nu)$ be arbitrary injections of sets. Write $I_\alpha$ and $I_\beta$ for the images of $j_\alpha$ and $j_\beta$, respectively. Let $b: D(\nu) \to I_\alpha \to D(\nu) - I_\beta$ be an arbitrary bijection and let $B_{ij}$ be the set of cells in the $i$th column of the diagram $D(\nu) - I_\beta$ whose images under $b$ lie in the $j$th column of $D(\nu) - I_\beta$. Let $\theta_{B_{ij}}: k \to \Lambda^{B_{ij}}(E) \otimes \Lambda^{h(B_{ij})}(E^*)$ be the map $\theta$ given by (5.4) where $r = |B_{ij}|$ and the tensor positions of $\Lambda^r(E)$ and $\Lambda^r(E^*)$ are indexed by the sets $B_{ij}$ and $b(B_{ij})$, respectively.

Given the data $(\lambda, \nu, j_\alpha, j_\beta, b)$, we construct the following $G$-equivariant map $\Xi_{\lambda, \nu, j_\alpha, j_\beta, b}$:

$$\Lambda^{D(\alpha)}(E) \otimes \Lambda^{D(\beta)}(E^*) \xrightarrow{\phi_{\alpha} \otimes \phi_{\beta} \otimes \theta_{B_{ij}}^{\otimes k \otimes 2}} \Lambda^{I_\alpha}(E) \otimes \Lambda^{I_\beta}(E^*) \otimes \bigotimes_{1 \leq i, j \leq v_1} (\Lambda^{B_{ij}}(E) \otimes \Lambda^{h(B_{ij})}(E^*))$$

$$\xrightarrow{\bigotimes_{i=1}^{r_1} (\Lambda^{I_{\lambda_i}}(E) \otimes \bigotimes_{j=1}^{B_{ij}}(E)) \otimes \bigotimes_{i=1}^{r_2} (\Lambda^{I_{\nu_i}}(E^*) \otimes \bigotimes_{j=1}^{B_{ij}}(E^*))} \Lambda^{D(\nu)}(E) \otimes \Lambda^{D(\nu)}(E^*) \xrightarrow{\phi} S_\beta(E \otimes E^*).$$

The third map groups the tensor positions by columns and the fourth is given by exterior multiplication within columns.

Let $K^*(\beta)$ denote the column strict tableau of shape $\beta$ in the alphabet $[n]^\ast$ whose $i$th row is comprised of $\beta_i$ copies of the letter $(n + 1 - i)^\ast$:

$$\xi_{\lambda, \nu, j_\alpha, j_\beta, b} = \Xi_{\lambda, \nu, j_\alpha, j_\beta, b}(e_{K(\alpha)} \otimes e_{K^*(\beta)}^\ast). \quad (5.6)$$

This polynomial has degree $|\nu|$ and weight $\lambda - (k^n)$ and is $U$-invariant, since it is the image of an obviously $U$-invariant vector under a $G$-equivariant map.
6. A BASIS FOR $C[\mathcal{R}]^U$

In the previous section we constructed an assortment of $U$-invariants $\xi_{\lambda, \nu, T} \in C[\mathcal{R}]$. A basis $\mathcal{B}$ for $C[\mathcal{R}]^U$ will be obtained by selecting some of these $U$-invariants, based on the combinatorial description of the index set $\mathcal{I}$ (cf. Remark 19).

Throughout this construction we adopt the notation at the end of Section 5. For each triple $(\lambda, \nu, T) \in \mathcal{I}$, let $\xi_{\lambda, \nu, T} \in C[\mathcal{R}]^U$ be the homogeneous polynomial of degree $|\nu|$ and weight $\lambda - (k^n)$ where $\lambda \in \mathcal{P}_k$, given by

$$\xi_{\lambda, \nu, T} = \xi_{\lambda, \nu, j_{\alpha}, j_{\beta}}$$

(cf. (5.6)) where $j_{\alpha}: D(\alpha) \rightarrow D(\nu)$ and $j_{\beta}: D(\beta) \rightarrow D(\nu)$ are the inclusion maps and $b_T: \nu/\alpha \rightarrow \nu/\beta$ is the bijection defined as follows. Recall that $T \in \text{LRT}(\lambda^+, \tilde{\nu}, \nu)$. Slice $T$ vertically just to the right of the $\nu_i$th column, obtaining right and left subtableaux $T_r$ and $T_l$, respectively.

Let $T_r$ be the rigid 180-degree rotation of $T_l$ inside the rectangular partition $(\nu_i^+)$, and let $T_i$ be the rigid 180-degree rotation of $T_j$ inside the rectangular partition $(\nu_i^+)$, respectively. Note that the shapes of $T_r$ and $T_l$ are $\alpha$ and $\nu/\beta$, respectively. Since $T$ is lattice, the shape of $T_r$ is a partition, and colword($T$) = colword($T_l$)colword($T_r$), it follows that $T_r$ is lattice of content $\alpha$, that is, $T_r = K(\alpha)$ (cf. Remark 8). Finally, let $b_T$ be the unique bijection that sends the cells in the $i$th row of $\nu/\alpha$, from right to left, to the cells in $T_l$ that contain the letter $i$, from right to left, for all $i$.

**Example 26.** Let $n = 5$, $\nu = (5, 3, 2, 1, 0)$, $\lambda = (5, 3, 1, 1, 0)$ and $T$ be the tableau given in Example 9. We have $k = 2$, $\alpha = (3, 1, -1, -1, -2)$, $\beta = (2, 1, 1, 0, 0)$, $\lambda^+ = a + (\nu_i^+) = (8, 6, 4, 3)$, and $\tilde{\nu} = (5, 4, 3, 2, 0)$. Note that $(\lambda, \nu, T) \in \mathcal{I}$. The tableaux $T_r$, $T_l$, and $T_i$ are given by

$$T_r = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 3 & 4 \end{array}$$

$$T_i = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array}$$

$$T_l = \begin{array}{cccc} 1 & 4 & 3 & 2 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}$$
The bijection $b_T$ is depicted by the following pair of tableaux. Cells that correspond under the bijection $b_T$ contain the same letter.

$$
\begin{array}{ccccccc}
& & & a_2 & a_1 & & \\
& & a_4 & a_3 & & a_7 & a_5 & a_3 \\
a_6 & a_5 & & & a_6 & a_4 & \\
a_7 & & & & a_1 & & \\
\end{array}
$$

**Remark 27.** The bijection $b_T$ satisfies the following properties:

(B1) If $s$ and $s'$ are in the same row of $\nu/\alpha$ with $s'$ to the west of $s$, the $b_T(s')$ is strictly west and weakly south of $b_T(s)$ in $\nu/\beta$. This follows by definition and the fact that $T$ is column strict.

(B2) If $s$ and $s'$ are in the same column of $\nu/\alpha$ with $s'$ south of $s$ then $b_T(s')$ is strictly north and weakly east of $b_T(s)$. This is proven using the lattice property of $T$ (cf. the Appendix).

(B3) The intersection of any column of $\nu/\beta$ with the image under $b_T$ of any column of $\nu/\alpha$ is a connected set of cells (cf. the Appendix).

(B4) The inverse bijection $b_T^{-1}: \nu/\beta \rightarrow \nu/\alpha$ also satisfies the properties (B1) and (B2), where $b_T$ is replaced by $b_T^{-1}$ and the roles of $\alpha$ and $\beta$ are switched (cf. the Appendix).

The polynomial $\xi_{\nu, \nu, T}$ can be expressed explicitly as a sum of bitableaux, by computing the image of the $U$-invariant $e_{K(\alpha)} \otimes e_{K^*(\beta)}$ under the map $\Xi_{\nu, \nu, T} = \Xi_{\alpha, \nu, j_\alpha, j_\beta, b_T}$ (cf. (5.5)). Using the definitions of the maps in (5.5) and careful bookkeeping, it follows that the polynomial $\xi_{\nu, \nu, T}$ is the sum of all bitableaux $(P' | Q')$ with $P'$ and $Q'$ filled with letters in the respective alphabets $[n]$ and $[n]^\alpha$, such that:

(T1) The restriction of the tableau $P'$ to the shape $\alpha$ is $K(\alpha)$.

(T2) The restriction of the tableau $Q'$ to the shape $\beta$ is $K^*(\beta)$.

(T3) If $P'(s) = r$ then $Q'(b_T(s)) = r^\ast$ for all $s \in \nu/\alpha$.

(T4) If $s$ and $s'$ are cells in the same column of $\nu/\alpha$ with $s'$ south of $s$ and $b_T(s)$ and $b_T(s')$ are in the same column of $\nu/\beta$ (so that necessarily $b_T(s')$ is north of $b_T(s)$ by (B2)) then $Q'(b_T(s)) > Q'(b_T(s'))$ in the alphabet $[n]^\beta$.

No extra signs appear since $j_\alpha$ and $j_\beta$ are inclusions and $b_T$ satisfies (B3).

**Example 28.** Let us return to Example 26. The vector $\xi_{\nu, \nu, T}$ is the sum of the bitableaux $(P' | Q')$ whose form is given below, where each of the letters $a_1$ through $a_7$ ranges freely over the alphabet $[5]$; condition (T4) is
vacuous since the bijection $b_1$ sends no pair of cells in the same column of $\nu/\alpha$ to the same column of $\nu/\beta$.

$$
P' = \begin{bmatrix} 1 & 1 & 1 & a_2 & a_1 \\ 2 & a_4 & a_3 \\ a_6 & a_5 \\ a_7 \end{bmatrix} \quad Q' = \begin{bmatrix} 5^* & 5^* & a_7^* & a_6^* & a_5^* \\ 4^* & a_5^* & a_4^* \\ 3^* & a_3^* \end{bmatrix}
$$

**Theorem 29.** The set

$$\mathcal{B} = \{ \xi_{\lambda, \nu, T} \mid (\lambda, \nu, T) \in I \}$$

forms a basis for $C[\{\tilde{q}\}]^\nu$.

**Proof.** By construction, $\mathcal{B}$ contains the correct number of $U$-invariants of each degree and weight. So it is enough to show that $\mathcal{B}$ is linearly independent. Fix partitions $\nu$ and $\lambda \in \mathcal{P}_k$ for some $k$. Let $a = \lambda - (k^*)$.

Let $V$ be the $a$th weight subspace of the subfactor $M_{\leq \nu}/M_{<\nu}$ of $C[\{\tilde{q}\}]$. By Theorem 23 and the weight space decomposition, it is enough to show that the set $(\xi_{\lambda, \nu, T} \mid T \in \text{LRT}(\lambda^*/\tilde{\nu}, \nu))$ is linearly independent in $V$. Let $W$ be the subspace of $V$ consisting of linear combinations of tableaux $(P|Q)$ where the content of $P$ equals $\nu$. Let $\pi : V \to W$ be the projection of $V$ onto $W$. It is enough to show that the set

$$\{ \pi(\xi_{\lambda, \nu, T}) \mid T \in \text{LRT}(\lambda^*/\tilde{\nu}, \nu) \}$$

is linearly independent in $W$. By Theorem 23, every element of $W$ can be expressed uniquely as a linear combination of bitableaux of the form $(K(\nu)|Q)$, where $Q$ is a column strict tableau of shape $\nu$ in the alphabet $[n]^*$; this follows from the fact that $K(\nu)$ is the unique column strict tableau of shape and content $\nu$.

Now fix $T \in \text{LRT}(\lambda^*/\tilde{\nu}, \nu)$. Recall the explicit expression of $\xi_{\lambda, \nu, T}$ as a sum of bitableaux:

$$\xi_{\lambda, \nu, T} = \sum (P'|Q')$$

where the bitableaux $(P'|Q')$ satisfy conditions (T1)–(T4). Applying the linear map $\pi$, we have

$$\pi(\xi_{\lambda, \nu, T}) = \sum (P'|Q')$$

(6.2)

where, in addition to conditions (T1)–(T4), the bitableau $(P'|Q')$ also satisfies the condition that content$(P') = \nu$. We may also assume that each column of $P'$ has distinct letters. Since $P'$ has shape and content $\nu$,
\( P' \) must be obtainable from \( K(\nu) \) by permuting some letters within columns. In the right hand side of (6.2) there is a unique bitableau whose left tableau is \( K(\nu) \); let \( Q_T \) be its right tableau. It is enough to prove the following assertions, which show that the vectors \( \pi(\xi_{\alpha,\nu,T}) \) for varying \( T \) are unitriangularly related to the standard bitableau basis of \( W \).

1. The bitableau \((K(\nu)|Q_T)\) is standard.
2. \((K(\nu)|Q_T)\) occurs with coefficient 1 as the leading bitableau in the expansion of \( \pi(\xi_{\alpha,\nu,T}) \) as a linear combination of standard bitableaux.
3. The map \( T \mapsto Q_T \) is injective.

The tableau \( Q_T \) has the following explicit description. The restriction of \( Q_T \) to the shape \( \beta \) equals \( K^*(\beta) \) by (T2) and the restriction of \( Q_T \) to the shape \( \nu/\beta \) is given by rotating the tableau \( T_I \) by 180 degrees and replacing each letter \( r \) by \( r^* \) in light of (T3). Since \( T_I \) is a column strict tableau in the alphabet \([n]\), it follows that the restriction of \( Q_T \) to the shape \( \nu/\beta \) is a column strict tableau in the alphabet \([n]^*\). The subtableau \( K^*(\beta) \) of \( Q_T \) is column strict and consists of letters in \([n]^*\) which are as small as possible, subject to the column strictness condition in the alphabet \([n]^*\). It follows that \( Q_T \) is column strict and that the bitableau \((K|Q_T)\) is standard. The injectivity of the map \( T \mapsto Q_T \) follows from the explicit description of \( Q_T \) just given. This proves 1 and 3.

Consider any nonzero bitableau \((P'|Q')\) other than \((K(\nu)|Q_T)\) in the right hand side of (6.2). \( P' \) must be obtainable from \( K(\nu) \) by a nontrivial permutation of letters within the columns of the skew shape \( \nu/\alpha \). Let \( Q' \) be the tableau obtained from \( Q \) by sorting its columns in strictly increasing order in the alphabet \([n]^*\). It is not hard to see that \( \text{colword}(Q') <_{\text{lex}} \text{colword}(Q_T) \) in the alphabet \([n]^*\), by (B2) and (T4). It follows from Theorem 23 that \((P'|Q')\) can be written as a linear combination of standard bitableaux, all of which are strictly less than \((K(\nu)|Q_T)\). Therefore the standard bitableau \((K(\nu)|Q_T)\) occurs in \( \pi(\xi_{\alpha,\nu,T}) \) as its leading term with coefficient 1, proving 2.

7. A BASIS FOR \( C[X_{(1^r)}]^U \)

Theorem 29 gave a basis \( \mathcal{B} \) of \( C[Q]^U \) indexed by the set \( I \) (cf. Remark 19). The goal of this section is to realize a basis \( \mathcal{B}_{(1^r)} \) for \( C[X_{(1^r)}]^U \) as a distinguished subset of \( \mathcal{B} \). By Theorem 21 we know that there exists a basis \( \mathcal{B}' \) for \( C[Q]^U \) indexed by the set \( I' \) (cf. Remark 22). The existence of such a basis is now used to find the desired subset \( \mathcal{B}_{(1^r)} \) of \( \mathcal{B} \). We define a bijection \( \Psi: I' \rightarrow I \). The set \( I' \) has an obvious subset \( I_0 \) that indexes a basis for \( C[X_{(1^r)}]^U \), namely, the set of triples of the form \((\lambda, S, (0^n))\).
We define $\mathcal{B}_{\lambda}$ to be the subset of $\mathcal{B}$ that is indexed by the subset $\Psi(I_0)$ of $I$, and show that $\mathcal{B}_{\lambda}$ is a basis for $C[X(\lambda)]^\Psi$.

Given the triple $(\lambda, S, m) \in I'$ where $\lambda \in \mathcal{P}_k$, define the triple $\Psi(\lambda, S, m) = (\lambda, \nu, T) \in I$ as follows. Let $\nu$ be the partition given by

$$
\nu_r - \nu_{r+1} = m_r + u_r(\text{ev}(S)) \quad \text{for } 1 \leq r \leq n - 1
$$

$$
\nu_n = m_n
$$

(7.1)

Of course $\nu_r - \nu_{r+1}$ is the number of columns of $\nu$ of length $r$. A quick calculation using (2.3) shows that $|\nu| = \text{charge}(S) + \sum_{r=1}^n m_r$, so that $\Psi$ preserves degree (cf. Remarks 19 and 22). The tableau $\text{ev}(S)$ is $\nu$-lattice by Lemma 11, so $S$ is $\nu$-lattice by Proposition 12. Let $S^+$ be the column strict tableau of shape $\lambda^+$ and content $(\nu_1^+, \ldots, \nu_n^+)$ obtained by adjoining the key tableau $K((\nu_1 - k)^o)$ to the left of $S$. In the Appendix it is shown that $S^+ \equiv K(\nu)$, so that we may define

$$
T = w_0(S^+ - K(\nu)),
$$

(7.2)

where $w_0$ is the plactic permutation operator for the longest element of the symmetric group on $n$ letters. The following lemma shows immediately that $\Psi(\lambda, S, m) = (\lambda, \nu, T)$ is a well-defined bijection.

**Lemma 30.** The map $S \mapsto w_0(S^+ - K(\nu))$ gives a bijection

$$
\text{LRT}_i(\lambda, (k^o)) \rightarrow \text{LRT}(\lambda^+/\nu, \nu).
$$

(7.3)

**Proof.** Cf. the Appendix. □

**Example 31.** Let $S' = S^+ - K(\nu)$. The calculation of the action of the plactic permutation operator $w_0$ on $S'$ is a lengthy process involving the successive actions of plactic adjacent transpositions. This example illustrates a shorter alternative method to compute $w_0 S'$. The crucial facts used here are $P(S') = K' = K(\nu_1, \nu_2 - 1, \ldots, 1)$ and $w_0 K(\nu) = K'$. The proof of Lemma 30 does not depend on the algorithm presented here. Let $Z$ be any standard tableau of shape $\nu$ in a separate alphabet.

1. Form the tableau $Z + S'$ of shape $\lambda^+$ whose restrictions to the shapes $\nu$ and $\lambda^+/\nu$ are $Z$ and $S'$, respectively. Perform the jeu de taquin (§32) that slides the skew tableau $S'$ into the cells of the Ferrers diagram $D(\nu)$ in the order determined by $Z$; the resulting tableau is $K'$. Let $Y$ be the standard tableau of shape $\lambda^+/\nu$ that indicates the order of cells vacated during this jeu.

2. Perform the jeu de taquin that slides the tableau $K(\nu)$ into the cells of $Y$. The resulting tableau is $w_0(S')$, and the cells are vacated in the order given by the standard tableau $Z$. 
Let $S$ be the tableau of Example 10 and $n = 5$, $k = 2$, $\nu = (53210)$, and $\tilde{\nu} = (54320)$. Depicted below are the starting and ending tableaux for the first jeu de taquin, where the starting inner tableau $Z$ is standard in the alphabet $a_1 < a_2 < \cdots < a_{14}$:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & a_3 & a_4 & a_5 \\
1 & 2 & 3 & 4 & 5 & a_7 & a_8 & a_9 \\
1 & 2 & 3 & 4 & 5 & a_{11} & a_{12} & 3 \\
1 & 2 & 3 & 4 & 5 & a_{13} & a_{14} & 4 \\
5 & 5 & 5 & 2 & 3 & 4 & 5 & a_3 & a_4 & a_5 \\
3 & 4 & 5 & a_7 & a_8 & a_9 \\
4 & 5 & a_2 & a_{12} \\
5 & a_1 & a_{11} & a_{14} \\
6 & a_{10} & a_{13} \\
\end{array}
\]

The next two tableaux are the starting and ending tableaux for the second jeu de taquin. Note that at the end one obtains the same inner tableau $Z$:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & a_3 & a_4 & a_5 \\
2 & 2 & 2 & a_7 & a_8 & a_9 \\
3 & 3 & a_2 & a_{12} \\
4 & a_1 & a_{11} & a_{14} \\
6 & a_{10} & a_{13} & a_1 & a_2 & a_3 & a_4 & a_5 & 1 & 1 & 1 \\
6 & a_7 & a_8 & a_9 & 1 & 2 \\
10 & a_{11} & a_{12} & 1 \\
13 & a_{14} & 2 & 3 \\
2 & 3 & 4 \\
\end{array}
\]

The following lemma characterizes the image under $\Psi$ of the triples of the form $(\lambda, S, (0^n)) \in \mathcal{I}$. Say that the (skew) column strict tableau $T$ is $p$-excessive if the $(n + 1 - r)$-th row of $T$ contains the letter $p + 1 - r$ for all $1 \leq r \leq p$.

**Lemma 32.** Let $T \in \text{LRT}(\lambda^+/\tilde{\nu}, \nu)$ and $\Psi(\lambda, S, m) = (\lambda, \nu, T)$.

1. $T$ is $p$-excessive if and only if $m_p > 0$.

2. Suppose $T$ is $p$-excessive. Let $\tilde{\nu}$ (resp. $\hat{\nu}$) be the partition obtained by removing a column of length $p$ (resp. $n - p$) from $\nu$ (resp. $\tilde{\nu}$). Let $\hat{T}$ be the skew tableau of shape $(\lambda^+/\nu)/\nu$ whose $(n + 1 - r)$-th row is obtained
from that of \( T \) by removing a letter \( p + 1 - r \), for \( 1 \leq r \leq p \). Then \( \hat{T} \) in \( \text{LRT}(\lambda^{-1} \langle 1^p \rangle) / \hat{\nu} \) and \( \Psi(\lambda, S, \hat{m}) = (\lambda, \hat{\nu}, T) \), where \( \hat{m}_i = m_i \) for \( i \neq p \) and \( \hat{m}_p = m_p - 1 \).

**Proof.** Cf. the Appendix.

The main algebraic lemma is the following.

**Lemma 33.** Suppose that \( T \in \text{LRT}(\lambda^+ / \hat{\nu}, \nu) \) is \( p \)-excessive. Then up to sign, the leading terms in the expansion into standard bitableaux of the polynomials \( \xi_{\lambda, v, T} \) and \( t_p \xi_{\lambda, \hat{v}, \hat{T}} \) agree.

Before giving the proof of Lemma 33 we derive some corollaries.

Let \( B^{(1^p)}_{\nu} \) be the subset of \( B \) consisting of the polynomials \( \xi_{\lambda, S} = \xi_{\Psi(\lambda, S, (1^p))} \) for all \( (\lambda, S) \in I_0 \) (cf. Remark 22). In light of Theorem 21, let \( B' \) the collection of polynomials

\[
\xi_{\lambda, S, m} = \xi_{\lambda, S} \prod_{r=1}^{n} t_{m_r}^{m_r}
\]  

(7.5)

for all \((\lambda, S, m) \in I'\).

**Corollary 34.** 1. \( B^{(1^p)}_{\nu} \) is a basis for \( C[X(1^p)]^V \).

2. \( B' \) is a basis for \( C[\Omega]^V \).

**Proof.** The leading bitableaux in the expansions of \( \xi_{\lambda, v, T} \) and \( \xi_{\lambda, S, m} \) into standard bitableaux agree up to sign, by Lemma 33 and induction. Then 2 follows immediately from Theorem 29. To prove 1, recall that Remark 22 shows that \( B^{(1^p)}_{\nu} \) contains the correct number of vectors of each weight and degree, so that it is enough to show that \( B^{(1^p)}_{\nu} \) spans \( C[\Omega]^V \). But this is certainly true, since each polynomial in \( B' - B^{(1^p)}_{\nu} \) is a multiple of some basic invariant and hence vanishes on \( X(1^p) \) by Theorem 21.

We now give the proof of Lemma 33.

**Proof.** The notation of the proof of Theorem 29 will be used here. Let \( \pi \) (resp. \( \hat{\pi} \)) be the projections associated with the shape \( \nu \) (resp. \( \hat{\nu} \)) and weight \( \lambda - (k^p) \). Both \( \xi_{\lambda, v, T} \) and \( t_p \xi_{\lambda, \hat{v}, \hat{T}} \) are \( U \)-invariants of weight \( \lambda - (k^p) \). Consider the defining expression of \( \xi_{\lambda, \hat{v}, \hat{T}} \) as a linear combination of bitableaux:

\[
\xi_{\lambda, \hat{v}, \hat{T}} = \sum (\hat{P}\hat{Q}).
\]  

(7.6)

The invariant \( t_p \) (cf. Theorem 21) can be written as a sum of single-column bitableaux as follows. If \( Y \subseteq [n] \) is given by \( Y = \{ y_1 < y_2 < \cdots < y_p \} \), let \( (Y|Y^*) \) be the bitableau of shape \( (1^p) \) whose left (resp. right) tableau
consists of the letters \( y_1, y_2, \ldots, y_p \) (resp. \( y^*_p, y^*_{p-1}, \ldots, y^*_1 \)) from top to bottom. Then we have the following formula for \( t_p \):

\[
t_p = (-1)^{p(p-1)/2} \sum_{Y \subseteq [n], |Y| = p} (Y|Y^*).
\]

(7.7)

Multiplying equations (7.6) and (7.7), we have

\[
t_p \xi_{\lambda}, \hat{\nu} \hat{\nu} = (-1)^{p(p-1)/2} \sum (P|Q)
\]

(7.8)

where a typical summand \( (P|Q) \) comes from adding the column \( Y \) (resp. \( Y^* \)) to the tableau \( P \) (resp. \( Q \)), for some subset \( Y \subseteq [n] \) with \( |Y| = p \) and some summand \( (P', Q') \) in Eq. (7.6). Suppose that the shape \( \hat{\nu} \) has \( m \) columns of length greater than or equal to \( p \). Since a bitableau is a product of determinants coming from the columns of the right and left tableaux, we may assume that \( P \) (resp. \( Q \)) is obtained by inserting the column \( Y \) (resp. \( Y^* \)) between the \( m \)th and \( (m + 1) \)-st columns of \( \hat{P} \) (resp. \( \hat{Q} \)). Suppose that \( \pi((P'|Q')) \neq 0 \). It is easy to see that \( Y = [p] \) and \( \pi((P|Q)) \neq 0 \). It follows from Lemma 35 below that the right tableau of the leading term of the expansion into standard bitableaux of \( \pi((P|Q)) \) is obtained from that of \( \pi((\hat{P}, \hat{Q})) \) by adding the letter \( r^* \) to the \( (p + 1 - r) \)-th row for \( 1 \leq r \leq p \) and sorting the rows into weakly increasing order. Therefore the leading terms of \( t_p \xi_{\lambda}, \hat{\nu} \hat{\nu} \) and \( \xi_{\lambda, \nu, T} \) agree up to sign.

**Lemma 35.** Let \( \hat{\nu} \) be a column strict tableau in the alphabet \([n]^*\) of shape \( \hat{\nu} = [p]^* \) the column containing the letters \( p^*, \ldots, 2^*, 1^* \) from top to bottom, \( m \) the number of columns of \( \hat{\nu} \) whose lengths are at least \( p \), \( Q \) the tableau obtained by placing the column \( C \) between the \( m \)th and \( (m + 1) \)-st columns of \( \hat{\nu} \) and \( \hat{K} = K(\hat{\nu}) \). Then the leading term of the expansion of \( (K|Q) \) into standard bitableaux is \( (K|Q') \), where \( Q' \) is obtained by sorting each of the rows of \( Q \) into weakly increasing order in the alphabet \([n]^*\).

**Proof.** It is enough to produce a sequence of tableaux \( Q = Q_{m+1}, Q_{m+2}, \ldots \) such that all of the bitableaux \( (K|Q_i) \) have the same leading term when expanded as a sum of standard bitableaux and the first \( i - 1 \) columns of \( Q_i \) and \( Q' \) agree for all \( i > m \). Let \( c_r \) be the leftmost column in \( Q' \) containing the letter \( r^* \), for \( 1 \leq r \leq p \). Since the tableau \( Q \) is column strict and \( C \) consists of the largest \( p \) letters in the alphabet \([n]^*\), it follows that \( Q' \) is a column strict tableau and that \( m + 1 = c_1 \leq c_2 \leq \cdots \leq c_p \).

For \( s > m + 1 \) let \( Q_s \) be the tableau obtained from \( Q_{s-1} \) by switching the first \( k_s \) entries in the \((s - 1)\)-st and \( s \)th columns, where \( k_s \) is the number of indices \( r \) such that \( c_r \geq s \). It is easy to see from this definition that the
first \(i - 1\) columns of \(Q\) and \(Q_i\) agree for all \(i > m\). Assume that the
bitableaux \((K|Q)\) and \((K|Q_i)\) have the same leading term for \(m + 1 \leq i < s\). It is enough to show that the bitableau \((K|Q_{s-1})\) can be expressed as

\[
(K|Q_{s-1}) = (K|Q_s) + \sum c_j(K|T_j)
\]

where the \(T_j\) have strictly increasing columns, the constants \(c_j\) are in the
set \((-1, 0, 1)\), and \text{colword}(T_j) < \text{colword}(Q_s).\) The reader can verify that
such a relation is obtained by applying Lemma 36 to the \((s - 1)\)-st and \(s\)th
columns of \(Q_{s-1}\) using the index \(k = k_s.\)

\textbf{Lemma 36 (quadratic relations).} Let \(K = K(\nu), T\) a tableau of shape \(\nu,\)
\(C_1\) and \(C_2\) two of the columns of \(T\) (viewed as words read from top to
bottom), and \(k\) a nonnegative integer, such that \(k \leq |C_2| \leq |C_1|\). Then

\[
(K|T) = \sum_s (K|T_s)
\]

where \(S\) runs over all subwords of \(C_2\) of length \(k\) and \(T_s\) is obtained from \(T\) by
exchanging the \(i\)th letter in \(C_2\) with the \(i\)th letter in \(S\) for all \(1 \leq i \leq k\).

\textbf{Proof.} By Remark 25, this relation follows from the well-known quadratic relations for the Schur module.

\textbf{Example 37.} Here is the sequence of tableaux that occurs in the
straightening described in Lemma 35 for \(n = 5, p = 3\) and the tableau \(\hat{Q}\)
given below. It is easy to verify that \(m = 1, (c_1, c_2, c_3) = (2, 3, 4)\) and
\((k_3, k_4, k_5) = (2, 1, 0).\) The letters in the column \([3]^n\) are underlined.

\[
\begin{array}{cccccc}
5^* & 5^* & 4^* & 2^* & \hat{Q} = & 4^* & 3^* & 5^* & 4^* & 2^* \\
4^* & 3^* & 3^* & 1^* & Q = & Q_2 = & 4^* & 2^* & 3^* \\
1^* & 2^* & 1^* & 1^* & 5^* & 5^* & 4^* & 3^* & 2^* \\
3^* & 3^* & 2^* & 1^* & Q_3 = Q_4 = & Q_5 = & 4^* & 3^* & 2^* \\
3^* & 1^* & 1^* & 1^* & & & 3^* & 1^* \\
1^* & 1^* & 1^* & 1^* & & & 1^* \\
\end{array}
\]
Example 38. The relation of Lemma 36 used in the proof of Lemma 35 applied to the tableau $Q_2$ in the previous example is given by

\[(K|Q_2) = (K|Q_3) + (K|J_1) + (K|J_2)\]

where $K = K(5,3,2,1)$ and $J_1$ and $J_2$ are given by

\[
J_1 = \begin{array}{cccc}
5^* & 5^* & 3^* & 4^* \\
4^* & 2^* & 1^* & 3^* \\
3^* & 3^* & 1^* & \\
1^* & & &
\end{array}
\]

\[
J_2 = \begin{array}{cccc}
5^* & 3^* & 2^* & 4^* \\
4^* & 5^* & 1^* & 3^* \\
3^* & 3^* & &
\end{array}
\]

To get the relation exactly as described in Lemma 35, delete the bitableau $J_2$ (which is zero) and sort the columns of the tableaux $J_1$ into increasing order in the alphabet $[5]^*$, obtaining $J_1'$ and a sign. Then

\[(K|Q_2) = (K|Q_3) - (K|J_1').\] (7.10)

8. BASES FOR $\mathbb{C}[X_\mu]^Y$

8.1. Basis Conjecture for $\mathbb{C}[X_\mu]^Y$

Recall that the index set $I_0$ for the basis $\mathcal{B}_{\mathbb{C}[y]}$ is given by the pairs $(\lambda, S)$ where $\lambda \in \mathcal{P}_k$ and $S \in CST(\lambda, (k^n))$. For each such tableau $S$ we associate a partition of $n$ called the catabolism type $\text{cattype}(S)$ of $S$. This is a generalization of a construction of Lascoux and Schützenberger [21]. Suppose that $S$ contains the rectangular key tableau $K = K(k^m)$ for the positive integer $m$. Note that this always holds for $m = 1$. Let $S_u$ and $S_l$ be the upper and lower subtableaux obtained by slicing the skew tableau $S - K$ just after the $m$th row. The catabolism type $\text{cat}_{(k^m)}(S)$ is defined by $P(\text{word}(S_u), \text{word}(S_l))$. The catabolism type of $S$ is the sequence of positive integers $\text{cattype}(S) = \tau$ defined by the properties:

(BC1) $\tau_1$ is the largest number $m$ such that $S$ contains the subtableau $K(k^m)$.

(BC2) The catabolism type of the tableau $\text{cat}_{(k^m)}(S)$ is $(\tau_2, \tau_3, \ldots)$, with respect to the alphabet $[\tau_1 + 1, n]$. 
Example 39. Let \( k = 3 \) and \( n = 6 \). A tableau \( S \) of catabolism type \( \tau = (3, 2, 1) \) is given below.

\[
\begin{array}{cccccc}
1 & 1 & 1 & 4 & 4 & 5 \\
2 & 2 & 2 & 5 & 5 & 6 \\
3 & 3 & 3 & 6 & 6 & 6 \\
\end{array}
\]

\[
S - K(3^3) =
\begin{array}{cccccc}
\square & \square & \square & 4 & 4 & 5 \\
\square & \square & \square & 5 & 5 & 6 \\
\square & \square & \square & 4 & 6 & 6 \\
\end{array}
\]

\[
\text{cat}_{(3^3)}(S) = 5 \quad 5 \quad 5 \\
\]

\[
\text{cat}_{(3^3)}(S) - K(0^3, 3^2) =
\begin{array}{cccccc}
\square & \square & \square & 6 & 6 \\
\square & \square & \square & 6 \\
\end{array}
\]

\[
\text{cat}_{(3^3)}^\ast \text{cat}_{(3^3)}^\ast \text{cat}_{(3^3)}^\ast S = 6 \quad 6 \quad 6 \\
\text{cat}_{(3^3)}^\ast \text{cat}_{(3^3)}^\ast S = \emptyset
\]

Corollary 45 below asserts that the catabolism type is a partition.

Let \( I^\mu \) be the subset of \( I_0 \) (cf. Remark 22) consisting of the pairs \((\lambda, S)\) such that \(\text{cattype}(S) \geq \mu\), and let \( \mathcal{B}_\mu \) be the subset of \( \mathcal{B}^\mu \) given by the polynomials \( z_{\lambda, S} \) where \((\lambda, S) \in I^\mu \) (cf. (7.5)).

Conjecture 40. \( \mathcal{B}_\mu \) is a basis for \( \mathbb{C}[X^\mu] \).

8.2. Hook Partitions

Let us first prove Conjecture 40 when \( \mu \) is a hook \((n - p, 1^p)\). In this case the variety \( X^\mu \) is the set of nilpotent matrices in \( \mathfrak{g} \) with rank at most \( p \). We rely on Hesselink’s theorem [14] that \( X^\mu \) is a complete intersection in the determinantal variety of \( C_p \) of matrices in \( \mathfrak{g} \) of rank at most \( p \), cut out by the first \( p \) basic \( G \)-invariants \( t_r \) for \( 1 \leq r \leq p \).

It is necessary to study an appropriate restriction of the bijection \( \Psi \).

Lemma 41. Let \((\lambda, S) \in I_0 \) with \( \lambda \in \mathcal{P}_k \) and let \( \Psi(\lambda, S, (0^n)) = (\lambda, \nu, T) \), where \( \Psi: I^0 \to I^1 \) is the bijection given in (7.1) and (7.2). Then \(\text{cattype}(S) \geq (n - p, 1^p)\) if and only if \( \nu \) has at most \( p \) nonzero parts.

Proof. The following are equivalent:

1. \(\text{cattype}(S) \geq (n - p, 1^p)\).
2. \( S \mid_{[1, n - p]} = K(k^{n-p}) \).
3. \( S \mid_{[1, n - p]} \) is lattice of content \((k^{n-p})\).
4. \( u_r(S) = 0 \) for \( 1 \leq r \leq n - p - 1 \).
5. \( u_r(\text{ev}(S)) = 0 \) for \( p + 1 \leq r \leq n - 1 \).
6. \( \nu \) has no column of length strictly exceeding \( p \).
The equivalence of each assertion with the next, is given by definition, Remark 8, Lemma 11, Proposition (12), and (7.1).

**Theorem 42.** \( \mathfrak{B}_{(n-p, 1^p)} \) is a basis for \( \mathbb{C}[X_{(n-p, 1^p)}]^\circ \).

**Proof.** By Lemma 41 it is enough to show that \( \mathbb{C}[C_p]^\circ \) has a basis given by the collection \( \mathfrak{B}^n \) of polynomials \( \xi_{\lambda, S, m} \in \mathfrak{B}^n \) (cf. (7.5)) with \( S \in I_{(n-p, 1^p)} \) and \( m_i = 0 \) for \( i > p \). It follows from Theorem 23 that \( \mathbb{C}[C_p]^\circ \) has a basis consisting of the standard bitableaux with at most \( p \) rows. Every polynomial in \( \mathfrak{B}^n - \mathfrak{B}^p \) is a linear combination of bitableaux that have more than \( p \) rows, and hence vanishes on \( C_p^\circ \). Thus \( \mathfrak{B}^n \) spans \( \mathbb{C}[C_p]^\circ \). It follows from Lemma 33 that the polynomials in \( \mathfrak{B}^n \) have distinct leading bitableaux with at most \( p \) rows. Therefore \( \mathfrak{B}^n \) is independent.

8.3. Two Part Partitions

Now let us consider the case where \( \mu \) is a two part partition \( (n-p, p) \) where \( 2p \leq n \). The variety \( X_{(n-p, p)} \) is the set of \( n \times n \) matrices with square zero and rank at most \( p \). The defining ideal of the variety \( X_{(n-p, p)} \) is generated by the \( (p+1) \times (p+1) \) minors of the generic matrix \( (t_{ij})_{1 \leq i, j \leq n} \) and the "ideal of traces," generated in degree two by the matrix entries of the square of the generic matrix, that is, the elements of the form \( \sum_{r=s}^{n} t_{is} t_{sr} \) for all \( 1 \leq r, s \leq n \). These elements of degree two generate an ideal that is the linear span of the polynomials in \( \mathbb{C}[\mathfrak{B}] \) of the form

\[
\sum_{i=1}^{n} (P^i | Q^i)
\]

where \( P \) and \( Q \) are tableaux of the same partition shape in the alphabets \([n]\) and \([n]^\circ\), respectively, and \( P^i \) (resp. \( Q^i \)) is obtained from \( P \) (resp. \( Q \)) by replacing the entry in a fixed cell \( s \) (resp. \( s' \)) by the letter \( i \) (resp. \( i' \)).

The coordinate ring \( \mathbb{C}[X_{(n-p, p)}] \) has the multiplicity-free decomposition [36]

\[
\mathbb{C}[X_{(n-p, p)}] = \bigoplus_{i(a) \leq p} S_{(a_1, a_2, ..., a_p, a_{p-2}, -a_p, -a_{p-1}, ..., -a_1)}(E)
\]

**Example 43.** Let \( n = 7 \) and \( k = 3 \). The unique tableau \( S \) of shape \((6,5,4,3,2,1)\) and catabolism type \((4,3)\) is given below.

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\[ (8.3) \]
Lemma 44. Let $a$ be a dominant integral weight with $|a| = 0$ and let $\lambda \in \mathcal{P}_k$ such that $\lambda - (k^n) = a$.

(1) A column strict tableau $S$ of shape $\lambda$ and content $(k^n)$ has two part catabolism type if and only if its $j$th and $(2k + 1 - j)$-th columns are respectively initial and final complementary subintervals of the alphabet $[n]$ for $1 \leq j \leq k$. Hence there is at most one tableau of shape $\lambda$ whose catabolism type has two parts, and such a tableau exists if and only if $a = a^*$.

(2) A tableau $S$ has two part catabolism type if and only if the corresponding $U$-invariant $\zeta_{\lambda,S}$ has no traces, that is, $\nu = \alpha = \beta$ where $\Psi(\lambda, S, (0^n)) = (\lambda, \nu, T)$ in the notation of Sections 6 and 7.

Proof. First consider the backward implication in (1). Let $n - p$ and $p$ be the lengths of the $k$th and $(k + 1)$-st columns of $S$, respectively. Since $S$ has partition shape, its $k$th column must be weakly longer than its $(k + 1)$-st, that is, $n - p \geq p$. Furthermore the first $k$ columns of $S$ must have length at least $n - p$. By hypothesis they all contain the initial interval $[1, n - p]$ of $[n]$. It follows that the first part of the catabolism type of $S$ is equal to $n - p$. Let $\alpha$ be the shape of the tableau $S_n$ given in the definition of the catabolism type. Note that $\ell(\alpha) = p$. The shape $\gamma$ of $S_n$ and $\alpha$ are related by $\gamma_i = k - \alpha_{p+1-i}$ for $1 \leq i \leq p$. In terms of these partitions, $S_l = K(0^{n-p}, \gamma)$ and $S_n = K(0^{n-p}, \alpha, \alpha_{p-1}, \ldots, \alpha_1)$. A direct computation shows that $P(\text{word}(S_n), \text{word}(S_i)) = K(0^{n-p}, k^p)$, using the fact that two strictly decreasing words commute under Knuth equivalence if one is a subword of the other. Thus the catabolism type of $S$ is $(n - p, p)$.

To prove the forward implication in (1), let $S$ be a column strict tableau of shape $\lambda$ and content $(k^n)$ with catabolism type $(n - p, p)$. This means that $S_{[1, n-p]} = K(0^{n-p})$ and that $P(\text{word}(S_n), \text{word}(S_i)) = K(0^{n-p}, k^p)$. By Lemma 11, the juxtaposition $\text{word}(S_n) \text{word}(S_i)$ is lattice in the alphabet $[n - p + 1, n]$, so that its final subword $\text{word}(S_i)$ is also. Let $\gamma = (\gamma_1, \ldots, \gamma_p)$ be the shape of the subtableau $S_i$ (where some of these parts may be zero). Then $S_i = K(0^{n-p}, \gamma)$ by Remark 8. Furthermore $\gamma_1 < k$, for otherwise $S$ would contain the larger rectangular key tableau $K(k^{p+1})$.

Applying the reverse complement operator $\#$ with respect to the alphabet $[n - p + 1, n]$ to $\text{word}(S_n) \text{word}(S_i)$, it follows from Theorem 2 (1) and Remark 8 that $S_u$ is the tableau $K(0^{n-p}, k - \gamma_1, k - \gamma_2, \ldots, k - \gamma_p)$, the unique column strict tableau of shape $\alpha = (k - \gamma_p, k - \gamma_{p-1}, \ldots, k - \gamma_1)$ and content $(0^{n-p}, \alpha, \alpha_{p-1}, \ldots, \alpha_1)$, or equivalently, the unique column strict tableau of shape $\alpha$ whose $j$th column consists of a final subinterval of $[n]$. It follows that $S$ has the desired form. Note that $S_{u'}$, which has at most $n - p$ rows by definition, has exactly $n - p$ rows since $k - \gamma_1 > 0$, so that $n - p \geq p$. 
To finish (1), note that if $S$ exists then

$$a = \lambda - (k^n) = (\alpha_1, \alpha_2, \ldots, \alpha_p, 0^{n-2p}, -\alpha_p, -\alpha_{p-1}, \ldots, -\alpha_1),$$

so that $a = a^s$. Conversely, let $a$ be a dominant weight with $a = a^s$. The partition $\alpha$ is given by the positive parts of $a$, and $p = l(\alpha)$. Let $\gamma$ and $\alpha$ be related as above. It has been shown that the column strict tableau $S$ described in (1) has shape $\lambda$ and catabolism type $(n-p, p)$.

To prove (2), using the explicit form of $S$ determined above, it is easy to see that $u(S)$ is the number of the first $k$ columns of $S$ that have length $r$, for $1 \leq r \leq n - 1$. By Proposition 12, $u(S) = u_{a-r}(ev(S))$. By the definition of $\Psi$ (cf. (7.1)), the number of columns of $\nu$ of length $n - r$ agrees with the number of the first $k$ columns of $S$ of length $r$ and the total number of columns $n_1$ of $\nu$ is $k$. It follows that the shape of $S$ is $\lambda$ and that the partition $\bar{\nu}$ is the shape of the first $k$ columns of $S$ (or of $ev(S)$).

Therefore, the shape $\lambda$ is obtained by placing the shapes $\bar{\nu}$ and $\nu$ side by side. It follows that the partitions $\alpha$ and $\beta$ (defined by $a = \alpha + \beta^*$ and $l(\alpha) + l(\beta) \leq n$) are both equal to $\nu$, and the $U^\mu$-invariant $\xi_{\lambda, S} = \xi_{\lambda, \nu, T}$ is given by the bitableau $(K(\nu) | K^*(\nu))$.

**Corollary 45.** The catabolism type is a partition.

**Proof.** Let $S$ have shape $\lambda$, content $(k^n)$, and catabolism type $\tau$. By induction applied to $\text{cat}_{k^{r+1}}(S)$ it follows that $\tau_2 \geq \tau_3 \geq \cdots$. Now $S \mid_{[1, \tau_1 + \tau_2]}$ has catabolism type $(\tau_1, \tau_2)$ since K\text{}nuth equivalence is preserved under restriction to intervals. But then $\tau_1 \geq \tau_2$ by the proof of Lemma 44.

Conjecture 40 for $\mu = (n-p, p)$ follows immediately from Lemma 44 and the decomposition (8.2) of $\mathbb{C}[X_{(a-p, p)}]$.

**Theorem 46.** $\mathcal{B}_{(a-p, p)}$ is a basis for $\mathbb{C}[X_{(a-p, p)}]^U$.

**Remark 47.** In the proofs of Theorems 42 and 46, one can replace either the proof of spanning or of linear independence by the following explicit conjectural formula for the Poincaré polynomial of isotypic components of $\mathbb{C}[X_{\lambda}]$:

$$P_{\mu, \lambda}(q) = \sum_{\text{shape}(S) = \lambda, \text{content}(S) = (k^n), \text{cattype}(S) \geq \mu} q^{|\text{charge}(S)|}$$

(8.4)

The special case $k = 1$ is more or less known [21] although no proof yet exists in the literature. Technically, in order to make the connection with [21] one must use the identity (for $k = 1$) $P_{\mu, \lambda}(q) = q^{\sum (-1)^{\mu} K_{\lambda, \mu}(q^{-1})}$ where $K_{\lambda, \mu}(q)$ is the Kostka–Foulkes polynomial. Under this correspondence the direction of slicing in our definition is orthogonal to that in [21].
The polynomials $P_{\mu}(q)$ have a defining recurrence given in [40] (6.6). The authors have a sign-reversing, weight-preserving involution that shows that the right hand side of (8.4) satisfies this recurrence when $\mu$ is a hook or has two parts. This proof can be altered slightly to give a combinatorial proof of (4.13). This cancellation is different from that in the proof in Lynne Butler’s thesis [6], which clarifies and completes the original argument of Lascoux and Schützenberger [24]. The authors believe this cancellation will prove not only (8.4) for all $\mu$, but also a generalization of (8.4) for the Poincaré polynomial of isotypic components of the twist of the coordinate ring $C[x_\mu]$ by certain vector bundles [37]. These Poincaré polynomials are $q$-analogues of Littlewood-Richardson coefficients associated to sequences of partitions.

The form of (8.4) implies that there are polynomials $R_{\mu}(q) \in \mathbb{Z}[q]$ such that $P_{\mu}(q) = \sum_{\nu \preceq \mu} R_{\nu}(q)$. It is a consequence of [28] that such a formula must hold; see [3].

9. UNIVERSAL BASES FOR $k[X_\mu]$

9.1. A $\mathbb{Z}$-form for $\mathbb{C}[X_\mu]$

For any commutative ring $R$, we define a graded $R$-algebra and $GL(R)$-module $A_{\mu, R}$ with the property $A_{\mu, \mathbb{Z}} \otimes \mathbb{Z} R = A_{\mu, R}$ for all $R$ and $A_{\mu, k} = k[X_\mu]$ for any algebraically closed field $k$. This shows that any basis for $A_{\mu, \mathbb{Z}}$ is universal, meaning that it also gives a basis of $A_{\mu, R}$ for all $R$ and hence for $k[X_\mu]$ for all $k$. Furthermore it is clear that the actions of the diagonal matrix group are preserved, so that the graded character of $k[X_\mu]$ is independent of the characteristic of the field $k$.

Fix a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s > 0)$ of $n$. Let $E_{\mathbb{Z}}$ be the free $\mathbb{Z}$-module of rank $n$ with basis $(e_1, \ldots, e_n)$. Denote by $\mathcal{F}_{\mathbb{Z}}$ the partial flag scheme whose $\mathbb{Z}$-valued points are the partial flags of free $\mathbb{Z}$-submodules of $E_{\mathbb{Z}}$. The scheme $\mathcal{F}_{\mathbb{Z}}$ is constructed by patching together affine spaces. Its dimension equals $d(\mu) = \sum_{i=1}^{s} \mu_i (n - \mu_2 - \cdots - \mu_s)$. The scheme $\mathcal{F}_{\mathbb{Z}}$ is constructed by patching together the open sets $U_I$, each isomorphic to the affine space over $\mathbb{Z}$ of dimension $d(\mu)$. The datum $I$ is a sequence $I = (I_{\mu_1} \subset I_{\mu_1 + \mu_2} \subset \cdots \subset I_{\mu_1 + \cdots + \mu_s} \subset \{1, n\})$. The cardinality of $I_{\mu_1 + \cdots + \mu_s}$ is $\mu_1 + \cdots + \mu_s$.

For each $I$ the set $U_I$ may be identified with the subset of flags of $E_{\mathbb{Z}}$. Namely, define the vectors $v_j \in E_{\mathbb{Z}}$ by the formula

$$v_j = e_j + \sum_{s \in I_{\mu_1 + \cdots + \mu_s}} x_j e_s$$
for $\mu_1 + \cdots + \mu_n + 1 \leq j \leq \mu_1 + \cdots + \mu_n + 1$. The functions $x_{j,x}$ are the coordinate functions in $U$. Given a point corresponding to an $n$-tuple $(v_1, \ldots, v_n)$, associate the flag $F$, where $F_i$ is the submodule of $E_2$ generated by $v_1, \ldots, v_{\mu_i + \cdots + \mu_j}$.

This rule also tells how to identify parts of sets $U_i$ and $U_j$; the points in $U_i$ and $U_j$ are patched together when the corresponding flags are equal.

Next, define the subscheme $Z$ of $\mathcal{F}_m \times \text{End}_z(E_2)$ by

$$Z = \{(F, A) \mid A(F_1) \subset F_{1-i}, i = 1, \ldots, s\}.$$  

We represent the endomorphism $A$ by its matrix $A = (a_{i,j})$. This means $Ae_j = \sum_{i=1}^n a_{i,j}e_i$. Let us fix $I$. In the open subset $U_i \times \text{End}_z(E_2)$ it is easy to write down explicit equations of $Z$. These equations are obtained by writing $Av_j$ as the linear combination of the vectors $v_j$ from the appropriate $\mu$-blocks. The number of these equations equals the codimension of $Z$ in $\mathcal{F}_m \times \text{End}_z(E_2)$, which equals $\sum_{i=1}^n \mu_i (n - \mu_1 - \cdots - \mu_{i-1})$. The equations correspond to certain entries $a_{i,j}$. To understand this correspondence, let us define the index $\text{ind}(i)$ of the element $i \in [1, n]$ to be the smallest number $m$ such that $i \in I_{\mu_1 + \cdots + \mu_n}$.

Then the defining equations of $Z$ correspond to entries $a_{i,j}$ such that $\text{ind}(i) \geq \text{ind}(j)$. The equation corresponding to $a_{i,j}$ is of the form

$$a_{i,j} = \sum_{s,t; \text{ind}(s) < \text{ind}(i)} a_{s,t} f_{s,t},$$

where $f_{s,t}$ is a polynomial in the coordinates $x_{s,t}$.

This means that for the projection $p_2: Z \to \mathcal{F}_m$, there is a covering of the base for which the preimage of each open affine set $U_i$ is given by a polynomial ring over $\text{Spec}(O(U))$ in some of the variables $a_{i,j}$.

Define $X_{m,z}$ to be the image of the projection $q_2: Z \to \text{End}_z(E_2)$. Now let

$$\mathcal{F}_m = \mathcal{F}_m \times \text{Spec}_z \text{Spec} R,$$

$$\text{End}_z(E_2) = \text{End}_z(E_2) \times \text{Spec}_z \text{Spec} R,$$

$$Z = Z \times \text{Spec}_z \text{Spec} R.$$

There are natural projections $p_2: Z \to \mathcal{F}_m$ and $q_2: Z \to \text{End}_z(E_2)$. Define $X_{m,R}$ to be the image of $q_2$. When $R$ is an algebraically closed field $k$, it shown in [28] that $X_{m,k}$ is the nilpotent conjugacy class closure $X_{m}$ defined over $k$. Moreover, $q_2(\mathcal{O}(Z_k)) = \mathcal{O}(X_k)$, and in [28] it is shown that

$$H^i(Z_k, \mathcal{O}(Z_k)) = 0 \text{ for } i > 0.$$  (9.1)
Taking global sections, we have

$$H^0(Z_k, \mathcal{O}(Z_k)) \cong k[X_\mu]$$  \hspace{1cm} (9.2)

since $X_\mu$ is affine.

Let $A_{\mu, R}$ be the $R$-algebra of global sections $H^0(Z_R, \mathcal{O}(Z_R))$. By [10, Section 6.6] there is an exact sequence

$$0 \to H^i(Z_k, \mathcal{O}(Z_k)) \otimes_k k \to H^i(Z_R, \mathcal{O}(Z_R))$$

$$\to \text{Tor}_1^R(H^{i+1}(Z_k, \mathcal{O}(Z_k)), k)$$  \hspace{1cm} (9.3)

for algebraically closed fields $k$. By (9.3) it follows that $H^i(Z_k, \mathcal{O}(Z_k)) = 0$ for $i > 0$. Setting $i = 0$ in (9.3) and using $i = 1$ in (9.1) yields the isomorphism $H^0(Z_k, \mathcal{O}(Z_k)) \otimes_k k \cong H^0(Z_R, \mathcal{O}(Z_R))$. It follows that

$$A_{\mu, Z} \otimes_k k \cong A_{\mu, k}$$  \hspace{1cm} (9.4)

for any field $k$. In particular, if $k$ is algebraically closed, (9.2) shows that

$$A_{\mu, Z} \otimes_k k \cong k[X_\mu].$$  \hspace{1cm} (9.5)

9.2. Good Filtrations

Let $k$ be an algebraically closed field. We employ the theory of good filtrations to construct bases of $k[X_\mu]$ from suitable sets of $U$-invariants, again for special $\mu$.

Let $V$ be a finite dimensional $G$-module over $k$. A good filtration [7] of $V$ is a chain of $G$-submodules

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_m = V$$  \hspace{1cm} (9.6)

such that each factor $V_j/V_{j-1}$ is isomorphic to a Schur module $S_{\mu_j}(E)$.

Some facts on good filtrations:

(F1) All Schur modules have a good filtration.

(F2) If $V$ is a $G$-module with a filtration $(V_j)$ such that $V_j/V_{j-1}$ has a good filtration for all $j$, then so does $V$. In particular, if each of a finite family of $G$-modules has a good filtration then so does their direct sum.

(F3) If each of a finite family of $G$-modules has a good filtration then so does their tensor product [7, 39]. An explicit good filtration for the tensor product of two Schur modules is given in [4].

(F4) Although the Schur modules $S_{\mu}(E)$ are not in general irreducible in positive characteristics, the notion of the “multiplicity” $m_{\mu}(V)$ of the
Schur module $S_a(E)$ in a $G$-module $V$ still makes sense if $V$ has a good filtration. Define the nonnegative integers $m_a(V)$ by

$$\text{ch}(V) = \sum_a m_a(V) \text{ch}(S_a(E)).$$

The formal character can be written this way since it is additive on exact sequences and $V$ has a good filtration. Such an expression is necessarily unique. The multiplicity $m_a(V)$ is also given by the number of $j$ such that $V_j/V_{j-1} \cong S_a(E)$, where $\{V_j\}$ is any good filtration of $V$.

**Proposition 48** ([7, Lemma 3.2.3]). Let $V$ be a finite dimensional $G$-module with a good filtration, $V \leq a$ (resp. $V < a$) the span of all submodules of $V$ that have a good filtration whose factors are isomorphic to Schur modules $S_a(E)$ such that $b \leq a$ (resp. $b < a$). Then $V \leq a/V < a \cong S_a(E)^{b m_a(V)}$.

The following technical lemma gives a construction of a basis for a $G$-module with a good filtration, given a suitable set of $U$-invariants realized as images under $G$-equivariant maps of highest weight vectors in tensor products of exterior powers. This will be used to construct bases for the coordinate rings $k[X]$ for special $\mu$.

For a dominant weight $a$, write $a = \alpha + \beta^*$ where $\alpha$ and $\beta$ are partitions with $l(\alpha) + l(\beta) \leq n$.

**Lemma 49.** Let $V$ be a finite dimensional $G$-module with a good filtration, $a$ a dominant weight, $m = m_a(V)$, and $c_a \in \Lambda^{D(\alpha)}(E) \otimes \Lambda^{D(\beta)}(E^*)$ the vector given by $c_a = e_{K(a)}^* \otimes e_{K^*(\beta)}^*$. Suppose that $\{\Xi_i \mid 1 \leq i \leq m\}$ is a family of $G$-equivariant maps

$$\Xi_i : \Lambda^{D(\alpha)}(E) \otimes \Lambda^{D(\beta)}(E^*) \to V$$

such that the set $\{\Xi(c_i) \mid 1 \leq i \leq m\}$ is linearly independent. Then for each $i$, $\Xi_i$ induces an embedding $\Xi_i : S_a(E) \to V \leq a/V < a$ such that $\Xi = \bigoplus \Xi_i$; $S_a(E)^{b m_a} \to V \leq a/V < a$ is an isomorphism. Furthermore, a basis for $V \leq a/V < a$ is given by

$$\Xi_i(e_{P_i} \otimes e_{Q_i}^*)$$

where $1 \leq i \leq m$ and $(P_i, Q_i)$ is a sequence of complementary pairs of tableaux of type $(\alpha, \beta^*)$.

**Proof.** Consider the $G$-module $M = \Lambda^{D(\alpha)}(E) \otimes \Lambda^{D(\beta)}(E^*)$. Since $M$ is the tensor product of Schur modules, it has a good filtration by (F1) and (F3). It follows from the LR rule that $m_a(M) = 1$ and that $m_a(M) > 0$ implies $b < a$. Since a $G$-equivariant map is also equivariant for the action of the torus $T$ of diagonal matrices in $G$, it follows that $1 \text{Im}(\Xi_i) \subseteq V \leq a$. 
Recall the natural projection $d_a$ defined by (3.11). Applying a similar argument to each of the direct sums of exterior powers that generate the kernel of the map $d_a$ (cf. 3.3), it follows that $\Xi(d_a) \subseteq \mathcal{V}_a$. Thus $\Xi$ induces a map $\Xi: S_\lambda(E) \to \mathcal{V}_a$. In light of Proposition 48 the map $\Xi = \bigoplus_{i=1}^n \Xi_i$ may be viewed as a $G$-equivariant endomorphism of $\bigoplus_{i=1}^n S_i(E)$. However, the only $G$-equivariant endomorphisms of $S_i(E)$ are given by scalar multiplication $7, 1.5.3$. This implies that $J$ is the highest weight vector $d_a(c_a)$ in $S_a(E)$. By hypothesis, the set $\{\Xi_i(v) \mid 1 \leq i \leq m\}$ is linearly independent and consists of vectors of weight $a$. But the weight $a$ does not occur the module $\mathcal{V}_a$. Therefore $\Xi(v) \mid 1 \leq i \leq m$ is linearly independent and $\Xi$ is an isomorphism. The basis for $\mathcal{V}_a/\mathcal{V}_a$ is a consequence of Remark 16.

**Theorem 50.** (1) The $G$-module $k[\Omega]_d = S_\lambda(E \otimes E^*)$ has a basis given by the polynomials of the form $\Xi_{\lambda, \nu, T}(e_p \otimes e_\nu^{\otimes d})$, where $(\lambda, \nu, T) \in I, |\nu| = d$ and $(P, Q)$ is a complementary pair of type $(\alpha, \beta^*)$.

(2) When the partition $\mu$ has the form $(n - p, p)$ or $(n - p, 1^p)$ the $G$-module $k[X_\mu]_d$ has a basis given by the subset of the above polynomials such that $\Psi(\lambda, \nu, T) = (\lambda, S, 0^\gamma)$ where $(\lambda, S) \in I_\mu$.

**Proof.** These results will follow from Lemma 49. The multiplicities $m_\mu(k[\Omega]_d)$ and $m_\mu(k[X_\mu]_d)$ are independent of the characteristic of $k$, by (9.5).

(1) The module $k[\Omega]_d$ has an explicit good filtration, given as follows. The coarse filtration of $k[\Omega]_d$ given by the submodules $M_{\lambda, \nu, T}$ has filter quotients isomorphic to $S_{\lambda}(E) \otimes S_\nu(E^*)$ for $|\nu| = d$, by Corollary 24. But these factors have explicit good filtrations (cf. Property (F) of good filtrations).

Fix a dominant integral weight $a$ with $|a| = 0$; these are the only weights such that $m_a(k[\Omega]_d) > 0$. Let $\lambda \in \mathcal{P}_k$ such that $\lambda - (k^\gamma) = a$. Consider the maps $\Xi_{\lambda, \nu, T}$ for $(\lambda, \nu, T) \in I$ with $|\nu| = d$. By the proof of Theorem 29, the polynomials $\xi_{\lambda, \nu, T}$ are linearly independent, since they have distinct leading bitableaux that have coefficient $1$. Therefore Lemma 49 applies, proving (1).

(2) For any $\mu$, the module $k[X_\mu]_d$ has a good filtration [8, 28]. Suppose that $\mu$ is the hook $(n - p, 1^p)$ or the two part partition $(n - p, p)$. In arbitrary characteristic, the methods of the proofs of Theorems 46 and 42 show that the elements of $\mathcal{B}_\mu$ in degree $d$ and weight $a$ are $m_a(k[X_\mu]_d)$ linearly independent $U$-invariants, Lemma 49 then applies.
APPENDIX

Except for the proof of Theorem 4, this section contains proofs that were deferred from Sections 6 and 7. We emphasize that all of the results related to the Littlewood–Richardson rule (with the exception of the notion of $p$-excessiveness) are all well known in one form or another.

Proof sketch of Theorem 4. (2) and (6) hold by definition. The proof of (3) is straightforward. So is that of (4), which is seen by computing the action of $s_r$ on a pair of words that differ by an elementary Knuth equivalence. To prove (7) it may be assumed that $\rho = s_r$ and that $u$ is a single letter, in which case (7) may be verified directly. Let $n$ be the largest letter of $w$. For later use, it may be directly verified that in the notation of Theorem 2,

$$(s,w)\# = s_{n-r}(w\#) \quad \text{for } 1 \leq r \leq n-1. \quad (A.1)$$

For (5) it may be assumed that $\rho = s_r$. Suppose $n > r + 1$. Let $1 \leq i_1 < \cdots < i_m \leq N$ be the positions of $n$ in $w$ where $N$ is the length of $w$. Let $Q'(w |_{[n-1]})$ be the recording tableau whose alphabet is the complement of the set $\{i_1, \ldots, i_m\}$ in $[N]$. It is well known in the folklore of the Robinson–Schensted–Knuth correspondence that

$$Q(w) = P\left( \text{word}(Q'(w |_{[n-1]}))i_1i_2 \cdots i_m \right).$$

By (6) and induction, $Q'(w |_{[n-1]}) = Q(s_r(w |_{[n-1]})) = Q'(w |_{[n-1]})$. It follows that $Q(s_r w) = Q(w)$. So it may be assumed that $n = r + 1$. Now $\text{ev}(Q(s_r w)) = Q((s_r w)\#) = Q(s_r(w\#))$ and $\text{ev}(Q(w)) = Q(w\#)$ by Theorem 2(3) and (10.1), so it may be assumed that $n = 1$. By the previous reduction it may be assumed that $n = 2$. In this easy situation (5) may be verified directly. For (1) the only nontrivial relation to check is $s_a s_b s_a w = s_b s_a s_b w$ where $b = a + 1$. Let $I$ be the set of positions in $w$ occupied by letters not in the interval $(a, b, b + 1)$. By (6), $I$ is also the corresponding sets of positions for $s_a s_b s_a w$ and $s_b s_a s_b w$. Thus it may be assumed that $w$ is a word in the alphabet $(a, b, b + 1)$. For convenience let $a = 1$ and $b = 2$. By the bijectivity of the Robinson–Schensted–Knuth correspondence it is enough to check that $P(s_1 s_2 s_1 w) = P(s_2 s_1 s_2 w)$. In particular we may replace $w$ by any Knuth-equivalent word. First choose $w$ to be the column-reading word of a column-strict tableau of partition shape. Since any column of height 3 must remain invariant under both $s_1$ and $s_2$, such columns may be removed so that the tableau $w$ has two rows. Now choose $w$ to be the row-reading word of a two-rowed tableau in the alphabet $(1, 2, 3)$. Let $w = uv$ where $v$ and $u$ are the first and second rows of this tableau, so that $u$ contains no ones. By (7) we may replace $w = uv$ by $uu$
and then by a previous reduction by the row-reading word of $P(vu)$. But note that the second row of the tableau $P(vu)$ can only contain threes. So now assume that $w = u'v'$ where $u'$ and $u'$ are the first and second rows of a column-strict tableau in the alphabet $(1, 2, 3)$ such that $u'$ consists only of threes. A gain by (7) we may replace $w$ by $v'u'$ which is a weakly increasing word. But a word $w$ is weakly increasing if and only if $Q(w)$ is a one-row tableau. By (2) and (5) this means both $s_3s_2s_1w$ and $s_2s_1s_2w$ are weakly increasing words of the same content and hence must be equal.

We require yet another description of the Littlewood--Richardson multiplicities. The following notion of pictures is taken from Zelevinsky [42] whose definition was inspired by that in [17]. Consider the partial orders on the set $\mathbb{Z}^2$ given by

$$(i_1, j_1) \trianglelefteq_s (i_2, j_2) \quad \text{if } i_1 \leq i_2 \text{ and } j_1 \leq j_2$$

$$(i_1, j_1) \trianglelefteq_s (i_2, j_2) \quad \text{if } i_1 \leq i_2 \text{ and } j_1 \geq j_2.$$ 

Note that partition diagrams (resp. skew diagrams) are precisely the finite subsets of $\mathbb{Z}^2$ that are initial (resp. convex) with respect to the order $\trianglelefteq_s$. Consider the following linear extensions of the partial order $\trianglelefteq_s$:

$$(i_1, j_1) < s_w(i_2, j_2) \quad \text{if } i_1 < i_2 \text{ or } i_1 = i_2 \text{ and } j_1 > j_2$$

$$(i_1, j_1) < s_w(i_2, j_2) \quad \text{if } j_1 > j_2 \text{ or } j_1 = j_2 \text{ and } i_1 < i_2.$$ 

Let $D$ and $D'$ be finite subsets of $\mathbb{Z}^2$. A picture is a bijection $f: D \rightarrow D'$ such that for all $x, y \in D$ with $x \trianglelefteq_s y$, $f(x) \leq_s f(y)$, and for all $x, y \in D'$ with $x \trianglelefteq_s y$, $f^{-1}(x) \leq_s f^{-1}(y)$. Let $\text{PIC}(D, D')$ be the set of pictures $D \rightarrow D'$.

**Theorem 51** [42]. Let $D$ and $D'$ be skew diagrams. Then

$$\langle S_D(E), S_{D'}(E) \rangle = |\text{PIC}(D, D')|.$$

Let us compare Theorem 51 with Corollary 18 using standard tableaux as an intermediate set. The following definition is due to White [41]; this particular description is taken from [31].

Let the reverse row labelling $RL(D)$ of the skew shape $D$ be the labelling of the cells of $D$ by the numbers 1 through $|D|$ that proceeds from right to left in the first row, then right to left in the second row, etc.
If $Q$ is a standard tableau, let $\text{row}_Q(i)$ and $\text{col}_Q(i)$ be the row and column indices for the cell of $Q$ that contains $i$. Say that a (skew) standard tableau $Q$ is $RL(D)$-compatible if it has $|D|$ cells and

1. If $j$ is just to the left of $i$ in the same row of $RL(D)$, then $\text{row}_Q(j) \leq \text{row}_Q(i)$ and $\text{col}_Q(j) > \text{col}_Q(i)$, and

2. If $j$ lies just below $i$ in the same column of $RL(D)$, then $\text{row}_Q(j) > \text{row}_Q(i)$ and $\text{col}_Q(j) \leq \text{col}_Q(i)$.

Let $\text{RCMP}(D, D')$ be the set of $RL(D')$-compatible tableaux of shape $D$.

**Example 52.** Let $D' = (53210)$. $RL(D')$ and a $RL(D')$-compatible tableau $Q$ of shape $(86443)$ are given below.

$$
RL(D') = \begin{array}{cccc}
5 & 4 & 3 & 2 \\
8 & 7 & 6 \\
10 & 9 \\
11
\end{array}
$$

$$
Q = \begin{array}{cccc}
\Box & \Box & \Box & 3 \\
\Box & \Box & \Box & 2 \\
\Box & \Box & 7 & 10 \\
\Box & 6 & 9 & 11
\end{array}
$$

**Remark 53.** It can be shown from the definitions that the sets $\text{PIC}(D, D')$ and $\text{RCMP}(D, D')$ are in bijection by sending the map $f \in \text{PIC}(D, D')$ to the tableau $Q \in \text{RCMP}(D, D')$ such that the letters $Q(s)$ and $RL(D')(f(s))$ coincide for all $s \in D$ [11].

**Remark 54.** The notion of picture and compatible tableau have equivalent “transpose” formulations as well. Replacing the total order $\leq_{SW}$ in the definition of picture by $\leq_{SW}$ produces the same set of maps. Let $\text{CL}(D)$ be the reverse column labelling of $D$, given by labelling the cells of $D$ with the numbers 1 through $|D|$, from top to bottom in each column, starting with the rightmost. Say that the standard tableau $Q$ is $CL(D)$-compatible if the conditions in the definition of $RL(D)$-compatibility hold with $RL(D)$ replaced by $CL(D)$. Let $\text{CCMP}(D, D')$ be the set of $CL(D')$ compatible tableaux of shape $D$. There is a bijection $\text{PIC}(D, D') \rightarrow \text{CCMP}(D, D')$ with $f \mapsto Q$ such that the letters $Q(s)$ and $CL(D')(f(s))$ coincide for all $s \in D$. More general reading orders are considered in [11].

Given a (skew) column strict tableau $T$, the Schensted standardization [34] $\text{std}(T)$ of $T$ is the standard tableau $S$ of the same shape as $T$ obtained by replacing the 1's in $T$ from left to right, then the 2's in $T$ from left to right, etc., by the numbers 1 through $\text{shape}(T)$. The tableau $Q$ in Example 52 is the standardization of the tableau $T$ in Example 9.

**Remark 55.** Let $\lambda/\mu$ and $\sigma/\tau$ be skew shapes. It is shown in [35] that the standardization map gives a bijection

$$
\text{LRT}_\tau(\lambda/\mu, \sigma - \tau) \rightarrow \text{CMP}(\lambda/\mu, \sigma/\tau)
$$
Thus the notions of a Littlewood–Richardson tableau, a compatible tableau, and a picture are all equivalent.

Here is a convenient criterion for a tableau with weakly increasing rows to be column strict and \( \tau \)-lattice.

**Proposition 56.** Let \( w = \cdots w^2 w^1 \) be the concatenation of weakly increasing words \( w^i \).

1. \( w \) is the row-reading word of a column strict tableau of shape \( \lambda / \mu \) (where \( w^r \) is the \( r \)th row of the tableau) if and only if the total number of letters in the subword \( w^r \) equals \( \lambda_r - \mu_r \), and

\[
\mu_r + m_2(w^r) + m_3(w^r) + \cdots + m_{r-1}(w^r) \\
\geq \mu_{r+1} + m_2(w^{r+1}) + m_3(w^{r+1}) + \cdots + m_{r}(w^{r+1}) \quad \text{(A.2)}
\]

for all \( i, r \geq 1 \), where \( m_i(w^r) \) is the multiplicity of the letter \( i \) in the subword \( w^r \). This condition says that the letters \( i \) in row \( r+1 \) lie beneath letters strictly less than \( i \) in row \( r \).

2. \( w \) is \( \tau \)-lattice of content \( \sigma - \tau \) if and only if the total multiplicity of the letter \( i \) in all of the subwords \( w^r \) is equal to \( \sigma_i - \tau_i \), and

\[
\tau_i + m_2(w^1) + m_3(w^2) + \cdots + m_i(w^{i-1}) \\
\geq \tau_{r+1} + m_{2r+1}(w^1) + m_{r+1}(w^2) + \cdots + m_{r+1}(w^r) \quad \text{(A.3)}
\]

for all \( i, r \geq 1 \).

*Change of notation.* Let us introduce new notation for the recording tableau for the column insertion version of the Robinson–Schensted–Knuth correspondence. For a sequence \( w = \cdots w^2 w^1 \) of weakly increasing words \( w^r \), let \( Q(w) \) denote the column strict tableau such that \( \text{shape}(Q(w) \mid _{w^r}) = \text{shape}(P(w^r \cdots w^2 w^1)) \) for all \( r \geq 1 \). This supercedes the definition of \( Q(w) \) in Section 2.

We now give an assortment of bijections between sets of LR tableaux.

1. There is a bijection

\[
\text{LRT}_r(\lambda / \mu, \sigma - \tau) \rightarrow \text{LRT}_\mu(\sigma / \tau, \lambda - \mu) \quad \text{(A.4)}
\]

defined by \( T \rightarrow U \) where \( U \) has shape \( \sigma / \tau \) and weakly increasing rows, such that the multiplicity of the letter \( j \) in the \( i \)th row of \( U \) equals the multiplicity of the letter \( j \) in the \( i \)th row of \( T \). This is a well-defined bijection by Proposition 56. It has the property that \( P(U) = Q(T) \) and \( Q(U) = P(T) \). If \( T \) represents a picture \( f \) (via Remarks 53 and 55) then \( U \) represents the picture \( f^{-1} \).
2. Suppose \( \sigma \) is contained in the rectangular partition \((m^n)\). Let \( \tilde{\sigma} \) be the partition \((m^n) - \sigma^+ = (m - \sigma_1, \ldots, m - \sigma_k)\). We claim that there is a bijection

\[
\text{LRT}(\lambda/\mu, \sigma) \to \text{LRT}_\tilde{\sigma}(\lambda/\mu, (m^n) - \tilde{\sigma})
\]  

(A.5)
given by \( U = w_0 T \), where \( w_0 \) is the plactic operator corresponding to the longest permutation on \( n \) letters. Note that \( T \) is a column strict tableau of shape \( \lambda/\mu \) and content \( \sigma \) if and only if \( U \) is a column strict tableau of shape \( \lambda/\mu \) and content \((m^n) - \tilde{\sigma}\), by Theorem 4 (3) and (2). Also \( P(U) = P(w_0 T) = w_0 P(T) \) by Theorem 4 (4). Note that \( T \) is lattice of content \( \sigma \) if and only if \( U \) is lattice of content \((m^n) - \tilde{\sigma}\) (by direct calculation), if and only if \( P(U) = K(\sigma) \) (by Remark 8), if and only if \( U \) is \( \tilde{\sigma} \)-lattice of content \((m^n) - \tilde{\sigma}\) (by Theorem 4 (4)). Note that \( T \) is lattice if and only if \( U \) is lattice of content \((m^n) - \tilde{\sigma}\) (by Theorem 4 (4)), if and only if \( T \) is lattice of content \( \sigma \) (by Remark 8), if and only if \( U \) is lattice of content \((m^n) - \tilde{\sigma}\) (by Theorem 4 (4)), if and only if \( T \) is lattice of content \( \sigma \) (by Theorem 4 (4)).

3. Given the skew diagrams \( D \) and \( D' \) having \( n \) rows and \( m \) columns, respectively, define the diagram \( D \otimes D' \) to be the skew shape given by the union \((D + (0, m)) \cup (D' + (n, 0))\) of the union of the translation of \( D \) by \( m \) columns to the right, with the translation of \( D' \) by \( n \) columns downwards. If \( T \) and \( T' \) are column strict tableaux of shapes \( D \) and \( D' \), respectively, then let \( T \otimes T' \) be the obvious column strict tableau of shape \( D \otimes D' \). Then a bijection

\[
\text{LRT}_r(\lambda/\mu, \sigma - \tau) \to \text{LRT}(\lambda/\mu \otimes \tau, \sigma)
\]  

(A.6)
is given by \( T \mapsto T \otimes K(\tau) \). It is immediate that this gives a well-defined injection. Conversely, if \( T \otimes T' \in \text{LRT}(\lambda/\mu \otimes \tau, \sigma) \) with \( T \) and \( T' \) of shapes \( \lambda/\mu \) and \( \tau \), respectively, then \( \text{word}(T \otimes T') = \text{word}(T) \text{word}(T') \) is lattice. This implies that \( T' \) is lattice. But the shape of \( T' \) is the partition \( \tau \), so \( T' = K(\tau) \) by Remark 8. It follows that \( T \) must be \( \tau \)-lattice of content \( \sigma \), showing that the map is surjective.

Finally, suppose that \( D \) and \( D' \) are skew shapes such that their side-by-side juxtaposition \( DD' \) is also a skew shape. Let \( \sigma/\tau \) be a third skew shape. Then there is an injection

\[
\text{LRT}_r(DD', \sigma - \tau) \to \text{LRT}_r(D \otimes D', \sigma - \tau)
\]  

(A.7)
defined as follows. Any tableau in \( \text{LRT}_r(DD', \sigma - \tau) \) can be written uniquely as the side-by-side juxtaposition \( ST' \) of skew tableaux \( S \) and \( T \) of shapes \( D \) and \( D' \), respectively. Then let \( ST' \mapsto S \otimes T \). The tableaux \( ST' \)}
and $S \otimes T$ have the same column-reading word and $S \otimes T$ is column strict, so the map is a well-defined injection by Lemma 11 and (K2) in Section 2. The image of this injection is clearly the set of tableaux $S \otimes T \in \text{LRT}_{\varsigma}(D \otimes D', \sigma/\tau)$ such that $ST$ is a column strict tableau of shape $DD'$. In particular, if $D$ is the 180-degree rotation of the diagram of a partition and $D'$ is the diagram of a partition, then the map is bijective, since any letter in the $r$th row of $S$ (resp. $T$) is at most $r$ (resp. at least $r$), by the column strictness and shape of $S$ (resp. $T$).

**Proof.** Proof of Lemma 30 and the properties of the bijection $\beta_\tau$ given in Remark 27: Let $(\lambda, S, m) \in I'$ with $\lambda \in \mathcal{P}_k$. Let $a = \lambda - (k^n)$ and $\alpha$ and $\beta$ be partitions such that $l(\alpha) + l(\beta) \leq n$ and $\alpha + \beta^* = a$. Recall that $S$ is a column strict tableau of shape $\lambda$ and content $(k^n)$ and $m \in \mathbb{Z}^n_+$. The partition $\nu$ is defined by (7.1). The partition $\lambda^\tau$ was defined by $\lambda^\tau = a + (\nu^2_1) = \lambda + ((\nu_1 - k^n)_1)$. Note that the part of $D(\lambda^\tau)$ that lies outside the rectangle $R$ is a translate of $D(\alpha)$. Let $\beta$ be the partition obtained from the 180-degree rotation of the skew diagram $R - D(\beta)$ where $R = (\nu^2_1)$. Consider the following sets of tableaux.

1. $\text{LRT}_{\varsigma}(\lambda, (k^n))$.
2. $\text{LRT}_{\varsigma}(\lambda^\tau, (R + \tilde{\nu}) - \tilde{\nu})$.
3. $\text{LRT}_{\varsigma}(\lambda^\tau/\tilde{\nu}, R - \tilde{\nu})$.
4. $\text{LRT}(\lambda^\tau/\nu, n)$.
5. $\text{LRT}(\beta/\tilde{\nu} \otimes \alpha, \nu)$.
6. $\text{LRT}_{\alpha}(\beta/\tilde{\nu}, \nu - \alpha)$.
7. $\text{PIC}(\beta/\tilde{\nu}, \nu/\alpha)$.
8. $\text{PIC}(\nu/\alpha, \beta/\tilde{\nu})$.

The maps going from each set to the next are as follows. (1) $\rightarrow$ (2): $S \rightarrow S^+$ where $S^+$ is obtained by adjoining the tableau $K((\nu_1 - k^n))$ to the left of $S$. Clearly $S^+$ has the correct shape and content. Using column-reading words, it is easy to check that $u_r(S^+) = u_r(S)$ for all $1 \leq r \leq n - 1$. Thus $S^+$ is $\tilde{\nu}$-lattice by Lemma 11. (2) $\rightarrow$ (3): This map is given by the removal of the subtableau $K(\tilde{\nu})$. To show this is a well-defined bijection, we realize it in a different way by the following composition of bijections: $P \in \text{LRT}_{\varsigma}(\lambda^\tau, R)$ to $Q \in \text{LRT}(R + \tilde{\nu}/\tilde{\nu}, \lambda^\tau)$ to $R \in \text{LRT}(R/\tilde{\nu} \otimes \tilde{\nu}, \lambda^\tau)$ to $U \in \text{LRT}_{\beta}(\lambda^\tau/\tilde{\nu}, \nu)$ given by the maps (LR-inverse), (A.7), the removal of the right tensor factor $K(\tilde{\nu})$, and (LR-inverse). We claim that every tableau $P \in \text{LRT}_{\varsigma}(\lambda^\tau, R)$ contains $K(\tilde{\nu})$ as a subtableau. Note that $P$ contains $K(\tilde{\nu})$ if the multiplicity of the letter $r$ in the $r$th row of $P$ is at least $\tilde{\nu}$, for all $r$, which holds if the same is true for $Q$ by Proposition 56. $R$ must have the
form \( \hat{R} \otimes K(\tilde{\nu}) \), so \( Q = \hat{R}K(\tilde{\nu}) \). This shows that \( Q \) has the desired property and hence that \( P \) contains \( K(\tilde{\nu}) \). In passing from \( Q \) to \( \hat{R} \) the letters of \( K(\tilde{\nu}) \) are removed, so the LR-inverse \( U \) of \( \hat{R} \) differs from the LR-inverse \( P \) of \( Q = \hat{R}K(\tilde{\nu}) \) by the removal of \( \tilde{\nu} \) letters \( r \) from the \( r \)th row. That is, \( U \) is obtained from \( P \) by the removal of the subtableau \( K(\tilde{\nu}) \).

The bijections from set (1) to (4) prove Lemma 30. Given \( T \in \text{LRT}(\lambda^+ / \tilde{\nu}, \nu) \) there is a corresponding picture in set (6). The bijection \( b_T : \nu/\alpha \rightarrow \nu/\beta \) is given by composing this picture with the 180-degree rotation, which sends the skew diagram \( \beta/\tilde{\nu} \) into \( \nu/\beta \). The properties (B2), (B3), and (B4) of Remark 27 follow from Remark 53, 54, and the definition of a picture.

We now begin the proof of Lemma 32. Let \((\lambda, S, m) \in I' \) and \( \Psi(\lambda, S, m) = (\lambda, \nu, T) \).

First, note that if \( T \) is \( p \)-excessive or if \( m_p > 0 \), then \( \nu \) has a column of length \( p \). If \( m_p > 0 \) then by (7.1) \( \nu_p - \nu_{p+1} > 0 \) so that \( \nu \) has a column of length \( p \). If \( T \) is \( p \)-excessive, then the letter 1 appears in the \((n + 1 - p)\)th row of \( T \). The column strictness of \( T \) implies that the cell just above this letter 1 is in the inner shape \( \tilde{\nu} \) of the skew shape \( \lambda^+/\tilde{\nu} \) of \( T \). It follows that \( \tilde{\nu} \) has a column of length \( n - p \). Likewise \( \nu \) has a column of length \( p \).

Thus for either direction in the proof of Lemma 32, we may assume that \( \nu \) has a column of length \( p \) and that \( \tilde{\nu} \) has a column of length \( n - p \). Let \( \nu \) (resp. \( \tilde{\nu} \)) be the partitions obtained by removing a column of length \( p \) (resp. \( n - p \)) from \( \nu \) (resp. \( \tilde{\nu} \)). If \( T \) is a \( p \)-excessive tableau of shape \( D = \lambda^+/\tilde{\nu} \), let \( T^\nu \) denote the tableau of shape \( D = (\lambda^+ - (1^n))/\tilde{\nu} \) obtained by removing the letter \( p + 1 - r \) from the \((n + 1 - r)\)th row of \( T \).

We require Sagan and Stanley's internal row insertion [33]. Define the operator \( J \) on column strict tableaux \( T \) of the skew shape \( \lambda^+ / \tilde{\nu} \) as follows. Let \( T^\nu = T \). Let \( T_{r-1} \) be the tableau resulting from the internal insertion on the tableau \( T_r \) at the cell \((n + 1 - r, \tilde{\nu}_{r+1-r} + 1)\), for \( r \) going from \( p \) down to 1. Define \( J(T) = T_0 \).

**Example 57.** Consider the triple \((\lambda, \nu, T)\) in Example 26. The tableau \( T \) is \( p \)-excessive for \( p = 3 \). The underlined letter in \( T_r \) indicates the starting cell for the internal row insertion that produces \( T_{r-1} \).

\[
\begin{array}{cccccc}
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & 1 & 1 & 1 \\
\emptyset & \emptyset & \emptyset & \emptyset & 1 & 2 \\
\emptyset & \emptyset & 1 & 1 & 1 & 2 \\
\emptyset & 2 & 3 \\
2 & 3 & 4 \\
\end{array}
\]
Lemma 58. Suppose $T$ is a column strict tableau of shape $D$ and content $\nu$. Then $T$ is $p$-excessive and lattice if and only if $J(T) = K(1^p) \otimes T'$ where $T'$ is lattice of shape $D$ and content $\nu$. In this case $T' = T^\circ$.

Proof. If $T$ is $p$-excessive, then it is not hard to see that for each $1 \leq r \leq p$, the bumping path of the internal insertion that takes $T_r$ to $T_{r-1}$ consists of the letters $1, 2, \ldots, p$ (cf. Example 57), and that $J(T) = K(1^p) \otimes T^\circ$. The latticeness of $T^\circ$ follows from that of $T$ by Proposition 56. Conversely, if $T'$ is lattice of content $\nu$, then

$$P(T) = P(J(T)) = P(K(1^p)T') = P(K(1^p)P(T')) = P(K(1^p)K(\nu)) = K(\nu). \quad (A.8)$$

The first equality holds by Lemma 61 (1). The second holds by assumption. The third holds by the definition of $P(\ )$. The fourth holds by Remark 8. The fifth equality follows from a trivial direct computation. From (A.8) and Remark 8 it follows that $T$ is lattice. A careful examination of the calculation of $J(T)$ shows that the only way that one obtains the output tableau $K(1^p) \otimes T'$ is if the bumping path in passing from $T_r$ to $T_{r-1}$ consists of the letters $1, 2, \ldots, p$, for each $1 \leq r \leq p$. It follows that $T = T^\circ$. 

\[ T_2 = \begin{array}{cccc|cc}
\square & \square & \square & \square & 1 & 1 & 1 \\
\square & \square & \square & \square & 1 & 2 \\
\square & \square & \square & \square & 1 & 3 \\
\square & \square & 1 & 2 & 4 \\
2 & 2 & 4 \\
3 \\
\end{array} \]

\[ T_1 = \begin{array}{cccc|c}
\square & \square & 3 \\
\square & \square & 2 & 4 \\
\square & 1 & 2 \\
\square & \square & \square & \square \\
2 & 3 \\
\end{array} \]

\[ T_0 = \begin{array}{cccc|c}
\square & \square & 3 \\
\square & \square & 2 & 4 \\
\square & 1 \\
\square & 2 \\
\square & 3 \\
\end{array} \]
had to be $p$-excessive. The fact that $T' = T''$ follows from the forward direction, since we now know that $T$ is $p$-excessive.

Say that the tableau $U$ is $p$-antiexcessive if the leftmost letter in the $(n + 1 - r)$-th row of $U$ is $n + 1 - r$, for $1 \leq r \leq p$. Note the asymmetry between the definitions of $p$-excessiveness and $p$-antiexcessiveness. If $U$ is $p$-antiexcessive of shape $D$, let $U'$ denote the tableau of shape $D$ obtained by removing the letter $n + 1 - r$ from the $(n + 1 - r)$-th row of $U$ for $1 \leq r \leq p$.

**Lemma 59.** Let $U$ be a column strict tableau of shape $D$ and content $\text{rev}(\nu)$. Then $U$ is $p$-antiexcessive and $\tilde{\nu}$-lattice if and only if $J(U) = K(0^{n-p}, 1^p) \otimes U'$ where $U'$ is $\tilde{\nu}$-lattice of shape $D$ and content $\text{rev}(\tilde{\nu})$. In this case $U' = U$.

**Proof.** Define the sequence of tableaux $U = U_p, U_{p-1}, \ldots, U_0 = J(U)$ as in the definition of $J$. Suppose that $U$ is $p$-antiexcessive and $\tilde{\nu}$-lattice. It is not hard to see that for $1 \leq r \leq p$, the bumping path of the internal insertion that computes $U_{r-1}$ from $U_r$ consists of the letters $n + 1 - p, \ldots, n$ (always occurring at the left end of their respective rows) and that $J(U) = K(0^{n-p}, 1^p) \otimes U'$ (cf. Example 60 below). All of the properties of $U' = U''$ are trivially verified except its $\tilde{\nu}$-latticeness, which follows from Proposition 56. Conversely, if $U'$ is $\tilde{\nu}$-lattice of content $\text{rev}(\tilde{\nu})$, then

$$P(U) = P(J(U)) = P(K(0^{n-p}, 1^p)U') = P(K(0^{n-p}, 1^p)P(U'))$$

$$= P(K(0^{n-p}, 1^p)K(\text{rev}(\tilde{\nu}))) = K(\text{rev}(\nu)). \quad (A.9)$$

The first equality holds by Lemma 61 (1). The second holds by assumption. The third holds by the definition of $P(\cdot)$. The fourth holds by Remark 8. The fifth holds by a direct computation. From (A.9) and Remark 8 it follows that $T$ is lattice. Again by examining the computation of $J(U)$ it follows that $U$ is $p$-antiexcessive. From the forward direction we see that $U' = U$.

**Example 60.** Let $(\lambda, \nu, T)$ be as in Example 57. We calculate $J(U)$ for the tableau $U = w_0(T)$, which has shape $D$, content $\text{rev}(\nu)$, and is $\tilde{\nu}$-lattice.

$$U = U_3 = \begin{array}{cccccccc}
\Box & \Box & \Box & \Box & \Box & 2 & 5 & 5 \\
\Box & \Box & \Box & \Box & 3 & 4 \\
\Box & \Box & 4 & 4 \\
5 & 5 & 5 \\
\end{array}$$
Suppose that $T$ is $p$-excessive. Then $J(T) = K(1^p) \otimes T^\hat{\lambda}$ and $T^\hat{\lambda} \in LRT(D, \hat{\mu})$, by Lemma 58. We have $(\lambda, \hat{\nu}, T^\hat{\lambda}) \in \mathcal{I}$. Let $\Psi^{-1}(\lambda, \hat{\nu}, T^\hat{\lambda}) = (\lambda, \hat{S}, \hat{m})$. Let $S^+$ (resp. $\hat{S}^+$) be the tableaux obtained by adjoining the tableau $S((\nu - k)^{\mu})$ (resp. $\hat{S}((\hat{\nu} - k)^{\mu}) = S((\nu - k - 1)^{\mu})$) to the left of $S$ (resp. $\hat{S}$). We have $w(\nu) = S^+ - K(\nu)$ and $w(\hat{\nu}) = \hat{S}^+ - K(\hat{\nu})$ by (7.2) and the fact that $w_0$ is an involution. We want to apply the converse direction of Lemma 59 to the tableaux $U = w(\nu)$ and $U' = w(\hat{\nu})$. First the hypotheses for $U'$ in Lemma 59 must be verified. We have shape$(U') = \text{shape}(T^\hat{\lambda}) = \hat{D}$ by Theorem 43. Now $P(U') = P'(\nu) = w(\nu)$ by Theorem 44 and Remark 8. This shows that $U'$ is $\hat{\nu}$-lattice, again by Remark 8.

Next, to apply Lemma 59 we must show that $J(U') = K(0^{0-p}, 1^p) \otimes U'$; that is, the two tableaux $J(w(\nu))$ and $W = K(0^{0-p}, 1^p) \otimes w(T^\hat{\lambda})$ agree. To check this, by the bijectivity of the Robinson–Schensted–Knuth correspondence it suffices to verify that the two tableaux have the same $P$ and $Q$ tableaux

$$P(J(w(\nu))) = P(w(\nu)) = w(\nu) = K(\nu) = K(\text{rev}(\nu)).$$

(A.10)
by Lemma 61 (1), Theorem 4 (4), and Remark 8. We have
\[ P(\omega_0(T^*)) = w_0(P(T^*)) = w_0(K(\hat{\nu})) = K(\text{rev}(\hat{\nu})) \quad (A.11) \]

by Theorem 4 (4), Remark 8, and direct computation. Finally we have
\[
P(W) = P\left(K(0^{n-p},1^p)P(\omega_0(T^*))\right)
= P\left(K(0^{n-p},1^p)K(\text{rev}(\hat{\nu}))\right) = K(\text{rev}(\nu)) \quad (A.12)
\]

by the definition of \( P(\cdot) \), (A.11), and direct computation. By (A.10) and (A.12) we see that \( P(J(\omega_0(T))) = K(\text{rev}(\nu)) = P(W) \), so that the \( P \) tableaux of \( J(\omega_0(T)) \) and \( W \) agree. Let \( Q = Q(J(\omega_0(T))) \). We know that both the shapes of \( Q \) and \( P(J(\omega_0(T))) \) are equal to \( \nu \). Now
\[
Q = Q(J(\omega_0(T))) = Q(w_0(J(T))) = Q(J(T)) = Q(K(1^p) \otimes T^*) = Q(T^*) + V
\]

(A.13)

where \( V \) is the skew tableau of shape \( \nu/\hat{\nu} \) whose \( r \)th row consists of the single letter \( n + r \) for \( 1 \leq r \leq p \). (A.13) holds by Lemma 61 (2), Theorem 4 (5), and the fact that \( \text{word}(K(1^p)) \) is a strictly decreasing sequence, which implies that the letters \( n + 1 \) through \( n + p \) must appear in distinct rows of \( Q \) from top to bottom in the shape \( \nu/\hat{\nu} = \text{shape}(Q)/\text{shape}(Q(T^*)) \). We know that \( \text{shape}(Q(W)) = \text{shape}(P(W)) = \nu \). We have
\[
Q(W) = Q(K(0^{n-p},1^p) \otimes w_0(T^*)) = Q(w_0(T^*)) + V = Q(T^*) + V.
\]

(A.14)

again due to Theorem 4 (5) and the fact that \( \text{word}(K(0^{n-p},1^p)) \) is a strictly decreasing word. But (A.13) and (A.14) show that \( Q(J(\omega_0(T))) = Q(T^*) + V = Q(W) \), so that the \( Q \) tableaux of \( J(\omega_0(T)) \) and \( W \) agree. Therefore \( J(\omega_0(T)) = W = K(0^{n-p},1^p) \otimes w_0(T^*) \). Finally Lemma 59 applies, with \( U = w_0(T) \) and \( U' = w_0(T^*) \). It follows that \( w_0(T) \) is \( p \)-antiexcessive and that \( (w_0(T))^* = w_0(T^*) \). We have \( (S^* - K(\hat{\nu}))^* = S^* - K(\hat{\nu}) \). It follows that \( S = S \). Then by (7.1), \( \hat{m}_i = m_i \) for \( i \neq p \) and \( 0 \leq \hat{m}_p = m_p - 1 \), so that \( m_p > 0 \).

(\( = \)): Conversely, suppose that \( m_p > 0 \). Let \( \hat{m} \in \mathbb{Z}^n_+ \) be defined by \( \hat{m}_i = m_i \) for \( i \neq p \) and \( \hat{m}_p = m_p - 1 \). Let \( \Psi(\lambda, S, \hat{m}) = (\lambda, \nu, T') \) where \( T' \in \text{LRT}(\hat{D}, \nu) \). It is enough to show that the tableaux \( J(T) \) and \( Y = K(1^p) \otimes T' \) are equal. For then an application of the converse direction of Lemma 58 using \( T \) and \( T' \) would show that \( T \) is \( p \)-excessive and \( T' = T^* \).

To show \( J(T) = Y \) it is enough to prove that \( P(J(T)) = P(Y) \) and
shape(\(J(T)\)) = shape(Y). We have

\[
P(J(T)) = P(T) = K(\nu) = P(K(1^p)K(\nu))
= P(K(1^p) \otimes T') = P(Y)
\tag{A.15}
\]

by Lemma 61 (1), Remark 8, and direct computation. Let \(S^+\) (resp. \(\hat{S}^+\)) be the tableau obtained by adjoining the tableau \(K((\nu_k - k)^n)\) (resp. \(K((\hat{\nu} - k)^n) = K((\nu_k - k - 1)^n)\)) to the left of \(S\). Then we have \(w_0(T) = S^+ - K(\nu)\) and \(w_0(T') = \hat{S}^+ - K(\hat{\nu})\) by (7.2) and the fact that \(w_0\) is an involution. It is not hard to see that \(w_0(T)\) is \(p\)-antiexcessive and that \(w_0(T') = (w_0(T))'\). By Lemma 59 with \(U = w_0(T)\) and \(U' = w_0(T')\) it follows that \(J(w_0(T)) = K(0^{a-p}, 1^p) \otimes w_0(T')\). We have

\[
Q(J(T)) = Q(w_0(J(T))) = Q(J(w_0(T))) = Q(K(0^{a-p}, 1^p) \otimes w_0(T'))
= Q(w_0(T')) + V = Q(T') + V = Q(K(1^p) \otimes T') = Q(Y),
\tag{A.16}
\]

with \(V\) as before. (A.16) holds by Theorem 4 (5), Lemma 61 (2), and the fact that word(\(K(0^{a-p}, 1^p)\)) and word(\(K(1^p)\)) are strictly decreasing sequences. By (A.15) and (A.16), \(P(J(T)) = P(Y)\) an \(Q(J(T)) = Q(Y)\), so that \(J(T) = Y\). This concludes the proof of Lemma 32.

**Lemma 61.** (1) Internal insertions preserve Knuth equivalence.

(2) Internal insertion commutes with plactic permutations.

**Proof.** Let \(T\) be a column strict tableau of shape \(\lambda/\mu\) in the alphabet \(A\), \(w\) a permutation on the set \(A\), and \(s\) a cell of the form \((i, \mu_i + 1)\). Let \(I\) be the internal insertion operation at \(s\). Let \(S'\) be any column strict tableau of shape \(\mu \cup (s)\) in an alphabet \(B\) whose letters are declared to be less than those of \(A\). Let \(x\) be the letter of \(B\) ejected and \(S\) the tableau resulting from the reverse row insertion on \(S'\) at \(s\). We have \(P(Sx) = S'\). Now let \(a\) be any word in the alphabet \(B + A\) such that \(P(a) = S + T\), the column strict tableau of shape \(\lambda\) in the alphabet \(B + A\) whose restrictions to the alphabets \(B\) and \(A\) are given by \(S\) and \(T\), respectively. Then it follows that \(P(ax) = S' + I(T)\) (cf. [33]); that is, internal row insertions can be emulated by ordinary row insertions using an alphabet augmented by smaller letters. Restricting the equation \(P(ax) = S' + I(T)\) to the alphabet \(A\), we have \(T \equiv I(T)\), proving (1). Now \(P(wa) = w(S + T) = S + w(T)\) by Theorem 4 (4) and (6). Again we have \(P((wa)x) = S' + I(w(T))\). On the other hand, \(P((wa)x) = P(w(ax)) = wP(ax) = w(S' + I(T)) = S' + w(I(T))\) by Theorem 4 (4) and (6). Therefore \(w(I(T)) = I(w(T))\), proving (2).
REFERENCES