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Discrete Morse theory on graphs ☆

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ABSTRACT

We characterize the topology of a graph in terms of the critical elements of a discrete Morse function defined on it. Besides, we study the structure and some properties of the gradient vector field induced by a discrete Morse function defined on a graph. Finally, we get results on the number of non-homologically equivalent excellent discrete Morse functions defined on some kind of graphs.

1. Introduction

Discrete Morse theory was introduced by R. Forman [5] as a purely combinatorial version of classical or smooth Morse theory. This approach has proven to be a powerful tool to study the topology of a general *cw*-complex. In our point of view, discrete Morse theory has two basic advantages over the smooth setting: mainly due to its discrete nature it obtains analogous results to the classical one but in a straightforward and less complicated way [5] and besides, it turns out to be very suitable to adapt results in a computational way [7]. There is a growing number of researchers who are finding different applications of this theory to solve problems in many areas, from denoising digital data sets [6,9], to establishing links with complexes of graphs [3] just to cite some of them.

The authors are particularly interested in the extension of this theory to the non-compact case. In this sense we have obtained a generalized version of Morse inequalities for infinite graphs [1] and for triangulated and non-compact surfaces [2]. In this work the decreasing monotonous behaviour of a discrete Morse function at the ends plays an outstanding role, in fact, they are acting as a kind of critical simplices at the end of the considered complex and this is the reason to unify both concepts with the notion of critical element.

The goal of this paper is to study some aspects of the extension of discrete Morse theory to the non-compact 1-dimensional case, namely, the extension to infinite graphs. We begin presenting in Section 2 the basic notions and results concerning infinite discrete Morse theory on graphs. In Section 3 we introduce the notions of excellent discrete Morse function, homology equivalence for this kind of functions and homological sequences. Section 4 is devoted to the study of

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the structure of the set of all gradient paths induced by a discrete Morse function defined on a graph. This is the discrete analogue of the smooth notion of flow lines of the field induced by a Morse function. By using the notion of tree rooted in a 0-critical element, we decompose this set as a disjoint union of such trees and hence, it can be seen as a forest. Later on, we include in Section 5 the exposition and proof of a result which characterizes those (infinite) graphs which admit a discrete Morse function with a given number of critical elements. Our infinite version of the discrete Morse inequalities and an explicit definition of the desired function are the basic tools that we use to prove it. Notice that this result is looking for the basic goal of classical Morse theory, that is, getting links between the topology of a manifold and the critical points of a smooth Morse function defined on it. Besides, in this section we introduce the notion of optimal discrete Morse function defined on an infinite graph and we prove that every graph admits an optimal discrete Morse function. Finally, Section 6 is devoted to carry out in the discrete setting a study which has been done in the smooth setting by Nicolaescu [11] for S^1. Our goal in this section is counting the number of homology classes of excellent discrete Morse functions defined on a graph with a given number of critical simplices in the cases that the graph is a tree, a finite wedge of cycles or the union of a finite wedge of cycles and a tree.

2. Preliminaries

Through all this paper, we only consider infinite graphs which are locally finite. Given such a graph G, we introduce here the basic notions of discrete Morse theory [5]. A **discrete Morse function** is a function $f : G \to \mathbf{R}$ such that, for any p-simplex $\sigma \in G$:

(M1) $card\{\tau^{(p+1)} > \sigma / f(\tau) \leqslant f(\sigma)\} \leqslant 1$.
(M2) $card\{\upsilon^{(p-1)} < \sigma / f(\upsilon) \geqslant f(\sigma)\} \leqslant 1$.

A p-simplex $\sigma \in G$ is said to be **a critical simplex** with respect to f if:

(C1) $card\{\tau^{(p+1)} > \sigma / f(\tau) \leqslant f(\sigma)\} = 0$.
(C2) $card\{\upsilon^{(p-1)} < \sigma / f(\upsilon) \geqslant f(\sigma)\} = 0$.

A value of a discrete Morse function on a critical simplex is called **critical value**.

Given $c \in \mathbb{R}$ the **level subcomplex** $G(c)$ is the subcomplex of G consisting of all simplices τ with $f(\tau) \leqslant c$, as well as all of their faces, that is,

$$G(c) = \bigcup_{f(\tau) \leqslant c} \bigcup_{\sigma \leqslant \tau} \sigma.$$

A **ray** is an infinite sequence of simplices

$$v_0, e_0, v_1, e_1, \ldots, v_r, e_r, v_{r+1}, \ldots$$

verifying that the 0-simplices v_i and v_{i+1} are faces of the 1-simplex e_i, for any $i \in \mathbf{N} \cup \{0\}$.

Two rays contained in an infinite graph are said to be **equivalent** or **cofinal** if both coincide from a common 0-simplex.

If there is a discrete Morse function f defined on G, a **decreasing ray** is a ray satisfying

$$f(v_0) \geqslant f(e_0) > f(v_1) \geqslant f(e_1) > \cdots \geqslant f(e_r) > f(v_{r+1}) \geqslant \cdots.$$

A **critical element** of f on G is either a critical simplex or a decreasing ray.

Given a discrete Morse function defined on G, we say that a pair of simplices $(v < e)$ is in the **gradient vector field** induced by f if and only if $f(v) \geqslant f(e)$.

Given a gradient vector field V on G, a V**-path** is a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \ldots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}, \ldots, \tag{1}$$

such that, for each $i \geqslant 0$, the pair $(\alpha_i^{(p)} < \beta_i^{(p+1)}) \in V$ and $\beta_i^{(p+1)} > \alpha_{i+1}^{(p)} \neq \alpha_i^{(p)}$.

Two discrete Morse functions f and g defined on a simplicial complex M are *equivalent* if every pair of simplices $\alpha^{(p)}$ and $\beta^{(p+1)}$ in M such that $\alpha^{(p)} < \beta^{(p+1)}$ verify that

$$f(\alpha) < f(\beta) \quad \text{if and only if} \quad g(\alpha) < g(\beta).$$

The next result states that any two equivalent discrete Morse functions have the same gradient vector field and conversely.

Theorem 2.1. *([4]) Two discrete Morse functions f and g defined on a simplicial complex M are equivalent if and only if f and g induce the same gradient vector field.*

Given a discrete vector field V defined on an (infinite) graph, it is easy to prove that if V does not contains any closed V-path then, there exits a (proper) discrete Morse function (in fact, infinitely many) such that $V = V_f$.

Following the main goal of classical Morse theory, that is, looking for links between the topology of a manifold and the critical points of a Morse function defined on it, the authors proved the following result, which generalizes to the infinite 1-dimensional case the well known Morse inequalities.

Theorem 2.2. *([1]) Let G be an infinite graph and let f be a discrete Morse function defined on G such that the numbers of critical i-simplices of f with $i = 0, 1$, denoted by $m_i(f)$, are finite and the number of non-cofinal decreasing rays, denoted by d_0, is finite too. Then*:

(i) $m_0(f) + d_0 \geqslant b_0$ *and* $m_1(f) \geqslant b_1$, *where* b_i *denotes the ith Betti number of G with* $i = 0, 1$.
(ii) $b_0 - b_1 = m_0(f) + d_0 - m_1(f)$.

3. Excellent discrete Morse functions on graphs

As we will see in Sections 4 and 5, the topological properties of a graph are deeply related to the qualitative properties of the discrete Morse functions defined on it. Roughly speaking, these properties are essentially the number of critical elements of the functions and the changes on the topology of their level subcomplexes which are detected by the behaviour of its Betti numbers. Thus, it seems reasonable to consider two discrete Morse functions defined on a graph as indistinguishable if these data sets are the same for both functions. For this reason, it is convenient to deal with functions whose critical values, that is, its values on the critical simplices, are different and we assume that two such functions are equivalent if their level subcomplexes have the same homology.

Definition 3.1. A discrete Morse function is called **excellent** if all its critical values are different.

Definition 3.2. Two excellent discrete Morse functions f and g defined on G with critical values $a_0 < a_1 < \cdots < a_{m-1}$ and $c_0 < c_1 < \cdots < c_{m-1}$ respectively will be called **homologically equivalent** if for all $i = 0, \ldots, m-1$ the level subcomplexes $G(a_i)$ and $G(c_i)$ have the same Betti numbers.

Definition 3.3. Let f be an excellent discrete Morse function defined on a connected graph G with m critical simplices and critical values $a_0, \ldots, a_{m-1}$. We denote the level subcomplexes $G(a_i)$ by G_i for all $i = 0, 1, \ldots, m-1$. The **homological sequences** of f are the two sequences $B_0, B_1 : \{0, 1, \ldots, m-1\} \to \mathbb{N}$ containing the homological information of the level subcomplexes $G_0, \ldots, G_{m-1}$, that is, $B_0(i) = b_0(G_i) = dim(H_0(G_i))$ and $B_1(i) = b_1(G_i) = dim(H_1(G_i))$ for each $i = 0, 1, \ldots, m-1$.

Remark 3.4. The homological sequences of f satisfy

$$B_0(0) = B_0(m-1) = b_0 = 1, \qquad B_0(i) > 0, \qquad \left|B_0(i+1) - B_0(i)\right| = 0 \text{ or } 1;$$
$$B_1(0) = 0, \qquad B_1(m-1) = b_1, \qquad B_1(i+1) - B_1(i) = 0 \text{ or } 1.$$

Lemma 3.5. *For each $i = 0, 1, \ldots, m-2$ it holds one and only one of the following identities*:

(H1) $B_0(i) = B_0(i+1)$.
(H2) $B_1(i) = B_1(i+1)$.

Proof. Since every interval $(a_i, a_{i+1}]$ contains a unique critical value, the level subcomplexes G_i and G_{i+1} are homologically different, but only one of the Betti numbers of G_i and G_{i+1} are the same. □

It is interesting to point out that identity (H2) of the above lemma reveals the appearance of a new connected component or the join of two different connected components in the process of obtention of G by level subcomplexes. Analogously, identity (H1) reveals the creation of a new 1-cycle of G in this process.

Notice that two excellent discrete Morse functions are homologically equivalent if and only if their homological sequences are the same.

4. The gradient vector field of a discrete Morse function on a graph

The qualitative properties of a discrete Morse function f are reflected by its induced gradient vector field V_f. Thus, in many situations we do not need to consider the values of f but we just deal with that field. In fact, the authors proved in [4] that two different discrete Morse functions f and g defined on a graph verify $V_f = V_g$ if and only if they have the same sets of critical elements. Note that this result is no longer true for locally finite simplicial complexes with dimension greater than or equal to two. Moreover, the above characterization is equivalent to the following condition: If $\sigma < \tau$ then $f(\sigma) < f(\tau)$ if and only if $g(\sigma) < g(\tau)$.

Now we are going to study the structure of the gradient field of a discrete Morse function f defined on a graph G. It can be easily proved that a gradient vector field V does not contain closed V-paths.

Definition 4.1. Given a 0-critical element in G, that is, a critical vertex v or a decreasing ray r, we say that a vertex w of G is **rooted** in v (respectively in r) if there exists a finite V-path joining w and v (respectively w and some vertex of r).

Note that if a vertex w is rooted in a decreasing ray r, then there exists a decreasing ray r' starting from w which is equivalent to r.

Proposition 4.2. *Let G be an infinite graph and let f be a discrete Morse function defined on G. It holds that*:

1. *Given w any vertex of G, there is a unique 0-critical element on which w is rooted.*
2. *Given any 0-critical element (v or r), the set of all V-paths rooted in it is a tree called* ***the tree rooted in*** *v* ***or*** *r and denoted by T_v or T_r.*
3. *Any two of such rooted trees are disjoint.*

Proof. 1. Let us suppose that there exists a vertex w of G rooted in two 0-critical elements p and q (these critical elements can be vertices or decreasing rays). Then there exist two finite V-paths $we_1v_1e_2v_2\cdots e_rv_r$ and $w\bar{e}_1u_1\bar{e}_2u_2\cdots\bar{e}_su_s$ where $v_r = p$ or a vertex in p if p is a decreasing ray and $u_s = q$ or a vertex in q if q is a decreasing ray. As f is a discrete Morse function, it is not possible that both $f(w) \geqslant f(e_1)$ and $f(w) \geqslant f(\bar{e}_1)$ if $e_1 \neq \bar{e}_1$. Therefore $e_1 = \bar{e}_1$ and $v_1 = u_1$, and reasoning in the same way in each vertex we obtain that the two V-paths are the same.

2. Let p be a 0-critical element and let us consider the union T_p of all V-paths rooted in p. If there exists a cycle C in T_p for some vertex v in C there would exist two different V-paths joining v with p, but this is not possible by (1). Thus, T_p is a tree since it contains no cycle.

3. If there exist two trees rooted in different 0-critical elements p and q such that $T_p \cap T_q \neq \emptyset$, any vertex in the intersection would be rooted in both p and q and that is not possible by (1). $\square$

Remark 4.3. If F is the union of all rooted trees in G, it is easy to prove that F is a tree if and only if there is a unique 0-critical element of f in G.

Theorem 4.4. *Under the above definitions and notations, F can be obtained by removing all critical edges of f on G.*

Proof. Let $e_1, \dots, e_{m_1}$ be the 1-critical elements of the discrete Morse function f on an infinite graph G and we set $H = G - \{e_1, \dots, e_{m_1}\}$. As F and H are spanning subgraphs of G, to prove that $F = H$ is enough to prove that these subgraphs have the same edges.

If e is an edge in F, it is not possible that $e = e_i$ for any $i = 1, \dots, m_1$ since e_i cannot be in a V-path because $f(e_i)$ is greater than the values of f on the vertices of e_i. Then e is an edge of H.

Conversely, if e is an edge in H, e is not a critical edge ($e \neq e_i$ for every $i = 1, \dots, m_1$). Then, if $e = uv$ it is not possible both $f(u) \geqslant f(e)$ and $f(v) \geqslant f(e)$. We can suppose, for example, that $f(u) \geqslant f(e) > f(v)$. So we have that u and v are rooted in the same 0-critical element p because adding u and e to the V-path joining v with p we get a V-path joining u with p. Thus, e is an edge of F since e is in the tree rooted in p and F is the union of all rooted trees. $\square$

Definition 4.5. Let f be a discrete Morse function defined on an infinite graph G with m critical elements. The **forest generated by** r 0-critical elements $v_1, \dots, v_r$ and s superfluous critical edges $e_1, \dots, e_s$ is the forest consisting of:

- the trees T_{v_i} with $1 \leqslant i \leqslant r$, and
- the edges e_i with $1 \leqslant i \leqslant s$.

Remark 4.6. Observe that the forest F is the forest generated by all the 0-critical elements of f.

5. The critical elements of a discrete Morse function on a graph

Once the generalized Morse inequalities have been introduced, we are in condition to extend the notion of optimality for discrete Morse function defined on infinite graphs. Classically optimal Morse functions are those on which Morse inequalities became equalities. We shall use this idea in the next definition.

Definition 5.1. Let f be a discrete Morse function defined on an (infinite) graph G. We say that f is **optimal** if $m_0(f) + d_0 = b_0(G)$ and $m_1(f) = b_1(G)$.

In fact, as an easy consequence of Theorem 2.2 we have that the two conditions of the above definition can be reduced to just any of them as the next result states.

Proposition 5.2. *Let f be a discrete Morse function defined on an (infinite) graph G. The following conditions are equivalent:*

(i) *f is optimal.*
(ii) $m_0(f) + d_0 = b_0(G)$.
(iii) $m_1(f) = b_1(G)$.

Proposition 5.3. *Every connected graph G admits an optimal discrete Morse function.*

Proof. We will define a discrete Morse function on G with $m = b_1(G) + 1$ critical elements. It is known (see [10]) that if T is a spanning tree in G there exists a bijection between the set of basic cycles of G and the set of $b_1(G)$ edges $e_1, \ldots, e_{b_1}$ not in T.

First we take a spanning tree T of G. If we choose a root vertex v_0 in T, let $f : T \to \mathbb{R}$ be defined as follows

- if v is a vertex in the level t of T, we set $f(v) = t$, and
- if uv is an edge in T, we set $f(uv) = \max\{f(u), f(v)\}$.

Now, if we extend f to each edge $e_i = u_i v_i$, $1 \leqslant i \leqslant b_1$, by defining $f(e_i)$ such that $f(e_i) > \max\{f(u_i), f(v_i)\}$, we obtain a discrete Morse function f on G whose critical elements are v_0 and the edges $e_1, \ldots, e_{b_1}$. □

Taking into account the values of a discrete Morse function defined on a graph, it is interesting to consider two different kinds of critical simplices. Basically, we want to distinguish between those critical simplices which arise forced by the topology of the considered graph and those which are introduced by the non-optimality of the function. Given a non-optimal discrete Morse function, by considering the ordered family of level subcomplexes associated to its critical values, we can control how a critical edge arises, that is, either it appears because it is completing a homology cycle or it appears due to the need to join two different connected components.

Definition 5.4. Let f be an excellent discrete Morse function defined on a connected graph G with critical values $a_0 < \cdots < a_{n-1}$. We say that a critical vertex v is an **essential vertex** if $f(v)$ is the global minimum of f on G, that is, $f(v) = a_0$. One critical edge e_i with $f(e_i) = a_i$ is an **essential edge** if $B_1(i) - B_1(i-1) = 1$. Otherwise, if a critical simplex is not an essential one, we say that it is a **superfluous or cancellable simplex**.

Notice that it is straightforward to prove that a critical edge e_i is essential if and only if e_i is completing a 1-cycle which represents a basic element of $H_1(G)$ not considered until this point.

It interesting to point out that the concepts of cancellable and essential critical simplices are strongly matched to the topology of the studied graph G. So in this sense, we can say that essential critical simplices are those whose existence is forced by the homology groups of G. These ideas are going to be made precise in the following results.

Proposition 5.5. *Let f be a discrete Morse function defined on a connected graph G such that $b_1 < +\infty$. If e is an essential critical edge of f on G with vertices v and w, then there exists a spanning tree T in G such that $e \notin T$ and $e + \widehat{vw}$ is a basic cycle of $H_1(G)$, where $\widehat{vw}$ is the unique path joining v and w in T.*

Proof. By means of Theorem 4.4, it holds that if all critical edges of f on G are removed, then we obtain a forest. Since every essential critical edge is given by every independent cycle of $H_1(G)$, it follows that if we just remove all essential critical edges, then we do not obtain any new connected component and hence, we get a spanning tree.

Since there is a bijection between the independent cycles of $H_1(G)$ and those edges of G such that do not belong to a spanning tree contained in G (see [10]), we obtain that every essential edge characterizes an independent cycle of $H_1(G)$. This is precisely the 1-cycle obtained by gluing the essential critical edge e with the two gradient paths starting from e and merging at some vertex. □

Proposition 5.6. *Let G be a connected graph with $b_1 < +\infty$. Then a discrete Morse function f on G is optimal if and only if all its critical edges are essential.*

Proof. By Proposition 5.5 the number of essential edges of f is b_1 and by Proposition 5.2 the optimality condition is equivalent to $m_1 = b_1$. Thus, we conclude that f is optimal if and only if there are no critical edges except the essential ones. □

The next result shows us the topological consequences of the existence of a discrete Morse function with no critical simplices.

Proposition 5.7. *Let G be a connected graph. If G admits a discrete Morse function with no critical simplices then G is an infinite tree.*

Proof. Let f be a discrete Morse function on G with no critical simplices. Then by Theorem 2.2 we have $0 + d_0 = m_0(f) + d_0 \geqslant b_0 = 1$ and $0 = m_1(f) \geqslant b_1$. Thus, $b_1 = 0$ and so G is a tree. Moreover, from $b_0 - b_1 = m_0(f) + d_0 - m_1(f)$, we also have $1 = d_0$, that is, f has one decreasing ray on G, so that G must be infinite. □

Remark 5.8. If we only assume $m_1 = 0$, we get that G is a forest.

Now, we are going to characterize a graph by taking into account the total number of critical elements of a discrete Morse function defined on it.

Theorem 5.9. *An infinite connected graph G admits a discrete Morse function with m critical elements if and only if*:

(i) *If m is odd, then G is either a tree or $G = C \cup F$, where C is a finite subgraph containing $2h$ independent cycles which are a basis for $H_1(G)$ with $h \leqslant \lceil \frac{m}{2} \rceil$, F is a finite forest with at least an infinite tree and every tree in F intersects C in a unique vertex.*
(ii) *If m is even, $G = C \cup F$ where C is a finite subgraph containing $2h + 1$ independent cycles which are a basis for $H_1(G)$ with $h \leqslant \lceil \frac{m}{2} \rceil - 1$ and F is a forest in the same conditions as above.*

Proof. Let us assume that m is odd (the case m is even is analogous). So, we may assume that $m = 2j + 1 = m_0 + d_0 + m_1$.

Now, by means of Theorem 2.2, we get that $1 - b_1 = m_0 + d_0 - m_1$ and adding both equalities we get that $b_1 = 2j + 2 - 2(m_0 + d_0) = 2h$. Hence, b_1 is even. Moreover, by using Theorem 2.2, it holds that $m_0 + d_0 \geqslant 1$ which implies that $0 \leqslant b_1 \leqslant 2j$, that is, $0 \leqslant h \leqslant j = \lceil \frac{m}{2} \rceil$.

Conversely, let us suppose that G is the union of $2h$ independent cycles C and a forest F. By means of the proof of Proposition 5.3, we can obtain an optimal discrete Morse function f on G with $m = 1 + 2h$. If $m = 1 + 2j > 1 + 2h$, then it is possible to get a new (non-optimal) discrete Morse function $\widehat{f}$ starting from f and introducing $j - h$ new pairs of critical vertices and edges. It can be done by selecting a non-critical edge e of f on G. Thus, one of its two bounding vertices v satisfies $f(e) \leqslant f(v)$. Then we define $\widehat{f} = f$ on $G - \{v, e\}$ and $\widehat{f}(e) > \widehat{f}(v)$ so $\widehat{f}$ has two new critical simplices: v and e. By repeating this procedure $j - h$ times, we finally get the desired function. □

Remark 5.10. Notice that $m = 1$ implies that G is a tree.

6. Counting the number of discrete Morse functions on a graph

In this section we shall obtain the number of elements of the set of classes of homologically equivalent discrete Morse functions on certain graphs. In the differentiable setting, this calculation was done for S^1 in [11].

Theorem 6.1. *The number of homology equivalence classes of excellent discrete Morse functions with $m = b_0 + b_1 + 2k$ critical elements on a graph G is*:

1. *C_k if G is a tree,*
2. *$C_k\binom{m-1}{2k}$ if $G = \bigvee^{b_1} S^1 \cup T_1 \cup \cdots \cup T_r$ where the T_i are trees,*
3. *$C_k\binom{m-2}{2k}$ if $G = \bigvee^{b_1} S^1$;*

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ denotes the kth Catalan number, $\bigvee^{b_1} S^1$ denotes the union of b_1 copies of S^1 by a common vertex and every tree T_i intersects $\bigvee^{b_1} S^1$ in a unique vertex.

Proof. 1. If G is a tree, then $b_0 = 1$ and $b_1 = 0$. Therefore B_1 is a sequence of zeros. By Lemma 3.5 we have $B_0(i+1) \neq B_0(i)$ for every i, then $B_0(i+1) - B_0(i) = \pm 1$ for each $i = 0, 1, \ldots, m-2$. So, the sequence B_0 is as a walk in $\mathbb{Z}_{>0}$ starting and ending at 1, with length $m - 1 = 2k$ and steps of size ± 1. But it is known (see [8]) that the number of such walks is the kth Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$. So there are at most C_k homology equivalence classes in this case.

In order to prove that there are exactly C_k classes, we will construct an excellent discrete Morse function f on G such that its sequence B_0 is equal to a walk $n_0, n_1, \ldots, n_{2k}$.

First, by subdividing sufficiently many times we can get that the number of simplices is greater than m. Next, we choose the m simplices which will be the critical elements of the Morse function: we select $k + 1$ 0-simplices $v_1, \ldots, v_{k+1}$ and we take k 1-simplices e_l in the following way: If the unique path joining two 0-simplices v_i and v_j does not contain any other selected 0-simplex, e_l is some 1-simplex in this path. We shall denote the sets of selected vertices and edges by V_c and E_c respectively.

Notice that if we remove the k selected 1-simplices of G, we obtain a forest F with $k+1$ trees and each of these trees contains exactly one selected 0-simplex. So, for each 0-simplex v_i, we can consider the tree of F containing v_i as a tree rooted in v_i, denoted T_{v_i}. Observe that, by means of by Proposition 4.2, once we have constructed the Morse function on G, the tree T_{v_i} is equal to the tree rooted in the 0-simplex v_i.

We will get the function f following the next steps:

Step 0 We begin with one 0-simplex $p_0 \in V_c$ which will be the global minimum of f and we define $f(p_0) = 0$. Next, we define f on the tree T_{p_0} by levels as we did in the proof of Proposition 5.3 obtaining a Morse function on the tree $T_0 = T_{p_0}$, with one critical element p_0 and whose sequence B_0 has only one element which is $n_0 = 1$.

Step 1 As we have $n_1 = 2$, we take a new 0-simplex $p_1 \in V_c$ such that there is a unique 1-simplex in E_c contained in the unique path joining p_0 and p_1. We set $f(p_1) = f(p_0) + 1$ and next, we define f on T_{p_1} by levels as it was done in Step 0. Thus, we get an excellent discrete Morse function on the subgraph $T_1 = T_{p_0} \cup T_{p_1}$ of G whose associated sequence is $B_0 = (n_0, n_1) = (1, 2)$, since T_1 is not connected because $T_{p_0} \cap T_{p_1} = \emptyset$.

Step $j+1$ Suppose that we have already defined f on the subgraph

$$T_j = T_{p_0} \cup T_{p_1} \cup \cdots \cup T_{p_{i_r}} \cup \{p_{i_{r+1}}, \ldots, p_{i_j}\}$$

of G whose associated homological sequence is $B_0 = (n_0, n_1, \ldots, n_j)$. Now, we check if the number of connected components must increase or decrease and then, we extend f:

- If $n_{j+1} - n_j = 1$ we take a 0-simplex $p_{j+1} \in V_c$ not in T_j verifying that for some critical 0-simplex p_{i_t} in T_j there is a unique 1-simplex in E_c contained in the unique path joining p_{i_t} and p_{j+1}. We set

$$f(p_{j+1}) = \max\{f(p_0), \ldots, f(p_j)\} + 1.$$

Now, we define f on $T_{p_{j+1}}$ as in Step 1 and we take $T_{j+1} = T_j \cup T_{p_{j+1}}$.

- If $n_{j+1} - n_j = -1$ we take a 1-simplex $p_{j+1} = uv$ in E_c not in T_j such that there exist two critical 0-simplices in T_j joined by a path in G including p_{j+1}. We set

$$f(p_{j+1}) = \max\{f(p_0), \ldots, f(p_j), f(u), f(v)\} + 1$$

and $T_{j+1} = T_j \cup \{p_{j+1}\}$.

It is easy to check that f is an excellent discrete Morse function on T_{j+1} whose homological sequence is $B_0 = (n_0, n_1, \ldots, n_j, n_{j+1})$.

In the last step, we need to consider a 1-simplex in E_c since $n_{2k} = 1$ and n_{2k-1} must be 2 ($n_{2k} - n_{2k-1} = -1$). Notice that in this step the function is defined on the whole of G.

At the end of this construction we obtain an excellent discrete Morse function on G whose sequence B_0 is $n_0, n_1, \ldots, n_{2k}$.

2. Let us now consider that G is the union of b_1 copies of S^1 and r trees $T_1, \ldots, T_r$, that is,

$$G = \bigvee^{b_1} S^1 \cup T_1 \cup \cdots \cup T_r.$$

In the homological sequences of an excellent discrete Morse function on G, we can see that there exist exactly b_1 values of t such that $B_1(t+1) - B_1(t) = 1$. By Lemma 3.5, for these values of t we have $B_0(t) = B_0(t+1)$. Thus, the homological sequences B_0 and B_1 obtained in this case are

$$\begin{array}{llllllllllll} B_0: & n_0, & \ldots, & n_{t_1}, & n_{t_1}, & n_{t_1+1}, & \ldots, & n_{t_{b_1}}, & n_{t_{b_1}}, & n_{t_{b_1}+1}, & \ldots, & n_{2k}, \\ B_1: & 0, & \ldots, & 0, & 1, & 1, & \ldots, & b_1 - 1, & b_1, & b_1, & \ldots, & b_1. \end{array} \tag{2}$$

To count these sequences, we remove the copies of n_{t_i} for $i = 1, \ldots, b_1$ in the sequence B_0 and we obtain a walk of length $2k$ like in the case of trees (there are C_k different such walks). On the other hand, the sequence B_1 is determined by the position of the b_1 1-simplices which are added to complete the copies of S^1. Therefore, there are $C_{b_1}^{m-1} = \binom{m-1}{b_1} = \binom{m-1}{2k}$ different sequences since the first critical element must be a 0-simplex, and the number of homology equivalence classes in this case is less than or equal to the number of different pairs of sequences B_0 and B_1, namely $C_k\binom{m-1}{2k}$.

In order to prove the equality, given sequences B_0 and B_1 like in Eq. (2), we can construct an excellent discrete Morse function f on G with these homological sequences. Again, we begin subdividing G to have enough simplices. Let us choose the m simplices which will be the critical elements of the Morse function in this way: we select the 0-simplex p which joins the copies of S^1, one 1-simplex e_i for each S^1 and the remaining selected simplices in one of the trees, for example T_1 (again, by subdividing sufficiently many times if necessary, we can get that the number of simplices of T_1 is greater than $2k$). As we have seen before, we can construct an excellent discrete Morse function g on the tree $T = G - \{e_1, e_2, \ldots, e_{b_1}\}$ whose sequence B_0 is

$$n_0, n_1, \ldots, n_{t_1}, n_{t_1+1}, \ldots, n_{2k}$$

and we can suppose that g has its global minimum on p. Moreover, in order to get this function, we have previously decomposed T in $k+1$ trees rooted in 0-simplices q_j and k 1-simplices q_l, where $q_0, \dots, q_{2k}$ are the critical elements of g with critical values $c_j = g(q_j)$ for $i = 0, \dots, 2k$.

Next, starting from g, let us construct a new excellent function f on G having the given homological sequences. The critical elements of f are those of g, $q_0, \dots, q_{2k}$, together with the 1-simplices $e_1, \dots, e_{b_1}$, where every edge e_i must be between in q_{t_i} and $q_{t_{i+1}}$, that is, $f(q_{t_i}) < f(e_i) < f(q_{t_{i+1}})$.

First, let us set $f = g$ on the forest F_1 generated by $q_0, q_1, \dots, q_{t_1}$. So, the first $t_1 + 1$ critical elements of f and g are the same, that is, $p_j = q_j$ for $j \leqslant t_1$.

In order to obtain $B_1(t_1+1) = 1$ at this step, we need to complete a copy of S^1. So the next critical element of f must be e_1. Therefore, we set $p_{t_1+1} = e_1$ and we define $f(e_1) = \max\{f(u_1), f(v_1), c_{t_1}\} + 1$. Notice that $e_1 = u_1 v_1$ and since u_1 and v_1 belong to F_1, f is already defined on them.

In the forest F_2 generated by $q_{t_1+1}, \dots, q_{t_2}$, we set $f = g + C_1$ where $C_1 = f(e_1) - c_{t_1+1} + 1$. Now, the new critical elements of f are $p_j = q_{j-1}$ for $t_1 + 1 \leqslant j \leqslant t_2$.

So we get that the critical values of f will be different. In fact, we have $f(q_{t_1}) < f(e_1) < f(q_{t_1+1})$ since $f(e_1) > c_{t_1} = f(q_{t_1})$ and

$$f(q_{t_1+1}) - f(e_1) = \big(c_{t_1+1} + f(e_1) - c_{t_1+1} + 1\big) - f(e_1) = 1 > 0.$$

In a similar way, we define f as follows: $f(e_i) = \max\{f(u_i), f(v_i), f(q_{t_i})\} + 1$ where $e_i = u_i v_i$ and $f = g + C_i$ on the forest F_i generated by $q_{t_{i-1}+1}, \dots, q_{t_i}$ being C_i a suitably chosen constant to assure that the critical values are different.

Besides obtaining different critical values, we have the following relations between the level subcomplexes of the tree $T = G - \{e_1, e_2, \dots, e_{b_1}\}$ and G:

$$G(a_j) = T(c_j), \quad \text{for } 0 \leqslant j \leqslant t_1,$$
$$G(a_j) = T(c_{j-r}) \cup \{e_1, \dots, e_r\}, \quad \text{for } \begin{cases} t_r + 1 \leqslant j \leqslant t_{r+1}, \\ 1 \leqslant r \leqslant b_1 - 1 \end{cases}$$

and

$$G(a_j) = T(c_{j-b_1}) \cup \{e_1, \dots, e_{b_1}\}, \quad \text{for } t_{b_1} + 1 \leqslant j \leqslant m,$$

where $a_j = f(p_j)$ are the critical values of f. Therefore, we obtain an excellent discrete Morse function f on G whose homological sequences are the given ones.

In consequence, the number of homology equivalence classes of excellent discrete Morse function for this type of graphs is $C_k\binom{m-1}{2k}$.

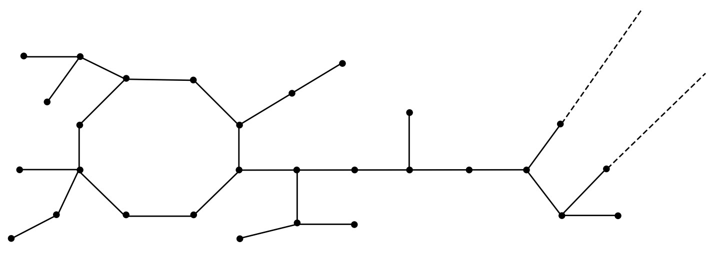
3. If G is the wedge of some copies of S^1, then the homological sequences satisfy $B_0(m-1) = B_0(m) = 1$, $B_1(m) = b_1$ and $B_1(m) - B_1(m-1) = 1$. That is, every excellent discrete Morse function on G reaches its global maximum on a critical 1-simplex e, which completes one of the copies of S^1.

If we remove e from G, we obtain the union of the wedge of copies of S^1 and trees or just a tree if $G = S^1$.

Notice that two excellent discrete Morse functions f and g on G are homologically equivalent if and only if their restrictions to $G - \{e\}$ are homologically equivalent. Then, the number of homology equivalence classes of excellent discrete Morse functions with m critical elements on G is equal to the number of these equivalence classes for $m-1$ critical elements and $G - \{e\}$. Thus, if $b_1 = 1$ we have $m = 2k+2$, and we obtain $C_k = C_k\binom{2k}{2k} = C_k\binom{m-2}{2k}$ equivalence classes, since $G - \{e\}$ is a tree. Besides, if $b_1 > 1$ we obtain $C_k\binom{m-2}{2k} = C_k\binom{(m-1)-1}{2k}$ equivalence classes. □

In the following example we clarify the constructions described in the above theorem:

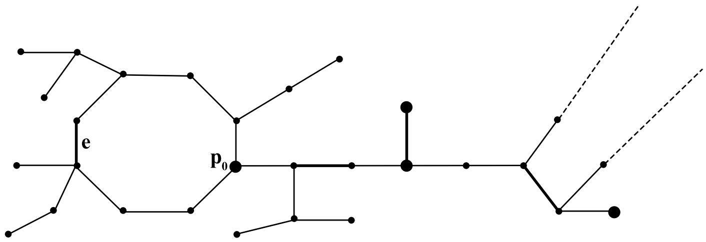
Example 6.2. Let us define an excellent discrete Morse function on the graph G of the figure below where dotted lines are rays:



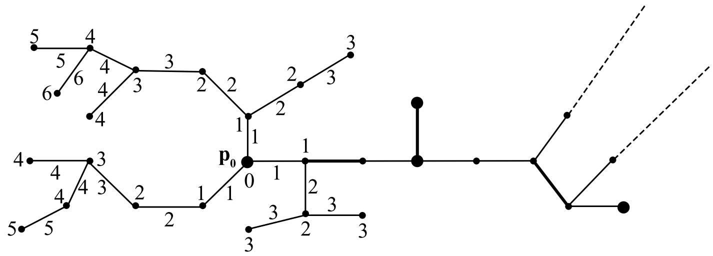
with 8 critical elements and whose homological sequences are

B_0: 1, 2, 3, 2, 2, 3, 2, 1,
B_1: 0, 0, 0, 0, 1, 1, 1, 1.

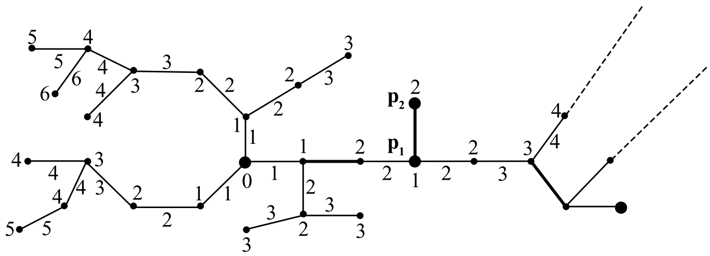
Let C be the unique cycle in G and let I be the infinite tree such that $C \cap I$ is a unique vertex p_0. We begin selecting the critical elements: we take p_0, the edge e in S^1 and the 6 simplices of I shown in dark in the picture below:



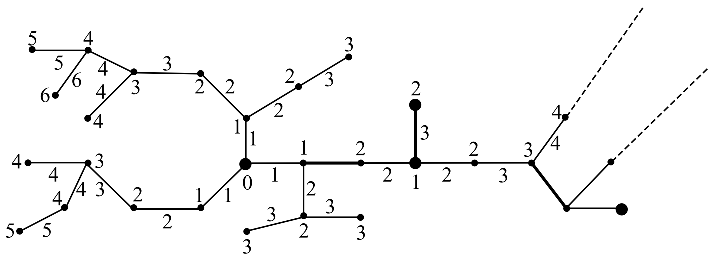
First, we can define an excellent discrete Morse function g on the tree $T = G - \{e\}$ in several steps. In the first step, we define the Morse function g on T_{p_0} by levels:



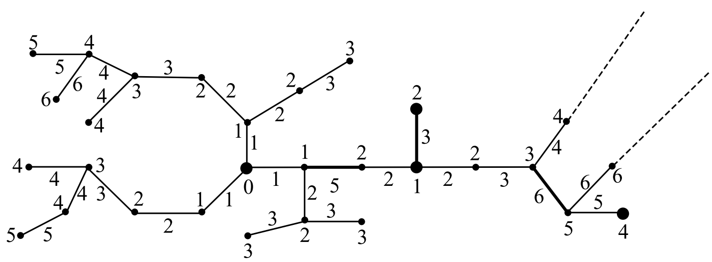
Then, we define g on $T_{p_1} \cup T_{p_2}$:



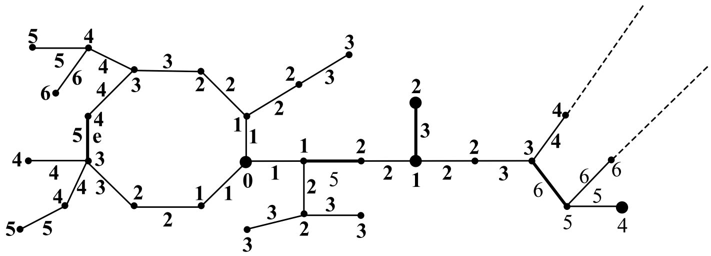
Next, we define g on one selected 1-simplex uv between two 0-simplices already considered using increasing critical values and setting $f(uv) > \max\{f(u), f(v)\}$:



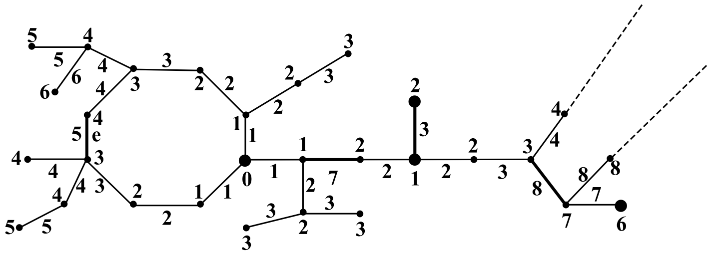
In the following steps we define g on a tree or a 1-simplex depending on whether $B_0(i+1) - B_0(i) = 1$ or -1:



On the forest F generated by the first 4 critical elements of g, we set $f = g$ and we assign to e a value greater than the values of f on its vertices and on the last critical element:



Finally on $G - (F \cup \{e\})$ we set $f = g + C$ where $C = f(e) - 4 + 1 = 2$:



As we can see in the last picture, the excellent Morse function f has the given homological sequences.

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