Existence of local strong solutions for motions of electrorheological fluids in three dimensions

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Received 13 September 2005; accepted 27 February 2006

Abstract

We prove for three-dimensional domains the existence of local strong solutions to systems of nonlinear partial differential equations with \( p(\cdot) \)-structure, \( p_\infty \leq p(\cdot) \leq p_0 \), and Dirichlet boundary conditions for \( p_\infty > \frac{9}{5} \) without restriction on the upper bound \( p_0 \). In particular this result is applicable to the motion of electrorheological fluids.

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Keywords: Electrorheological fluids; Strong solutions; Nonlinear partial differential equation

1. Introduction

Electrorheological fluids are special viscous liquids that change their viscosity rapidly when an electric field is applied. The motion of such fluids is governed by a system of nonlinear partial differential equations which read (cf. [1,2]).

\[
\begin{align*}
\text{div} (E + P) &= 0, \\
curl E &= 0, \\
\rho_0 \frac{\partial v}{\partial t} - \text{div} S + \rho_0 [\nabla v] v + \nabla \phi &= \rho_0 f + [\nabla E] P, \\
\text{div} v &= 0,
\end{align*}
\]

where \( E \) is the electric field, \( P \) the polarisation, \( \rho_0 \) the density, \( v \) the velocity, \( S \) the extra stress tensor, \( \phi \) the pressure and \( f \) the mechanical force. The above system has to be supplemented with constitutive relations for \( P \) and \( S \) and with boundary and initial conditions. According to the model in [1] the extra stress \( S \) is given by

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1 Here and in the following we use the notation \( [\nabla u] = \left( \frac{\partial u_i}{\partial x_j} \right)_{i=1,2,3} \), where the summation convention over repeated indices is used and little Latin indices run from 1 to 3.
Hence we study the existence of strong solutions to the system
\[(1b)\]
for all \(B\). It is shown in \([3,4]\) that \(S\) has under suitable assumptions on the coefficients \(a_{ij}\) and on the material function \(p \in W^{1,\infty}\) the following properties: strong monotonicity
\[
\partial_{kl} S_{ij}(D, E) B_{ij} B_{kl} \geq c \left(1 + |D|^2 \right)^\frac{p(|E|^2)-2}{2} |B|^2,
\]
coercivity
\[
S(D, E) : D \geq c \left(1 + |D|^2 \right)^\frac{p(|E|^2)-2}{2} |D|^2,
\]
growth conditions
\[
|\partial_{kl} S(D, E)| \leq C \left(1 + |D|^2 \right)^\frac{p(|E|^2)-1}{2},
\]
\[
|\partial_n S(D, E)| \leq C \left(1 + |D|^2 \right)^\frac{p(|E|^2)-1}{2} \left(1 + \ln \left(1 + |D|^2 \right) \right)
\]
for all \(B, D \in X := \{D \in \mathbb{R}^{3 \times 3}, \text{tr} D = 0\}\) and \(E \in \mathbb{R}^3\). Here we have used the abbreviations \(\partial_{kl} S_{ij} := \frac{\partial S_{ij}}{\partial D_{ij}}\) and \(\partial_n S_{ij} := \frac{\partial S_{ij}}{\partial E_{ij}}\). The constants \(c, C\) depend on \(|E|^2\) and \(\|p\|_{1,\infty}\). It can be easily seen that \(S\) can be approximated by \(S^\lambda\) with the following properties (cf. \([17-19,2]\))
\[
\partial_{kl} S_{ij}^\lambda(D, E) B_{ij} B_{kl} \geq c \left(1 + |D|^2 \right)^\frac{p(|E|^2)-2}{2} |B|^2,
\]
\[
S^\lambda(D, E) : D \geq c \left(1 + |D|^2 \right)^\frac{p(|E|^2)-2}{2} |D|^2,
\]

Thus we divide the equations through the constant density \(\rho_0\) and obtain new variables \(S, E\) and \(\phi\).
\[ |\partial_D S^k(D, E)| \leq C \left( \frac{1 + |D|^2}{1 + \lambda |D|^2} \right)^{\frac{\rho(|E|^2) - 2}{2}}, \]

\[ |\partial_D S^k(D, E)| \leq C \left( \frac{1 + |D|^2}{1 + \lambda |D|^2} \right)^{\frac{\rho(|E|^2) - 2}{2}} \left( 1 + |D|^2 \right)^{\frac{1}{2}} \times \left( 1 + \ln \left( \frac{1 + |D|^2}{1 + \lambda |D|^2} \right) \right) \]

for all \( B, D \in X \) and \( E \in \mathbb{R}^3 \). The constants \( c, C \) are independent of \( \lambda \). We remark that due to the inequality \( 1 \leq \frac{1+y}{1+y'\lambda} \leq \frac{1}{\lambda} \), which holds for all \( \lambda \in (0, 1) \) and all \( y \geq 0 \), we get that for each \( \lambda \) there exist constants \( c_\lambda, C_\lambda \) which depend on \( \lambda, p_0 \) and \( p_\infty \) such that \( S^k \) has the properties (4) with \( p \equiv 2 \). For a detailed discussion of the underlying model and structure, the relevant functions spaces, especially Lebesgue and Sobolev spaces with variable exponents and notations we refer the reader to [2].

In the following we consider instead of the concrete model (2) a more general situation, namely

Assumption 1. \( S \) can be locally uniformly approximated by \( S^k \) with the properties (5).

Besides the above example it can be easily checked that stress tensors \( S \) which are derived from a potential, i.e. \( S_{ij}(D) := \frac{\partial \phi(D)}{\partial D_{ij}} \), fulfill the Assumption 1 (cf. [20,21]).

In [2] it is shown amongst other things that the system (3) possesses weak solutions if \( \frac{9}{5} < p_\infty \leq \rho(|E|^2) \) and local strong solutions if \( \frac{9}{5} < p_\infty \leq \rho(|E|^2) \leq p_0 < 6 \). In the present paper we remove the artificial upper bound \( p_0 < 6 \). Acerbi and Mingione proved in their paper [5] partial regularity, if \( \rho(|E|^2) > \frac{9}{5} \). In a series of papers, Kaplický, Málek and Stará showed the existence of a local strong solution in two dimensions for constant \( p > \frac{6}{5} \). This solution is actually \( C^{1,\alpha}_{\text{loc}} \) (cf. [19,22,20]). In the unsteady case with constant \( p \) satisfying \( \frac{9}{4} \leq p < 3 \), the existence of a strong solution up to the boundary was proved by Málek, Nečas and Růžička in [23].

**Theorem.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain, let \( S \) fulfill Assumption 1, \( f \in L^{q'}(\Omega) \) with \( q = \min(2, p_\infty) \) and \( p_\infty > \frac{9}{5} \). Then there exists a local strong solution of the problem (3), i.e. \( v \in W^{2,\tilde{p}(\cdot)}_{\text{loc}}(\Omega) \cap V_3^{p(\cdot),\text{loc}} \cap V_{p(\cdot)} \) with \( \tilde{p}(\cdot) := \min(p(|E(\cdot)|^2), 2) \) solves

\[
\int_{\Omega} S(D(v), E) : D(\varphi) \, dx + \int_{\Omega} |\nabla v| \cdot \varphi \, dx = \int_{\Omega} (f + \chi E [\nabla E] E) \cdot \varphi \, dx, \quad \forall \varphi \in V_{p(\cdot)}.
\]

Further, for all \( U \subset \subset \Omega \) we have

\[
\int_{U} \left( 1 + |D(v)|^2 \right)^{\frac{\rho(|E|^2) - 2}{2}} |D(\nabla v)|^2 \, dx < \infty.
\]

**Remark 1.** From the above theorem we conclude that \( v \in W^{2,\tilde{p}(\cdot)}_{\text{loc}}(\Omega) \).

In order to prove this theorem we construct approximate solutions \( v^k \) as solutions of problem (3) with \( S \) substituted by \( S^k \). For this approximate problem it holds that:

**Proposition 2.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain, \( f \in L^2(\Omega) \). Further, let \( E \in W^{1,\infty}(\Omega) \). Then there exists a local strong solution \( v^k \in W^{2,2}_{\text{loc}}(\Omega) \cap V_2 \) of the problem

\[
-\text{div} S^k(D(v^k), E) + [\nabla v^k] + \nabla \varphi = f + \chi E [\nabla E] E \quad \text{in} \ \Omega, \quad (6a)
\]

\[
\text{div} v^k = 0 \quad \text{in} \ \Omega, \quad (6b)
\]

\[
v^k = 0 \quad \text{on} \ \partial \Omega. \quad (6c)
\]

**Proof.** We get the existence of a weak solution \( v^k \in V_2 \) with the help of Brezis’ theorem on pseudomonotone operators (cf. [24,25,2]). By the standard difference quotient technique one sees that \( v^k \in W^{2,2}_{\text{loc}} \) (cf. [2,17,18]) holds. □
2. Proof of the theorem

Let \( \psi^\lambda \in W_{loc}^{2,2}(\Omega) \cap V_2 \) be the approximate solution given by the above proposition. By testing the weak formulation of (6a) with \( \psi^\lambda \) we conclude with the help of (5b) and using that \( \int_{\Omega} [\nabla \psi^\lambda] \psi^\lambda \, dx = 0 \)

\[
\int_{\Omega} \left( \frac{1 + |D(\psi^\lambda)|^2}{1 + \lambda |D(\psi^\lambda)|^2} \right)^{p(E^2) - 2} |D(\psi^\lambda)|^2 \, dx \leq C(f, E),
\]

uniformly in \( \lambda \) (cf. [2,18]). Thus we get from Lemma 4\(^4\) with \( \alpha = \frac{1}{2} \)

\[
\int_{\Omega} |D(\psi^\lambda)|^\tilde{p}(\cdot) \, dx \leq C(f, E),
\]

where \( \tilde{p}(x) = \min(p(|E(x)|^2), 2) \). Now we want to multiply (6a) by a function which is locally equal to \( -\Delta \psi^\lambda \). In order to avoid problems with the pressure we choose \( \psi := \text{curl} (\xi^{2\alpha} \text{curl} \psi^\lambda) = -\xi^{2\alpha} \Delta \psi^\lambda + h \),

where \( \xi \) is a usual cut-off function, \( \alpha > 1 \) big enough and \( h := \nabla \xi \times \text{curl} \psi^\lambda \xi^{2\alpha - 1} \). Since \( h \) is linear in \( \nabla \psi^\lambda \) and \( \nabla \xi \), and the estimates

\[
|h| \leq C \xi^{2\alpha - 1} |\nabla \psi^\lambda|, \quad |\nabla h| \leq C \xi^{2\alpha - 1} |\nabla^2 \psi| + C \xi^{2\alpha - 2} |\nabla \psi^\lambda|
\]

are true with \( C = C(\nabla \xi, \nabla^2 \xi) \), we have at our disposal a test function \( \psi \) which is weakly divergence-free and possesses pointwise estimates for the correction term. We therefore have

\[
\int_{\Omega} \text{div} S^\lambda(\psi^\lambda, E) \cdot (\Delta \psi^\lambda \xi^{2\alpha} - h) \, dx - \int_{\Omega} [\nabla \psi^\lambda] \psi^\lambda \cdot (\Delta \psi^\lambda \xi^{2\alpha} - h) \, dx + \int_{\Omega} f \cdot (\Delta \psi^\lambda \xi^{2\alpha} - h) \, dx
\]

\[
+ \chi E \int_{\Omega} [\nabla E] \psi \cdot (\Delta \psi^\lambda \xi^{2\alpha} - h) \, dx =: \sum_{i=1}^4 I_i = 0.
\]

Using a density argument in order to perform the partial integration we obtain

\[
I_1 = \int_{\Omega} \text{div} S^\lambda(\psi^\lambda, E) \cdot (\xi^{2\alpha} \Delta \psi^\lambda - h) \, dx
\]

\[
= \int_{\Omega} \frac{\partial}{\partial x_k} S_{ij}^\lambda(\psi^\lambda, E) D_{ij} \left( \frac{\partial \psi^\lambda}{\partial x_k} \right) \xi^{2\alpha} \, dx + 2\alpha \int_{\Omega} S_{ij}^\lambda(\psi^\lambda, E) D_{ij} \left( \frac{\partial \psi^\lambda}{\partial x_k} \right) \xi^{2\alpha - 1} \, dx
\]

\[
- \alpha \int_{\Omega} S_{ij}^\lambda(\psi^\lambda, E) \left( \Delta \psi^\lambda \xi^{2\alpha - 1} \frac{\partial \xi}{\partial x_j} + \Delta \psi^\lambda \xi^{2\alpha - 1} \frac{\partial \xi}{\partial x_i} \right) \, dx + \int_{\Omega} S^\lambda(\psi^\lambda, E) : D(h) \, dx
\]

\[
=: J_1 + J_2 + J_3 + J_4.
\]

With the chain rule, the term \( J_1 \) can be written as

\[
J_1 = \int_{\Omega} \partial_{lm} S_{ij}^\lambda(\psi^\lambda, E) D_{lm} \left( \frac{\partial \psi^\lambda}{\partial x_k} \right) D_{ij} \left( \frac{\partial \psi^\lambda}{\partial x_k} \right) \xi^{2\alpha} \, dx + \int_{\Omega} \partial_n S_{ij}^\lambda(\psi^\lambda, E) \frac{\partial E_n}{\partial x_k} D_{ij} \left( \frac{\partial \psi^\lambda}{\partial x_k} \right) \xi^{2\alpha} \, dx
\]

\[
=: J_{1,1} + J_{1,2}.
\]

Using the monotonicity condition (5a) we obtain

\[
J_{1,1} \geq c \int_{\Omega} \left( \frac{1 + |D(\psi^\lambda)|^2}{1 + \lambda |D(\psi^\lambda)|^2} \right)^{p(E^2) - 2} |D(\nabla \psi^\lambda)|^2 \xi^{2\alpha} \, dx
\]

\[
=: c I_p^\lambda(\nabla \psi^\lambda) \xi^{\alpha}.
\]

\(^4\) Please note that all technical lemmata related to the properties of the operator \( S \) are collected in the Appendix.

\(^5\) This test function, which is divergence-free, was used in [26] to study Bingham-type fluids and earlier in the two-dimensional situation by [19, 22,20].
This term will give us all the information and all other terms have to be estimated. With the Sobolev embedding theorem, Korn’s inequality, estimate (7) and the inequalities (21), (23) and (24) (cf. Lemmas 5, 7 and 9) we get the following lower bound for $I_p^\lambda$:

\[
\left( \int_\Omega \left( \frac{1 + |Dv^\lambda|^2}{1 + \lambda|Dv^\lambda|^2} \right)^\frac{3\min(p,2)}{2} |Dv^\lambda|^{6\xi} \sigma \text{d}x \right)^{\frac{1}{3}} + \left( \int_\Omega |\nabla v^\lambda|^3 \tilde{\beta}(\lambda) \xi \sigma \text{d}x \right)^{\frac{1}{3}}
\]

\[+ \int_\Omega \nabla^{2}\lambda |\tilde{\beta}(\xi)\alpha \tilde{\beta}(\xi)\text{d}x + \|\nabla^{2}\lambda \xi \sigma\|_2^2 \leq CI_p^\lambda |D(\nabla v^\lambda)|^{\xi \sigma} + C(f, E), \tag{10}\]

where $\tilde{\beta}(x) = \min(2, p(|E(x)|^2))$ and $\tilde{\beta}_\sigma := \inf_{\text{supp} \xi} \tilde{\beta}$. The constants $C, C(E, f)$ are independent of $\lambda$. From the growth of $\partial_n S_k$ (cf. (5c)) we obtain that the term $J_{1,2}$ can be estimated by

\[
C \int_\Omega \left( \frac{1 + |Dv^\lambda|^2}{1 + \lambda|Dv^\lambda|^2} \right)^{\frac{\min(p,2)}{2}} \ln \left( \frac{1 + |Dv^\lambda|^2}{1 + \lambda|Dv^\lambda|^2} \right) |Dv^\lambda| |D(\nabla v^\lambda)|^{\xi \sigma} \text{d}x + C \leq \delta \left( \int_\Omega \left( \frac{1 + |Dv^\lambda|^2}{1 + \lambda|Dv^\lambda|^2} \right)^{\frac{3\min(p,2)}{2}} |Dv^\lambda|^{6\xi} \sigma \text{d}x \right)^{\frac{1}{3}} + \delta I_p^\lambda |D(\nabla v^\lambda)|^{\xi \sigma} + C(\delta, f, E),
\]

where we used similar calculations as in the proof of Lemma 5. Note that the first two terms can be absorbed with the help of (10). We notice that the terms $J_2 - J_4$ can be estimated with the help of the growth condition for $S_k$, the Cauchy–Schwarz inequality and estimate (7) as follows:

\[
|J_2 + J_3 + J_4| \leq C \int_\Omega \left( \frac{1 + |Dv^\lambda|^2}{1 + \lambda|Dv^\lambda|^2} \right)^{\frac{\min(p,2)}{2}} |Dv^\lambda| |D(\nabla v^\lambda)|^{\xi \sigma - 1} \text{d}x + C(f, E) \leq \delta I_p^\lambda |D(\nabla v^\lambda)|^{\xi \sigma} + C(f, E).
\]

We estimate the convective term through

\[
|I_2| \leq \delta \|\nabla^2 v^\lambda \xi \sigma\|_{\tilde{\beta}_\sigma}^2 + c(\nabla \xi) \left( \int_\Omega |\nabla v^\lambda|^3 |\tilde{\beta}_\sigma - |\nabla v^\lambda| \tilde{\beta}_\sigma \text{d}x \right)^{\frac{2}{p_\sigma}} + C \int_\Omega |\nabla v^\lambda|^2 |\nabla v^\lambda|^{\xi \sigma - 1} \text{d}x. \tag{11}\]

Now we have to distinguish two cases, namely $p_\sigma \leq 2$ and $p_\sigma \geq 2$. In the case $p_\sigma \leq 2$ the second term in (11) can be written as

\[
\left( \int_\Omega |\nabla v^\lambda|^3 |\tilde{\beta}_\sigma - |\nabla v^\lambda| \tilde{\beta}_\sigma \text{d}x \right)^{\frac{2}{p_\sigma}} = \left( \int_\Omega |\nabla v^\lambda|^2 + (p_\sigma - 2) |\nabla v^\lambda|^{\frac{p_\sigma - 2}{2}} \text{d}x \right)^{\frac{2}{p_\sigma}}. \tag{12}\]

In (10) we have at our disposal the $L^{3p_\sigma}$-norm of $\nabla v^\lambda$ raised to the power $p_\sigma$ ($p_\sigma \leq 2 \Rightarrow \tilde{\beta}(x) = p(x) \geq p_\sigma$) and further we know that $\nabla v^\lambda \in L^{p_\sigma}$ and $v^\lambda \in L^{(p_\sigma)^*}$ are bounded uniformly in $\lambda$ (cf. (8)). Thus we can handle the right-hand side in (12) by Hölder’s inequality and the additive splitting of $p'$ in $z$, $p' - z \in (0, p_\sigma')$ if the following relations are satisfied:

\[
\frac{2}{p_\sigma'} - p_\sigma = p_\sigma, \quad \frac{z}{3p_\sigma} + \frac{p' - z}{p_\sigma} + \frac{p'}{p_\sigma'} \leq 1.
\]

The first condition comes from the fact that we want to absorb the term with the $L^{3p_\sigma}$ norm on the left-hand side and the powers must fit. The second condition comes from Hölder’s inequality. That leads for three-dimensional domains
to the condition \( z = \frac{p-p_-}{2} = \frac{p_-^2}{2(p_- - 1)} \)

\[
p_+ > \frac{9}{2},
\]

(13)

which coincides with the condition that ensures the existence of weak solution with the theory of monotone operators. However, we also need to apply Young’s inequality to absorb one term on the left-hand side and thus we have to modify the above argument slightly and introduce a parameter \( s > 1 \). We get with Hölder’s inequality \((6s/p_-', p_- / p_-/(1 - p_-/(2s)), 3(p_- - 1)/(3 - p_-), \frac{3s(p_- - 1)}{6s - (s+1)p_-})\), where \( 1 < s < \frac{p_-}{p_- - 4p_-} \), and with Young’s inequality

\[
\left( \int_\Omega |\nabla \lambda^\alpha|^2 |\lambda|^2|\xi|^{2\alpha-1} \right)^{\frac{p_-}{2}} = \left( \int_\Omega |\nabla \lambda^{p_-/(2s)}| |\nabla \lambda|^{3p_-/(2s)} |\lambda|^2|\xi|^{2\alpha-1} \right)^{\frac{p_-}{2}}
\]

\[
\leq \left( \int_\Omega |\nabla \lambda|^{6s/(s+1)p_-} \right)^{\frac{2s}{2}} \left( \int_\Omega |\nabla \lambda|^{3p_-/(2s)} \right)^{\frac{2}{2}} \delta \left( \int_\Omega |\nabla \lambda|^{3p_-/(2s)} \right)^{\frac{3}{2}} + C_{\delta}.
\]

We easily estimate the last term in (11) in a similar way:

\[
\int_\Omega |\nabla \lambda^\alpha|^2 |\lambda|^{2\alpha-1} \right)^{\frac{p_-}{2}} = \int_\Omega |\nabla \lambda|^{p_- + (2 - p_-)} |\lambda|^2|\xi|^{2\alpha-1} \right)^{\frac{p_-}{2}}
\]

(14)

if

\[
\frac{p_-}{3p_-} + \frac{2 - p_-}{p_-} + \frac{1}{p_-} \leq 1 \Leftrightarrow p_- \geq \frac{3}{2}.
\]

To handle the powers of the cut-off function we need \( p_- > \frac{3}{2} \). Altogether the requirement which has to be satisfied is \( p_- > \frac{9}{2} \). In the case \( p_- \geq 2 \) we use Hölder’s inequality with (6, 2, 3):

\[
\left| \int_\Omega |\nabla \lambda| |\nabla \lambda| |\lambda|^\alpha |\xi|^{2\alpha} \right| + \int_\Omega |\nabla \lambda| |\nabla \lambda| |\lambda|^\alpha |\xi|^{2\alpha-1} \right| \right|
\]

\[
\leq \left( \int_\Omega |\nabla \lambda|^{6s/(s+1)p_-} \right)^{\frac{6}{s+1}} \left( \int_\Omega |\nabla \lambda|^{(2\alpha-1)} \right)^{\frac{7}{s+1}} \delta \left( \int_\Omega |\nabla \lambda|^{3p_-/(2s)} \right)^{\frac{3}{2}} + C_{\delta},
\]

(15)

where we also used Young’s inequality with (2, 2), Sobolev’s embedding theorem and estimate (7).

The term \( I_3 \) is easy and we have by Hölder’s and Young’s inequality

\[
|I_3| \leq \delta \|\nabla \lambda^\alpha \|_{\tilde{\rho}_-}^2 + C_{\delta} \|\nabla \lambda^\alpha \|_{\tilde{\rho}_-}^2 + C \|\nabla \lambda^\alpha \|_{\tilde{\rho}_-}^2.
\]

(16)

\( I_4 \) can be treated analogously to \( I_3 \). Altogether we therefore get from (9)–(16) the uniform estimate with respect to \( \lambda \):

\[
\|\nabla \lambda^\alpha \|_{\tilde{\rho}_-, \tilde{\rho}_-}^2 + \|\nabla \lambda^\alpha \|_{\tilde{\rho}_-, \tilde{\rho}_-}^2 + \|\nabla \lambda^\alpha \|_{\tilde{\rho}_-, \tilde{\rho}_-}^2 \leq c(f, E, \nabla \lambda, p_+).
\]

(17)
Thus $D(v^2)$ converges pointwise to $D(v)$ by compact embedding. With Fatou’s lemma we derive from the inequalities (7), (10) and (17) that

$$
\int_{\Omega} \left( 1 + |D(v)|^2 \right)^{\frac{p(|E|^2-1)}{2}} |D(v)|^2 \, dx + \int_{\Omega} \left( 1 + |D(v)|^2 \right)^{\frac{2(p(|E|^2-1))}{2}} |D(v)|^6 \xi^6 \, dx \leq C(f, E) \quad (18)
$$

From Lemma 4 with $\alpha = \frac{3}{2}$ we therefore get

$$
\int_{\Omega} |D(v)|^{3p(|E|^2)} \xi^6 \, dx \leq C(f, E).
$$

Further, (17) yields that for all $U \subset \subset \Omega$

$$
\int_{U} \left( 1 + |D(v)|^2 \right)^{\frac{p(|E|^2-1)}{2}} |D(\nabla v)|^2 \, dx \leq C. \quad (19)
$$

Indeed, set $f(\lambda, x, u) := \left( \frac{1+|u|^2}{1+\lambda|u|^2} \right)^{\frac{p(|E|^2-1)}{2}}$ and $F(\lambda, x, u, z) := f(\lambda, x, u)|z|^2$. Let $U \subset \subset \Omega$. For all $\epsilon > 0$ there exists a set $K \subset U$ such that $u^\lambda := D(v^\lambda) \Rightarrow u := D(v)$ uniformly in $K$, $u, z := \nabla u$ are continuous on $K$ and $|U - K| \leq \epsilon$. This is possible with the help of the theorems of Egorov and Lusin. Because $2(z_1, z_2) \leq |z_1|^2 + |z_2|^2$, one sees that $|z_1|^2 - |z_2|^2 \geq 2(z_2, z_1 - z_2)$.

$$
C \geq \int_{U} F(\lambda, x, u_\lambda, z_\lambda) \, dx \geq \int_{K} F(\lambda, x, u_\lambda, z_\lambda) \, dx \geq \int_{K} F(\lambda, x, u_\lambda, z) + (F(\lambda, x, u_\lambda, z_\lambda) - F(\lambda, x, u_\lambda, z)) \, dx \geq \int_{K} (F(\lambda, x, u_\lambda, z, z_\lambda - z).
$$

The second integral on the right-hand side converges to 0 due to the weak convergence of $z_\lambda \rightharpoonup z$ in $L^{\hat{p}}$ and the above uniform convergence of $u_\lambda$. With the help of Fatou’s lemma, the claim (19) is proven for $\epsilon \to 0$. Thus all statements of the theorem are proven, if we apply Vitali’s theorem and use a diagonal sequence argument (cf. [27]).

**Remark 3.** We remark that our solution is locally continuous, i.e. $v \in C^\alpha_{\text{loc}}(\Omega)$ and if $p_\infty > 3$ actually $C^\beta(\overline{\Omega})$ with suitable $\alpha, \beta$.

**Appendix. Auxiliary results**

Here we present some technical results, mostly estimates related to $I^\lambda_p$.

**Lemma 4.** For $y, p, \alpha \geq 0$ it holds that

$$
y^{2ap} \leq 1 + c(1 + y^2)^{\alpha(p-2)} y^{4a}.
$$

If $y, \alpha \geq 0$, $0 \leq p \leq 2$ and $0 \leq \lambda \leq 1$ then

$$
y^{2ap} \leq 1 + c \left( \frac{1+y^2}{1+\lambda y^2} \right)^{\alpha(p-2)} y^{4a}.
$$

**Proof.** The only non-obvious case we have to look at is $0 \leq p \leq 2$ and $y \geq 1$:

$$
(1+y^2)^{\alpha(p-2)} y^{4a} \geq (2y^2)^{\alpha(p-2)} y^{4a} = 2^{\alpha(p-2)} y^{2ap}.
$$

The second inequality follows easily from the first and $\frac{1+y^2}{1+\lambda y^2} \leq 1+y^2$ which holds for all $\lambda \in [0, 1]$ and all $y \geq 0$. $\square$
Lemma 5. Let \( \xi \) be a usual cut-off function, \( \lambda \in [0, 1] \) and \( p \in W^{1, \infty} \) with \( p \geq p_\infty > 1 \), then

\[
\left\| \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)| \xi \right\|_r^2 \leq C \left( I_{p'}(D(\nabla u)) + \left\| \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)| \right\|_2^2 \right) + C,
\]

holds with \( 1 \leq r \leq 2^* = \frac{2n}{n-2} \) for \( n \geq 3 \); in particular, \( r \leq 6 \) for \( n = 3 \).

Proof. We calculate

\[
\nabla \left( \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)| \right) \leq C(p', \nabla E) \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} \ln \left( 1 + |D(u)|^2 \right) |D(u)|
\]

\[
+ C \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} \frac{1 - \lambda}{1 + |D(u)|^2} \frac{|D(u)|}{1 + |D(u)|^2} |D(\nabla u)|
\]

\[
+ C \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(\nabla u)|
\]

\[
\leq C \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)|^s + C_s
\]

\[
+ C \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(\nabla u)|,
\]

for arbitrary \( s > 1 \) where we used the estimate \( \ln \left( 1 + \frac{\lambda^2}{1 + \lambda^2 + \lambda y} \right) \leq \left( \frac{\lambda^{p-2}}{1 + \lambda^2 + \lambda y} \right)^\epsilon + C_\epsilon \) for every \( \epsilon > 0 \) and \( p \geq p_\infty > 1 \). The constant \( C \) may depend on \( p', E \) and \( \nabla E \). Since \( W^{1, 2}_0(\Omega) \) is embedded into \( L^r(\Omega) \), for every \( r \leq \frac{2n}{n-2} \) if \( \Omega \subset \mathbb{R}^n \) is bounded, we get using also Poincaré’s inequality

\[
\left\| \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)| \xi \right\|_r \leq C \left\| \nabla \left( \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)| \right) \xi \right\|_2^2
\]

\[
\leq C \left\| \nabla \left( \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)| \right) \xi \right\|_2^2
\]

\[
+ C \left\| \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)| \nabla \xi \right\|_2^2
\]

\[
\leq C \left\| \nabla \left( \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)| \right) \xi \right\|_2^2
\]

\[
+ C \left\| \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)| \nabla \xi \right\|_2^2
\]

\[
\leq \left\| \left( 1 + |D(u)|^2 \right)^{\frac{p(E^2)-2}{4}} |D(u)| \right\|_2^2.
\]
Therefore we get using (20) and interpolation of $L^2$ between $L^2$ and $L^r$, $2 < 2s < r$, Young’s inequality and $|\xi| \leq 1$,
\[
\left\| \frac{1 + |D(u)|^2}{1 + \lambda |D(u)|^2} \right\| \frac{p(\beta^{2s-2})}{4} |D(u)|^{\xi} \leq C \left( 1 + I_{\beta}(D(\nabla u)\xi) + \left\| \frac{1 + |D(u)|^2}{1 + \lambda |D(u)|^2} \right\| \right)^{2}.
\]
\[
\text{Remark 6. Using estimate (7) and Lemma 4 we get from Lemma 5, with } r = 6,
\[
\left( \int_{\Omega} \left( 1 + |D(v^{\lambda})|^2 \right) \frac{\beta^{2s-2}}{4} |D(v^{\lambda})|^{\xi} \right)^{1/2} d\lambda + \left( \int_{\Omega} |\nabla v^{\lambda}|^{3} |\xi|^{6\alpha} d\lambda \right)^{1/3} \leq C I_{\beta}^{\lambda}(D(\nabla v^{\lambda})\xi^\alpha) + C(f, E).
\]

\textbf{Lemma 7. Let } w \in L^\infty(\Omega) \text{ be a positive weight, i.e. } w > 0 \text{ a.e. Then the following inequality is true:}
\[
\int_{\Omega} |D(\nabla u)|^{\tilde{\rho}(\cdot)} \xi^\alpha d\lambda \leq C(p_0, p_\infty) \int_{\Omega} w^{\tilde{\rho}(\cdot)} + w^{\frac{\tilde{\rho}(\cdot)-2}{2}} |D(\nabla u)|^{2} \xi^{2\alpha} d\lambda.
\]
\text{Note that if } w \geq 1 \text{ or } w \geq \epsilon > 0 \text{ a.e., then}
\[
\int_{\Omega} |D(\nabla u)|^{\tilde{\rho}(\cdot)} \xi^\alpha d\lambda \leq C(p_0, p_\infty, \epsilon) \int_{\Omega} w^{\tilde{\rho}(\cdot)} + w^{\frac{\tilde{\rho}(\cdot)-2}{2}} |D(\nabla u)|^{2} \xi^{2\alpha} d\lambda.
\]
\textbf{Proof. If } \tilde{\rho}(x) < 2 \text{ use Young’s inequality with } (\frac{2}{2-\tilde{\rho}(x)}, \frac{2}{\tilde{\rho}(x)})
\[
|D(\nabla u)(x)|^{\tilde{\rho}(x)} = w(x)^{\tilde{\rho}(x)-2} |\tilde{\rho}(x)| |D(\nabla u)|^{\tilde{\rho}(x)} \leq C(p_0, p_\infty) w(x)^{\tilde{\rho}(x)-2} + w(x)^{\frac{\tilde{\rho}(x)-2}{2}} |D(\nabla u)(x)|^{2}.
\]
If \( \tilde{\rho}(x) = 2 \), obviously
\[
|D(\nabla u)(x)|^{2} \leq w(x) + |D(\nabla u)(x)|^{2}.
\]
If we now integrate over \( \Omega \) the first inequality follows. The second assertion follows from the first and \( \omega \geq \epsilon \). \( \square \)

\textbf{Remark 8. If we take } w = \left( \frac{1+|D(\nabla v)|^2}{1+\lambda|D(\nabla v)|^2} \right) \text{ in (22) and use estimate (7), } \frac{1+x^2}{1+\lambda x^2} \leq 1 + x^2 \text{ and Lemma 10 we get from inequality (22)}
\[
\int_{\Omega} |\nabla^2 v^\lambda|^{\tilde{\rho}(\cdot)} \xi^\alpha d\lambda \leq C I_{\beta}^{\lambda}(D(\nabla v^\lambda)\xi^\alpha) + C(f, E).
\]

\textbf{Lemma 9. Let } q < 2 \text{ and } u \in W^{2,q}, \text{ then}
\[
\|\nabla u\xi^\alpha\|_{q}^{2} \leq C \|1 + |D(u)|^{2}\|_{q}^{2} I_{q}(D(\nabla u)\xi^\alpha).
\]
\textbf{Proof. By Hölder’s inequality with } (\frac{2}{2-q}, \frac{2}{q}) \text{ we directly get}
\[
\int_{\Omega} |D(\nabla u)|^{q} \xi^{aq} d\lambda = \left( \int_{\Omega} \left( 1 + |D(u)|^{2} \right)^{\frac{2-q}{2}} \left( 1 + |D(u)|^{2} \right)^{\frac{q-2}{2}} |D(\nabla u)|^{q} \xi^{aq} d\lambda \right)^{\frac{2}{q}} \leq \|(1 + |D(u)|^{2})\|_{q}^{\frac{2-q}{2}} \int_{\Omega} \left( 1 + |D(u)|^{2} \right)^{\frac{q-2}{2}} |D(\nabla u)|^{2} \xi^{2\alpha} d\lambda \leq \|(1 + |D(u)|^{2})\|_{q}^{\frac{2-q}{2}} I_{q}(D(\nabla u)\xi^\alpha).
\]
Now we use Lemma 10 to get the desired result. \( \square \)
Lemma 10. The terms $|D(\nabla u)|$ and $|\nabla^2 u|$ are equivalent, i.e. there exist constants $c, C$, such that $c|\nabla^2 u| \leq |D(\nabla u)| \leq C|\nabla^2 u|$. 

Proof. The assertion follows from the algebraic identity

$$\partial_{ij}u_k = D_{jk}(\partial_i u) - \frac{1}{2}\partial_{ik} u_j + D_{ik}(\partial_j u) - \frac{1}{2}\partial_{jk} u_i = D_{jk}(\partial_i u) + D_{ik}(\partial_j u) - D_{ij}(\partial_k u).$$

References