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Compactifying the Picard Scheme

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Compactifications of Picard schemes have been studied by many authors using different methods. In [CJ], we announced a treatment modeled on Grothendieck's construction of the relative Picard scheme. Below we provide the details and also obtain some new finiteness theorems.

Igusa [I], inspired to some extent by Néron [Ne], was the first to study explicitly a compactification of a Picard scheme. He began with a Lefschetz pencil of hyperplane sections on a smooth surface (a general member is a smooth curve and finitely many members have a node as their only singularity). He defined the compactification for a singular member as the limit of the Jacobians of the smooth members using Chow coordinates (and Chow's construction [Ch] of the Jacobian). He proved that his compactification was intrinsic in the sense that, whenever the singular curve was expressed as a limit of nonsingular curves, its compactified Jacobian was the limit of the Jacobians.

Mayer and Mumford [MM] announced an intrinsic characterization of Igusa's compactified Jacobian as a component of the moduli space of rank-1, torsion-free sheaves. They said that such a compactification could be constructed for any integral curve using geometric invariant theory. D'Souza [D] obtained the relative compactified Jacobian for a family of integral curves over a Henselian (Noetherian) local ring with separably closed residue field by this method, and moreover he proved that it is flat and that its geometric fibers are integral

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local complete intersections when all the singularities of the curves are simple nodes or simple cusps.

In [AIK, (9)] it was shown that the relative compactified Jacobian of a family is flat and that its geometric fibers are integral local complete intersections whenever the family can be embedded in a family of smooth surfaces, recovering D'Souza's result in particular. By contrast, an example is given [AIK, (13)] to show that the compactified Jacobian may be reducible even for a curve that is a complete intersection in projective 3-space.

Namikawa [Na] obtained, using complex-analytic methods, a relative compactified Jacobian for a family of stable curves over $\mathbb{C}$. Seshadri and Oda [SO] obtained, using geometric invariant theory, various compactified Jacobians for a reduced but reducible curve over a field.

Below we work with a proper, finitely presented family X/S over an arbitrary base scheme S. The key to our approach is a theory of linear equivalence of quotients of a fixed flat sheaf F. Two quotients of F are considered to be linearly equivalent if they have the same "pseudo-Ideal" locally over S. We represent the corresponding functor $\mathbf{Lin\ Syst}_{(I,F)}$ by a twisted family of projective spaces $\mathbb{P}(H(I,F))$ associated to a manageable sheaf $H(I,F)$ in the case that I is a simple sheaf, where "simple" means that I is flat and on the fibers its global endomorphisms are the constants. Our usage of the term "simple" was inspired by Narasimhan and Seshadri's [NS, Definition 2.1, p. 541].

Assuming the family X/S is flat and projective with integral and Cohen–Maclaulay geometric fibers, and forming a quotient modulo linear equivalence, we construct a natural *quasi-projective* scheme $\mathrm{Pic}^{-\theta}_{(X/S)(\text{ét})}$; it represents the étale sheaf $\mathbf{Pic}^{-\theta}_{(X/S)(\text{ét})}$ of flat sheaves whose fibers are torsion-free, rank-1, and Cohen–Macaulay with Hilbert polynomial θ. (As is conventional, we denote the scheme or algebraic space representing a functor $\mathbf{P}$ by P.) In dimension 1, this scheme is projective, but in dimension greater than 1 it is not, because Cohen–Macaulayness is not a closed condition. On the other hand, we do represent by a *proper* algebraic space, the larger functor $\mathbf{Pic}^{=\theta}_{(X/S)(\text{ét})}$ of all flat sheaves whose fibers are rank-1, torsion-free with Hilbert polynomial θ, assuming only that the geometric fibers of X/S are integral (and not that X/S is flat). We plan in [C II] to represent $\mathbf{Pic}^{=\theta}_{(X/S)(\text{ét})}$ by a scheme under these same hypotheses. The construction will be based on the method Mumford used in [CS, Lectures 19–21] to form a quotient to construct the Picard scheme of a smooth surface.

Some important results on base-change theory are presented in Section 1. Essential to our theory of compactification is the sheaf $H(I,F)$. We recall its definition and basic properties, and we give a criterion for it to be locally free. (Its existence is proved for locally projective maps in [EGA III_2, 7.7.8]. Its existence for proper maps is stated there without proof. We use the latter result in our discussion of linear systems and conjugate systems but not in proving the main representation theorems.) We also prove some basic results for local Ext's. Most of the work comes in defining the base-change map (1.8) and proving the property of exchange (1.9). We obtain the latter using a lovely, general result [OB, 2.2]. It was in fact in this way that we got started. However, the property of exchange for local Ext's could also be obtained by extending the ideas of [EGA IV_3, Sect. 12.3; HC, Appendix], and this line of reasoning would yield a stronger result, namely, base-change in a neighborhood assuming a surjection along a fiber.

The second section introduces a new finiteness notion, strong quasi-projectivity. An S-scheme X is *strongly quasi-projective* if it is a finitely presented subscheme of a $\mathbb{P}(E)$, where E is a locally free O_S-Module with a constant finite rank. Strong quasi-projectivity is useful because projectivity is not a local property on the base.

We show (2.6) that $\text{Quot}^{\phi}_{(F/X/S)}$ is strongly quasi-projective if X/S is strongly quasi-projective, under a mild condition on F (automatically satisfied for $F = O_X$). The existence of $\text{Quot}^{\phi}_{(F/X/S)}$ as a locally projective scheme is well known, although no detailed proof has yet appeared in print. Grothendieck gave an outline [FGA 221–11] and Mumford worked it out in detail [CS, Lecture 15] in a special case, the Hilbert scheme for a smooth surface over a field. However, a careful look at Grothendieck's construction yields the strong finiteness.

We carry out and strengthen one of Grothendieck's constructions of the quotient for a flat and proper equivalence relation. This construction uses the Hilbert scheme and we are able to obtain strong finiteness. The basic idea goes back at least to Chow [Ch] and Matsusaka [M], who used Chow coordinates in place of the Hilbert scheme; the idea may go back to Castelnuovo (see [M, p. 51] and also [Z, p. 104]). Grothendieck's construction has never before appeared in print even in outline, although it was mentioned by Grothendieck [FGA 232–13]. It was briefly outlined privately by Mumford in 1967. Paying careful attention gives a strong finiteness theorem for the quotient (2.8), apparently not possible using quasi-sections and not expected even in this case.

Section 3 contains some rudimentary facts we use later about rank-1, torsion-free sheaves on an integral, algebraic scheme. Lemma (3.4) is the key to our finiteness results for the compactification.

Section 4 includes a generalization $\mathbf{Lin\ Syst}_{(I,F)}$ of the functor $\mathbf{Lin\ Syst}_I$ presented in [ASDS], which in turn generalizes a corresponding functor for I invertible introduced by Grothendieck [FGA, 232–10] and presented in detail by Mumford [CS, Chap. 13]. The representability of $\mathbf{Lin\ Syst}_{(I,F)}$ for I simple and F flat is established in (4.2); the basic ideas are found in [ASDS, 15] but are clarified and generalized here.

In Section 5 the basic functors are introduced and studied. The functor $\mathbf{Spl}_{(X/S)}$ of simple sheaves is proved separated for the étale topology, and we work with the associated sheaf. The étale subsheaves of relatively torsion-free, rank-1 sheaves, of pseudoinvertible sheaves, and of invertible sheaves are proved open, retrocompact subfunctors of $\mathbf{Spl}_{(X/S)(\text{et})}$. These functors are the targets of the "Abel map" and its restrictions. The sources are appropriate open, retrocompact subschemes of $\mathcal{Q}\mathrm{uot}_{(F/X/S)}$, where almost any F will do. The Abel map sends a quotient of F to the class of its pseudo-Ideal. The fibers of the Abel map are linear systems of quotients of F. Using the representation theorem for $\mathbf{Lin\ Syst}_{(I,F)}$ and the freeness criterion for $H(I,F)$, we prove that the Abel map is proper and finitely presented, compute its relative dimension and give criteria for its smoothness and projectivity.

The two main representation theorems are proved in Section 6. The key result is Proposition (6.2), which contains almost all the work. The representing boils down to forming a quotient of an appropriate open, retrocompact subscheme of a suitable $\mathrm{Quot}_{(F/X/S)}$ by linear equivalence. From our study of the Abel map, we conclude that the equivalence relation is representable, smooth, and proper. Then the quotient theorem (2.8) gives the desired representability by a strongly quasi-projective S-scheme.

The two main representation theorems are derived from (6.2). The first (6.3) asserts that the summand of the relative Picard functor $\mathbf{Pic}^{\theta}_{(X/S)(\acute{e}t)}$ is representable by a strongly quasi-projective S-scheme when X/S is flat and projective with geometrically integral fibers. This strengthens Grothendieck's theorem [FGA, 232]; see also [A1, p. 22 bottom]), which asserts only that the scheme $\mathrm{Pic}^{\theta}_{(X/S)(\acute{e}t)}$ exists and is locally quasi-projective. Our second theorem (6.6) asserts that $\mathbf{Pic}^{-\theta}_{(X/S)(\acute{e}t)}$ is representable by a strongly quasi-projective S-scheme when X/S is flat and projective with geometrically integral, Cohen–Macaulay fibers. In this case, the sheaf F of (6.2) is taken to be the dualizing sheaf ω.

In Section 7 we work "on the other side" with conjugate systems instead of linear systems. (The term "conjugate" was chosen because a common way in which one quotient G of F is turned nontrivially into another one is via an automorphism of F.) In this way we obtain a smooth equivalence relation on a retrocompact, open subscheme $S\text{-div}_{(F/X/S)}$ of $\mathrm{Quot}_{(F/X/S)}$, and the quotient is the étale sheaf $\mathbf{Spl}_{(X/S)(\mathrm{et})}$ of simple sheaves. The equivalence relation is not proper, but Artin's theorem [A2, Corollary 6.3] implies that the quotient is representable by an algebraic space $\mathrm{Spl}_{(X/S)(\mathrm{et})}$. No checking of axioms is necessary here; that work is already done in Artin's proof. As a corollary we get that $\mathbf{Pic}^{=\theta}_{(X/S)(\acute{e}t)}$ is, at least, representable by a finitely presented, proper algebraic space. Mumford's example [FGA, 236–01] shows it is not always a scheme.

The final section contains our main results; they deal with the case that X/S is a family of integral curves. In this case, the functors $\mathbf{Pic}^{-}_{(X/S)(\acute{e}t)}$ and $\mathbf{Pic}^{=}_{(X/S)(\acute{e}t)}$ coincide; they are representable by a disjoint union of projective schemes P_n, and P_n parametrizes the torsion-free, rank-1 sheaves with Euler characteristic n. We give a rather precise description of the Abel map $\mathcal{A}_\omega$ from $\mathrm{Quot}_{(\omega/X/S)}$ to $\mathrm{Pic}^{-}_{(X/S)(\acute{e}t)}$ in (8.4), where ω is the dualizing sheaf. It turns out somewhat surprisingly that $\mathrm{Quot}_{(\omega/X/S)}$ is the most natural source for the Abel map; the statements are natural generalizations of familiar statements for the map from the symmetric powers of the curve to the Jacobian in the smooth case.

We give, on the other hand, a more precise form of the D'Souza–Rego theorem (8.6), which asserts that the Abel map from $\mathrm{Hilb}_{(X/S)}$ to $\mathrm{Pic}^{-}_{(X/S)(\text{ét})}$ is smooth in degree $\geqslant 2p - 1$ if and only if X/S is Gorenstein. Though there is no statement yet in print, Rego mentioned in a preprint [Re] that D'Souza proved the Abel map to be smooth at a nonspecial point if X/k is Gorenstein using formal deformation theory. Rego proved the converse for large degree by studying the action of $\mathrm{Pic}_{(X/k)}$ on its boundary in $\mathrm{Pic}^{-}_{(X/k)}$. Our proofs are quite different, being more global in nature.

We construct a natural embedding of X/S into $P_{(-p)}$ where p is the arithmetic genus. The embedding generalizes the usual map in the smooth case, giving the Albanese property of the Jacobian, and as expected, it is an isomorphism for $p = 1$. We end with an example (inspired by [H]) of the compactified Picard scheme of a locally projective, but nonprojective family of nodal cubis.

1. Some Base-Change Theory

(1.1) (The O_S-Module $H(I, F)$). Let $f: X \to S$ be a finitely presented, proper morphism of schemes, and let I and F be two locally finitely presented O_X-Modules, with F flat over S. Then there exist a locally finitely presented O_S-Module $H(I, F)$ and an element $h(I, F)$ of $\mathrm{Hom}_X(I, F \neq_S H(I, F))$ which represent the (covariant) functor,

$$M \mapsto \mathrm{Hom}_X(I, F \otimes_S M),$$

defined on the category of quasi-coherent O_S-Modules M, and the formation of the pair commutes with base change; in other words, the Yoneda map defined by $h(I, F)$,

$$y: \mathrm{Hom}_T(H(I, F)_T, M) \to \mathrm{Hom}_{X_T}(I_T, F \otimes_S M), \tag{1.1.1}$$

is an isomorphism for every S-scheme T and every quasi-coherent O_T-Module M.

Indeed, the representability is a local quastion on the base S; hence we may assume S is affine. Then, by [EGA IV_3, 8.8.2(ii), 8.5.2(ii), 8.10.5(xiii), and 11.2.6(ii)], there exists a finite-type $\mathbb{Z}$-scheme S_0 such that X, I, and F come by base-change from an analogous triple X_0, I_0, and F_0 over S_0. Since S_0 is Noetherian, a pair $(H(I_0, F_0), h(I_0, F_0))$ representing the functor over S_0 exists and its formation commutes with arbitrary base-change. (The representability results from [EGA III_2, 7.7.8, 7.7.9] in case f is locally projective, and its compatibility with locally Noetherian base-changes is proved in [EGA III_2, 7.7.9]. See [ASDS, (12)] for a proof that the formation of $Q(F) = H(O_X, F)$ commutes with arbitrary base-change; the proof for $H(I, F)$ is analogous.)

For any invertible O_X-Module L, there is a canonical isomorphism,

$$H(I \otimes L, F \otimes L) = H(I, F), \tag{1.1.2}$$

because tensoring by L gives a map,

$$\mathrm{Hom}_X(I, F \otimes_S M) \to \mathrm{Hom}_X(I \otimes L, (F \otimes L) \otimes_S M),$$

with an inverse given by tensoring by L^{-1}.

The O_S-Module $H(I, F)$ is obviously functorial in I and F; it is covariant in I and contravariant in F. Moreover it is clearly right exact in each variable. In particular, the functor

$$N \mapsto H(I \otimes_S N, F)$$

is covariant and right exact. So we have a canonical isomorphism,

$$H(I, F) \otimes N = H(I \otimes_S N, F). \tag{1.1.3}$$

(1.2) Lemma. *Let X be a scheme and let I be an arbitrary O_X-Module. Then there exists a surjection $J \to I$ in which J is an O_X-Module such that for each affine morphism $g : Y \to X$ and each quasi-coherent O_Y-Module F, the pullback g^*J is acyclic for the functor* $\mathrm{Hom}_Y(-, F)$.

Proof. For any element f in any stalk of I, there is an affine neighborhood U of the stalk and an element $g \in \Gamma(U, I)$ whose image in the stalk is equal to f. So there are a family of affine open sets U and a surjection $J = \coprod J_U \to I$, where J_U denotes the extension by zero $(j_U)_!(O_X \mid U)$, where j_U denotes the inclusion of U in X. Then g^*J is equal to $\coprod J_{g^{-1}U}$ because pullback commutes with direct sum and with extension by zero. Hence we have

$$\begin{aligned}\mathrm{Hom}_Y(g^*J, F) &= \prod \mathrm{Hom}_Y(J_{g^{-1}U}, F)\\ &= \prod \mathrm{Hom}_{g^{-1}U}(O_{g^{-1}U}, F \mid g^{-1}(U))\\ &= \prod \Gamma(g^{-1}U, F).\end{aligned}$$

Therefore we have

$$\mathrm{Ext}^q_Y(g^*J, F) = \prod H^q(g^{-1}U, F \mid g^{-1}U).$$

Since $g^{-1}U$ is affine and F is quasi-coherent, the right-hand side is equal to zero for $q > 0$.

(1.3) Theorem. *Let $f : X \to S$ be a finitely presented, proper morphism of schemes, and let I and F be locally finitely presented, S-flat O_X-modules. Assume the relation,*

$$\mathrm{Ext}^1_{X(s)}(I(s), F(s)) = 0,$$

holds for some point $s \in S$. Then there exists an open, retrocompact neighborhood U of s such that $H(I, F) \mid U$ is locally free with a finite rank.

Proof. Retrocompact means the inclusion map is quasi-compact [EGA O_I, 2.4.1]. Obviously the notion is stable under base-change and obviously every subset of a locally Noetherian space is retrocompact.

The assertion is clearly local on S, so we may assume S is affine. It then follows from [EGA IV_3, Sect. 8] and the compatibility of $H(I, F)$ with base-change (1.1) that we may assume S is Noetherian. Finally it suffices to show $H(I, F)$ is free at s, because it is locally finitely presented [EGA O_I, 5.4.1]. So we may assume S is the spectrum of a Noetherian local ring A and s is the closed point of S.

Consider the functor,

$$T(M) = \mathrm{Ext}^1_X(I, F \otimes_S \tilde{M}).$$

from the category of finitely generated A-modules M to itself. (Note that $T(M)$ is finitely generated because f is proper and S is Noetherian [GD, IV, 3.2, p. 74].)

We shall now show that $T(k(s))$ is equal to zero. Consider an exact sequence,

$$0 \to K \to J \to I \to 0,$$

in which J is as specified in (1.2). Since I is S-flat, the sequence remains exact when restricted to $X(s)$. So it yields a commutative diagram with exact rows,

$$\begin{array}{ccccccc}
\mathrm{Hom}_X(J, j_* F(s)) & \longrightarrow & \mathrm{Hom}_X(K, j_* F(s)) & \longrightarrow & \mathrm{Ext}^1_X(I, j_* F(s)) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & & & \\
\mathrm{Hom}_{X(s)}(j^* J, F(s)) & \longrightarrow & \mathrm{Hom}_{X(s)}(j^* K, F(s)) & \longrightarrow & \mathrm{Ext}^1_{X(s)}(j^* I, F(s)) & \longrightarrow & 0
\end{array}$$

where j is the inclusion map of the closed fiber $X(s)$ into X. The two vertical maps are the adjunction isomorphisms. Now, j^*I is equal to $I(s)$, and $\mathrm{Ext}^1_{X(s)}(I(s), F(s))$ is equal to zero. Hence $\mathrm{Ext}^1_X(I, j_* F(s))$ is equal to zero. However, the latter Ext is just $T(k(s))$.

Since T is half-exact and since $T(k(s))$ is equal to zero, $T(M)$ is equal to zero for every finitely generated A-module M [OB, 2.1] or [EGA III_2, 7.5.3]). Therefore the functor $M \mapsto \mathrm{Hom}_X(I, F \otimes_S \tilde{M})$ is exact. Thus the functor $M \mapsto \mathrm{Hom}_S(H(I, F), \tilde{M})$ is exact. Hence $H(I, F)$ is free.

(1.4) Lemma. *Let $f: X \to S$ be a finitely presented morphism of affine schemes, and let I be an S-flat, finitely presented O_X-Module. Then there exists an exact sequence*

$$0 \to K \to J \to I \to 0, \tag{1.4.1}$$

with K and J finitely presented and with J free.

Proof. By [EGA IV_3, Sect. 8], there exists a Noetherian affine scheme such that all the data descend to S_0. Since X_0 is Noetherian, there exists a sequence like (1.4.1) on X_0. (On X, we can construct such a sequence with K finitely generated [CA, *I*, Sect. 2.8, Lemma 9, p. 21]; on X_0 any finitely generated Module is finitely presented.) Since I is flat, the pullback of the sequence on X_0 is the desired sequence on X.

(1.5) LEMMA. *Let X_0 be an affine scheme, and let $S = \varprojlim S_\lambda$ be a projective limit of S_0-schemes S. Let $f_0 : X_0 \to S_0$ be a finitely presented morphism, let I_0 and F_0 be locally finitely presented O_{S_0}-Modules, and let $X = \varprojlim(X_\lambda)$, $I = \varprojlim(I_\lambda)$, and $F = \varprojlim(F_\lambda)$ be the natural limits induced. Fix an integer q. Assume that I_0 is S_0-flat if $q \geqslant 1$ and that X_0 is S_0-flat if $q \geqslant 2$. Then there is a canonical isomorphism,*

$$\varinjlim \mathbf{Ext}^q_{X_\lambda}(I_\lambda, F_\lambda) = \mathbf{Ext}^q_X(I, F).$$

Proof. The assertion is local, so we may assume X_0 and the S_λ are affine. The proof now proceeds by induction on $q \geqslant 0$.

For $q = 0$, the assertion results from [EGA IV_3, 8.5.2 (i)].

Consider the case $q = 1$. By (1.4) there exists on X_0 an exact sequence,

$$0 \to K_0 \to J_0 \to I_0 \to 0, \tag{1.5.1}$$

with J_0 and K_0 finitely presented and with J_0 free. Since I_0 is S_0-flat, (1.5.1) induces analogous exact sequences on the X_λ and X. They yield diagrams with exact rows and commutative right squares because the J_λ are acyclic,

$$\begin{array}{ccccccc} \mathbf{Hom}_{X_\lambda}(J_\lambda, F_\lambda) & \longrightarrow & \mathbf{Hom}_{X_\lambda}(K_\lambda, F_\lambda) & \longrightarrow & \mathbf{Ext}^1_{X_\lambda}(I_\lambda, F_\lambda) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \vdots & & \\ \mathbf{Hom}_X(J, F) & \longrightarrow & \mathbf{Hom}_X(K, F) & \longrightarrow & \mathbf{Ext}^1_X(I, F) & \longrightarrow & 0. \end{array} \tag{1.5.2}$$

Induced are the dotted maps. The result for $q = 0$ now yields the result for $q = 1$.

Consider the case $q \geqslant 2$. The sequence (1.5.1) yields diagrams,

$$\begin{array}{ccc} \mathbf{Ext}^{q-1}_{X_\lambda}(K_\lambda, F_\lambda) & \xrightarrow{\sim} & \mathrm{Ext}^q_{X_\lambda}(I_\lambda, F_\lambda) \\ \downarrow & & \vdots \\ \mathbf{Ext}^{q-1}_X(K, F) & \xrightarrow{\sim} & \mathbf{Ext}^q_X(I, F). \end{array} \tag{1.5.3}$$

Induced are the dotted maps.

Since X_0 is S_0-flat, the free O_{X_0}-Module J_0 is also S_0-flat. Hence since I_0 is S_0-flat, K_0 is also. Therefore, by induction on q, the left-hand vertical maps in

(1.5.3) induce an isomorphism in the limit. Hence, so do the right-hand vertical maps.

The dotted arrows in (1.5.2) and (1.5.3) do not depend on the choice of the exact sequence (1.5.1) because any two such sequences are homotopic since J_0 is free.

(1.6) LEMMA. *Let A be a ring, let B a finitely presented A-algebra, and let M and N be finitely presented B-modules. Fix an integer q. Assume that M is A-flat if $q \geqslant 1$ and that B is A-flat if $q \geqslant 2$. Then there is a canonical isomorphism,*

$$\mathrm{Ext}_B^q(M, N)^\sim = \mathbf{Ext}_B^q(\tilde{M}, \tilde{N}).$$

Proof. The case $q = 0$ is proved in [EGA I, 1.3.12, (ii)]. The rest of the proof is straightforward and similar to that of the preceding lemma.

(1.7) LEMMA. *Let $f : X \to S$ be a finitely presented morphism of schemes, and let I and F be locally finitely presented O_X-Modules. Fix an integer q. Assume that I is S-flat if $q \geqslant 1$ and that f is flat if $q \geqslant 2$. Then there exists, for each base-change morphism $g : T \to S$ and each quasi-coherent O_T-Module M, a canonical "adjunction" isomorphism,*

$$\mathbf{Ext}_X^q(I, (1 \times g)_*(F \otimes_S M)) \xrightarrow{\sim} (1 \times g)_* \mathbf{Ext}_{X_T}^q(I_T, F \otimes_S M). \quad (1.7.1)$$

It is compatible with further base-change and with passage to limits like those in (1.5). (*If the formation of* $(1 \times g_0)_*(F_0 \otimes_{S_0} M_0)$ *does not commute with the transition maps* $S_\lambda \to S_0$, *then the type of limit is slightly different from that in* [EGA IV$_3$, Sect. 8] *but is a natural generalization of it.*)

Proof. For $q = 0$, the isomorphism (1.7.1) comes from the usual adjunction isomorphism [EGA 0_I, 4.4.3.1]. The compatibilities are straightforward. For general q, the construction is straightforward, following the line of reasoning of (1.5). The compatibilities follow, similarly, from those for $q = 0$.

(1.8) (The base-change map for local Ext's). Let $f : X \to S$ be a finitely presented morphism of schemes, and let I and F be locally finitely presented O_X-Modules. Let $g : T \to S$ be a morphism, and let M be a quasi-coherent O_T-Module.

The canonical map,

$$F \to (1 \times g)_*(1 \times g)^* F,$$

induces a map,

$$\mathbf{Ext}_X^q(I, F) \otimes_S M \to \mathbf{Ext}_X^q(I, (1 \times g)_*(1 \times g)^* F) \otimes_S M. \quad (1.8.1)$$

On the other hand, writing out the canonical map $R(O_T) \otimes_S M \to R(M)$ with

$$R(M) = (1 \times g)^* \mathbf{Ext}^q_X(I, (1 \times g)_*(F \otimes_S M)),$$

we get

$$\mathbf{Ext}^q_X(I, (1 \times g)_*(1 \times g)^*F) \otimes_S M \to (1 \times g)^* \mathbf{Ext}^q_X(I, (1 \times g)_*(F \otimes_S M)). \tag{1.8.2}$$

Assume that I is S-flat if $q \geqslant 1$ and that f is flat if $q \geqslant 2$. Then composing (1.8.1) and (1.8.2) with the adjoint of (1.7.1) yields a canonical *base-change map*,

$$b^q(M)\colon \mathbf{Ext}^q_X(I, F) \otimes_S M \to \mathbf{Ext}^q_{X_T}(I_T, F \otimes_S M).$$

It is straightforward to check that $b^q(M)$ commutes with restriction to an open subscheme of S, to a subscheme $\operatorname{Spec}(O_S)$, and to other localizations of S.

It is straightforward to check that $b^q(M)$ is compatible with further base-change and with passage to limits like those in (1.5).

If the base-change $g : T \to S$ is flat, then the base-change map $b^q(O_T)$ is an isomorphism. Indeed, this assertion is local on S, X, and T, and it is true in the affine case by (1.6) and [GD IV, 3.1, p. 73].

(1.9) THEOREM (property of exchange for local Ext's). *Let $f : X \to S$ be a finitely presented morphism of schemes, and let I and F be locally finitely presented O_X-Modules. Assume F is S-flat. Fix an integer q. Assume* (a) *I is S-flat if $q \geqslant 1$ and* (b) *f is flat if $q \geqslant 2$. Fix a point $s \in S$ and a point $x \in X(s)$. Assume that the base-change map to the fiber,*

$$b^q(k(s))\colon \mathbf{Ext}^q_X(I, F) \otimes_S k(s) \to \mathbf{Ext}^q_{X(s)}(I(s), F(s)),$$

is surjective at x. Then,

(i) *For every map $g : T \to S$ and every quasi-coherent O_T-Module M, the base-change map $b^q(M)$ is an isomorphism at every point of* $(1 \times g)^{-1}(x)$.

(ii) *The following three statements are equivalent:*

(1) *$b^{q-1}(k(s))$ is surjective at x.*

(2) *$b^{q-1}(M)$ is an isomorphism at every point of $(1 \times g)^{-1}(x)$ for every g and every M.*

(3) *$\mathbf{Ext}^q_X(I, F)$ is S-flat at x.*

Proof. (i) Clearly we may assume S and X are affine. Write S as a limit, $S = \varprojlim S_\lambda$, where each S_λ is the spectrum of a finitely generated $\mathbb{Z}$-algebra. We may assume by [EGA IV_3, Sects. 8, 11] that for each λ, there exist a finitely presented S_λ-scheme X_λ and locally finitely presented O_{X_λ}-Modules I_λ and F_λ

descending X, I, and F, satisfying the properties $(a)_\lambda$ and $(b)_\lambda$, analogous to (a) and (b), and with F_λ flat over S_λ.

Consider the maps,

$$b^q(k(s_\lambda)): \mathbf{Ext}^q_{X_\lambda}(I_\lambda, F_\lambda) \otimes_{S_\lambda} k(s_\lambda) \to \mathbf{Ext}^q_{X(s_\lambda)}(I(s_\lambda), F(s_\lambda)),$$

where s_λ is the image of s in S_λ. Their limit is equal to $b^q(k(s))$ by virtue of (1.5). Now, $\mathbf{Ext}^q_{X(S)}(I(s), F(s))_x$ is finitely generated because $X(s)$ is Noetherian and $I(s)$ and $F(s)$ are locally finitely generated [GD, IV, 3.2 (i), p. 74]. Since $b^q(k(s))_x$ is surjective, there exists a μ such that the image of $b^q(k(s_\mu))_{x_\mu}$ contains elements whose images in $\mathbf{Ext}^q_{X(S)}(I(s), F(s))_x$ generate, where x_μ denotes the image of x in X_μ. However, the map,

$$\mathbf{Ext}^q_{X(s_\mu)}(I(s_\mu), F(s_\mu)) \otimes_{k(s_\mu)} k(s) \to \mathbf{Ext}^q_{X(s)}(I(s), F(s)),$$

is an isomorphism because this base-change map is flat. Hence these elements generate $\mathbf{Ext}^q_{X(S_\mu)}(I(s_\mu), F(s_\mu))_x$. Therefore $b^q(k(s_\mu))$ is surjective at x_μ. Thus all the hypotheses descend, and so we may assume S is Noetherian.

Let $g: T \to S$ be a morphism, and let M be a quasi-coherent O_T-Module. To check that $b^q(M)$ is an isomorphism at every point of $(1 \times g)^{-1}x$, we may clearly assume $S = \operatorname{Spec}(O_s)$, $X = \operatorname{Spec}(O_x)$, and $T = \operatorname{Spec}(O_t)$ for $t \in g^{-1}(s)$.

Define a functor from the category of O_s-modules N to the category of O_x-modules,

$$R(N) = \operatorname{Ext}^q_{O_x}(I_x, F_x \otimes_{O_s} N).$$

It is easy to see that R commutes with direct limits. Moreover, if N is finitely generated, then $R(N)$ is also finitely generated [GD IV, 3.2 (i), p. 74].

Since $b^q(k(s))_x$ is surjective, the natural map,

$$R(O_s) \otimes_{O_s} k(s) \to R(k(s)),$$

is surjective. Moreover, the (unique) maximal ideal of O_x contracts to the (unique) maximal ideal of O_s. Therefore, by [OB, 4.1], the map,

$$R(O_s) \otimes_{O_s} N \to R(N), \tag{1.9.1}$$

Writing out (1.9.1) for $N = M_t$, we get

$$\operatorname{Ext}^q_{O_x}(I_x, F_x) \otimes_{O_s} M_t \xrightarrow{\sim} \operatorname{Ext}^q_{O_x}(I_x, F_x \otimes_{O_s} M_t). \tag{1.9.2}$$

On the other hand, taking the stalk at x of the adjunction isomorphism (1.7.1), we get

$$\operatorname{Ext}^q_{O_x}(I_x, F_x \otimes_{O_s} M_t) \xrightarrow{\sim} \operatorname{Ext}^q_{O_x \otimes O_t}(I_x \otimes_{O_s} O_t, F_x \otimes_{O_s} M_t). \tag{1.9.3}$$

Putting together (1.9.2) and (1.9.3), we see that $b^q(M)$ is an isomorphism.

(ii) The implication (1) $\Rightarrow$ (2) holds by (*i*). For the implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1), clearly we may assume $S = \operatorname{Spec}(O_s)$. Let $0 \to M' \to M \to M'' \to 0$ be an arbitrary exact sequence of quasi-coherent O_S-Modules, and consider the following diagram, with two commutative squares and exact lower sequence:

$$\begin{array}{ccccccc}
\mathbf{Ext}_X^{q-1}(I,F) \otimes M & \xrightarrow{u} & \mathbf{Ext}_X^{q-1}(I,F) \otimes M'' & \to & \mathbf{Ext}_X^{q}(I,F) \otimes M' & \xrightarrow{v} & \mathbf{Ext}_X^{q}(I,F) \otimes M \\
\downarrow{\scriptstyle b^{q-1}(M)} & & \downarrow{\scriptstyle b^{q-1}(M'')} & & \simeq\downarrow{\scriptstyle b^{q}(M')} & & \simeq\downarrow{\scriptstyle b^{q}(M)} \\
\mathbf{Ext}_X^{q-1}(I,F \otimes M) & \to & \mathbf{Ext}_X^{q-1}(I,F \otimes M'') & \to & \mathbf{Ext}_X^{q}(I,F \otimes M') & \to & \mathbf{Ext}_X^{q}(I,F \otimes M)
\end{array}$$

The maps $b^q(M')$ and $b^q(M)$ are isomorphisms at x by (i).

Assume (2). Then, in particular, $b^{q-1}(M)$ and $b^{q-1}(M'')$ are isomorphisms at x. On the other hand, u is surjective by the right-exactness of tensor product. Hence v is injective at x. Therefore, since every O_s-module N is the stalk of some quasi-coherent S-module M (indeed, take $M = \tilde{N}$), (3) holds.

Assume (3). Then v is injective at x. Take $M = O_s$ and $M'' = k(s)$, which is permissible because s is now a closed point. Then $b^{q-1}(M)$ is obviously an isomorphism. Hence (1) holds.

(1.10) THEOREM. *Let $f: X \to S$ be a finitely presented, proper morphism of schemes, and let I and F be locally finitely presented O_X-Modules. Assume F is S-flat. Fix an integer q. Assume that I is S-flat if $q \geqslant 1$ and that f is flat if $q \geqslant 2$. Then,*

(i) *Let V denote the set of $s \in S$ where we have*

$$\mathbf{Ext}_{X(s)}^{q}(I(s), F(s)) = 0.$$

Then V is open and retrocompact, and for each base-change $g: T \to S$ factoring through V and for each quasi-coherent O_T-Module M, we have

$$\mathbf{Ext}_{X_T}^{q}(I_T, F \otimes_S M) = 0.$$

(ii) *Fix an integer c and let V denote the set of $s \in S$, where we have*

$$\mathbf{Ext}_{X(s)}^{p}(I(s), F(s)) = 0 \qquad \text{for} \quad p = c+1, c-1.$$

Then V is open and retrocompact, the restriction,

$$\mathbf{Ext}_X^{c}(I, F) \mid f^{-1}(V),$$

is locally finitely presented and flat over V, and the map $b^p(M)$ is an isomorphism for every base-change $g: T \to S$ and for every quasi-coherent O_T-Module M.

(iii) *Consider the following functor on the category of quasi-coherent O_S-Modules N:*

$$N \mapsto \mathbf{Ext}^q_X(I, F \otimes N).$$

(a) *If this functor is right-exact, then the map $b^q(M)$ is an isomorphism for every base-change $g : T \to S$ and for every quasi-coherent O_T-Module M.*

(b) *If this functor is exact, then $b^q(M)$ and $b^{q-1}(M)$ are isomorphisms for every g and every M and $\mathbf{Ext}^q_X(I, F)$ is S-flat.*

Proof. (i) Let U denote the set of $x \in X$, where we have

$$\mathbf{Ext}^q_{X(f(x))}(I(f(x)), F(f(x))) = 0.$$

Then the proof of [EGA IV_3, 12.3.4] shows that U is open and retrocompact, although this is not fully stated. (In fact, modified a little, the proof shows that, for all quasi-coherent O_S-Modules N, we have

$$\mathbf{Ext}^q_X(I, F \otimes N) \mid U = 0.) \tag{1.10.1}$$

It is easy to prove that, because f is proper and finitely presented, the set V of points $s \in S$ such that $f^{-1}(x)$ lies in U is open and retrocompact. The assertion now follows from (1.9(i)) or from (1.10.1).

(ii) By (i) the set V is open and retrocompact. By (1.9(ii)) applied twice, first with $q = c + 1$ and then with $q = c$, the restricted **Ext** is flat over V and the map $b^q(M)$ is an isomorphism for every g factoring through V and for every M.

Finally, the assertion of local finite presentation is local on S and is compatible with base-change. So we may assume S is affine and by [EGA IV_3, Sects. 8, 11] Noetherian. Then the assertion holds by [EGA 0_{III}, 12.3.3].

(iii) (a) Let s be an arbitrary point of S, and consider the canonical morphism$_t$

$$g : T = \mathrm{Spec}(k(s)) \to S.$$

Obviously g is quasi-compact and quasi-separated; hence, $g_* k(s)$ is quasi-coherent [EGA *I*, 6.7.1]. Consider the exact sequence,

$$O_S \xrightarrow{u} g_* k(s) \longrightarrow \mathrm{Coker}(u) \longrightarrow 0,$$

in which u is the comorphisms of g. The terms of the sequence are all quasi-coherent. Hence, by hypothesis, the induced sequence,

$$\mathbf{Ext}^q_X(I, F) \to \mathbf{Ext}^q_X(I, F \otimes g_* k(s)) \to \mathbf{Ext}^q_X(I, F \otimes \mathrm{Coker}(u)),$$

is exact. Localizing at s, we get an exact sequence,

$$\mathbf{Ext}^q_{X_s}(I_s, F_s) \to \mathbf{Ext}^q_{X_s}(I_s, F_s \otimes k(s)) \to 0,$$

because, obviously, $\mathrm{Coker}(u)$ is zero at s. Hence, in view of the adjunction isomorphism (1.7.1),

$$\mathrm{Ext}^q_{X_s}(I_s, F_s \otimes k(s)) \xrightarrow{\sim} \mathrm{Ext}_{X(s)}(I(s), F(s)),$$

the base-change map to the fiber $b^q(k(s))$ is surjective. Since $b^q(k(s))$ is surjective for every $s \in S$, assertion (a) holds by (1.9,(iii)).

(b) The hypothesis that $\mathbf{Ext}^p_X(I, F \otimes N)$ is exact in N for $p = q$ obviously implies that it is right-exact in N for $p = q, q-1$. (In fact, the two are equivalent.) Hence by (a) the map $b^p(M)$ is an isomorphism for every g and every M for $p = q, q-1$. In particular, taking g to be the canonical map, $\mathrm{Spec}(k(s)) \to S$, we get that $b^p(k(s))$ is surjective for every $s \in S$ for $p = q, q-1$. Hence $\mathbf{Ext}^q_X(I, F)$ is flat by (1.9(ii)).

2. Quotients

(2.1) Definition. A morphism of schemes $f: X \to S$, or X/S, will be called *strongly quasi-projective* (resp. *strongly projective*) if it is finitely presented and if there exists a locally free O_S-Module E with a constant finite rank such that X is S-isomorphic to a (retrocompact) subscheme (resp. closed subscheme) of $\mathbb{P}(E)$.

(2.2) Examples. (i) A finitely presented, quasi-projective (resp. projective) morphism $f: X \to S$ is strongly quasi-projective (resp. strongly projective) if S is quasi-compact and quasi-separated and admits an ample sheaf, for example, if S is affine or quasi-affine.

Indeed, S can be embedded in an S-scheme $\mathbb{P}(F)$, where F is a quasi-coherent, locally finitely generated O_S-Module [EGA II, 5.3.2]. Now, F is a quotient of a locally free O_S-Module E with a constant rank because S is quasi-compact and quasi-separated and admits an ample sheaf [EGA IV, 1.7.14]. Thus X can be embedded in a suitable $\mathbb{P}(E)$.

(ii) A flat, finitely presented, projective morphism $f: X \to S$ is strongly projective if there exist a relatively very ample sheaf $O_X(1)$ and an integer $n \geqslant 1$ such that $h^0(X(s), O_{X(s)}(n))$ is bounded and $h^1(X(s), O_{X(s)}(n))$ is zero for all $s \in S$.

Indeed, $f_* O_X(n)$ is locally free with a bounded rank on S. Hence, adding appropriate free summands $O_S^{\oplus r}$ on the various connected components of S

produces a locally free O_S-Module E with a constant rank and a surjection $E \to f_* O_X(n)$. Hence, since $O_X(n)$ is relatively very ample [EGA II, 4.4.9(ii)], it defines an S-embedding [EGA II, 4.4.4],

$$X \hookrightarrow \mathbb{P}(f_* O_X(n)) \hookrightarrow \mathbb{P}(E).$$

Thus X/S is strongly projective.

(iii) Let $f: X \to S$ be a flat, finitely presented, projective morphism whose geometric fibers are reduced, connected, and equidimensional. Fix a relatively very ample sheaf $O_X(1)$. Assume the fibers $X(s)$ have only a finite number of distinct Hilbert polynomials. Then f is strongly projective.

Indeed, we shall show below that there exists an integer m given by a universal polynomial in the coefficients of its Hilbert polynomial such that each $O_{X(s)}$ is m-regular. Then the assertion will follow from (ii).

To complete the proof, we may assume S is the spectrum of an algebraically closed field. Since X is reduced and connected we have $h^0(X, O_X) = 1$. So $h^0(X, O_X(-1))$ is equal to zero [SGA 6, 6.5, p. 655]. Hence it follows from [SGA 6, 2.10, p. 630] that O_X is a $(0, \deg(X))$-sheaf if X is one-dimensional. Assume $\dim(X) \geqslant 2$. Then by Bertini's theorem there exists a reduced, connected, equidimensional hyperplane section Y of X and the coefficients of its Hilbert polynomial are among those of the Hilbert polynomial of X [SGA 6, 1.7, p. 620]. So, by induction on dim (X), clearly Y is a $(0, \ldots 0, \deg(X))$-sheaf. Hence X is a $(0, \ldots, 0, \deg(X))$-sheaf. Therefore a suitable m exists so that O_X is m-regular [SGA 6, 1.11, p. 621].

(iv) (pointed out privately by Lønsted) A proper, flat, finitely presented family of Gorenstein, geometrically integral curves with the same arithmetic genus $p \neq 1$ is strongly projective. Indeed, $\omega_{X/S}^{\otimes 3}$ is very ample if $p \geqslant 2$ and $\omega_{X/S}^{-1}$ is if $p = 0$, where $\omega_{X/S}$ is the dualizing sheaf (6.5); hence strong projectivity holds by (iii).

By contrast for $p = 1$ the corresponding statement fails: There is a locally trivial, proper but nonprojective family of nodal cubics over the projective line; moreover, each finite set of points lies in some affine, open subset ([*H*]; see also Example (8.11)).

(2.3) LEMMA (flattening). *Let $f: X \to S$ be a finitely presented, locally projective morphism of schemes, and let F be a locally finitely presented O_X-Module. Let $\phi(n) \in \mathbb{Q}[n]$ be a polynomial. Then there is a retrocompact subscheme Z of S such that a map $T \to S$ factors through Z if and only if F_T is T-flat with Hilbert polynomial ϕ on the fibers.*

Proof. The assertion is clearly local on the base, so we may assume S is affine. By [EGA IV_3, Sect. 8] there is a Cartesian diagram,

$$\begin{array}{ccc} X & \xrightarrow{u \times 1} & X_0 \\ \downarrow & \square & \downarrow \\ S & \xrightarrow{u} & S_0 \end{array}$$

with S_0 Noetherian and X_0 projective over S_0, and there is a coherent O_{X_0}-Module F_0 whose pullback to X is equal to F.

There exists a locally closed subscheme Z_0 of S_0 such that a map $R \to S_0$ factors through Z_0 if and only if $(F_0)_R$ is flat over R with Hilbert polynomial ϕ by [FGA, Lemma 3.4, p. 221–14]. (Mumford [CS, Lecture 8] gives a more detailed discussion but deals only with Noetherian R.)

Set $Z = u^{-1}(Z_0)$. Then Z is a retrocompact subscheme of S, and a map $T \to S$ factors through Z if and only if F_T is T-flat with Hilbert polynomial ϕ on the fibers.

(2.4) LEMMA. *Let X be a projective scheme over a field, and fix a very ample sheaf $O_X(1)$. Let $0 \to I \to F \to G \to 0$ be an exact sequence of coherent O_X-Modules. Let*

$$\chi(F(n)) = \sum_{i=0}^{r} f_i \binom{n+i}{i} \quad \text{and} \quad \chi(G(n)) = \sum_{i=0}^{r} g_i \binom{n+i}{i}$$

denote the Hilbert polynomials of F and G. Assume F is a b-sheaf for $b = (b_0, \ldots, b_r)$. Then I, F, and G are m-regular for all $m \geqslant m_0$, where m_0 is the value of a universal polynomial in the b_i, f_i, g_i. (For the definitions of b-sheaf and m-regular, see [SGA 6, 1.5, *p.* 619, *and* 1.1, *p.* 616].)

Proof. Clearly there is a relation,

$$\chi(I(n)) = \sum (f_i - g_i) \binom{n+i}{i}.$$

Moreover, I is also a b-sheaf by [SGA 6, 1.6(ii), p. 619] because it is a subsheaf of a b-sheaf. So, there exists an integer m_1 given by a universal polynomial in the b_i, f_i, and g_i such that I and F are m-regular for all $m \geqslant m_0$ by [SGA 6, 1.11, p. 621].

For each m and each q, there is an exact sequence,

$$H^q(X, F(m-q)) \to H^q(X, G(m-q)) \to H^{q+1}(X, I(m+1-q-1)).$$

Hence G is also m-regular for all $m \geqslant m_0$.

(2.5) DEFINITION. Let $f: X \to S$ be a finitely presented morphism of schemes, and let F be a locally finitely presented O_X-Module. Define the *pseudo-Ideal* $I(G)$ of an S-flat quotient G of F as the kernel of the canonical surjection

$F \to G$. (Note that the formation of $I(G)$ commutes with base-change because G is S-flat.)

Define a functor $\mathbf{Quot}_{(F/X/S)}$ as follows. For each S-scheme T, let

$$\mathbf{Quot}_{(F/X/S)}(T)$$

be the set of locally finitely presented, T-flat quotients of F_T whose support is proper and finitely presented over T.

Let ϕ be a polynomial (with rational coefficients). Define a subfunctor $\mathbf{Quot}^{\phi}_{(F/X/S)}$ of $\mathbf{Quot}_{(F/X/S)}$ as follows. For each S-scheme T, let

$$\mathbf{Quot}^{\phi}_{(F/X/S)}(T)$$

be the set of $G \in \mathbf{Quot}_{(F/X/S)}(T)$ with Hilbert polynomial ϕ on each fiber.

(2.6) THEOREM. *Let $f: X \to S$ be a strongly projective (resp. strongly quasi-projective) morphism of schemes, and let F be a locally finitely presented O_X-Module. Assume F is isomorphic to a quotient of an O_X-Module of the form $f^*B \neq O_X(\nu)$ for some ν, where B is a locally free O_S-Module with a constant finite rank. Fix a polynomial ϕ. Then the functor $\mathbf{Quot}^{\phi}_{(F/X/S)}$ is representable by a pair $(\mathbb{Q}, \mathbb{G})$, where $\mathbb{Q} = \mathrm{Quot}^{\phi}_{(F/X/S)}$ is a strongly projective (resp. strongly quasi-projective) S-scheme and $\mathbb{G}$ is the universal member of $\mathbf{Quot}^{\phi}_{(F/X/S)}(\mathbb{Q})$).*

Say X is S-isomorphic to a subscheme of $\mathbb{P}(E)$, where E is a locally free O_S-Module with a constant finite rank. Then for $m \geqslant m_0$, where m_0 is the value of a universal polynomial in the integers rank (B), rank (E), ν, and the coefficients of ϕ, the direct image $(f_Q)_*\mathbb{G}(m)$ is locally free with rank $\phi(m)$ and there exists an embedding

$$\mathbb{Q} = \mathrm{Quot}^{\phi}_{(F/X/S)} \to \mathbb{P}\left(\bigwedge^{\phi(m)} (B \otimes \mathrm{Sym}_{\nu+m}(E))\right)$$

such that the following formula holds:

$$O_{\mathbb{Q}}(1) = \det((f_{\mathbb{Q}})_*\mathbb{G}(m)).$$

Proof. The proof proceeds by steps. In Steps I–V, we assume X is closed in $\mathbb{P}(E)$. In Step VI we derive the general case from this one.

Step I. $\mathbf{Quot}^{\phi}_{(F/X/S)}$ is a closed subfunctor of $\mathbf{Quot}^{\phi}_{((h^*B)(\nu)/\mathbb{P}(E)/S)}$, where $h: \mathbb{P}(E) \to S$ denotes the structure morphism.

Proof. Let T be an S-scheme and let G be an element of $\mathrm{Quot}^{\phi}_{(h^*B)(\nu)/\mathbb{P}(E)/S)}(T)$ We must show that there is a closed subscheme T_0 of T such that a morphism $R \to T$ factors through T_0 if and only if G_R defines an element of $\mathrm{Quot}^{\phi}_{(F/X/S)}(R)$. This assertion is clearly local on T and compatible with base-change. So by [EGA IV_3, Sect. 8] we may assume T is affine and Noetherian.

Let K denote the kernel of the canonical map, $(h_T^* B_T)(\nu) \to F_T$, and let $u : K \to G$ denote the induced map. Clearly G_R defines a quotient of F_R if and only if u_R is equal to zero. By (1.1) the map u_R is equal to zero if and only the corresponding map,

$$v_R = y^{-1}(u)_R : H(K, G)_R \to O_R,$$

is equal to zero. Finally, by [EGA I, 9.7.9.1] there exists a closed subscheme $Z(v)$ of T such that $R \to T$ factors through $Z(v)$ if and only if v_R is equal to zero.

Step II. By Step I we may assume $X = \mathbb{P}(E)$ and $F = (f^*B)(\nu)$. In particular, both X and F are now S-flat. Set

$$\mathbf{A} = \mathbf{Quot}^{\phi}_{(F/X/S)}.$$

The sheaf F has the same Hilbert polynomial on every fiber of X/S, namely, $\chi(F(s)(n)) = c\binom{e+\nu+n}{n}$, where c and $(e+1)$ are the ranks of B and E. Moreover, F is clearly a b-sheaf with $b = (0,\dots, 0, c)$. Hence by (2.4) there exists an integer m_0, given by a universal polynomial in c, e, ν and the coefficients of ϕ, such that, for each S-scheme T and for every quotient $G \in \mathbf{A}(T)$, and for each integer $m \geqslant m_0$, both G and its pseudo-Ideal I are m-regular on the fibers. Fix an $m \geqslant m_0$, and set

$$\mathcal{G} = \mathrm{Grass}_{\phi(m)}(f_* F(m)).$$

Define a map of functors,

$$\Phi : \mathbf{A} \to \mathcal{G},$$

as follows. Let T be an S-scheme and take $G \in \mathbf{A}(T)$. Since G is m-regular on the fibers, $(f_T)_*G(m)$ is locally free with rank $\phi(m)$. Since I is m-regular on the fibers, $R^1(f_T)_*I(m)$ is equal to zero. So $(f_T)_*G(m)$ defines a $\phi(m)$-quotient of $(f_T)_*F_T(m)$, hence a T-point $\Phi(G)$ of $\mathcal{G}$ because the formation of $f_*F(m)$ commutes with base-change.

Let Q denote the universal $\phi(m)$-quotient of $f_*F(m)$ on $\mathcal{G}$. Then, on $\mathcal{G}$, there is a natural exact sequence,

$$0 \longrightarrow K \xrightarrow{u} (f_{\mathcal{G}})_*(F_{\mathcal{G}}(m)) \longrightarrow Q \longrightarrow 0,$$

where K is the pseudo-ideal of Q and u is the natural inclusion followed by the base-change isomorphism. The adjoint of u gives rise to an exact sequence,

$$f_{\mathcal{G}}^*K \xrightarrow{u^*} F_{\mathcal{G}}(m) \longrightarrow H(m) \longrightarrow 0,$$

on $X \times_S \mathcal{G}$.

Step III. Let $g : T \to \mathcal{G}$ be an S-morphism, and let G be an element of $\mathbf{A}(T)$. Set $G' = (1 \times g)^*H$. Then G' is equivalent to G as a quotient of F_T if and only if the $\phi(m)$-quotients g^*Q and $(f_T)_*G(m)$ of $(f_*F(m))_T$ are equivalent.

Proof. Suppose g^*Q and $(f_T)_*G(m)$ are equivalent $\phi(m)$-quotients. Then there is a diagram with exact rows and commutative right-hand square,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & g^*K & \xrightarrow{g^*(u)} & g^*(f_{\mathcal{G}})_*F_{\mathcal{G}}(m) & \longrightarrow & g^*Q & \longrightarrow & 0 \\
 & & \downarrow{\simeq} & \overset{v}{\searrow} & \downarrow{\simeq} & & \downarrow{\simeq} & & \\
0 & \longrightarrow & (f_T)_*I(m) & \longrightarrow & (f_T)_*F_T(m) & \longrightarrow & (f_T)_*G(m) & \longrightarrow & 0,
\end{array}$$

where I is the pseudo-ideal of G and the middle map is the base-change isomorphism. The bottom row is exact because, since I is m-regular on the fibers, $R^1(f_T)_*I(m)$ is equal to zero. Hence the dotted isomorphism making the left-hand square commutative exists.

Taking the adjoint of the lower left-hand triangle yields the commutative diagram,

$$\begin{array}{ccc}
(f_T)^*g^*K & = & (1 \times g)^*f_G^*K \\
\downarrow{\simeq} & \overset{v^\#}{\searrow} & \downarrow{(u^\#)_T} \\
(f_T)^*(f_T)_*I(m) & \longrightarrow & F_T(m).
\end{array}$$

Since I is m-regular on the fibers, the canonical map,

$$(f_T)^*(f_T)_*I(m) \to I(m),$$

is surjective by base-change theory and by [SGA 6, XIII, 1.3(iii), p. 616]. So the image of the lower horizontal map is equal to $I(m)$. Hence the quotient of $F_T(n)$ it defines is $G(m)$. On the other hand, $G'(m)$ is clearly equal to $\operatorname{coker}((u^\#)_T)$. Thus G' is equivalent to G.

For the converse, start with the diagram with exact rows and commutative right square,

$$\begin{array}{ccccccccc}
 & & (f_{\mathcal{G}}^*K)_T & \xrightarrow{(u^\#)_T} & F_T(m) & \longrightarrow & G'(m) & \longrightarrow & 0 \\
 & & \vdots & \searrow & \downarrow{=} & & \downarrow & & \\
0 & \longrightarrow & I(m) & \longrightarrow & F_T(m) & \longrightarrow & G(m) & \longrightarrow & 0.
\end{array}$$

Induced is the dotted vertical map making the left-hand square commutative. Taking the adjoint of the lower left-hand triangle yields the diagram with exact rows and commutative left square,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & g^*K & \xrightarrow{g^*u} & g^*(f_{\mathscr{G}})_*F(m) & \longrightarrow & g^*Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \simeq & & \downarrow & & \\ 0 & \longrightarrow & (f_T)_*I(m) & \longrightarrow & (f_T)_*(F_T(m)) & \longrightarrow & (f_T)_*G(m) & \longrightarrow & 0, \end{array}$$

in which the middle map is the base-change isomorphism. Hence the dotted vertical map exists and is surjective. It is an isomorphism because its source and target are both locally free with rank $\phi(m)$. Thus g^*Q and $(f_T)_*G(m)$ are equivalent $\phi(m)$-quotients.

Step IV. Let A be the finitely presented subscheme of $\mathscr{G}$ such that a map $g : T \to \mathscr{G}$ factors through A if and only if H_T is T-flat with Hilbert polynomial ϕ; it exists by (2.3). Then we have $H_A \in \mathbf{A}(A)$ and the pair (A, H_A) represents $\mathbf{A}$.

Proof. The first assertion is clear; so, H_A defines a map of functors,

$$a : A(T) \to \mathbf{A}(T).$$

Take any $g \in A(T)$. By Step III with $G = (1 \times g)^*H$, the quotients g^*Q and $(f_T)_*G(m)$ of $F_T(m)$ are equivalent. So the image $a(g) = (1 \times g)^*H$ determines the quotient of g^*Q, so also g. Thus, a is injective.

Take any $G \in \mathbf{A}(T)$. Then, by Step II, the map $g = \Phi(G) : T \to \mathscr{G}$ is defined such that g^*Q is equivalent to $(f_T)_*G(m)$. By Step III, the element G is equivalent to $(1 \times g)^*H$. Therefore $(1 \times g)^*H$ is flat with Hilbert polynomial ϕ. Hence g factors through A, and so $a(g)$ is equal to G. Thus a is surjective, so bijective.

Step V. We have

$$f_*F(m) = B \otimes \mathrm{Sym}_{\nu+m}(E)$$

by the projection formula [EGA 0_I, 5.4.8] and by Serre's explicit computation [EGA III_1, 2.1.12]. So the Plücker morphism is closed embedding,

$$\mathscr{G} \hookrightarrow \mathbb{P}\left(\bigwedge^{\phi(m)} (B \otimes \mathrm{Sym}_{\nu+m}(E))\right),$$

$$Q \mapsto \bigwedge^{\phi(m)} Q.$$

Hence A is strongly quasi-projective, and the final assertion holds. Finally A is strongly projective because the valuative criterion [EGA I, 5.5.8] is satisfied [EGA IV_2, 2.8.1].

Step VI. The quasi-projective case.

Proof. By Step V, the functor $\mathbf{Quot}^{\phi}_{(B_{\mathbb{P}(E)}(\nu)/\mathbb{P}(E)/S)}$ is representable by a strongly projective S-scheme, and $\mathbf{Quot}^{\phi}_{(F/X/S)}$ is clearly a subfunctor of

$\mathbf{Quot}^{\phi}_{(B_{\mathbb{P}(E)}(\nu)/\mathbb{P}(E)/S)}$. So it suffices to show that it is a locally closed subfunctor. This assertion is local on S and compatible with base-change, so we may assume S is affine and, by [EGA IV$_3$ Sect. 8], Noetherian.

Let $\bar{X}$ denote the scheme-theoretic closure of X in $\mathbb{P}(E)$ (it exists by [EGA I, 6.10.6]), and let $j : X \to \bar{X}$ be the inclusion. Let $\bar{F}$ denote the image of the canonical map $B_{\bar{X}}(\nu) \to j_*(F)$. Since j is quasi-compact, $j_*(F)$ is quasi-coherent and $j_*(F) | X$ is equal to F [EGA I, 6.9.2]. Then the image $\bar{F}$ of the canonical map $B_{\bar{X}}(\nu) \to j_*(F)$ is locally finitely generated, so locally finitely presented because S is Noetherian. Clearly $\bar{F} \mid X$ is equal to F.

Clearly $\mathbf{Quot}^{\phi}_{(F/X/S)}$ is a subfunctor of $\mathbf{Quot}^{\phi}_{(\bar{F}/\bar{X}/S)}$. Moreover $\mathbf{Quot}^{\phi}_{(\bar{F}/\bar{X}/S)}$ representable by a closed subscheme $\bar{Q}$ of $\mathbf{Quot}^{\phi}_{(B_{\mathbb{P}(E)}(\nu)/\mathbb{P}(E)/S)}$ by Step I.

Set $\bar{Q} = \mathbf{Quot}^{\phi}_{(\bar{F}/\bar{X}/S)}$. Let $\bar{G}$ denote the universal quotient of $\bar{F}$ on $\bar{X} \times \bar{Q}$, and let $p : \bar{X} \times {}_S\bar{Q} \to \bar{Q}$ denote the projection. Since p is proper, the subset $Q = \bar{Q} - p([\bar{X} - X) \times \bar{Q}] \cap \mathrm{Supp}\,(G))$ is open in $\bar{Q}$. Clearly a map $g : T \to \bar{Q}$ factors through Q if and only if the relation,

$$(1 \times g)^{-1}(\mathrm{Supp}(G)) \cap [(\bar{X} - X) \times T] = \varnothing,$$

holds. Since the support of $(1 \times g)^*G$ is equal to $(1 \times g)^{-1}\,(\mathrm{Supp}(G))$ by (EGA 0_I, 5.2.4.1], the map g factors through Q if and only if the corresponding element of $\mathbf{Quot}^{\phi}_{(\bar{F}/\bar{X}/S)}(T)$ lies in $\mathbf{Quot}^{\phi}_{(X/X/S)}(T)$. Thus Q represents $\mathbf{Quot}^{\phi}_{(F/X/S)}$.

(2.7) Corollary. *Let $f : X \to S$ be a finitely presented, locally projective (resp. locally quasi-projective) morphism of schemes, and let F be a locally finitely presented O_X-Module. Then* $\mathbf{Quot}_{(F/X/S)}$ *is representable by a disjoint union of locally finitely presented, locally projective (resp. locally quasi-projective) S-schemes.*

Proof. This assertion is local on S [EGA 0_I, 4.5.5], so we may assume S is affine and f is projective. The assertion now follows easily from (2.6), from Example (2.2(i)), and from [EGA 0_I, 4.5.4].

(2.8) Corollary. *Let $f : X \to S$ be a strongly projective (resp. strongly quasi-projective) morphism of schemes. Then for any polynomial $\phi \in \mathbb{Q}[T]$, the functor* $\mathbf{Hilb}^{\phi}_{(X/S)}$ *is representable by a strongly projective (resp. strongly quasi-projective) S-scheme.*

Proof. The assertion follows immediately from (2.6) with $F = O_X$, with $B = O_S$, and with $\nu = 0$.

(2.9) Theorem. *Let $f : X \to S$ be a strongly quasi-projective morphism of schemes, and let R be a flat, finitely presented, proper equivalence relation on X. Assume the fibers of $p_2 : R \to X$ have only a finite number of Hilbert polynomials* *for an embedding of X into $\mathbb{P}(E)$, where E is a locally free O_S-Module with a constant rank. Then R is effective, the quotient map $q : X \to (X/R)$ is strongly projective and faithfully flat, and $h : (X/R) \to S$ is strongly quasi-projective.*

Proof. Step I. Set $H = \coprod \mathrm{Hilb}^{\phi}_{(X/S)}$, where ϕ ranges over the finitely many Hilbert polynomials of p_2; the S-scheme H exists and is strongly projective by (2.8). Let W denote the universal subscheme of $X \times_S H$. Since R is a flat, finitely presented, proper subscheme of $X \times_S X$, there is a unique map $g: X \to H$ such that the following equation holds:

$$(1 \times g)^{-1}(W) = R. \tag{2.9.1}$$

Step II. Let T be an S-scheme and let x_1, x_2 be two T-points of X. Write $x_1 \sim x_2$ whenever $(x_1, x_2) \in R(T)$ holds. Then we have

$$x_1 \sim x_2 \qquad \text{if and only if} \quad g(x_1) = g(x_2).$$

Proof. Set $R_i = (1 \times x_i)^{-1}(R) \subset X \times T$. Then $g(x_1) = g(x_2)$ holds if and only if $R_1 = R_2$ holds, so if and only if $R_1(T') = R_2(T')$ holds for all T-schemes T'.

Clearly we have the relation,

$$R_i(T') = \{(x, t) \in (X \times T)(T') \mid x \sim x_i(t)\}.$$

Suppose $x_1 \sim x_2$ holds. Then for $(x, t) \in R_1(T')$ we have $x \sim x_1(t) \sim x_2(t)$. Since R is transitive, we have $(x, t) \in R_2(T')$. Thus $R_1 \subset R_2$ holds. So, since R is symmetric, $R_1 = R_2$ holds. Hence $g(x_1) = g(x_2)$ holds.

Suppose $g(x_1) = g(x_2)$ holds. Then $R_1(T) = R_2(T)$ holds. Since R is reflexive, we have $(x_1, \mathrm{id}) \in R_1(T)$. So we have $(x_1, \mathrm{id}) \in R_2(T)$. Thus $x_1 \sim x_2$ holds.

Step III. For each S-scheme T and for $x_1, x_2 \in X(T)$, we have

$$x_1 \sim x_2 \qquad \text{if and only if} \quad (x_1, g(x_2)) \in W(T).$$

Furthermore, Γ_g is a finitely presented, closed subscheme of W.

Proof. The first assertion follows immediately from Equation (2.9.1.). Since H/S is separated, Γ_g is a closed subscheme of $X \times_S H$. Then, since R is reflexive, the first assertion implies the second.

Step IV. The projection p: $W \to H$ is faithfully flat and quasi-compact, and Γ_g descends to a finitely presented subscheme Z of H.

Proof. The projection p is flat and quasi-compact by definition of $\mathbf{Hilb}_{(X/S)}$. Since R is reflexive and so nonempty, p is surjective, so faithfully flat. So, to descend Γ_g, it suffices to show that $\Gamma_g \times_H W$ and $W \times_H \Gamma_g$ coincide in

$W \times_H W$ by [SGA 1, VIII, Corollary 1.9, p. 200]. For each S-scheme T, there are formulas,

$$(W \times_H \Gamma_g)(T) = \{(x_1, g(x_2), x_2, g(x_2)) \mid x_1 \sim x_2\},$$

$$(\Gamma_g \times_H W)(T) = \{(x_1, g(x_1), x_2, g(x_1)) \mid x_2 \sim x_1\},$$

by Step III. Hence, $W \times_H \Gamma_g$ and $\Gamma_g \times W$ coincide by Step II. Finally, since finite presentation descends down a faithfully flat, quasi-compact map [EGA IV$_2$, 2.7.1], and since Γ_g is isomorphic to X, the scheme Z is finitely presented.

Step V. The map $g\colon X \to H$ factors through Z, and $X \times_Z X$ is equal to R. Moreover, the induced map $g\colon X \to Z$ is faithfully flat, finitely presented, and strongly projective.

Proof. Since Z is the result of descending Γ_g, there is a diagram with Cartesian squares and exact rows,

$$\begin{array}{ccccc} \Gamma_g \times_H \Gamma_g & \rightrightarrows & \Gamma_g & \longrightarrow & Z \\ \downarrow & & \downarrow & \square & \downarrow \\ W \times_H W & \rightrightarrows & W & \xrightarrow{p} & H \end{array}$$

where the vertical maps are embeddings. Hence g factors through Z, and $X \times_Z X$ is clearly equal to R.

The map $p\colon W \to H$ is finitely presented and proper by definition of $\mathbf{Hilb}_{(X/S)}$. Since W is a subscheme of $X \times_S H/H$ and since X/S is strongly projective, p is therefore strongly projective. By Step IV, p is faithfully flat. Therefore, g also has these desirable properties.

Step VI. The theorem holds, and the induced map $g\colon X \to Z$ is equal to the quotient map $X \to (X/R)$.

Proof. Since g is faithfully flat (Step V) and since it is obviously quasi-compact, it is universally an effective epimorphism [SGA I, VIII, Corollary 5.3, p. 213]. Therefore, since $X \times_Z X$ is equal to R (Step V), the map $g\colon X \to Z$ is a quotient map of X by R by Step V. Finally, $f\colon (X/R) \to S$ is strongly quasi-projective because Z is a finitely presented subscheme of the strongly quasi-projective S-scheme H.

(2.10) Corollary. *Let $f\colon X \to S$ be a locally projective morphism of schemes, and let R be a flat, finitely presented, proper equivalence relation on X. Then R is effective, the quotient map is faithfully flat, finitely presented and proper, and the quotient (X/R) is locally quasi-projective over S.*

Proof. The assertion is obviously local on the base so we may assume S is affine. Then f is strongly quasi-projective (2.2(i)) and the assertion results from (2.9).

3. Rank-1, Torsion-Free Sheaves

(3.1) Lemma. *Let X be a geometrically integral, algebraic scheme over a field k, and let I be a coherent O_X-Module. Then,*

(i) *I is rank-1, torsion-free (that is, I satisfies S_1 and is generically isomorphic to O_X) if and only if I is reduced (that is,* [EGA IV_2, 3.2.2], *I has no embedded components, and for each generic point x of* $\operatorname{Supp}(I)$, *we have length $(I_x) = 1$) and* $\operatorname{Supp}(I)$ *is equal to X.*

(ii) *For any field extension k' of k, the pullback I' of I to $X \otimes_k k'$ is rank-1, torsion-free if and only if I is rank-1, torsion-free.*

Proof. Both assertions are obvious from the definitions.

(3.2) Lemma. *Let X be a projective scheme over an algebraically closed field. Fix an embedding of X into a projective space and let Y be a general hyperplane section of X.*

(i) *Let $0 \to F \to G \to H \to 0$ be an exact sequence of coherent O_X-Modules. Then the restriction,*

$$0 \to F \mid Y \to G \mid Y \to H \mid Y \to 0, \tag{3.2.1}$$

is exact.

(ii) *For coherent O_X-Modules I and F, the canonical map*

$$\mathbf{Hom}_X(I, F) \mid Y \to \mathbf{Hom}_Y(I \mid Y, F \mid Y)$$

is an isomorphism.

Proof. (i) Using the snake lemma, it is easy to see that for any Y avoiding $\operatorname{Ass}(H)$, Sequence (3.2.1) is exact.

(ii) Construct a presentation $E_1 \to E_0 \to I \to 0$ with each E_i a locally free O_X-Module with finite rank (for example, E_i may have the form $O_X(-m_i)^{\otimes M_i}$). The presentation gives rise to a commutative diagram,

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathbf{Hom}_X(I,F) \mid Y & \longrightarrow & \mathbf{Hom}_X(E_0, F) \mid Y & \longrightarrow & \mathbf{Hom}_X(E_1, F) \mid Y \\
& \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathbf{Hom}_Y(I \mid Y, F \mid Y) & \to & \mathbf{Hom}_Y(E_0 \mid Y, F \mid Y) & \to & \mathbf{Hom}_Y(E_1 \mid Y, F \mid Y).
\end{array}$$

The top row is exact by (i), the bottom row is obviously exact, and the two right-hand vertical maps are clearly isomorphisms. Hence the left-hand vertical map is an isomorphism.

(3.3) LEMMA. *Let X be an integral, projective scheme over an algebraically closed field. Fix an embedding of X into a projective space. Let I be a nonzero coherent O_X-Module. Then I is rank-1, torsion-free if and only if there exist an integer m and an embedding of I into $O_X(m)$. Moreover, if I is rank-1, torsion-free, then $I \mid Y$ is also rank-1, torsion-free for any general hyperplane section Y of X.*

Proof. If I is isomorphic to a subsheaf of $O_X(m)$, then clearly I is rank-1, torsion-free.

Assume I is rank-1, torsion-free. Then there exists an integer m such that $\mathbf{Hom}_X(I, O_X)(m)$ is generated by its global sections. Since $\mathbf{Hom}_X(I, O_X)$ is obviously nonzero at the generic point of X, there exists a nonzero O_X-homomorphism $u: I \to O_X(m)$. Since X is integral and I is rank-1, torsion-free, u is injective.

The second assertion results from the first and from (3.2(i)).

(3.4) PROPOSITION. *Let X be an integral, r-dimensional projective scheme over an algebraically closed field, with $r \geqslant 1$. Fix a very ample sheaf $O_X(1)$. Let J and F be rank-1, torsion-free O_X-Modules. Set*

$$\chi(J(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i} \qquad \text{and} \qquad \chi(F(n)) = \sum_{i=0}^{r} c_i \binom{n+i}{i}.$$

(i) *There is a formula,*

$$c_r = \deg(X).$$

(ii) (a) *Assume the relation,*

$$\chi(F(n)) \leqslant \chi(J(n)) \qquad \textit{for all} \quad n \gg 0.$$

Then every nonzero map u: $J \to F$ is an isomorphism.

(b) *Assume the relation,*

$$\chi(F(n)) < \chi(J(n)) \qquad \textit{for all } n \gg 0.$$

Then $\mathrm{Hom}_X(J, F)$ *is equal to zero.*

(iii) *Fix an integer μ satisfying*

$$\mu > \mu_0 = (c_{r-1} - a_{r-1} - a_r)/a_r\,.$$

Set

$$H = \mathbf{Hom}_X(J, F) \qquad \textit{and} \qquad b = (0, \ldots, 0, \deg(X)).$$

Then $H(-\mu)$ is a b-sheaf. Moreover, H is m-regular for $m \geqslant m_0$, where m_0 is the value of a universal polynomial in the integer μ and the coefficients of the Hilbert polynomial of H.

Proof. (i) There is a nonempty open set U of X such that $F \mid U$ is free with rank 1 by [EGA 0_I, 5.4.1]. By Bertini's theorem, there is a reduced, zero-dimensional linear space section Y of X contained in U. Then since the coefficients of a Hilbert polynomial slide down under hyperplane slicing [SGA 6, 1.7, p. 620], we have

$$\chi((F \mid Y)(n)) = c_r .$$

Since $F \mid Y$ is isomorphic to O_Y, the coefficient c_r is therefore equal to $h^0(Y, O_Y)$, so to $\deg(X)$.

(ii) (a) Since F is torson-free, u is nonzero at the generic point of X. Hence, since J is rank-1 and torsion-free, u is injective. Thus u defines an exact sequence,

$$0 \longrightarrow J \xrightarrow{u} F \longrightarrow \mathrm{Coker}(u) \longrightarrow 0.$$

The sequence and the hypothesis yield the relations,

$$\chi(\mathrm{Coker}(u)(n)) = \chi(F(n)) - \chi(J(n)) \leqslant 0 \qquad \text{for all} \quad n \gg 0.$$

Now, by Serre's theorem [EGA III_1, 2.2.2(iii)], we have the formula,

$$\chi(\mathrm{Coker}(u)(n)) = h^0(X, \mathrm{Coker}(u)(n)) \qquad \text{for all} \quad n \gg 0.$$

Since $h^0(X, \mathrm{Coker}(u)(n))$ can never be negative, it must therefore be zero. So, since $\mathrm{Coker}(u)(n)$ is generated by its global sections for $n \gg 0$ by Serre's theorem, $\mathrm{Coker}(u)$ is equal to zero. Thus u is an isomorphism.

Assertion (b) is an immediate consequence of (a).

(iii) The proof that $H(-\mu)$ is a b-sheaf proceeds by induction on r. The leading coefficients of $\chi(F(n))$ and $\chi(J(\mu + 1)(n))$ are equal by (i). Therefore the leading coefficient of $\chi(F(n)) - \chi(J(\mu + 1)(n))$ is equal to $c_{r-1} - (a_{r-1} + a_r(\mu + 1))$ by an easy computation. (All the coefficients $a_{\nu,i}$ of $\chi(J(\nu)(n))$ are given by the formula,

$$a_{\nu,i} = \sum_{j=0}^{r-i} a_{j+i} \binom{\nu - 1 + j}{j}.$$

See [SGA 6, 2.10, p. 630] where, unfortunately, a misprint occurs.) The hypothesis on μ implies that this leading coefficient is strictly negative. Hence $H(-\mu - 1) = \mathbf{Hom}_X(J(\mu + 1), F)$ has no nonzero global sections by (ii,b). Thus we have

$$\Gamma(X, H(-\mu)(-1)) = 0. \qquad (3.4.1)$$

For $r = 1$, it now follows from (3.4.1) and [SGA 6, 1.8, p. 620] that $H(-\mu)$ is a $(0, \deg(X))$-sheaf. For $r \geqslant 2$, take a general (integral) hyperplane section Y of X. Since the coefficients of a Hilbert polynomial slide down under hyperplane slicing [SGA 6, 1.7, p. 620], the condition on $J \mid Y$ and $F \mid Y$ analogous to $\mu > \mu_0$ is just the condition $\mu > \mu_0$; so it is satisfied. Moreover, $J \mid Y$ and $F \mid Y$ are rank-1, torsion-free by (3.3). So, by induction on r, the O_Y-Module $\mathbf{Hom}_Y(J \mid Y, F \mid Y)(-\mu)$ is a $(0, \ldots, 0, \deg(Y))$-sheaf. Now, $\mathbf{Hom}_Y(J \mid Y, F \mid Y)$ is isomorphic to $H \mid Y$ by (3.2). Therefore $H(-\mu)$ is a $(0, \ldots, 0, \deg(X))$-sheaf.

The final assertion now follows immediately from the main theorem on b-sheaves [SGA 6, 1.11, p. 621].

(3.5) PROPOSITION. *Let X be a projective, integral curve over an algebraically closed field. Let p denote the arithmetic genus, and let ω denote the dualizing sheaf. Fix an integer d. Then,*

(i) *For each of the following three properties, there exists an invertible O_X-Module L satisfying it:*

(a) $h^0(X, L) = 1$ *and* $h^1(X, L) = d + 2 - p$ *if* $p - 2 \leqslant d \leqslant 2p - 2$.

(b) $h^0(X, L) = p - 1 - d$ *and* $h^1(X, L) = 0$ *if* $d \leqslant p - 2$.

(c) $h^0(X, L) = 0$ *and* $h^1(X, L) = d + 1 - p$ if $p - 1 \leqslant d \leqslant 2p - 2$.

(ii) *There exists a rank-*1*, torsion-free O_X-Module I of the form $I = \omega \otimes L$, with L invertible, satisfying the condition*

(d) $h^0(X, I) = p - d$ *and* $h^1(X, I) = 1$ *if* $0 \leqslant d \leqslant p$.

(iii) *For every rank-*1*, torsion-free O_X-Module I, the following statements hold:*

(e) $\chi(I(n)) = n \deg(X) + \chi(I)$.

(f) $\chi(I) < p - 1$ *implies* $h^0(X, I) = 0$.

(g) $\chi(I) \geqslant p - 1$ *implies either* $h^1(X, I) = 0$ *or I is isomorphic to* ω.

Proof. (i) The proof of (a) and (b) proceeds by descending induction on d. For $d = 2p - 2$, take $L = O_X$. Then $h^0(X, L) = 1$ and $h^1(X, L) = p$ hold because X is integral.

Let L be an invertible sheaf satisfying the appropriate conditions for d. Let x be a smooth, closed point of X, and set

$$M = L \otimes \mathcal{M}_x^{-1},$$

where $\mathcal{M}_x$ denotes the Ideal of x. Tensoring the exact sequence,

$$0 \to \mathcal{M}_x \to O_X \to k(x) \to 0,$$

with M and taking cohomology, we obtain the long exact sequence,

$$0 \longrightarrow H^0(X, L) \longrightarrow H^0(X, M) \xrightarrow{e} k(x) \longrightarrow H^1(X, L) \xrightarrow{u} H^1(X, M) \longrightarrow 0. \quad (3.5.1)$$

If $d \leqslant p - 2$ holds, the conditions on L obviously imply the conditions on M appropriate for $d - 1$. Thus (b) will hold if (a) holds for $d = p - 2$.

Assume $d > p - 2$. We shall choose x carefully so that the map u in (3.5.1) is not injective. Then the conditions on L will obviously imply the conditions on M appropriate for $d - 1$. Thus (a) will hold, and so (b) will too.

The map dual to u is, by [GD, IV, 5.5, p. 81], equal to the map

$$\mathrm{Hom}(L, \omega) \leftarrow \mathrm{Hom}(M, \omega) = \mathscr{M}_x \mathrm{Hom}(L, \omega),$$

induced by the inclusion of L into M. Since $H^1(X, L)$ has dimension $d + 2 - p > 0$, there is a nonzero element v in $\mathrm{Hom}(L, \omega)$. Since ω has rank 1 at the generic point η of X, the nonzero map $v: L \rightarrow \omega$ is surjective at η, so surjective on an open set U. Take x from U. Then clearly v does not lie in $\mathrm{Hom}(M, \omega)$. Thus u is not injective.

By (a) or (b) with $d = p - 2$, there exists an invertible sheaf M with $h^0(X, M) = 1$ and $h^1(X, M) = 0$. Let v be a nonzero element of $H^0(X, M)$ and let x be a smooth point of X, where $v(x) \neq 0$ holds (x exists for the same reason as it did for $v: L \rightarrow \omega$ above). Set $L = M \otimes \mathscr{M}_x$. Then in sequence (3.5.1), the scalar $e(v)$ is nonzero (it is $v(x)$) and so the map e is surjective; since $h^0(X, M) = 1$ holds, e is an isomorphism. So we have

$$h^0(X, L) = h^1(X, L) = 0.$$

The construction of L in (c) proceeds by ascending induction on d. For $d = p - 1$, the construction was just made. Assume we have M with $h^0(X, M) = 0$ and $h^1(X, M) = d + 1 - p$. Set $L = M \otimes \mathscr{M}_x$ for any smooth point x of X. Then (3.5.1) clearly yields $h^0(X, L) = 0$ and $h^1(X, L) = d + 2 - p$. Thus (c) holds.

(ii) Let L be an invertible sheaf satisfying $h^0(X, L) = 1$ and $h^1(X, L) = l + 2 - p$ with $l = 2p - 2 - d$; such an L exists by (a). Set $I = \omega \otimes L^{-1}$. Then $H^0(X, I)$ is clearly equal to $\mathrm{Hom}_X(L, \omega)$. So by duality we have

$$h^0(X, I) = h^1(X, L) = p - d.$$

On the other hand, we have canonical isomorphisms,

$$\begin{aligned} H^0(X, L)^\vee &= \mathrm{Ext}^1_X(L, \omega) && \text{(duality [GD IV, 5.6, p. 81])} \\ &= H^1(X, \mathbf{Hom}_X(L, \omega)) && \text{(L invertible [GD IV, 2.6, p. 72])} \\ &= H^1(X, I). \end{aligned}$$

Therefore we have the formula,

$$h^1(X, I) = 1.$$

Thus (d) is satisfied.

(iii) Statement (e) follows immediately from (3.4(i)). Statement (f) now follows from (3.4(ii, b)) with O_X for J and with I for F because $\chi(O_X) = 1 - p$. Similarly statement (g) follows from (3.4(ii, a)) with I for J and with ω for F because $H^1(X, I)^\vee$ is equal to $\mathrm{Hom}_X(I, \omega)$ and $\chi(\omega)$ is equal to $p - 1$ by duality.

4. Linear Systems

(4.1) Definition. Let $f\colon X \to S$ be a morphism of schemes, and let I and F be two locally finitely presented O_X-Modules. Define a subfunctor $\mathbf{Lin\ Syst}_{(I,F)}$ of $\mathbf{Quot}_{(F/X/S)}$ as follows: For each S-scheme T, let

$$\mathbf{Lin\ Syst}_{(I,F)}(T)$$

be the set of $G \in \mathbf{Quot}_{(F/X/S)}(T)$ such that there exists an invertible O_T-Module N and an isomorphism,

$$I(G) \simeq I \otimes_S N.$$

(4.2) Theorem. *Let $f\colon X \to S$ be a proper finitely presented morphism of schemes, and let I and F be two locally finitely presented O_X-Modules. Assume that F is S-flat and that, for each S-scheme T for which I_T is T-flat, the canonical map,*

$$\sigma_T^\times\colon O_T^\times \to (f_T)_* \,\mathbf{Isom}_{X_T}(I_T, I_T),$$

is an isomorphism.

Then the functor $\mathbf{Lin\ Syst}_{(I,F)}$ is representable by an open, retrocompact subscheme U of the family of projective spaces $\mathbb{P}(H(I,F))$ associated to the locally finitely presented O_S-Module $H(I,F)$. Moreover, the universal member C of $\mathbf{Lin\ Syst}_{(I,F)}(U)$ fits into an exact sequence,

$$0 \to I_U \otimes O_p(-1) \to F_U \to C \to 0.$$

Furthermore U is equal to P if and only if, for each geometric point s of S, every nonzero $O_{X(s)}$-homomorphism $I(s) \to F(s)$ is injective.

Proof. For each S-scheme T and each invertible O_T-Module M, there are natural isomorphisms,

$$\kappa\colon \mathrm{Hom}_T(H(I,F)_T, M) \xrightarrow{\sim} \mathrm{Hom}_{X_T}(I, F \otimes_S M) \xrightarrow{\sim} \mathrm{Hom}_{X_T}(I \otimes_S M^{-1}, F_T).$$

The existence of the first is a basic property (1.1) of $H(I, F)$; the second is the canononical isomorphism. So, to each T-point of $\mathbb{P}(H(I, F))$, that is [EGA II, 4.2.3], to each isomorphism class of pairs (M, q) where M is an invertible O_T-

Module and $q\colon H(I,F)_T \to M$ is a surjection, there corresponds an isomorphism class of pairs (M, u), where $u = \kappa(q)$ is an O_{X_T}-homomorphism from $I \otimes_S M^{-1}$ to F_T satisfying $u(t) \neq 0$ for all $t \in T$. Conversely each such isomorphism class arises from a unique T-point of $\mathbb{P}(H(I, F))$ because a map $v\colon H(I,F)_T \to M$, where M is an invertible O_T-Module, is surjective if $v(t)$ is nonzero for each $t \in T$ by Nakayama's lemma.

On the other hand, a quotient F of F_T/T in $\mathbf{Lin\,Syst}_{(I,F)}(T)$ gives rise to an isomorphism class of pairs (N^{-1}, v), where N is an invertible O_T-Module and

$$v\colon I \otimes_S N \xrightarrow{w} I(G) \longrightarrow F_T$$

is an O_{X_T}-homomorphism. The O_T-Module N and the isomorphism w exist by definition of $\mathbf{Lin\,Syst}_{(I,F)}$; the isomorphism class of (N^{-1}, v) is independent of the choices of N and w because by [ASDS, (5)], the functor $N \mapsto I \otimes_S N$ is fully faithful under the hypotheses at hand.

For a quotient G of F_T/T in $\mathbf{Lin\,Syst}_{(I,F)}(T)$, each fiber $v(t)$ for $t \in T$ of such a map v is injective because G is T-flat. On the other hand, the injectivity on the fibers of a map $v\colon I \otimes_S N \to F_T$, where N is an invertible O_T-Module, is equivalent to the flatness of its cokernel [EGA IV$_3$, 11.3.7]. Consequently $\mathbf{Lin\,Syst}_{(I,F)}(T)$ is equal to the set of pairs (M, u) such that $u(t)$ is injective for all $t \in T$.

The final assertion now follows from the preceding characterizations of $\mathbf{Lin\,Syst}_{(I,F)}$ and $\mathbb{P}(H(I,F))$ as the sets of isomorphism classes of pairs (M, u) with, respectively, $u(t)$ injective and $u(t)$ nonzero for all $t \in T$.

To prove the first assertion, consider the tautological map,

$$\alpha\colon H(I, F)_P \to O_P(1),$$

and the O_{X_P}-homomorphism,

$$\beta = \kappa(\alpha)\colon I \otimes_S O_P(-1) \to F_P\,.$$

The points p of P such that along $X(p)$ the cokernel of β is flat and the kernel of β is surjective form an open subset U by [EGA IV$_3$, 11.3.7]; moreover, although it is not stated, the proof shows that U is retrocompact. Then $C = \mathrm{Coker}(\beta) \mid U$ is an element of $\mathbf{Lin\,Syst}_{(I,F)}(U)$, and it is easily seen to be universal.

(4.3) Lemma. *Let $f\colon X \to S$ be a quasi-compact, quasi-separated morphism of schemes. Let I and F be two quasi-coherent O_X-Modules, and assume I is locally finitely presented. Set*

$$N = f_*\mathbf{Hom}_X(I, F).$$

Assume that the natural map,

$$\sigma\colon O_S \to f_*\mathbf{Hom}_X(I, I),$$

is an isomorphism. Then the following conditions are equivalent:

(a) *N is invertible and the natural map*

$$u\colon I \underset{S}{\otimes} N \to F$$

is an isomorphism.

(b) *There exist an invertible sheaf N on S and an isomorphism,*

$$I \underset{S}{\otimes} N \simeq F$$

(c) *I and F are isomorphic locally over S.*

(d) *There exists a faithfully flat morphism $T \to S$ such that the pullbacks I_T and F_T are isomorphic.*

Proof. The only nontrivial implication is (d) $\Rightarrow$ (a). Assume (d). Since T/S is flat and since I is locally finitely presented, it is easy to see that the natural map,

$$\sigma_T\colon O_T \to (f_T)_* \mathbf{Hom}_{X_T}(I_T, I_T),$$

is also an isomorphism (see [EGA 0_I, 5.7.6; I, 9.3.3]). Similarly, the base-change map,

$$N_T \to (f_T)_* \mathbf{Hom}_{X_T}(I_T, F_T),$$

is an isomorphism. Therefore (d) implies that N_T is trivial. An easy and well-known lemma now implies that N itself is invertible (since N_T is invertible and T/S is faithfully flat). Moreover, the natural map u of (a) becomes an isomorphism when pulled back to T, so u itself is an isomorphism.

(4.4) *Remark.* The functor $\mathbf{Lin\,Syst}_{(I,F)}$ is often separated for the faithfully flat topology. It is separated under the hypotheses of (4.2) by descent theory because it is representable (4.2). It is also separated if the canonical map,

$$\sigma_T\colon O_T \to (f_T)_* \mathbf{Hom}_{X_T}(I_T, I_T),$$

is an isomorphism whenever $\mathbf{Lin\,Syst}_{(I,F)}(T)$ is nonempty by the implication (d) $\Rightarrow$ (b) of (4.3). Moreover, the first case is a special case of the second if $f_*\mathbf{Hom}_X(I, I)$ is locally finitely generated in view of (4.5(ii)) below.

On the other hand, the implication (b) $\Rightarrow$ (a) of (4.3) shows there is a canonical choice for the pair (N^{-1}, v) in the proof of (4.2). Similarly there exists a canonical choice for N and the isomorphism $I(G) \cong I \otimes_S N$ in (4.1) if the canonical map σ_T is an isomorphism.

Finally, in the notation of (4.3), if N is invertible and if (b) holds, then both σ_S and u are isomorphisms by (4.5(i)) below.

(4.5) *Remark.* Let $f\colon X \to S$ be a morphism of ringed spaces, and let I be an O_X-Module. Consider the canonical maps,

$$\sigma_S\colon\ O_S \to f_* \,\mathbf{Hom}_X(I, I),$$

$$\sigma_S^{\times}\colon O_S^{\times} \to f_* \,\mathbf{Isom}_X(I, I).$$

(i) σ_S is an isomorphism if and only if $f_*\mathbf{Hom}(I, I)$ is invertible.

(ii) Assume S is a local-ringed space. Then,

(a) If $\sigma_S^{\times}$ is injective, then σ_S is injective.

(b) If $\sigma_S^{\times}$ is surjective and if $f_*\mathbf{Hom}_X(I, I)$ is locally finitely generated then σ_S is surjective.

Proof. (i) The "only if" implication is trivial. Consider the "if." The assertion is local on S, so we may assume $\mathrm{Hom}_X(I, I)$ is freely generated by an O_X-homomorphism v. Then $\mathrm{id}_I = av$ holds for some $a \in \Gamma(S, O_S)$. Since a and v commute, both are isomorphisms. Since $\mathrm{Hom}_X(I, I)$ is isomorphic to $\Gamma(S, O_S)$, the element a is therefore a unit. Hence σ_S is an isomorphism.

(ii) (a) Take an element a of the stalk $\ker(\sigma_S)_s$ for some $s \in S$. Then $1 + a$ is a unit. Since $\sigma^{\times}(1 + a)$ is equal to $\sigma^{\times}(1)$ and since $\sigma^{\times}$ is injective, a is equal to zero.

(b) Take any $s \in S$ and any element b of $f_*\mathbf{Hom}_X(I, I)_s$ and let B be the $O_{S,s}$-algebra b generates. The $k(s)$-algebra B/m_sB is a finite dimensional $k(s)$-vector space because B is finitely generated; hence, since it is commutative, B/m_sB is a product of Artinian local rings, $A_1 \times \cdots \times A_n$. Moreover, σ_S induces a map,

$$\sigma_S(s)\colon k(s) \to A_1 \times \cdots \times A_n .$$

Since B is a finitely generated $O_{S,s}$-module, every maximal ideal of B contains m_s. It follows that every unit of B/m_sB is the residue class of a unit of B. Hence $\sigma_S(s)^{\times}$ is surjective because $\sigma_S^{\times}$ is. Therefore n is equal to 1. Consequently B is a local ring.

If b belongs to the maximal ideal of B, then $1 + b$ is a unit. So $1 + b$ belongs to the image of $\sigma_S^{\times}$; so b belongs to the image of σ_S. If b is not in the maximal ideal of B, then b is a unit; so b belongs to the image of $\sigma_S^{\times}$, so to that of σ_S. Thus σ_S is surjective.

5. The Abel Map

(5.1) Definition. Let $f\colon X \to S$ be a morphism of schemes, and let I be an O_X-Module. Then I will be called **simple over** S, or S-**simple**, if I is locally finitely presented and flat over S and if the canonical map,

$$\sigma_T\colon O_T \to f_{T*}\,\mathbf{Hom}_{X_T}(I_T, I_T),$$

is an isomorphism for each S-scheme T.

(5.2) Proposition. *Let $f\colon X \to S$ be a finitely presented, proper morphism of schemes, and let I be a locally finitely presented, S-flat O_X-Module. Then there exists an open, retrocompact subscheme U of S such that a morphism $T \to S$ factors through U if and only if I_T is T-simple.*

Proof. By (1.1) there are a locally finitely presented O_S-Module $H = H(I, I)$ and an isomorphism,

$$y\colon \mathbf{Hom}_S(H, O_S) \simeq f_*\,\mathbf{Hom}_X(I, I).$$

Set

$$u = y^{-1}(id_I)\colon H \to O_S\,.$$

Since the formation of (H, y) commutes with base change (1.1), the fiber $u(s)$ is nonzero for each point s of S. Hence $u(s)$ is surjective for each s. So by Nakayama's lemma u is surjective.

Since u is surjective, clearly $\mathrm{Ker}(u)$ is locally finitely generated and the formation of $\mathrm{Ker}(u)$ commutes with base-change. Set

$$U = S - \mathrm{Supp}(\mathrm{Ker}(u)).$$

Then U is an open subset [EGA I, 5.2.2(iv)]. It is easy to see that U is retrocompact by descending to the Noetherian case à la [EGA IV_3, Sect. 8]. Consider U as an open subscheme. Then clearly a morphism $R \to S$ factors through U if and only if u_R is an isomorphism.

Fix a map $T \to S$. Assume it factors through U. Then for all $R \to T$, the map u_R is an isomorphism. Therefore, consideration of y_R shows that $(f_R)_*\mathbf{Hom}_{X_R}(I_R, I_R)$ is generated by id_{I_R}. So $\sigma_R\colon O_R \to (f_R)_*\mathbf{Hom}_{X_R}(I_R, I_R)$ is an isomorphism. Thus I_T is T-simple.

Suppose now that I_T is T-simple. Fix a point $t \in T$. Then the map,

$$\sigma(t)\colon k(t) \to \mathrm{Hom}_{X(t)}(I(t), I(t)),$$

is an isomorphism. So $\mathrm{Hom}_{k(t)}(H(t), k(t))$ is a one-dimensional vector space. It follows that $\mathrm{Ker}(u(t))$ is equal to zero. Since $\mathrm{Ker}(u)$ is locally finitely generated and since the formation of $\mathrm{Ker}(u)$ commutes with base-change, Nakayama's lemma implies that the stalk $\text{Ker}(u_T)_t$ is equal to zero. Hence $\text{Ker}(u_T)$ is equal to zero. So u_T is an isomorphism. Hence the map $T \to S$ factors through U.

(5.3) COROLLARY. *Let $f: X \to S$ be a finitely presented, proper morphism of schemes, and let I be a locally finitely presented, S-flat O_X-Module. Then I is S-simple if (and only if) the canonical map,*

$$\sigma(s): k(s) \to \text{Hom}_{X(s)}(I(s), I(s)),$$

is an isomorphism (i) *for each each point $s \in S$, or equivalently,* (ii) *for each geometric point s of S.*

Proof. Each of (i) and (ii) implies that the open subscheme U of (5.2) is equal to S.

(5.4) LEMMA. *Let X be a proper, R_0, irreducible scheme over an algebraically closed field k. Let I be an S_1, coherent O_X-Module whose stalk I_η at the generic point η is isomorphic to $O_{X,\eta}$. (These conditions are satisfied, for example, when X is integral and I is rank-1, torsion-free.) Then I is simple.*

Proof. Set $K = O_{X,\eta}$; it is a field because X satisfies R_0. Since I satisfies S_1, clearly $\mathbf{Hom}_X(I, I)$ satisfies S_1. Hence $\text{Hom}_X(I, I)$ is contained in the generic fiber $\mathbf{Hom}_X(I, I)_\eta$. Since I_η is isomorphic to K, the ring $\text{Hom}_X(I, I)$ is isomorphic to a subring of $\text{Hom}_K(K, K) = K$. Consequently, $\text{Hom}_X(I, I)$ is an integral domain. On the other hand, $\text{Hom}_X(I, I)$ is a finite dimensional vector space over k because X is proper over k. Hence $\text{Hom}_X(I, I)$ is equal to k because k is algebraically closed. Thus I is simple.

(5.5) DEFINITION. Let $f: X \to S$ be a morphism of schemes. Define a functor $\mathbf{Spl}_{(X/S)}$ as follows: For each S-scheme T, let

$$\mathbf{Spl}_{(X/S)}(T)$$

denote the set of equivalence classes of T-simple O_{X_T}-Modules I, where I and J are considered equivalent if there exist an invertible O_T-Module N and an isomorphism,

$$I \underset{S}{\otimes} N \cong J.$$

As is conventional for any functor, we let

$$\mathbf{Spl}_{(X/S)(\text{Zar})}(\text{resp. } \mathbf{Spl}_{(X/S)(\text{ét})}, \text{ resp. } \mathbf{Spl}_{(X/S)(\text{fppf})}, \text{ resp. } \mathbf{Spl}_{(X/S)(\text{fppf})})$$

denote the associated sheaf of $\mathbf{Spl}_{(X/S)}$ in the Zariski (resp. étale, resp. fppf, resp. fpqc) topology.

(5.6) PROPOSITION. *Let $f: X \to S$ be a finitely presented, proper morphism of schemes. Then*

(i) $\mathbf{Spl}_{(X/S)}$ *is a separated presheaf for the* fpqc *topology; in other words, the canonical map from* $\mathbf{Spl}_{(X/S)}$ *to its associated sheaf for the* fpqc *topology is a monomorphism.*

(ii) *There are canonical monomorphisms,*

$$\mathbf{Spl}_{(X/S)} \hookrightarrow \mathbf{Spl}_{(X/S)(\mathrm{Zar})} \hookrightarrow \mathbf{Spl}_{(X/S)(\text{ét})} \hookrightarrow \mathbf{Spl}_{(X/S)(\mathrm{fppf})} \hookrightarrow \mathbf{Spl}_{(X/S)(\mathrm{fppf})} .$$

(iii) *Let t be a geometric point of S. Then there is a formula,*

$$\mathbf{Spl}_{(X/S)}(k(t)) = \mathbf{Spl}_{(X/S)(\mathrm{fppf})}(k(t)).$$

In other words, every $k(t)$-point of $\mathbf{Spl}_{(X/S)(\mathrm{fppf})}$ *can be represented by a simple sheaf I on $X(t) = X \otimes_S k(t)$.*

Proof. Assertion (i) follows immediately from the implication (d) $\Rightarrow$ (b) of (4.3). Assertion (ii) follows immediately from (i) because sheaving preserves monomorphisms [SGA 3, IV, 4.4.1(iii), p. 205].

To prove (iii), let J on $X_R = X \times_S R$ represent a $k(t)$-point of $\mathbf{Spl}_{(X/S)(\mathrm{fppf})}$ for some surjective, fppf extension $R \to k(t)$. Since $k(t)$ is algebraically closed, R has a $k(t)$-rational point by Hilbert's Nullstellensatz. Then clearly the pullback of J to $X(t)$ represents the $k(t)$-point.

(5.7) DEFINITION. Let $f: X \to S$ be a projective morphism of schemes. Fix a relatively very ample sheaf $O_X(1)$ and a polynomial θ. Define a subfunctor $\mathbf{Spl}^{\theta}_{(X/S)(\text{ét})}$ of $\mathbf{Spl}_{(X/S)(\text{ét})}$ as follows: For each S-scheme T, let

$$\mathbf{Spl}^{\theta}_{(X/S)(\text{ét})}(T)$$

denote the classes in $\mathbf{Spl}_{(X/S)(\text{ét})}(T)$ having some representative I on an X_R, where $R \to T$ is a suitable surjective étale S-morphism, whose fibers $I(t)$ all have Hilbert polynomial θ. (It is clearly equivalent to require every possible representative I to have Hilbert polynomial θ on all fibers.)

(5.8) LEMMA. *Let $f: X \to S$ be a finitely presented, projective morphism of schemes. Fix a relatively very ample sheaf $O_X(1)$. Then,*

(i) *Let θ be a polynomial. Then* $\mathbf{Spl}^{\theta}_{(X/S)(\text{ét})}$ *is an open and closed subfunctor of* $\mathbf{Spl}_{(X/S)(\text{ét})}$.

(ii) *The subfunctors* $\mathbf{Spl}^{\theta}_{(X/S)(\text{ét})}$ *cover* $\mathbf{Spl}_{(X/S)(\text{ét})}$ *as θ runs through the set of polynomials.*

Proof. Let T be an S-scheme, choose a T-point of $\mathbf{Spl}_{(X/S)(\acute{e}t)}$, and let I be a representative for it on an X_R, where $g: R \to T$ is a suitable surjective, étale morphism. The set U of points $r \in R$, where $I(r)$ has Hilbert polynomial θ, is open and closed in R [EGA III$_2$, 7.9.11]. Set $V = g(U)$. Clearly $g^{-1}(V) = U$ holds. So V is open and closed [EGA IV$_2$, 2.3.12]. Clearly V represents the fibered product, $T \times \mathbf{Spl}^{\theta}_{(X/S)(\acute{e}t)}$. Thus (i) holds. Assertion (ii) is obvious.

(5.9) DEFINITION. Let $f: X \to S$ be a finitely presented morphism of schemes, whose geometric fibers are integral. An O_X-Module I will be called ***relatively torsion-free, rank*-1** (resp. ***relatively pseudo-invertible***) over S if it is locally finitely presented and S-flat and if the fiber $I(s)$ is a rank-1, torsion-free (resp. and Cohen–Macaulay) $O_{X(s)}$-Module for every geometric point s of S, or equivalently, for every point s of S.

(5.10) PROPOSITION. *Let $f: X \to S$ be a finitely presented, proper morphism of schemes, with integral geometric fibers. Then a relatively torsion-free, rank*-1 *(resp. relatively pseudo-invertible) O_X-Module is S-simple.*

Proof. The assertion follows immediately from (5.3) and (5.4).

(5.11) DEFINITION. Let $f: X \to S$ be a proper, finitely presented morphism of schemes with integral geometric fibers. Define two subfunctors $\mathbf{Pic}^{-}_{(X/S)}$ and $\mathbf{Pic}^{=}_{(X/S)}$ of $\mathbf{Spl}_{(X/S)}$ as follows: For each S-scheme T, let

$$\mathbf{Pic}^{-}_{(X/S)}(T) \qquad (\text{resp. } \mathbf{Pic}^{=}_{(X/S)}(T))$$

denote the classes in $\mathbf{Spl}_{(X/S)}(T)$ represented by relatively pseudo-invertible (resp. relatively torsion-free, rank-1) O_{X_T}-Modules.

For each polynomial θ and each subsheaf $\mathbf{P}$ of $\mathbf{Spl}_{(X/S)(\acute{e}t)}$, set

$$\mathbf{P}^{\theta} = \mathbf{P} \cap \mathbf{Spl}^{\theta}_{(X/S)(\acute{e}t)}.$$

For example, we get in this way open and closed subfunctors $\mathbf{Pic}^{\theta}_{(X/S)(\acute{e}t)}$ and $\mathbf{Pic}^{-\theta}_{(X/S)(\acute{e}t)}$.

(5.12) LEMMA. *Let $f: X \to S$ be a proper, finitely presented morphism of schemes, and let I be a locally finitely presented, S-flat O_X-Module. Then,*

(i) *The points s of S for which $I(s)$ is invertible form an open retrocompact subset of S.*

(ii) *Assume all the geometric fibers of f are integral with the same dimension r. Then,*

(a) *The points s of S for which $I(s)$ is rank*-1, *torsion-free form a retrocompact, open subset of S.*

(b) *The points s of S for which I(s) is pseudo-invertible form a retrocompact, open subset of S.*

Proof. (i) The assertion follows easily from [EGA IV_3, 12.3.1; EGA 0_I, 5.4.1].

(ii) While it is not stated in [EGA IV_3, 12.2.1], the reference used below, the proofs therein show that the various open subsets are retrocompact.

(a) By [EGA IV_3, 12.2.1(viii)], the set of points $s \in S$, where $I(s)$ is geometrically reduced, is open in S. Hence, by [EGA IV_3, 12.2.1(iv)], the set of points $s \in S$, where the dimension of each component of $\mathrm{Supp}(I(s))$ is equal to r, is open in S. So, by (3.1(i)), the set of points where $I(s)$ is torsion-free, rank-1 on the fiber $X(s)$ is open in S.

(b) This assertion follows immediately from (a) and the fact that the set of points $s \in S$, where $I(s)$ is Cohen–Macaulay, is open [EGA IV_3, 12.2.1(vii)].

(5.13) PROPOSITION. *Let $f\colon X \to S$ be a finitely presented, proper morphism of schemes. Then,*

(i) *Assume O_X is S-simple. Then $\mathbf{Pic}_{(X/S)(\text{ét})}$ is an open, retrocompact subsheaf of $\mathbf{Spl}_{(X/S)(\text{ét})}$.*

(ii) *Assume all the geometric fibers of f are integral with the same dimension r. Then,*

(a) *$\mathbf{Pic}^{-}_{(X/S)(\text{ét})}$ is a retrocompact, open subsheaf of $\mathbf{Pic}^{=}_{(X/S)(\text{ét})}$.*

(b) *$\mathbf{Pic}^{=}_{(X/S)(\text{ét})}$ is a retrocompact, open subsheaf of $\mathbf{Spl}_{(X/S)(\text{ét})}$.*

Proof. Clearly $\mathbf{Pic}_{(X/S)(\text{ét})}$ is a subfunctor of $\mathbf{Spl}_{(X/S)(\text{ét})}$ if O_X is S-simple. (Note that, for any invertible sheaf I on X, obviously $\mathbf{Hom}_X(I, I)$ is canonically isomorphic to O_X.)

Let T be an S-scheme, choose a T-point of $\mathbf{Spl}_{(X/S)(\text{ét})}$, and let I be a representative for it on an X_R, where $g\colon R \to T$ is a suitable surjective étale morphism. The set U (resp. U', resp. U'') of points $r \in R$, where $I(r)$ is torsion-free, rank-1, (resp. pseudo-invertible, resp. invertible) is open and retrocompact in R by (5.12(ii, a)) (resp. (5.12(ii, b)), resp. (5.12(i))). Since g is flat and locally finitely presented, the image $g(U)$ (resp. $g(U')$, resp. $g(U'')$) is open in T [EGA IV_2, 2.4.6], and it clearly represents the fibered product, $T \times \mathrm{Pic}^{=}_{(X/S)(\text{ét})}$ (resp. $T \times \mathrm{Pic}^{-}_{(X/S)(\text{ét})}$, resp. $T \times \mathrm{Pic}_{(X/S)(\text{ét})}$).

By definition of étale topology, we may take R of the form $R = \coprod R_\alpha$ such that the restriction, $R_\alpha \to g(R_\alpha)$, is étale and finitely presented and such that the $g(R_\alpha)$ form an open covering of T. Now, $g^{-1}(g(U))$ is clearly equal to U. Hence we have the relation,

$$g(R_\alpha \cap U) = g(R_\alpha) \cap g(U).$$

Since $U \cap R_\alpha$ is retrocompact in R and since $g \mid R_\alpha$ is quasi-compact, $g(R_\alpha) \cap g(U)$ is retrocompact in $g(R_\alpha)$. Hence $g(U)$ is retrocompact. The proofs that $g(U')$ and $g(U'')$ are retrocompact are similar.

(5.14) DEFINITION. Let $f\colon X \to S$ be a finitely presented, proper morphism of schemes, and let F be a locally finitely presented O_X-Module. Define a nested sequence of subfunctors of $\mathbf{Quot}_{F/X/S'}$

$$\mathbf{P\text{-}div}_{(F/X/S)} \subset \mathbf{Q\text{-}div}_{(F/X/S)} \subset \mathbf{Smp}_{(F/X/S)},$$

as the subfunctors consisting of those quotients whose pseudo-Ideals are, respectively, relatively pseudo-invertible, relatively torsion-free, rank-1, and relatively simple.

Assume f is projective. Fix a relatively very ample sheaf $O_X(1)$. For each polynomial ϕ and each subfunctor $\mathbf{D}$ of $\mathbf{Quot}_{(F/X/S)}$, set

$$\mathbf{D}^\phi = \mathbf{D} \cap \mathbf{Quot}^\phi_{(F/X/S)}.$$

For example, we get in this way open and closed subfunctors, $\mathbf{P\text{-}div}^\phi_{(F/X/S)}$ and $\mathbf{Q\text{-}div}^\phi_{(F/X/S)}$ and $\mathbf{Smp}^\phi_{(F/X/S)}$.

(5.15) PROPOSITION. *Let $f\colon X \to S$ be a finitely presented, locally projective morphism of schemes, whose geometric fibers are integral, and let F be a locally finitely presented O_X-Module. Then* $\mathbf{Smp}_{(F/X/S)}$ *(resp.* $\mathbf{Q\text{-}div}_{(F/X/S)}$, *resp.* $\mathbf{P\text{-}div}_{(F/X/S)}$*) is representable by a retrocompact, open subscheme* $\mathrm{Smp}_{(F/X/S)}$ *(resp.* Q-*div*$_{(F/X/S)}$, *resp.* P-*div*$_{(F/X/S)}$*) of* $\mathrm{Quot}_{(F/X/S)}$.

If f is strongly projective and if F is isomorphic to a quotient of an O_X-Module of the form $f^*B \otimes O_X(n)$ for some n, where B is a locally free O_S-Module with a constant, finite rank, then for any polynomial ϕ, the functor $\mathbf{Smp}^\phi_{(F/X/S)}$ *(resp.* $\mathbf{Q\text{-}div}^\phi_{(F/X/S)}$, *resp.* $\mathbf{P\text{-}div}^\phi_{(F/X/S)}$*) is representable by a strongly quasi-projective S-scheme* $\mathrm{Smp}^\phi_{(F/X/S)}$ *(resp.* $Q\text{-}div^\phi_{(F/X/S)}$, *resp.* $P - div^\phi_{(F/X/S)}$*).*

Proof. The first assertion follows immediately from (5.2) and (5.12). The second assertion follows from the first and the strong projectivity (2.7) of $\mathrm{Quot}^\phi_{(F/X/S)}$.

(5.16) (The Abel map). Let $f\colon X \to S$ be a proper, finitely presented morphism of schemes, and let F be a locally finitely presented O_X-Module. The map of functors,

$$\mathscr{A}_F = \mathscr{A}_{(F/X/S)}\colon \mathbf{Smp}_{(F/X/S)} \to \mathbf{Spl}_{(X/S)(\text{ét})}, \tag{5.16.1}$$

sending a quotient G of F to the equivalence class of its pseudo-Ideal $I(G)$, will be called the *Abel map associated to F.*

For a given simple sheaf I on $X \times_S T/T$, the fiber of $\mathscr{A}_F$ over I is the "(I, F_T)-

linear system" functor $\mathbf{Lin\ Syst}_{(I,F_T)}$ because $\mathbf{Spl}_{(X/S)}$ is separated for the étale topology; that is, there is a Cartesian diagram of functors,

$$\begin{array}{ccc} \mathbf{Lin\ Syst}_{(I,F_T)} & \longrightarrow & \mathbf{Smp}_{(F/X/S)} \\ \downarrow & \square & \downarrow \mathscr{A}_F \\ T & \longrightarrow & \mathbf{Spl}_{(F/X/S)(\text{ét})} \, . \end{array}$$

(5.17) LEMMA. *Let $f\colon X \to S$ be a proper, finitely presented morphism of schemes, and let F be an S-flat, locally finitely presented O_X-Module. Let T be an S-scheme, and let I be a T-simple O_{X_T}-Module. Then,*

(i) *There is a commutative diagram with Cartesian right square,*

$$\begin{array}{ccccc} \mathbb{P}(H(I,F_T)) & \xleftarrow{\ u\ } \supset & U & \longrightarrow & \mathbf{Smp}_{(F/X/S)} \\ \| & & \downarrow & \square & \downarrow \mathscr{A}_F \\ R & \xrightarrow[g]{} & T & \xrightarrow{\ \tau\ } & \mathbf{Spl}_{(X/S)(\text{ét})} \, , \end{array} \qquad (5.17.1)$$

where U is an open, retrocompact subscheme of R, where τ is the map defined by I, and where g denotes the structure map. Moreover, U represents the functor $\mathbf{Lin\ Syst}_{(I,F_T)}$, *and there exists an exact sequence on X_U,*

$$0 \to I \otimes_T L^{-1} \to F_U \to \mathbb{G}_U \to 0,$$

in which $\mathbb{G}$ is the universal quotient of $F_{\mathbb{Q}}$, with $\mathbb{Q} = \operatorname{Quot}_{(F/X/S)}$.

(ii) *Assume that the geometric fibers of f are integral and that I and F are relatively torsion-free, rank-*1. *Then the open subscheme U in* (5.17.1) *is equal to R.*

Proof. Assertion (i) follows from the representation theorem for $\mathbf{Lin\ Syst}_{(I,F_T)}$ and from diagram (5.16.2). Under the hypotheses of (ii), clearly for each geometric point t of T, every nonzero homomorphism from $I(t)$ to $F_T(t)$ is injective. Hence (ii) follows from (4.2) too.

(5.18) THEOREM. *Let $f\colon X \to S$ be a proper, finitely presented morphism of schemes, whose geometric fibers are integral, and let F be a locally finitely presented, S-flat O_X-Module. Let t be a geometric point of* $\mathbf{Spl}_{(X,S)(\text{ét})}$, *and let I be a representing $O_{X(t)}$-Module* (5.6(iii)). *Then,*

(i) *The fiber $\mathscr{A}_F^{-1}(t)$ has dimension,*

$$\dim(\mathscr{A}_F^{-1}(t)) = \dim_{k(t)}(\operatorname{Hom}_{X(t)}(I, F(t))) - 1,$$

provided that, if there exists a nonzero map from I to $F(t)$, then there exists an injective map from I to $F(t)$.

(ii) *The Abel map* $\mathscr{A}_F$ *is smooth along* $\mathscr{A}_F^{-1}$ *if the following relation holds*:

$$\mathrm{Ext}^1_{X(t)}(I, F(t)) = 0.$$

Proof. (i) The fiber $\mathscr{A}_F^{-1}(t)$ is equal to the open subset U of $\mathbb{P}(H(I, F(t)))$ representing $\mathbf{Lin\ Syst}_{(I, F(t))}$ by (5.17(i)). Suppose there exists no nonzero map from I to $F(t)$. Then the isomorphism (1.1.1),

$$y\colon \mathrm{Hom}_{k(t)}(H(I, F(t)), k(t)) \simeq \mathrm{Hom}_{X(t)}(I, F(t)), \tag{5.18.1}$$

shows that $\mathscr{A}_F^{-1}(t)$ is empty. Suppose there exists a nonzero map, and so an injective map $u\colon I \to F(t)$. Then $\mathrm{coker}(u)$ is a $k(t)$-point of $\mathbf{Lin\ Syst}_{(I, F(t))}$; so U is nonempty. Hence $\dim(U)$ is equal to $\dim_{k(t)}(H(I, F(t))) - 1$ because $\mathbb{P}(H(I, F(t)))$ is irreducible. The isomorphism (5.18.1) now yields the assertion.

(ii) Let T be an S-scheme, and take a T-point u of $\mathbf{Spl}_{(X/S)(\text{ét})}$ such that t factors through it. There exist an étale neighborhood $g\colon R \to T$ of t and an R-simple sheaf J on X_R which represents u. Since $\mathrm{Ext}^1_{X(t)}(J(t), F(t)) = 0$ holds, there exists a (Zariski) neighborhood R' of t in R such that $H(J, F_R)$ is locally free on R' with a finite rank by (1.3). Hence $\mathbb{P}(H(J, F_R))$ is smooth over R'. Finally, since smoothness descends down a faithfully flat morphism [EGA IV$_4$, 17.7.3(ii)], $\mathscr{A}_F$ is smooth over the image $g(R')$, which is a (Zariski) neighborhood of t in T, because [EGA IV$_2$, 2.4.6] g is flat and locally finitely presented. Thus $\mathscr{A}_F$ is smooth along $\mathscr{A}_F^{-1}(t)$.

(5.19) Lemma. *Let* $f\colon X \to S$ *be a projective morphism whose geometric fibers are integral, and fix a relatively very ample sheaf* $O_X(1)$. *Let* F *be a relatively rank-*1, *torsion-free* O_X*-Module, and assume the fibers of* F *have a single Hilbert polynomial* ψ. *Then for all* $m \geqslant m_0$, *where* m_0 *is the value of a universal polynomial in the coefficients of* ψ, *the family* $\mathscr{F}$ *of classes of fibers of* F *is* m*-regular, the* O_S*-Module* $B = f_*(F(m))$ *is locally free with rank* $\psi(m)$, *and the canonical map,* $(f^*B)(-m) \to F$, *is surjective.*

Proof. The first assertion follows from (3.4(iii)) applied with O_X for J. The second and third assertions follow from the first by standard base-change theory. (Note that an m-regular sheaf is generated by its global sections [SGA 6, 1.3(iii), p. 616].)

(5.20) Theorem. *Let* $f\colon X \to S$ *be a finitely presented, proper morphism of schemes, whose geometric fibers are integral. Let* F *be a relatively rank-*1, *torsion-free* O_X*-Module. Let* P *represent a subsheaf of* $\mathbf{Pic}^{=}_{(X/S)(\text{ét})}$. *Then,*

(i) *The restriction of the Abel map* $\mathscr{A}_F \mid P$ *is proper and finitely presented.*

(ii) *Assume that* f *is projective and that the fibers* $X(s)$ (*resp.* $F(s)$) *all have* *the same Hilbert polynomial* ξ *(resp.* ψ*). Then for each polynomial* θ*, the restriction* $\mathscr{A}_F \mid P^\theta$ *is strongly projective.*

(iii) *Assume there exists a universal* O_{X_P}*-Module* I *(that is, the pair* (P, I) *represents the given subsheaf). Then* $\mathscr{A}_F \mid P$ *is equal to the structure map of* $\mathbb{P}(H(I, F_P))$.

Proof. Assertion (iii) is an immediate consequence of (5.17(ii)). Since P is an étale sheaf, there exists a surjective, étale morphism $R \to P$ and a relatively torsion-free, rank-1 sheaf on X_R representing the identity map of P. Since assertion (i) descends down a surjective, étale map [EGA IV$_2$, 2.7.1(vi), (vii)], it follows from (5.17(ii)).

The hypotheses in assertion (ii) imply that $\mathbb{Q} = \mathrm{Quot}^{\phi}_{(F/X/S)}$, with $\phi = \psi - \theta$, is strongly projective over S by (2.6) in view of (5.19) and (2.2(iii)). Hence $\mathbb{Q}$ is embeddable in an appropriate $\mathbb{P}(E)$, so also in $\mathbb{P}(E) \times_S P^\theta$ by [EGA I, 5.1.8(ii)]. Since $\mathscr{A}_F \mid P^\theta$ is proper and finitely presented (i), it is strongly projective.

(5.21) *Remark.* Under the hypotheses of (5.20), the existence of a universal I in (iii) is a strong condition. For example, the existence of such an I for $\mathrm{Pic}^{=}_{(X/S)(\text{ét})}$ or $\mathrm{Pic}^{-}_{(X/S)(\text{ét})}$ or $\mathrm{Pic}_{(X/S)(\text{ét})}$, assuming these schemes exist, is easily seen to be equivalent to the assertion that the functor $\mathbf{Pic}^{=}_{(X/S)}$ or $\mathbf{Pic}^{-}_{(X/S)}$ or $\mathbf{Pic}_{(X/S)}$ is itself an étale sheaf. However, there does exist a universal sheaf for $\mathrm{Pic}^{=}_{(X/S)(\text{ét})}$, and so for $\mathrm{Pic}^{-}_{(X/S)(\text{ét})}$ and $\mathrm{Pic}_{(X/S)(\text{ét})}$, if the smooth locus of X/S admits a section. This assertion comes from a straightforward generalization of the theory of rigidification outlined in [FGA 232–05, 2.5]; it will be done in detail in [CII].

6. Representation by Schemes

(6.1) Lemma. *Let* $f: X \to S$ *be a strongly projective morphism of schemes. Let* $\mathscr{I}$, $\mathscr{F}$ *be two families of classes of coherent sheaves on the fibers of* X/S *(see* [FGA, 221–01, 2] *or* SGA 6, 1.12, p. 622]). *Assume* $\mathscr{I}$ *and* $\mathscr{F}$ *are b-families (resp. m-regular families) with only a finite number of distinct Hilbert polynomials (an m-regular family is one whose members are all m-regular for a given integer m). Then the classes of sheaves* $H_K = \mathbf{Hom}_{X_K}(I_K, F_K)$ *for* I_K *and* F_K *representing classes of* $\mathscr{I}$ *and* $\mathscr{F}$ *form a family* $\mathbf{Hom}(\mathscr{I}, \mathscr{F})$ *which is both a b-family and an m-regular family with only a finite number of distinct Hilbert polynomials.*

Proof. By hypothesis, X is S-isomorphic to a closed subscheme of $\mathbb{P}(E)$, where E is a locally free O_S-Module with a constant rank, say $(e + 1)$. Then the families $\mathscr{I}$ and $\mathscr{F}$ may be considered to be families of classes of coherent sheaves on the fibers of $\mathbb{P}_{\mathbb{Z}}^e/\mathbb{Z}$. Set $P = \mathbb{P}_{\mathbb{Z}}^e$.

By [SGA 6, 1.13, p. 623], the families $\mathscr{I}$ and $\mathscr{F}$ are limited; that is, there exists a $\mathbb{Z}$-scheme T of finite type and O_{P_T}-Modules I and F such that all the classes of $\mathscr{I}$ and $\mathscr{F}$ are represented by fibers $I(t)$ and $F(t)$ of P_T/T. Replacing T by a flattening stratification for F [CS, Lecture 8], we may assume F is flat over T.

Consider a presentation $E_1 \to E_0 \to I \to 0$ by locally free O_{P_T}-Modules E_i. Then for each $t \in T$ there is an exact sequence,

$$0 \to \mathbf{Hom}_{P(t)}(I(t), F(t)) \to \mathbf{Hom}_{P(t)}(E_0(t), F(t)) \to \mathbf{Hom}_{P(t)}(E_1(t), F(t)).$$

Now, the families $\mathbf{Hom}_{P(t)}(E_i(t), F(t))$ are limited by the sheaves $\mathbf{Hom}_P(E_i, F)$ because the formation of $\mathbf{Hom}_P(E_i, F)$ obviously commutes with base-change. Therefore, the family $\mathbf{Hom}(\mathscr{I}, \mathscr{F})$ is a family of Kernels of a morphism u between two coherent O_{P_T}-Modules. Replacing T by a flattening stratification for F and Coker(u), we may assume the formation of Ker(u) commutes with base-change. Hence $\mathbf{Hom}(\mathscr{I}, \mathscr{F})$ is limited by Ker(u), by [SGA 6, 1.13, p. 622]. The family $\mathbf{Hom}(\mathscr{I}, \mathscr{F})$ is both a b-family and an m-regular family with a finite number of distinct Hilbert polynomials.

(6.2) PROPOSITION. *Let $f: X \to S$ be a flat, finitely presented, projective morphism whose geometric fibers are integral with dimension r. Fix a relatively very ample sheaf $O_X(1)$. Assume the fibers of O_X have a single Hilbert polynomial ζ. Let F be a relatively rank-1, torsion-free O_X-Module, and assume the fibers of F have a single Hilbert polynomial ψ. Fix a polynomial θ and define an etale subsheaf $\mathbf{P}$ of $\mathbf{Pic}^{=}_{(X/S)(\acute{e}t)}$ as the sheaf associated to the following presheaf:*

$\mathbf{P}(T) =$ *the set of relatively torsion-free, rank-1 sheaves I on X_T/T satisfying, for all $t \in T$,*

(a) $\chi(I(t)(n)) = \theta(n)$,

(b) $\mathbf{Ext}^1_{X(t)}(I(t), F_T(t)) = 0$.

Then $\mathbf{P}$ is representable by a strongly quasi-projective S-scheme.

Proof. The proof proceeds by steps.

Step I. There exists an integer $m_0 \geqslant 0$ such that the following three families are m-regular for $m \geqslant m_0$: (a) the family $\mathscr{I}$ of classes of geometric points of $\mathbf{P}$, (b) the family $\mathscr{F}$ of classes of fibers of F, and (c) the family $\mathbf{Hom}(\mathscr{I}, \mathscr{F})$.

Proof. The assertion follows from (3.4(iii)) applied with O_X for J and from (6.1).

Step II. It is easy to check that we may replace F by $F(m_0)$ without changing $\mathbf{P}$ if we change ψ appropriately. Clearly now the families $\mathscr{F}$ and $\mathbf{Hom}(\mathscr{I}, \mathscr{F})$ are m-regular for $m \geqslant 0$.

Step III. Set $\mathbf{Z} = \mathscr{A}_F^{-1}(\mathbf{P})$ and set $\phi = \psi - \theta$. Then $\mathbf{Z}$ is representable by a retrocompact, open subscheme Z of $\mathbf{Q} = \mathrm{Quot}^{\phi}_{(F/X/S)}$.

Proof. First note that $\mathbb{Q}$ exists by (3.6). Now, obviously $\mathbf{Z}$ is a subfunctor of $\mathbb{Q}$. Let $\mathbb{G}$ denote the universal quotient of $F_{\mathbb{Q}}$ and set $I = I(\mathbb{G})$. Then, since F is relatively rank-1, torsion-free, clearly I is also. Moreover, the set Z of points $q \in \mathbb{Q}$, where

$$\mathbf{Ext}^1_{X(q)}(I(q), F_{\mathbb{Q}}(q)) = 0$$

holds, is open and retrocompact (1.10(i)). Obviously the open subscheme induced by $\mathbb{Q}$ on Z represents $\mathbf{Z}$.

Step IV. Let T be an S-scheme and let I be an O_{X_T}-Module representing a T-point t of $\mathbf{P}$. Then there is a Cartesian diagram,

$$\begin{array}{ccc} R = \mathbb{P}(H(I, F_T)) & \longrightarrow & Z \\ {\scriptstyle g}\downarrow & \square & \downarrow{\scriptstyle \mathscr{A}_F \mid Z} \\ T & \xrightarrow{\ t\ } & \mathbf{P} \end{array} \tag{6.2.1}$$

where g denotes the structure map, and $H(I, F_T)$ is locally free with finite rank and nowhere zero. Moreover, there is an exact sequence on X_R,

$$0 \to I \otimes_T L^{-1} \to F_R \to \mathbb{G}_R \to 0, \qquad \text{with } L = O_R(1), \tag{6.2.2}$$

in which $\mathbb{G}$ is the universal quotient of $F_{\mathbb{Q}}$, where $\mathbb{Q} = \mathrm{Quot}^{\phi}_{(F/X/S)}$.

Proof. The diagram and the sequence exist by (5.17).

By the hypothesis, $\mathbf{Ext}^1_{X(t)}(I(t), F_T(t))$ is equal to zero for each $t \in T$. Therefore, the "local to global" spectral sequence [GD IV, 2.4., p. 71] yields an isomorphism,

$$H^1(X(t), \mathbf{Hom}_{X(t)}(I(t), F_T(t))) \xrightarrow{\sim} \mathrm{Ext}^1_{X(t)}(I(t), F_T(t)).$$

Since $\mathbf{Hom}_{X(t)}(I(t), F_T(t))$ is 1-regular (Step II), this isomorphism yields the relation,

$$\mathrm{Ext}^1_{X(t)}(I(t), F_T(t) = 0 \qquad \text{for all} \quad t \in T.$$

Consequently, $H(I, F_T)$ is locally free with a finite rank by (1.3).

Since $\mathbf{Hom}_{X(t)}(I(t), F_T(t))$ is 0-regular (Step II), it is generated by its global sections. [SGA 6, 1.3(iii), p. 616]. Clearly it is nonzero at the generic point of $X(t)$. Therefore, $\mathrm{Hom}_{X(t)}(I(t), F_T(t))$ is nonzero for each $t \in T$. Hence, $H(I, F_T)$ is nowhere zero by (1.1.1).

Step V. The map $\mathscr{A}_F \mid Z\colon Z \to \mathbf{P}$ is an epimorphism of étale sheaves.

Proof. Let T be an S-scheme and let t be a T-point of $\mathbf{P}$. There exists a commutative diagram,

$$\begin{array}{ccccc} R = \mathbb{P}(H) & \xrightarrow{z} & & & \mathbf{Z} \\ \sigma \downarrow \quad \downarrow g & & & & \downarrow \mathscr{A}_F \mid Z \\ T'' \longrightarrow T' & \rightarrow & T & \xrightarrow{t} & \mathbf{P} \end{array} \tag{6.2.3}$$

in which the composition $c\colon T'' \to T$ is a surjective, étale morphism.

Indeed, take $T' \to T$ to be a surjective, étale morphism for which there exists a relatively torsion-free, rank-1 O_{X_R}-Module I representing t. Step IV gives the right-hand square in (6.2.3), with $H = H(I, F_R)$ a locally free, nowhere zero O_R-Module with a finite rank. The structure map g is clearly smooth and surjective, so it admits an étale quasi-section [EGA IV_4, 17.16.3(ii)], that is, a surjective étale morphism $T'' \to T$ and a map $\sigma\colon T' \to R$ such that the left-hand square commutes.

Diagram (6.2.3) yields the relation,

$$\mathscr{A}_F(z(\sigma)) = \mathbf{P}(c)(t).$$

Thus $\mathscr{A}_F \mid Z$ is an epimorphism.

Step VI. There exist a locally free O_S-Module E with a constant, finite rank and a quasi-compact embedding,

$$\mathbb{Q} = \mathrm{Quot}^{\phi}_{(F/X/S)} \to \mathbb{P}(E), \qquad \text{with} \quad \phi = \psi - \theta.$$

Moreover, let T be an S-scheme, and let I be the pseudo-Ideal of a member of $\mathbf{Z}(T)$. Then there is a canonical induced embedding,

$$R = \mathbb{P}(H(I, F_T)) \to \mathbb{P}(E_T),$$

and it has a constant degree on the fibers.

Proof. First, X is strongly projective by (2.2(iii)). Second, F is isomorphic to a quotient of an O_X-Module of the form $(f^*B)(\nu)$ for some ν, where B is a locally free O_S-Module with a constant finite rank by (5.19). Hence by (2.7) there exist an integer $m \geqslant m_0$, a locally free O_S-Module E with a constant finite rank, and an embedding of $\mathbb{Q}$ into $\mathbb{P}(E)$ such that $(f_{\mathbb{Q}})_*(\mathbb{G}(m))$ is locally free of rank $\phi(m)$, where $\mathbb{G}$ denotes the universal quotient of $F_{\mathbb{Q}}$, and such that the following formula holds:

$$O_{\mathbb{Q}}(1) = \det(f_{\mathbb{Q}})_*(\mathbb{G}(m)). \tag{6.2.4}$$

The fibers of I and F are all $(m+1)$-regular by Step I. Hence $(f_T)_*(I(m))$ and $f_*(F(m))$ are locally free of ranks $\theta(m)$ and $\psi(m)$ and their formations commute with base-change. Moreover, $R^1(f_T)_*(I(m))$ is equal to 0. So, using the projection formula [EGA III_1, 12.2.3.1], we obtain from sequence (6.2.2) an exact sequence of locally free O_R-Modules,

$$0 \to ((f_T)_*(I(m))) \otimes_R L^{-1} \to (f_*(F(m)))_R \to (f_R)_*(\mathbb{G}_R(m)) \to 0.$$

Taking determinants yields the formula,

$$\det(f_R)_*(\mathbb{G}_R(m)) = (\det(f_*F(m)))_R \otimes (\det(f_T)_* I(m))_R^{-1} \otimes L^{\otimes\theta(m)}$$

Therefore, formula (6.2.4) shows that the degrees of the fibers of R/S are all equal to $\theta(m)$.

Step VII. The equivalence relation $Z \times_{\mathbf{P}} Z \rightrightarrows Z$ is representable by an effective equivalence relation. Moreover, the quotient scheme P is strongly quasi-projective and the quotient map $q\colon Z \to P$ is an epimorphism of étale sheaves.

Proof. Since Z is an open, retrocompact subscheme of $\mathbb{Q} = \mathrm{Quot}^{\phi}_{(F/X/S)}$ by Step III and since $\mathbb{Q}$ is strongly quasi-projective by Step VI, the scheme Z is strongly quasi-projective.

Let $\mathbb{G}$ denote the universal quotient of $F_{\mathbb{Q}}$ and set $I = I(\mathbb{G}_Z)$. By Step IV, the sheaf I defines a Cartesian diagram.

$$\begin{array}{ccc} \mathbb{P}(H) & \longrightarrow & Z \\ \downarrow & \square & \downarrow \\ Z & \longrightarrow & \mathbf{P} \end{array}$$

where $H = H(I, F_Z)$ is a locally free, nowhere zero O_Z-Module with a finite rank. Thus the equivalence relation $Z \times_{\mathbf{P}} Z \rightrightarrows Z$ is represented by $\mathbb{P}(H) \rightrightarrows Z$. The latter is clearly smooth, surjective, and proper.

By Step II, the sheaves $\mathbf{Hom}_{X(z)}(I(z), F(z))$ are 0-regular. Moreover, these sheaves have only a finite number of Hilbert polynomials by (6.1). Therefore, the rank of $H(I, F_Z)$ is bounded. By Step VI, the degree of $\mathbb{P}(H)$ is constant. Therefore, the equivalence relation $\mathbb{P}(H) \rightrightarrows Z$ has only a finite number of Hilbert polynomials. Consequently, the equivalence relation is effective and the quotient P is strongly quasi-projective by (2.9).

Since the equivalence relation is smooth and surjective, the quotient map $q\colon Z \to P$ is smooth and surjective [EGA IV_4, 17.7.4(v); IV_2, 2.6.1(i)]. So it admits an étale quasi-section [EGA IV_4, 17.16.3]. Hence q is an epimorphism of étale sheaves.

Step VIII. The scheme P represents the functor $\mathbf{P}$. Indeed, both P and $\mathbf{P}$ are equal to the quotient of the equivalence relation $\mathbf{Z} \times_{\mathbf{P}} \mathbf{Z} \rightrightarrows \mathbf{Z}$ in the category of étale sheaves; hence they are equal.

(6.3) Theorem. *Let $f\colon X \to S$ be a flat, finitely presented, projective morphism of schemes, whose geometric fibers are integral with dimension r. Fix a relatively very ample sheaf $O_X(1)$. Assume the fibers of O_X have a single Hilbert polynomial ψ. Fix a polynomial θ. Then the Picard functor $\mathbf{Pic}^{\theta}_{(X/S)(\acute{e}t)}$ is representable by a strongly quasi-projective S-scheme.*

Proof. Clearly $\mathbf{Pic}^{\theta}_{(X/S)(\text{ét})}$ is a subsheaf of the étale sheaf $\mathbf{P}$ of (6.2) with $F = \theta_X$. By (5.13(i)), it is an open, retrocompact subfunctor of $\mathbf{P}$. Hence since $\mathbf{P}$ is representable by a strongly quasi-projective S-scheme (6.2), so is $\mathbf{Pic}^{\theta}_{(X/S)(\text{ét})}$.

(6.4) COROLLARY. *Let $f\colon X \to S$ be a flat, finitely presented, projective morphism of schemes whose geometric fibers are integral. Fix a very ample sheaf $O_X(1)$. Then $\mathbf{Pic}_{(X/S)(\text{ét})}$ is representable by a disjoint union of quasi-projective S-schemes, which represent the étale sheaves $\mathbf{Pic}^{\theta}_{(X/S)(\text{ét})}$.*

Proof. We may clearly assume S is connected. Then the fibers of O_X have a single Hilbert polynomial [EGA III_2, 7.9.4] and so the assertion results immediately from (6.3) and (5.8).

(6.5) (Dualizing sheaves). Let $f\colon X \to S$ be a flat, finitely presented, proper morphism of schemes, whose fibers $X(s)$ are Cohen–Macaulay with pure dimension r. Then there exists a flat, locally finitely presented O_X-Module $\omega = \omega_{X/S}$ whose restriction $\omega(s)$ to each fiber $X(s)$ is a dualizing sheaf (see [RD, Exercise 9.7, p. 298]). In fact, there exists a "trace map,"

$$\eta\colon R^r f_* \omega \to O_S,$$

which induces the trace map,

$$\eta(s)\colon H^r(X(s), \omega(s)) \to k(s),$$

on the fibers $X(s)$, and the pair (ω, η) is uniquely determined up to unique isomorphism. While (ω, η) has certain global dualizing properties [RD; DR, p. 161; DB], we shall need only duality on the fibers as developed in [GD].

The set of points s of S such that $X(s)$ is Gorenstein is equal to the set of points s of S such that $\omega(s)$ is invertible along $X(s)$. The latter set is open and retrocompact in S by (5.12(i)). Thus it is an open, retrocompact condition that the fibers be Gorenstein.

Assume X is a closed subscheme of $P = \mathbb{P}(E)$, where E is a locally free O_S-Module with a constant finite rank, say $(e + 1)$. Then ω is given by the formula,

$$\omega = \mathbf{Ext}^{e-r}_P(O_X, O_P(-e-1)). \tag{6.5.1}$$

By base-change theory (1.10), this formula defines an S-flat, locally finitely presented O_X-Module, whose formation commutes with base-change, because the other local $\mathbf{Ext}$'s vanish on the fibers by [GD IV, 5.1, p. 77; III, 5.22, p. 66]. Formula (6.5.1) can be used also to define a trace map η, and the uniqueness of the pair (ω, η) can be used to construct a global dualizing pair in the locally projective case [DB].

Let I be an S-flat, locally finitely presented O_X-Module. The "change of rings" spectral sequence [GD IV, 2.9.2, p. 72] degenerates, and it yields the formula,

$$\mathbf{Ext}^q_X(I, \omega) = \mathbf{Ext}^{q+e-r}_P(I, O_P(-e-1)). \tag{6.5.2}$$

Then we have

$$\mathbf{Ext}^q_X(I, \omega) = 0 \qquad \text{for} \quad q > r - \max\{\operatorname{depth} I(s)\} \tag{6.5.3}$$

by base-change theory (1.10(i)) because the right-hand side of (6.5.2) vanishes on the fibers by [GD III, 5.21, p. 66; 5.19, p. 65].

Assume S is the spectrum of a field. Then ω has finite injective dimension, in fact, injective dimension at most r, by (6.5.3). Now, take $I = k(x)$ for any closed point x of X. Then the right hand side of (6.5.2) is equal to zero for $q + e - r < e$ by [GD III, 3.13, p. 52] because O_P is Cohen–Macaulay. Hence the left-hand side of (6.5.2) is equal to zero for $q < r$, and so ω is Cohen–Macaulay with dimension r [GD III, 3.13; 3.15, p. 52]. Hence it is also torsion-free. If X is reduced, then ω is rank-1, torsion-free [GD I, 2.8, p. 8]. Hence for S arbitrary, if the fibers $X(s)$ are geometrically integral, then ω is relatively pseudo-invertible.

(6.6) Theorem. *Let $f\colon X \to S$ be a flat, finitely presented, projective morphism of schemes, whose geometric fibers are integral and Cohen–Macaulay with dimension r. Fix a relatively very ample sheaf $O_X(1)$. Assume the fibers of O_X have a single Hilbert polynomial ξ.* Fix a polynomial θ. Then the étale sheaf $\mathbf{Pic}^{-\theta}_{(X/S)(\text{ét})}$ *is representable by a strongly quasi-projective S-scheme.*

Proof. Fix a dualizing sheaf ω. It is a relatively rank-1, torsion-free O_X-Module (6.5). Its fibers have a single Hilbert polynomial, namely, $\psi(n) = (-1)^r \xi(-n)$, by duality. Moreover, $\mathbf{Pic}^{-\theta}_{(X/S)(\text{ét})}$ is a subfunctor of the functor $\mathbf{P}$ of (6.2) with $F = \omega$, because of (6.5.3). In fact, it is an open, retrocompact subfunctor by (5.13). Hence, since $\mathbf{P}$ is representable by a strongly quasi-projective S-scheme (6.2), so is $\mathbf{Pic}^{-\theta}_{(X/S)(\text{ét})}$.

(6.7) Corollary. *Let $f\colon X \to S$ be a flat, finitely presented, locally projective morphism of schemes, whose geometric fibers are integral and Cohen–Macaulay. Then*

(i) $\mathbf{Pic}^{-}_{(X/S)(\text{ét})}$ *is representable by a separated S-scheme that is locally finitely presented over S.*

(ii) *Assume f is projective and fix a very ample sheaf $O_X(1)$. Then* $\mathbf{Pic}^{-}_{(X/S)(\text{ét})}$ *is representable by a disjoint union of quasi-projective S-schemes, which represent the étale sheaves* $\mathbf{Pic}^{-}_{(X/S)(\text{ét})}$.

Proof. Assertion (i) is local on S; hence it is an immediate consequence of (ii). To prove assertion (ii), we may obviously assume S is connected. Then the fibers of O_X have a single Hilbert polynomial and the assertion results immediately from (6.6) and (5.8).

7. Representation by Algebraic Spaces

(7.1) Definition. Let $f: X \to S$ be a morphism of schemes. Let F and G be locally finitely presented O_X-Modules, and assume G is S-flat. Define a functor $\mathbf{Conj}_{(F,G)}$ as follows: For each S-scheme T, let

$$\mathbf{Conj}_{(F,G)}(T)$$

be the subset of $\mathbf{Quot}_{(F/X/S)}(T)$ of those quotients G' such that there exist an invertible O_T-Module M and an isomorphism,

$$G' \cong G \otimes_S M.$$

(7.2) Theorem. *Let $f: X \to S$ be a finitely presented, proper morphism of schemes, and let F and G be two locally finitely presented O_X-Modules. Assume that G is S-flat and that the canonical map,*

$$\sigma_T{}^\times: O_T{}^\times \to (f_T)_* \mathbf{Isom}_{X_T}(G_T, G_T),$$

is an isomorphism for each S-scheme T. Then $\mathbf{Conj}_{(F,G)}$ is representable by an open, retrocompact subscheme V of $\mathbb{P}(H(F, G))$.

Proof. (The proof is similar to that of (4.2).) Set $H = H(F, G)$. For each S-scheme T and each invertible O_T-Module M, there is a functorial isomorphism (1.1.1),

$$y: \mathrm{Hom}_T(H_T, M) \xrightarrow{\sim} \mathrm{Hom}_{X_T}(F_T, G \otimes_S M).$$

So to each T-point of $\mathbb{P}(H)$, that is, to each isomorphism class of pairs (M, q), where M is an invertible O_T-Module and $q: H_T \to M$ is a surjection [EGA II, 4.2.4], there corresponds an isomorphism class of pairs (M, u), where $u = y(q)$ is an O_{X_T}-homomorphism from F_T to $G \otimes_S M$ satisfying $u(t) \neq 0$ for all $t \in T$. Conversely, every such isomorphism class arises from a unique T-point of $\mathbb{P}(H)$ because a map $v: H_T \to M$, where M is an invertible O_T-Module, is surjective if it is nonzero for each $t \in T$, by Nakayama's lemma.

On the other hand, a quotient G' of F_T/T in $\mathbf{Conj}_{(F,G)}(T)$ gives rise to an isomorphism class of pairs (M, v), where M is an invertible O_T-Module and

$$v: G' \xrightarrow{\sim} G \otimes_S M$$

is an O_{X_T}-homomorphism. The isomorphism class of the pair (M, v) is independent of the choices of M and of v because by [ASDS, (5)] the functor $M \mapsto G \otimes_S M$ is fully faithful under the hypotheses at hand.

Consider the tautological map $\alpha: H_P \to O_P(1)$ on $P = \mathbb{P}(H)$ and set $\beta = y(\alpha)$. Then it is easy to see that

$$V = \{p \in \mathbb{P}(H) | \operatorname{Coker}(\beta)(p) = 0\}$$

is open and retrocompact and represents $\mathbf{Conj}_{(F,G)}$.

(7.3) LEMMA. *Let $f: X \to S$ be a finitely presented, projective morphism of schemes. Fix a relatively very ample sheaf $O_X(1)$, and let F be an S-flat, locally finitely presented O_X-Module. Set*

$$S_m = \{s \in S \mid F(s) \text{ is } m\text{-regular}\}.$$

Then S_m is open and retrocompact in S and contained in S_{m+1}, and S is covered by the S_m.

Proof. Since an m-regular sheaf is $(m + 1)$-regular [SGA 6, 1.3(i), p. 616], clearly S_m is contained in S_{m+1}. Also, every coherent $O_{X(s)}$-Module is m-regular for some m by Serre's theorem [EGA III_1, 2.2.2]; so the S_m cover S.

The remaining assertion, that S_m is open and retrocompact in S, is clearly local and compatible with base-change. So we may assume S is affine and by [EGA IV_3, Sects. 8, 11] Noetherian. Then S_m is automatically retrocompact.

Fix a point $s \in S_m$. Then $H^p(X(s), F(s)(m - q))$ is clearly equal to zero for $p \geqslant 1$ and $q \leqslant p$. So for each such pair of integers (p, q), there exists an open neighborhood $U_{p,q}$ of s such that the following relation holds [EGA III_2, 7.7.10]:

$$R^p f_* F(m - q) | U_{p,q} = 0.$$

Set $d = \max_{t \in S}\{\dim(X(t))\}$. Then the following relation holds [EGA III_1, 4.2.2]:

$$R^p f_* F(m - q) = 0 \qquad \text{for } p > d, \text{ for all } m \text{ and all } q.$$

Set

$$U = \bigcap_{d \geqslant p \geqslant q \geqslant 1} U_{p,q}.$$

Then we have the relation,

$$R^p f_* F(m - q) | U = 0 \qquad \text{for all } p \geqslant q \geqslant 1.$$

So $H^p(X(t), F(t)(m - q)) = 0$ holds for all $t \in U$ and $p \geqslant q \geqslant 1$ by [CS, Corollary $1\frac{1}{2}$, p. 52]. Thus $U \subset S_m$ holds. So S_m is open.

(7.4) THEOREM. *Let $f\colon X \to S$ be a locally projective, finitely presented morphism of schemes. Then $\mathbf{Spl}_{(X/S)(\acute{e}t)}$ is representable by a quasi-separated algebraic space locally finitely presented over S.*

Proof. We may assume S is affine and connected and f is projective, for the assertion is local on S.

Fix a relatively very ample sheaf $O_X(1)$. Let $\Sigma_m{}^\theta$ denote the subsheaf of $\mathbf{Spl}_{(X/S)(\acute{e}t)}$ consisting of those T-points represented by m-regular simple sheaves with Hilbert polynomial θ on every fiber. It follows easily from (7.3) and (5.8) that the subfunctors $\Sigma_m{}^\theta$ form an open covering of $\mathbf{Spl}_{(X/S)(\acute{e}t)}$ (see the proof of (5.13)). Hence by [EGA 0_I, 4.5.4] it suffices to represent the étale sheaf $\Sigma = \Sigma_m{}^\theta$.

Since an m-regular sheaf is generated by its global sections [SGA 6, 1.3(iii), p. 616], every sheaf representing a geometric point of Σ occurs as a quotient of $E = O_X(-m)^{\otimes \theta(m)}$. Let $\mathbf{Z}$ denote the subfunctor of $\mathbf{Quot}^\theta_{(E/X/S)}$ parametrizing the relatively simple quotients whose fibers are all m-regular. Then $\mathbf{Z}$ is representable by an open, retrocompact subscheme Z of $\mathrm{Quot}^\theta_{(E/X/S)}$ by (5.2) and (7.3).

The rest of the proof is analogous to Steps IV, V, VII, and VIII of (6.2).

Let $c\colon \mathbf{Z} \to \Sigma$ denote the map of functors sending a quotient G of E_T to its class in $\mathbf{Spl}_{(X/S)(\acute{e}t)}$. Then by definition of $\mathbf{Conj}_{(E_T, G)}$ and by the separatedness of (5.6(i)) of $\mathbf{Spl}_{(X/S)}$, there is a Cartesian diagram,

$$\begin{array}{ccc} \mathbf{Conj}_{(E_T,G)} & \longrightarrow & Z \\ \downarrow & \square & \downarrow c \\ T & \longrightarrow & \Sigma. \end{array}$$

So by (7.2) there is, as in Step IV, a Cartesian diagram,

$$\begin{array}{ccc} \mathbb{P}(H(E_T, G)) \supset V & \longrightarrow & Z \\ \downarrow & \square & \downarrow c \\ T & \longrightarrow & \Sigma \end{array} \qquad (7.4.1)$$

where V is an open, retrocompact subscheme of $\mathbb{P}(H(E_T, G))$. Moreover, $H(E_T, G)$ is locally free by (1.3) and it is nonzero, because G is m-regular on the fibers and because G cannot be zero on any fiber because it is S-simple.

As in Step V, the map $c\colon Z \to \Sigma$ is an epimorphism of étale sheaves. As in Step VII, the equivalence relation $Z \times_{\mathbf{P}} Z \rightrightarrows Z$ is representable by a smooth, finitely presented equivalence relation. Indeed, these assertions follow formally from the existence of diagram (7.4.1). Now, by reduction to the Noetherian case [EGA IV_3, Sect. 8] and by Artin's quotient theorem, [A2, 6.3, p. 184], such an equivalence relation is effective in the category of quasi-separated algebraic spaces. Moreover, the quotient map is smooth, so an epimorphism of étale sheaves. So, as in Step VIII, the functor Σ is representable by the quotient, an algebraic space locally of finite type over S.

(7.5) *Remark.* Narasimhan and Seshadri [NS, 12.3, p. 565] give an example showing that $\mathrm{Spl}_{(X/S)(\text{ét})}$ is not separated in general. Their example involves simple bundles that are not stable but of rank 2 and degree 1 on a smooth curve of genus $g \geqslant 3$ over $\mathbb{C}$.

(7.6) COROLLARY (A case of Artin's theorem [A1, 7.3; A2, Appendix 2]). *Let $f\colon X \to S$ be a locally projective, finitely presented morphism of schemes. Assume O_X is S-simple. Then $\mathbf{Pic}_{(X/S)(\text{ét})}$ is representable by an algebraic space, which is locally finitely presented over S.*

Proof. The sheaf $\mathbf{Pic}_{(X/S)(\text{ét})}$ is an open, retrocompact subsheaf of $\mathbf{Spl}_{(X/S)(\text{ét})}$ by (5.13(i)). Hence the assertion follows from (7.4).

(7.7) *Remark.* Grothendieck in [FGA, 236–01] presents Mumford's example, in which $\mathbf{Pic}_{(X/S)(\text{ét})}$ is not representable by a scheme. In this example, $\mathbf{Pic}_{(X/S)(\text{ét})}$ is representable by an algebraic space by virtue of (7.6).

(7.8) LEMMA. *Let S be the spectrum of a discrete valuation ring with generic point η, and let $f\colon X \to C$ be a projective morphism whose geometric fibers are integral and both have the same dimension.*

(i) *Let I_η be a rank-1, torsion-free $O_{X(\eta)}$-Module. Then there exists a relatively rank-1, torsion-free O_X-Module I whose generic fiber $I(\eta)$ is equal to I_η.*

(ii) *Let I and J be two relatively rank-1, torsion-free O_X-Modules whose generic fibers become isomorphic after a field extension of $k(\eta)$. Then I and J are isomorphic.*

Proof. (i) There exists an integer m and an embedding $u_\eta\colon I_\eta \to O_{X(\eta)}(m)$ by (3.3). Then [EGA IV_2, 2.8.1] there exists (a unique) flat extension C of Coker (u_η) to X. Take I to be the kernel of the canonical map $u\colon O_X(m) \to C$. The restriction of I to $X(\eta)$ is obviously equal to I_η.

Since S is regular, f is proper, and both fibers of f are integral with the same dimension, f is flat [Hi, 1.3]. Hence I is S-flat because C is S-flat. Also because C is S-flat, the closed fiber $I(s)$ is contained in $O_{X(s)}(m)$. Thus I is relatively torsion-free, rank-1.

(ii) Consider the coherent O_S-Module $H = H(I, J)$. For any S-scheme T, there is a functorial isomorphism (1.1.1),

$$y_T\colon \mathbf{Hom}_T(H_T, O_T) \simeq \mathbf{Hom}_{X_T}(I_T, J_T).$$

Therefore, the hypothesis implies that $\mathrm{Hom}(H_\eta, O_{S,\eta})$ is nonzero. Now, because S is the spectrum of a discrete valuation ring, H is equal to a direct sum $H = H_1 \oplus H_2$, where H_1 is free and H_2 is torsion. Since H_η is nonzero, H_1 is nonzero. So there exists a surjective map $v\colon H \to O_S$.

The map $y_S(v)\colon I \to J$ is nonzero on each fiber because $v\colon H \to O_S$ is nonzero on each fiber. Since $I(\eta)$ and $J(\eta)$ become isomorphic after a field extension, they have the same Hilbert polynomial. Therefore I and J have the same Hilbert polynomial on the special fiber [EGA III_2, 7.9.2] because they are S-flat. Consequently v is an isomorphism on each fiber by (3.4(ii, a)); so it is an isomorphism because J is S-flat.

(7.9) THEOREM. *Let $f\colon X \to S$ be a projective, finitely presented morphism of schemes, whose geometric fibers are integral and all have the same dimension. Fix a relatively very ample sheaf $O_X(1)$ and a polynomial θ. Then* $\mathbf{Pic}^{=\theta}_{(X/S)(\text{ét})}$ (resp. $\mathbf{Pic}^{-\theta}_{(X/S)(\text{ét})}$) *is representable by an algebraic space, proper and finitely presented (resp. separated and finitely presented) over S.*

Proof. The assertion is local on S, so we may assume S is affine and by [EGA IV_3, Sect. 8] Noetherian. Then by [CS, (ii), p. 58] the fibers of X/S have only a finite number of distinct Hilbert polynomials. Let $\mathscr{I}$ denote the family of classes of rank-1, torsion-free (resp. pseudo-invertible) coherent sheaves on the fibers of X/S with Hilbert polynomial θ. Then $\mathscr{I}$ is an m-regular family by (3.4(iii)) applied with $J = O_X$. Consequently by [SGA 6, 1.13, p. 623] it is limited. Since $\mathbf{Pic}^{=}_{(X/S)(\text{ét})}$ (resp. $\mathbf{Pic}^{-}_{(X/S)(\text{ét})}$) is an open, retrocompact subfunctor of $\mathbf{Spl}_{(X/S)(\text{ét})}$ by (5.13), it is representable by a quasi-separated algebraic space, locally finitely presented over S by (7.4). Since $\mathscr{I}$ is limited, $\mathbf{Pic}^{=}_{(X/S)(\text{ét})}$ is therefore finitely presented over S. Finally, it is proper (resp. separated) over S because the valuative criterion [EGA II, 7.3.8; I, 5.5.4] is satisfied (7.8).

8. CURVES

(8.1) THEOREM. *Let $f\colon X \to S$ be a locally projective, finitely presented, flat morphism of schemes, whose geometric fibers are integral curves. Then* $\mathbf{Pic}^{-}_{(X/S)(\text{ét})}$ *is represented by a disjoint union $\coprod P_n$ of S-schemes, $P_n = \mathbf{Pic}^{-}_{(X/S)(\text{ét})n}$, and P_n parametrizes the rank-1, torsion-free sheaves with Euler characteristic n on the fibers of X/S.*

Proof. The assertion is local on S, so we may assume f is projective and S is connected. Fix a relatively very ample sheaf $O_X(1)$. Then $\mathbf{Pic}^{-}_{(X/S)(\text{ét})}$ is representable by a disjoint union $\coprod P^\theta$, where P^θ parametrizes the relatively rank-1, torsion-free sheaves with Hilbert polynomial θ on the fibers by (6.2).

Let s be a geometric point of S. Since S is connected, $d = \deg(X(s))$ is independent of s. So, by (3.5(e)) a rank-1, torsion-free $O_{X(s)}$-Module has Hilbert polynomial θ if and only if it has Euler characteristic $\theta(0)$. So take $P_n = P^\theta$ with $\theta(m) = md + n$. These P_n are the desired S-schemes.

(8.2) (The dth component of the Abel map). Let $f\colon X \to S$ be a flat, locally projective, finitely presented morphism of schemes, whose geometric fibers are integral curves. Let F be a locally finitely presented, S-flat O_X-Module such that $F(s)$ is rank-1, torsion-free for each $s \in S$.

While in general $P\text{-div}_{(F/X/S)}$ is an open subscheme of $\text{Quot}_{(F/X/S)}$ by (5.15), in the present case we have an equality,

$$P\text{-div}^d_{(F/X/S)} = \text{Quot}^d_{(F/X/S)} \qquad \text{for} \quad d > 0. \tag{8.2.1}$$

Indeed, every nontrivial subsheaf of a rank-1, torsion-free sheaf on an integral scheme is obviously rank-1, torsion-free, and every torsion-free sheaf on a curve is Cohen–Macaulay since it satisfies S_1.

Assume $\chi(F(s))$ is independent of $s \in S$. For example, $\chi(F(s))$ is independent of s for $F = O_X$ and for $F = \omega$, the dualizing sheaf, if the fibers $X(s)$ all have the same arithmetic genus. Then the Abel map (5.16.1) clearly splits up into disjoint components including, in view of (8.2.1), maps

$$\mathscr{A}_F{}^d = \mathscr{A}^d_{(F/X/S)} \colon \text{Quot}^d_{(F/X/S)} \to P_n = \text{Pic}^-_{(X/S)(\text{ét})n}\,,$$

with $n = \chi(F(s)) - d$.

Let L be an invertible O_X-Module. It is evident that tensoring by L defines a commutative diagram,

$$\begin{array}{ccc} \text{Quot}^d_{(F/X/S)} & \xrightarrow{\sim} & \text{Quot}^d_{(F\otimes L/X/S)} \\ {\scriptstyle \mathscr{A}_F{}^d}\downarrow & & \downarrow{\scriptstyle \mathscr{A}^d_{F\otimes L}} \\ P_n & \xrightarrow{\sim} & P_m \end{array} \tag{8.2.2}$$

with $m = \chi(F \otimes L) - d$.

The top and bottom maps are isomorphisms because tensoring by L^{-1} defines inverses.

Suppose all the fibers $X(s)$ are Gorenstein curves with the same arithmetic genus p. Then the dualizing sheaf ω of X/S is invertible, and diagram (8.2.2) becomes

$$\begin{array}{ccc} \text{Hilb}^d_{(X/S)} & \xrightarrow{\sim} & \text{Quot}^d_{(\omega/X/S)} \\ {\scriptstyle \mathscr{A}^d_{X/S}}\downarrow & & \downarrow{\scriptstyle \mathscr{A}_\omega{}^d} \\ P_{1-p-d} & \xrightarrow{\sim} & P_{p-1-d}\,. \end{array} \tag{8.2.3}$$

This is the most important case of (8.2.2).

(8.3) (Index of Specialty). Let X be a projective, integral curve over an algebraically closed field k, and let F be a coherent O_X-Module. Let I be a rank-1, torsion-free O_X-Module, and define the *F-index of specialty* of I as the dimension of $\mathrm{Ext}^1_X(I, F)$. Let G be a nontrivial quotient of F, and define the *index of specialty* of G as the dimension of $\mathrm{Ext}^1_X(I(G)F)$. If I (resp. G) has F-index (resp. index) of specialty equal to zero, it will be called *F-nonspecial* (resp. *nonspecial*).

For example, for $F = O_X$, we recover the usual notion of index of specialty of a divisor D because $\mathrm{Ext}^1_X(O_X(-D), O_X)$ is clearly equal to $H^1(X, O_X(D))$ or, by duality, to $H^0(X, \omega(-D))$, where ω is the dualizing sheaf. On the other hand, for $F = \omega$, the index of specialty of a nontrivial quotient G of ω is equal to $h^0(X, I(G))$ by duality.

Let L be an invertible O_X-Module. Then $G \otimes L$ is a nontrivial quotient of $F \otimes L$ with pseudo-Ideal $I(G) \otimes L$. Tensoring by L leads to a canonical map,

$$\mathrm{Ext}^1_X(I(G), F) \to \mathrm{Ext}^1_X(I(G) \otimes L, F \otimes L),$$

with inverse defined by tensoring with L^{-1}. So the index of specialty of G is equal to the index of specialty of $G \otimes L$.

Now let $f\colon X \to S$ be a flat, finitely presented, locally projective morphism of schemes, whose geometric fibers are integral curves, and let F be an S-flat, locally finitely presented O_X-Module. It is easy to extend the definitions of indices of specialty and of nonspecialty to geometric points and to scheme-theoretic points of $\mathrm{Quot}^d_{(F/X/S)}$ and of $P_n = \mathrm{Pic}^{-}_{(X/S)(\text{ét})n}$. It is easy to check that these notions are preserved by the Abel map and by the isomorphism,

$$\mathrm{Quot}^d_{(F/X/S)} \xrightarrow{\sim} \mathrm{Quot}^d_{(F \otimes L/X/S)},$$

defined by tensoring by an invertible O_X-Module L.

In particular, if each fiber $X(s)$ is Gorenstein, then the dualizing sheaf ω of X/S is invertible and so tensoring by it induces an isomorphism,

$$\mathrm{Hilb}^d_{(X/S)} \xrightarrow{\sim} \mathrm{Quot}^d_{(\omega/X/S)},$$

preserving indices of specialty. Thus the first example in the second paragraph is essentially a particular case of the second example.

(8.4) THEOREM. *Let $f\colon X \to S$ be a flat, finitely presented, locally projective morphism whose geometric fibers are integral curves with the same arithmetic genus p. Let ω denote the dualizing sheaf* (5.22). *Fix an integer d and consider the* dth *piece of the Abel map,*

$$\mathscr{A}_\omega^d\colon \mathrm{Quot}^d_{(\omega/X/S)} \to P_{p-1-d} = \mathrm{Pic}^{-}_{(X/S)(\text{ét})(p-1-d)}.$$

(i) *$\mathscr{A}_\omega^d$ is surjective if and only if $d \geqslant p$ holds. In fact, the image of $\mathscr{A}_\omega^d$ omits a point of* $\mathrm{Pic}_{(X/S)(\text{ét})}$ *if $d < p$.*

(ii) *$\mathscr{A}_\omega^d$ is smooth with relative dimension $d - p$ over an ω-nonspecial point π of P_{p-1-d}.*

(iii) *Let π be a point of P_{p-1-d} in the closure of* $\mathrm{Pic}_{(X/S)(\acute{e}t)}$. *Assume there is a neighborhood U of π where all the fibers of $\mathscr{A}_\omega{}^d$ are nonempty and have the same dimension. Then $d \geqslant p$ holds and all the points of U are ω-nonspecial.*

(iv) *Let q be a point of* $\mathrm{Quot}^d_{(\omega/X/S)}$ *such that $\mathscr{A}_\omega{}^d(q)$ is in the closure of* $\mathrm{Pic}_{(X/S)(\acute{e}t)}$. *Assume $\mathscr{A}_\omega{}^d$ is flat at q. Then q is nonspecial.*

(v) *$\mathscr{A}_\omega{}^d$ is smooth if and only if $d \geqslant 2p - 1$ holds.*

Proof. Let t be a geometric point of P_{p-1-d} and let I denote the rank-1, torsion-free $O_{X(t)}$-Module representing it. By (5.18(i)) and duality, the dimension of the fiber over t is equal to

$$r = \dim(\mathrm{Hom}_{X(t)}(I, \omega(t))) - 1 = h^1(X(t), I) - 1.$$

(i) Assume $d \geqslant p$. Then $\chi(I) < 0$ holds, and this obviously implies $r \geqslant 0$. Since this holds for every t, therefore $\mathscr{A}_\omega{}^d$ is surjective.

On the other hand, for $d < p$, there exists an invertible $O_{X(t)}$-Module L with $\chi(L) = p - 1 - d$ and with $h^1(X(t), L) = 0$ by (3.5(b, c)). Thus the image of $\mathscr{A}_\omega{}^d$ does not contain $\mathrm{Pic}_{(X/S)(\acute{e}t)}$.

(ii) Take t to be a geometric point over π. By (5.18(ii)) the map $\mathscr{A}_\omega{}^d$ is smooth over π because the ω-index of specialty $\mathrm{Ext}^1_{X(t)}(I, \omega(t))$ is equal to zero by hypothesis. Moreover, we have,

$$r = -\chi(I) - 1 = d - p.$$

(iii) By hypothesis, U contains an open subset V of

$$P = \mathrm{Pic}_{(X/S)(\acute{e}t)} \cap P_{p-1-d} .$$

Clearly we may replace π by a point of V.

To prove $d \geqslant p$, clearly we may assume $S = \mathrm{Spec}(k)$, where k is the algebraic closure of $k(\pi)$. It is known that P is then irreducible. (Briefly, $\mathrm{Pic}^{\tau}_{(X/k)(\acute{e}t)}$ is equal to $\mathrm{Pic}^0_{(X/k)(\acute{e}t)}$ because every invertible sheaf can be represented by a divisor supported on the smooth locus, and any two smooth points are algebraically equivalent.) Since $\mathscr{A}_\omega{}^d$ is proper (5.20), its image A is closed. Since A contains V and since P is irreducible, A therefore contains P. Hence by (i) we have $d \geqslant p$.

Returning to the case of an arbitrary base S, let W denote the set of ω-nonspecial points of P_{p-1-d} . Then W is open. Indeed, represent the inclusion map of P_{p-1-d} into $\mathrm{Pic}^-_{(X/S)(\acute{e}t)}$ by a relative pseudo-invertible sheaf J on X_R/R, where R is a suitable étale covering of P_{p-1-d} . By upper-semicontinuity (see [EGA III_2 , 7.6.9(i)] for the locally Noetherian case, the general case can be reduced to it using [EGA IV_2 , Sects. 8, 11]), the set,

$$W' = \{w \in R \mid h^0(X(w), J(w)) = 0\},$$

is open in R. Clearly the image of W' in P_{p-1-d} is equal to W. Since a flat, locally finitely presented morphism is open [EGA IV_2, 2.4.6], W is therefore open.

It remains to show $W \supset U$. Clearly the points of U all have the same ω-index of specialty (namely, $p - d + r$, where r is the constant dimension of the fibers of $\mathscr{A}_\omega^d$ over U). Hence it suffices to prove W and U contain a common point. Now, W is open and by (3.5(c)) it contains a point of the fiber $P(\pi)$. On the other hand, U contains an open subset of $P(\pi)$. Since $P(\pi)$ is irreducible, W and U therefore contain a common point of $P(\pi)$.

(iv) By [EGA IV_3, 11.3.1], the Abel map is flat in a connected, open neighborhood V of q. The image U of V in P_{p-1-d} is open [EGA IV_3, 11.3.1] and connected. The fibers of $\mathscr{A}_\omega^d \mid V$ all have the same dimension since $\mathscr{A}_\omega^d \mid V$ is flat [EGA IV_3, 12.1.1(i)]. Since the fibers of $\mathscr{A}_\omega^d$ are projective spaces (5.17), so irreducible, all the fibers of $\mathscr{A}_\omega^d$ over U are nonempty and have the same dimension. The assertion now follows from (iii).

(v) For $d \geqslant 2p - 1$, we have $\chi(I) \leqslant -p$. So by (3.5(f)) we have $h^0(X, I) = 0$. Thus every t is ω-nonspecial and so $\mathscr{A}_\omega^d$ is smooth by (ii).

For the converse, we may assume S is the spectrum of an algebraically closed field. For each $d \leqslant 2p - 2$ there exist rank-1, torsion-free sheaves I with different values for $h^0(H, I)$ by (3.5(a–d)). Since P_{p-1-d} is connected [AIK, Proposition 11], $\mathscr{A}_\omega^d$ cannot be smooth.

(8.5) THEOREM. *Let $f\colon X \to S$ be a flat, finitely presented, locally projective morphism of schemes, whose geometric fibers are integral curves with the same arithmetic genus p. Fix an integer n and set*

$$P_n = \mathrm{Pic}^{-}_{(X/S)(\text{ét})n}\,.$$

(i) *P_n is finitely presented and locally projective over S.*

(ii) *If f is projective, then P_n is finitely presented and projective over S.*

(iii) *If f is projective and S is connected, then P_n is strongly projective over S.*

Proof. (i) The assertion is obviously local on S, so it follows from (ii).

(ii) To prove P_n is projective and finitely presented, we may clearly assume S is connected. So, assertion (ii) follows from (iii).

(iii) Since S is connected, f is strongly projective by (2.2(iii)).

Fix a relatively very ample sheaf $O_X(1)$. Let I be a relatively torsion-free, rank-1 sheaf on X_T/T. Then we have the formula,

$$\chi(I(t)(m)) = \deg(X(t))m + \chi(I(t)), \quad \text{for} \quad t \in T,$$

by (3.5(e)), and $\deg(X(t))$ is independent of t because S is connected. So we have the formula,

$$P_n = \mathrm{Pic}^{-\theta}_{(X/S)(\acute{e}t)} \qquad \text{with} \qquad \theta(m) = \deg(X(t))m + n.$$

Hence P_n is strongly quasi-projective by (6.6).

Fix m so that $\theta(m) < 0$ holds. Note that we have an isomorphism,

$$P_n \xrightarrow{\sim} P_{\theta(m)} \qquad \text{by} \qquad I \mapsto I(m).$$

Consider the dth component of the Abel map,

$$\mathscr{A}_\omega^{\,d}\colon \mathrm{Quot}^d_{(\omega/X/S)} \to P_{\theta(m)}\,, \qquad \text{for} \quad d = p - 1 - \theta(m),$$

where ω is the dualizing sheaf. It is surjective by (6.5(i)) because $d \geqslant p$ holds. Since $\mathrm{Quot}^d_{(\omega/X/S)}$ is projective, so proper, over S and since $\mathscr{A}_\omega^{\,d}$ is surjective, $P_{\theta(m)}$ is therefore universally closed over S [EGA I, 3.8.2(iv)]. Hence P_n is also.

Since P_n is strongly quasi-projective, it can be embedded into a projective S-scheme $\mathbb{P}(E)$, where E is a locally free O_S-Module with a constant, finite rank. Since P_n is universally closed over S, its inclusion map into $\mathbb{P}(E)$ is closed [EGA I, 3.8.2(vi)]. Thus it is strongly projective over S.

(8.6) THEOREM (D'Souza–Rego). *Let $f\colon X \to S$ be a flat, finitely presented, locally projective morphism of schemes, whose geometric fibers are integral curves with arithmetic genus p. Fix an integer $d > 0$, and consider the dth component of the Abel map,*

$$\mathscr{A}^d = \mathscr{A}^d_{(X/S)}\colon \mathrm{Hilb}^d_{(X/S)} \to P_{1-p-d} = \mathrm{Pic}^-_{(X/S)(\acute{e}t)(1-p-d)}\,.$$

Then the following conditions are equivalent:

(i) *$d \geqslant 2p - 1$ holds and each fiber $X(s)$ is Gorenstein.*

(ii) *$\mathscr{A}^d$ is smooth with relative dimension $d - p$.*

(iii) *Every fiber of $\mathscr{A}^d$ has the same dimension.*

(iv) *$d \geqslant 2p - 1$ holds and every fiber of $\mathscr{A}^d$ over a point of the closure of* $\mathrm{Pic}_{(X/S)(\acute{e}t)}$ *has the same dimension.*

Proof. The implication (i) $\Rightarrow$ (ii) follows immediately from (8.4(v)) and diagram (8.2.3). The implication (ii) $\Rightarrow$ (iii) is trivial.

To prove (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i), clearly we may assume S is the spectrum of an algebraically closed field k.

Assume (iii). By (3.5(a–d)) there exist rank-1, torsion-free sheaves on X with Euler characteristic $1 - p - d$ but different values of h^0 for each d with $0 \leqslant d \leqslant 2p - 2$. So by (5.18(i)) we have $d \geqslant 2p - 1$. Thus (iv) holds.

Assume (iv). Fix an invertible O_X-Module L with degree $(d-1)$; for example, take $L = O_X(d-1)$. Consider the exact sequence,

$$0 \to I(\Delta) \to O_{X\times X} \to O_\Delta \to 0,$$

of the diagonal subscheme Δ of $X \times X$. Then, for any O_X-Module M, there is an exact sequence,

$$\begin{aligned} &\mathbf{Hom}_{X\times X}(O_\Delta, L\otimes_k M) \to \mathbf{Hom}_{X\times X}(O_{X\times X}, L\otimes_k M) \\ &\quad\to \mathbf{Hom}_{X\times X}(I(\Delta), L\otimes_k M) \to \mathrm{Ext}^1_{X\times X}(O_\Delta, L\otimes_k M) \to 0, \end{aligned} \tag{8.6.1}$$

where $L\otimes_k M$ denotes $p_1^*L \otimes_{O_{X\times X}} p_2^*M$. Note that $\mathbf{Hom}_X(k(x), L)$ is equal to zero for all closed points x because L is invertible and X satisfies S_1; hence, the first term of (8.6.1) is zero by (1.10(i)).

Applying p_{2*} to (8.6.1) yields the exact sequence,

$$\begin{aligned} &0 \to p_{2*}(L\otimes_k M) \to p_{2*}\,\mathbf{Hom}_{X\times X}(I(\Delta), L\otimes_k M) \\ &\quad\to p_{2*}\,\mathbf{Ext}^1_{X\times X}(O_\Delta, L\otimes_k M) \to R^1p_{2*}(L\otimes_k M). \end{aligned}$$

Now, if for any closed point x of X we have $H^1(X, L) \neq 0$, then L is isomorphic to the dualizing sheaf ω by (3.5(g)) since $\chi(L) \geqslant p-1$ holds because L has degree $d-1$ and $d-1 \geqslant 2p-2$ holds by hypothesis; then X is Gorenstein. Otherwise, $R^1p_{2*}(L\otimes_k M)$ is equal to zero for all quasi-coherent M, and the functor,

$$M \mapsto p_{2*}(L\otimes_k M),$$

is exact by the property of exchange [EGA III_2, 7.7.5].

The hypothesis that every fiber of $\mathscr{A}^d$ had the same dimension, say r, implies that $H(I(\Delta), p_1^*L)(x)$ has the same dimension, $r+1$, for every point x of X by virtue of (5.18(i)) and (1.1). Since X is reduced, $H(I(\Delta), p_1^*L)$ is therefore locally free. Hence the functor,

$$M \mapsto p_{2*}\mathbf{Hom}_{X\times X}(I(\Delta), L\otimes_k M),$$

is exact. Therefore, the functor,

$$M \mapsto p_{2*}\,\mathbf{Ext}^1_{X\times X}(O_\Delta, L\otimes_k M), \tag{8.6.2}$$

is also exact.

Functor (8.6.2) is isomorphic to the functor,

$$M \mapsto \mathbf{Ext}^1_{X\times X}(O_\Delta, L\otimes_k M),$$

because $\mathbf{Ext}^1_{X\times X}(O_\Delta, L \otimes_k M)$ has support on Δ. Hence the latter is exact. Consequently $\mathbf{Ext}^1_{X\times X}(O_\Delta, p_1^*L)$ is flat over X, so a locally free O_Δ-Module, and its formation commutes with base-change (1.10(iii)). Therefore, for each closed point $x \in X$, its fiber is isomorphic to $\mathbf{Ext}^1_X(k(x), O_X)$, which is a sheaf concentrated at x. Hence the rank of $\mathbf{Ext}^1_X(k(x), O_X)$ is independent of x. This rank is equal to 1 at all smooth points, hence at all points. Thus X is Gorenstein and (i) holds.

(8.7) Lemma. *Let $f: X \to S$ be a flat, finitely presented, locally quasi-projective morphism of schemes. Then the diagonal $\Delta_{X/S}$ defines an isomorphism,*

$$X \xrightarrow{\sim} \mathrm{Hilb}^1_{(X/S)}.$$

Proof. The diagonal $\Delta_{X/S} \subset X \times_S X$ clearly belongs to $\mathbf{Hilb}^1_{(X/S)}(X)$. On the other hand, let Y be a T-point of $\mathbf{Hilb}^1_{(X/S)}$. Then each fiber $Y(t)$ is equal to $k(t)$ since $\chi(O_{Y(t)}(n))$ is equal to 1. Hence $Y \to T$ is a surjective, closed embedding [EGA IV$_3$, 8.11.5] because it is proper and finitely presented. Therefore, since $Y \to T$ is flat, it is an isomorphism. (Any flat, finitely presented, surjective, closed embedding $Y \to T$ is an isomorphism. Indeed, the formation of the Ideal commutes with base-change. Hence, its restriction to Y is equal to zero. So it is zero by Nakayama's lemma.) Hence Y is equal to the graph Γ_g of a morphism $g: T \to X$. Thus the pair $(X, \Delta_{X/S})$ represents $\mathbf{Hilb}^1_{(X/S)}$.

(8.8) Theorem. *Let $f: X \to S$ be a flat, finitely presented, locally projective morphism of schemes, whose geometric fibers are all integral curves with the same arithmetic genus $p > 0$. Then the first piece of the Abel map,*

$$\mathscr{A}^1: \mathrm{Hilb}^1_{(X/S)} \to P_{-p} = \mathrm{Pic}^-_{(X/S)(\text{ét})(-p)},$$

is a closed embedding, and it is canonically isomorphic to a closed embedding,

$$\alpha: X \to P_{-p}.$$

Moreover, $\alpha^{-1}(P_{-p}) \cap \mathrm{Pic}_{(X/S)(\text{ét})})$ is equal to the smooth locus of X/S.

Proof. The second assertion follows immediately from the first and from Lemma (8.7). For the last assertion, clearly we may assume S is the spectrum of an algebraically closed field. Then obviously α carries a closed point x of X to the class of its maximal Ideal $\mathscr{M}_x$. Since x is smooth if and only if $\mathscr{M}_x$ is invertible, the assertion holds.

Return to the case of an arbitrary S and consider the first assertion. Since $\mathscr{A}^1$ is proper and finitely presented (5.20(i)), it will be a closed embedding by [EGA IV$_3$, 8.11.5] if each of its geometric fibers is empty or consists of a single reduced point. Since each geometric fiber is a projective space (5.17), it suffices to assume S is the spectrum of an algebraically closed field k and it suffices to show that the presence of two distinct closed points in the same fiber of $\mathscr{A}^1$ implies p is equal to zero.

Clearly the two closed points of $\mathrm{Hilb}^1_{(X/k)}$ correspond to two closed points x and y of X whose maximal Ideals $\mathscr{M}_x$ and $\mathscr{M}_y$ are isomorphic. Since X is integral and since $\mathscr{M}_x$ and $\mathscr{M}_y$ are rank-1 and torsion-free, the isomorphism from $\mathscr{M}_x$ to $\mathscr{M}_y$ is given by multiplication by a rational function g on X; that is, we have the relation,

$$g \cdot \mathscr{M}_x = \mathscr{M}_y .$$

Since x and y are distinct points, $\mathscr{M}_x$ and $\mathscr{M}_y$ are therefore invertible.

The functions 1 and g in $\Gamma(X, \mathscr{M}_x^{-1})$ clearly generate $\mathscr{M}_x^{-1}$. So, by [EGA II, 4.2.3], they define a morphism $h\colon X \to \mathbb{P}_k{}^1$. Since x is the only pole of g and since it is a simple pole, g generates the function field of X [F, Proposition 4, p. 194]. Hence h is birational. Consequently h is an isomorphism. Thus p is equal to zero.

(8.9) *Example.* Let $f\colon X \to S$ be a flat, finitely presented, locally projective morphism whose geometric fibers are all integral curves with the same arithmetic genus p.

(i) Suppose the fibers $X(s)$ are smooth. Then clearly every torsion-free, rank-1 sheaf on $X(s)$ is invertible, and so we have

$$\mathrm{Pic}_{(X/S)(\text{ét})} = \mathrm{Pic}^{-}_{(X/S)(\text{ét})} ,$$

and for $p > 0$ the embedding $X \to P_{-p}$ in (8.8) is just the usual embedding associated with the Albanese property of the Jacobian. (See [FGA, 236–17, Theorem 3.3(iii)] for a general "Albanese" theory.)

(ii) Suppose $p = 0$. Then the fibers $X(s)$ are isomorphic to plane conics; so, since they are integral, they are smooth. Then we have

$$\mathrm{Pic}^{-}_{(X/S)(\text{ét})} = \mathrm{Pic}_{(X/S)(\text{ét})} = \mathbb{Z}_S$$

(although there is no universal sheaf unless X has the form $\mathbb{P}(E)$ for some locally free O_S-Module E with rank 2). In this case the first piece of the Abel map,

$$\mathscr{A}^1\colon \mathrm{Hilb}^1_{(X/S)} \to P_0 ,$$

is canonically isomorphic to the structure map, $f\colon X \to S$.

(iii) Suppose $p = 1$. Then the fibers of $X(s)$ are isomorphic to plane cubics; hence they are Gorenstein. Therefore the first piece of the Abel map is an isomorphism,

$$\mathscr{A}^1\colon \mathrm{Hilb}^1_{(X/S)} \xrightarrow{\sim} P_{-1} ,$$

because it is an embedding (8.8) and is smooth (8.6). So in this case, $\mathscr{A}^1$ is canonically isomorphic to a canonical isomorphism,

$$\alpha: X \cong P_{-1},$$

which carries the smooth locus of X/S onto $P_{-1} \cap \mathrm{Pic}_{(X/S)(\text{ét})}$ by (8.8).

(8.10) *Example* (inspired by [H]). Let Y be a nodal plane cubic over an algebraically closed field k. Set

$$P_n = \mathrm{Pic}^{-}_{(Y/k)(\text{et})n}.$$

It is well known (see, for example [Oo], Sect. 2]) that

$$\mathrm{Pic}^{0}_{(Y/k)} = P_0 \cap \mathrm{Pic}_{(Y/k)}$$

is canonically isomorphic to the multiplicative group $\mathbb{G}_m$. Hence the tensor-product action of $\mathrm{Pic}^0_{(Y/k)}$ on P_n yields a canonical action of $\mathbb{G}_m$ on P_n.

Transporting the action of $\mathbb{G}_m$ on P_{-1} via the isomorphism $\alpha: Y \cong P_{-1}$ of (8.9(iii)) yields an action of $\mathbb{G}_m$ on Y, given explicitly as follows. Let z be a closed point of Y. Then $\alpha(z)$ is represented by the maximal Ideal $\mathscr{M}_z$. Let g be a closed point of $\mathbb{G}_m$, and let L be a corresponding invertible sheaf on Y. Then clearly we have

$$\mathscr{M}_{g(z)} = \mathscr{M}_z \otimes L.$$

The action of $\mathbb{G}_m$ on Y induces via pullback a second action of $\mathbb{G}_m$ on each P_n. The "pullback" action is equal to the nth power of the "tensor-product" action. Indeed, fix a smooth closed point $y \in Y$. Then the maximal Ideal $\mathscr{M}_y$ is invertible, and so every closed point of P_n is represented by a sheaf of the form $\mathscr{M}_z \otimes \mathscr{M}_y^{\otimes(-n-1)}$, where z is a suitable closed point of Y. Let g be a closed point of $\mathbb{G}_m$ and let L be a corresponding invertible sheaf on Y. Then clearly we have

$$\begin{aligned} g^*(\mathscr{M}_z \otimes \mathscr{M}_y^{\otimes(-n-1)}) &= \mathscr{M}_{g^{-1}z} \otimes \mathscr{M}_{g^{-1}y}^{\otimes(-n-1)} \\ &= L^{\otimes n} \otimes \mathscr{M}_z \otimes \mathscr{M}_y^{\otimes(-n-1)}. \end{aligned}$$

Let S be an arbitrary k-scheme and fix an element $G \in H^1(S, \mathbb{G}_m)$. Consider the S-scheme $X = G \times_S (S \times_k Y)$. It is constructed as follows: Represent G by a Cech 1-cocycle $(G_{\alpha,\beta})$ with respect to a suitable open covering (U_α) of S; glue $Y \times U_\alpha$ to $Y \times U_\beta$ over $U_\alpha \cap U_\beta$ by applying $G_{\alpha,\beta}$. Clearly $\mathrm{Pic}^{-}_{(X/S)(\text{ét})n}$ can be obtained similarly, by gluing $P_n \times U_\alpha$ to $P_n \times U_\beta$ over $U_\alpha \cap U_\beta$. Hence we have the formula,

$$\mathrm{Pic}^{-}_{(X/S)(\text{ét})n} = G^{\otimes n} \times_k P_n. \tag{8.10.1}$$

In particular, there is an isomorphism,

$$\mathrm{Pic}^{-}_{(X/S)(\acute{e}t)_n} \cap \mathrm{Pic}_{(X/S)} \simeq G^{\otimes n} \times_k \mathbb{G}_m, \qquad (8.10.2)$$

because $P_n \cap \mathrm{Pic}_{(Y/k)}$ is isomorphic to $\mathbb{G}_m$ (in many ways if $n \neq 0$).

Suppose G has infinite order. Then X is not projective over S! In fact, any invertible sheaf N on X must have degree 0 on some fiber over S, for we may assume S is connected. Then the degree n of N on a fiber is independent of the fiber. So N defines a section of $G^{\otimes n} \times_k \mathbb{G}_m$ via the isomorphism (8.10.2). Hence $G^{\otimes n} \times_k \mathbb{G}_m$ is trivial. Therefore, $n = 0$.

For $n \neq 0$, by the same token, $\mathrm{Pic}^{-}_{(X/S)(\acute{e}t)_n}$ is not projective over S in view of (8.10.1) because $G^{\otimes n}$ also has infinite order and P_n is isomorphic to P_{-1}, so to Y. On the other hand, we have the formula,

$$\mathrm{Pic}^{-}_{(X/S)(\acute{e}t)_0} = P_0 \times_k S,$$

in view of (8.10.1).

References

[GD] A. Altman and S. Kleiman, "Introduction to Grothendieck Duality Theory," Lecture Notes in Mathematics No. 146, Springer-Verlag, New York/Berlin, 1970.

[ASDS] A. Altman and S. Kleiman, Algebraic systems of linearly equivalent divisor-like subschemes, *Compositio Math.* **29** (1974), 113–139.

[AIK] A. Altman, A. Iarrobino, and S. Kleiman, Irreducibility of the compactified Jacobian, *in* "Proceedings, Nordic Summer School NAVF, Oslo, Aug. 5–25, 1976," Noordhoff, Groningen, 1977.

[CJ] A. Altman and S. Kleiman, Compactifying the Jacobian, *Bull. Amer. Math. Soc.* **82** (1976), 947–949.

[CII] A. Altman and S. Kleiman, Compactifying the Picard scheme, II, *Amer. J. Math.* **101** (1979) ("Zariski volume"), 10–41.

[A1] M. Artin, Algebraization of formal moduli, I, *in* "Global Analysis," (D. C. Spencer and S. Iyanaga, Eds.), ("Kodaira volume"), pp. 21–71, Princeton Univ. Press, Princeton, N.J., 1969.

[A2] M. Artin, Versal deformations and algebraic stacks, *Invent. Math.* **27** (1974), 165–189.

[CA] N. Bourbaki, "Commutative Algebra," Hermann, Paris, and Addison-Wesley, Reading, Mass., 1972.

[Ch] W. L. Chow, The Jacobian variety of an algebraic curve, *Amer. J. Math.* **76** (1954), 453–476.

[DR] P. Deligne and M. Rapoport, Les schémas de modules de courbes elliptiques, *in* "Modular Functions of One Variable, II," pp. 143–316, Lecture Notes in Mathematics No. 349, Springer-Verlag, New York/Berlin, 1973.

[SGA3] M. Demazure and A. Grothendieck, "Schémas en Groupes, I," Lecture Notes in Mathematics No. 151, Springer-Verlag, New York/Berlin, 1970.

[D] C. D'Souza, "Compactification of Generalized Jacobians," Thesis, Tata Institute, Bombay, 1973.

[F] W. Fulton, "Algebraic Curves," Benjamin, New York, 1969.

[FGA] A. GROTHENDIECK, Technique de descente et théorèmes d'existence en géométrie algébrique, IV, V, VI, Séminaire Bourbaki, 221, 232, 236, (May 1961; February 1962; May 1962).

[EGA 0_I, I] A. GROTHENDIECK AND J. DIEUDONNÉ, "Eléments de géométrie algébrique," Vol. I, Springer–Verlag, New York/Berlin, 1971.

[EGA II–IV_4] A. GROTHENDIECK AND J. DIEUDONNÉ, "Eléments de géométrie algébrique" Inst. Hautes Études Sci. Publ. Math. Nos. 8, 11, 17, 20, 24, 28, 32 Publ. Math., Paris, 1961, 1961, 1963, 1964, 1965, 1966, 1967.

[SGA 1] A. GROTHENDIECK, "Revêtements étales et groupe fondamental," Lecture Notes in Mathematics No. 224, Springer–Verlag, New York/Berlin, 1971.

[RD] R. HARTSHORNE, "Residues and Duality," Lecture Notes in Mathematics No. 20, Springer–Verlag, New York/Berlin, 1966.

[Hi] H. HIRONAKA, Smoothing of algebraic cycles of small dimensions, *Amer. J. Math.* **90** (1968), 1–54.

[H] G. HORROCKS, Birationally ruled surfaces without embeddings in regular schemes, *Bull. London Math. Soc.* **3** (1971), 57–60.

[I] J. IGUSA, Fiber systems of Jacobian varieties, *Amer. J. Math.* **78** (1956), 171–199.

[SGA6] S. KLEIMAN, Les Théorèmes de Finitude pour le Foncteur de Picard, *in* "Théorie des Intersections et Théorème de Riemann–Roch" (P. Berthelot *et al.*, Eds.), Lecture Notes in Mathematics No. 225, Springer–Verlag, New York/Berlin, 1971.

[DB] S. KLEIMAN, Relative duality for quasi-coherent sheaves, *L'Enseignement Math.*, in press.

[HC] S. KLEIMAN AND K. LØNSTED, Basic on families of hyperelliptic curves, *Compositio Math.* **38** (1) (1979), 83–111.

[Mat] T. MATSUSAKA, On the algebraic construction of the Picard variety, II, *Japan. J. Math.* **22** (1952), 51–62.

[MM] A. MAYER AND D. MUMFORD, Further comments on boundary points, Amer. Math. Soc. Summer Institute, Woods Hole, Mass., 1964.

[CS] D. MUMFORD, "Lectures on Curves on an Algebraic Surface," Annals of Math. Studies No. 59, Princeton Univ. Press, Princeton, N.J., 1966.

[Na] Y. NAMIKAWA, A new compactification of the Siegal space and degeneration of Abelian varieties, II, *Math. Ann.* **221** (1976), 201–241.

[NS] M. NARASIMHAN AND C. SESHADRI, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* No. 2, **82** (1965), 540–567.

[Ne] A. NÉRON, Problèmes arithmétiques et géométriques rattachés à la notion du rang d'une courbe algébrique dans un corps, *Bull. Soc. Math. France* **80** (1952), 101–166.

[OB] A. OGUS AND G. BERGMAN, Nakayama's lemma for half-exact functors, *Proc. Amer. Math. Soc.* **31** (1972), 67–74.

[Oo] F. OORT, A construction of generalized Jacobian varieties by group extensions, *Math. Ann.* **147** (1962), 277–286.

[Ra] M. RAYNAUD, Spécialisation du foncteur de Picard, *Inst. Hautes Études Sci. Publ. Math.* **38** (1970), 27–76.

[Re] C. REGO, A note on one dimensional gorenstein rings, January 1977, preprint.

[SO] C. SESHADRI AND T. ODA, Compactifications of the generalized Jacobian variety, preprint.

[Z] O. ZARISKI, "Algebraic Surfaces," 2nd supplemented ed., Ergebnisse der Math., Vol. 61, Springer–Verlag, New York/Berlin, 1971.

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