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Hermitian representations of the extended affine Lie algebra $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$

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Abstract

We use the idea of free fields to obtain highest weight representations for the extended affine Lie algebra $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$ coordinatized by the quantum torus $\mathbb{C}_q$ and go on to construct a contravariant hermitian form. We further give a necessary and sufficient condition such that the contravariant hermitian form is positive definite.

Keywords: Extended affine Lie algebra; Quantum torus; Hermitian form; Highest weight representation

0. Introduction

Extended affine Lie algebras are a higher-dimensional generalization of affine Kac–Moody Lie algebras introduced in [13]. Even earlier than this Saito in [17] developed the notion of extended affine root systems in the study of singularity theory. It turns out that the nonisotropic root systems of extended affine Lie algebras are precisely Saito's extended affine root systems. Those Lie algebras and root systems have been further studied in [1–3], and among others. There are extended affine Lie algebras which allow not only Laurent polynomial algebra as coordinate

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algebra but also quantum torus (even a nonassociative torus) depending on the type of Lie algebra. The representations for extended affine Lie algebras and their cousins—toroidal Lie algebras have been studied widely in the past two decades.

In the representation theory of Lie algebras with a triangular decomposition, the existence of a highest weight vector and unitarizability are two fundamental assumptions. Let us first recall some definitions (see [14]). Suppose that $\mathfrak{g}$ is a complex Lie algebra. Let $U(\mathfrak{g})$ be its universal enveloping algebra. Let $\mathcal{B}$ be a subalgebra of $\mathfrak{g}$ (called a Borel subalgebra) and ω be an anti-linear anti-involution of $\mathfrak{g}$ such that

$$\mathcal{B} + \omega(\mathcal{B}) = \mathfrak{g}. \tag{0.1}$$

Let $\lambda : \mathcal{B} \to \mathbb{C}$ be a 1-dimensional representation of $\mathcal{B}$. A representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a highest weight representation with highest weight λ if there exists a vector $v_\lambda \in V$ with the following properties:

$$\pi\big(U(\mathfrak{g})\big)v_\lambda = V, \tag{0.2}$$

$$\pi(b)v_\lambda = \lambda(b)v_\lambda \quad \text{for any } b \in \mathcal{B}. \tag{0.3}$$

A hermitian form $(\cdot,\cdot)$ on V such that

$$(v_\lambda, v_\lambda) = 1, \tag{0.4}$$

$$\big(\pi(a)u, v\big) = \big(u, \pi\big(\omega(a)\big)v\big) \quad \text{for all } a \in \mathfrak{g}, \text{ and } u, v \in V \tag{0.5}$$

is called contravariant. One can show that, under some natural conditions, for any highest weight $\lambda : \mathcal{B} \to \mathbb{C}$ there exists a unique highest weight representation with a nondegenerate contravariant hermitian form. As pointed out in [14], the nontrivial problem is then whether this contravariant hermitian form is positive definite (the representation π is thus unitarizable).

The free fields construction was first given by Wakimoto [20] for the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ and in a great generality by Feigin and Frenkel [8] for the affine Lie algebras $\widehat{\mathfrak{sl}}_n$. The book [7] gave a detailed treatment for the free fields construction of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$.

In this paper we use the idea of free fields to give a new class of highest weight representations of the extended affine Lie algebra $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$ with respect to some natural Borel subalgebra, where $\mathbb{C}_q$ is the quantum torus (or the algebraic version of the irrational rotation algebra in the noncommutative geometry). This class of representations depends on an infinite family X of elements of $\mathrm{SL}_2(\mathbb{C})$ and one complex parameter μ and is realized on the commutative polynomial algebra $V = \mathbb{C}[x_{(m,n)}\colon (m,n) \in \mathbb{Z}^2]$ in terms of the Weyl algebra $W = \mathbb{C}[x_{(m,n)}, \partial/\partial x_{(m,n)}\colon (m,n) \in \mathbb{Z}^2]$ twisted by an action of the family X of elements of $\mathrm{SL}_2(\mathbb{C})$. This is the main result which is stated in Theorem 2.12. It may be noteworthy to point out that this realization involves operators which are cubic on standard generators of the twisted Weyl algebra. The construction of these representations is motivated by Wakimoto's works in particular the unpublished manuscript [19] where he considered the Lie algebra $\mathfrak{sl}_2(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])$. Our Theorem 3.6 provides a contravariant hermitian form for the $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$-module. To find out a necessary and sufficient condition for the contravariant hermitian form being positive definite (see Theorem 4.8), we employ the techniques developed by Jakobsen–Kac [15].

Throughout this paper, we denote the field of complex numbers, real numbers and the ring of integers by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$, respectively.

1. Extended affine Lie algebras

Let q be a nonzero complex number. A quantum 2-torus associated to q (see [16]) is the unital associative $\mathbb{C}$-algebra $\mathbb{C}_q[s^{\pm 1}, t^{\pm 1}]$ (or, simply $\mathbb{C}_q$) with generators $s^{\pm 1}$, $t^{\pm 1}$ and relations

$$ss^{-1} = s^{-1}s = tt^{-1} = t^{-1}t = 1 \quad \text{and} \quad ts = qst. \tag{1.1}$$

Then we have

$$\left(s^{m_1}t^{n_1}\right)\left(s^{m_2}t^{n_2}\right) = q^{n_1 m_2} s^{m_1+m_2} t^{n_1+n_2} \tag{1.2}$$

and

$$\mathbb{C}_q = \bigoplus_{m,n\in\mathbb{Z}} \mathbb{C}s^m t^n. \tag{1.3}$$

Define $\kappa : \mathbb{C}_q \to \mathbb{C}$ to be a $\mathbb{C}$-linear function given by

$$\kappa\left(s^m t^n\right) = \delta_{(m,n),(0,0)}. \tag{1.4}$$

Let d_s, d_t be the degree operators on $\mathbb{C}_q$ defined by

$$d_s\left(s^m t^n\right) = m s^m t^n, \quad d_t\left(s^m t^n\right) = n s^m t^n \tag{1.5}$$

for $m, n \in \mathbb{Z}$.

For the associative algebra $\mathbb{C}_q$ over $\mathbb{C}$, we have the matrix algebra $M_2(\mathbb{C}_q)$ with entries from $\mathbb{C}_q$. We will write $A(x) \in M_2(\mathbb{C}_q)$ for $A \in M_2(\mathbb{C})$ and $x \in \mathbb{C}_q$, where $A(x) = (a_{ij}x) \in M_2(\mathbb{C}_q)$ if $A = (a_{ij}) \in M_2(\mathbb{C})$. Let $\mathfrak{gl}_2(\mathbb{C}_q)$ be the Lie algebra associated to $M_2(\mathbb{C}_q)$ as usual. The Lie algebra $\mathfrak{gl}_2(\mathbb{C}_q)$ has a nondegenerate invariant form given by

$$\big(A(a), B(b)\big) = tr(AB)\kappa(ab), \quad \text{for } A, B \in M_2(\mathbb{C}),\ a, b \in \mathbb{C}_q. \tag{1.6}$$

We form a natural central extension of $\mathfrak{gl}_2(\mathbb{C}_q)$ as follows:

$$\widehat{\mathfrak{gl}_2(\mathbb{C}_q)} = \mathfrak{gl}_2(\mathbb{C}_q) \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t \tag{1.7}$$

with Lie bracket

$$\begin{aligned}
&\left[A\left(s^{m_1}t^{n_1}\right), B\left(s^{m_2}t^{n_2}\right)\right] \\
&\quad = A\left(s^{m_1}t^{n_1}\right)B\left(s^{m_2}t^{n_2}\right) - B\left(s^{m_2}t^{n_2}\right)A\left(s^{m_1}t^{n_1}\right) + tr(AB)\kappa\left(\left(d_s s^{m_1}t^{n_1}\right)s^{m_2}t^{n_2}\right)c_s \\
&\qquad + tr(AB)\kappa\left(\left(d_t s^{m_1}t^{n_1}\right)s^{m_2}t^{n_2}\right)c_t
\end{aligned} \tag{1.8}$$

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, $A, B \in M_2(\mathbb{C})$, where c_s and c_t are central elements of $\widehat{\mathfrak{gl}_2(\mathbb{C}_q)}$.

The derivations d_s and d_t can be extended to derivations on $\mathfrak{gl}_2(\mathbb{C}_q)$. Now we can define the semi-direct product of the Lie algebra $\widehat{\mathfrak{gl}_2(\mathbb{C}_q)}$ and those derivations:

$$\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)} = \widehat{\mathfrak{gl}_2(\mathbb{C}_q)} \oplus \mathbb{C}d_s \oplus \mathbb{C}d_t. \tag{1.9}$$

Next we extend the nondegenerate form on $\mathfrak{gl}_2(\mathbb{C}_q)$ to a symmetric bilinear form on $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$ as follows:

$$\big(A(a), B(b)\big) = tr(AB)\kappa(ab), \qquad (c_s, d_s) = (c_t, d_t) = 1, \tag{1.10}$$

all others are zero, for $A, B \in M_2(\mathbb{C})$, $a, b \in \mathbb{C}_q$.

Then $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$ is an extended affine Lie algebra of type A_1 with nullity 2. (See [1] and [3] for definitions.)

Let E_{ij} be the matrix whose (i, j)-entry is 1 and 0 elsewhere. Then, in $\widehat{\mathfrak{gl}_2(\mathbb{C}_q)}$, we have

$$\begin{aligned}
&\big[E_{ij}\big(s^{m_1}t^{n_1}\big), E_{kl}\big(s^{m_2}t^{n_2}\big)\big] \\
&\quad = \delta_{jk}q^{n_1m_2}E_{il}\big(s^{m_1+m_2}t^{n_1+n_2}\big) - \delta_{il}q^{n_2m_1}E_{kj}\big(s^{m_1+m_2}t^{n_1+n_2}\big) \\
&\qquad + m_1q^{n_1m_2}\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\delta_{n_1+n_2,0}c_s + n_1q^{n_1m_2}\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\delta_{n_1+n_2,0}c_t
\end{aligned} \tag{1.11}$$

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$.

The extended affine Lie algebra $\widetilde{\mathfrak{gl}_n(\mathbb{C}_q)}$ for $n \geqslant 2$ has been studied in [4–6,9–12,18], and among others.

2. Representations for $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$

In this section, we will construct $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$-modules by using Wakimoto's free fields [19,20]. Let

$$V = \mathbb{C}\big[x_{(m,n)}\colon (m,n) \in \mathbb{Z}^2\big] \tag{2.1}$$

be the (commutative) polynomial ring of infinitely many variables. The operators $x_{(m,n)}$ and $\partial/\partial x_{(m,n)}$ act on V as the usual multiplication and differentiation operators, respectively.

Given a family $X = \{X_{m,n}\colon (m,n) \in \mathbb{Z}^2\}$ of 2×2 lower triangular matrices, where

$$X_{m,n} = \begin{pmatrix} a_{(m,n)} & 0 \\ c_{(m,n)} & d_{(m,n)} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

for $(m,n) \in \mathbb{Z}^2$ (so $a_{(m,n)}d_{(m,n)} = 1$), we set

$$P_{\boldsymbol{A}} = a_{\boldsymbol{A}}\frac{\partial}{\partial x_{\boldsymbol{A}}}, \tag{2.2}$$

$$Q_{\boldsymbol{A}} = c_{\boldsymbol{A}}\frac{\partial}{\partial x_{\boldsymbol{A}}} + d_{\boldsymbol{A}}x_{\boldsymbol{A}} \tag{2.3}$$

for $\boldsymbol{A}=(m,n)\in\mathbb{Z}^2$. It is easy to see the following formula holds true.

$$x_{\boldsymbol{A}}=a_{\boldsymbol{A}}Q_{\boldsymbol{A}}-c_{\boldsymbol{A}}P_{\boldsymbol{A}}. \tag{2.4}$$

Lemma 2.5. *For $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\in\mathbb{Z}^2$, we have*

$$[P_{\boldsymbol{A}},P_{\boldsymbol{B}}]=0,\qquad [Q_{\boldsymbol{A}},Q_{\boldsymbol{B}}]=0,\qquad [P_{\boldsymbol{A}},Q_{\boldsymbol{B}}]=\delta_{\boldsymbol{A},\boldsymbol{B}}.$$

For a fixed $\mu\in\mathbb{C}$, define the following operators on V:

$$\begin{aligned}e_{12}(m_1,n_1)={}&-q^{-m_1n_1}\mu P_{(-m_1,-n_1)}\\&-\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m',n')\in\mathbb{Z}^2}}q^{n_1m'+nm_1+nm'}Q_{(m+m'+m_1,n+n'+n_1)}P_{(m,n)}P_{(m',n')},\end{aligned} \tag{2.6}$$

$$e_{21}(m_1,n_1)=Q_{(m_1,n_1)}, \tag{2.7}$$

$$e_{11}(m_1,n_1)=-\sum_{(m,n)\in\mathbb{Z}^2}q^{nm_1}Q_{(m+m_1,n+n_1)}P_{(m,n)}-\frac{1}{2}\mu\delta_{(m_1,n_1),(0,0)}, \tag{2.8}$$

$$e_{22}(m_1,n_1)=\sum_{(m,n)\in\mathbb{Z}^2}q^{mn_1}Q_{(m+m_1,n+n_1)}P_{(m,n)}+\frac{1}{2}\mu\delta_{(m_1,n_1),(0,0)}, \tag{2.9}$$

$$D_1=\sum_{(m,n)\in\mathbb{Z}}mQ_{(m,n)}P_{(m,n)}, \tag{2.10}$$

$$D_2=\sum_{(m,n)\in\mathbb{Z}}nQ_{(m,n)}P_{(m,n)} \tag{2.11}$$

for $m_1,n_1\in\mathbb{Z}$. Although $e_{11}(m_1,n_1)$, $e_{22}(m_1,n_1)$, $e_{12}(m_1,n_1)$, D_1 and D_2 are infinite sums, they are well defined as operators on V.

Now we can state our first result.

Theorem 2.12. *The linear map $\pi_{X,\mu}:\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}\to \operatorname{End} V$ given by*

$$\pi_{X,\mu}\big(E_{ij}\big(s^{m_1}t^{n_1}\big)\big)=e_{ij}(m_1,n_1),$$
$$\pi_{X,\mu}(d_s)=D_1,\qquad \pi_{X,\mu}(d_t)=D_2,\qquad \pi_{X,\mu}(c_s)=\pi_{X,\mu}(c_t)=0$$

for $m_1,n_1\in\mathbb{Z}$, $1\leqslant i,j\leqslant 2$, is a Lie algebra homomorphism.

Proof. Since the parameter q is involved in our construction (2.6) through (2.9), we shall handle the verifications in a few more details.

The following three identities are straightforward:

$$\begin{aligned}&\big[e_{11}(m_1,n_1),e_{22}(m_2,n_2)\big]\\&\quad=-\sum_{(m',n')\in\mathbb{Z}^2}q^{(n'+n_2)m_1+m'n_2}Q_{(m'+m_2+m_1,n'+n_2+n_1)}P_{(m',n')}\end{aligned}$$

$$+\sum_{(m,n)\in\mathbb{Z}^2} q^{nm_1+(m+m_1)n_2} Q_{(m+m_1+m_2,n+n_1+n_2)} P_{(m,n)}$$
$$=0,$$

$$\big[e_{11}(m_1,n_1),e_{21}(m_2,n_2)\big]=-q^{n_2m_1}Q_{(m_1+m_2,n_1+n_2)}=-q^{n_2m_1}e_{21}(m_1+m_2,n_1+n_2),$$

$$\big[e_{22}(m_1,n_1),e_{21}(m_2,n_2)\big]=q^{m_2n_1}Q_{(m_1+m_2,n_1+n_2)}=q^{m_2n_1}e_{21}(m_1+m_2,n_1+n_2).$$

$$\begin{aligned}
&\big[e_{11}(m_1,n_1),e_{11}(m_2,n_2)\big]\\
&\quad=\sum_{(m',n')\in\mathbb{Z}^2} q^{(n'+n_2)m_1+n'm_2} Q_{(m'+m_2+m_1,n'+n_2+n_1)} P_{(m',n')}\\
&\qquad-\sum_{(m,n)\in\mathbb{Z}^2} q^{nm_1+(n+n_1)m_2} Q_{(m+m_1+m_2,n+n_1+n_2)} P_{(m,n)}\\
&\quad=-q^{n_2m_1}\bigg\{-\sum_{(m',n')\in\mathbb{Z}^2} q^{n'(m_1+m_2)} Q_{(m'+m_1+m_2,n'+n_1+n_2)} P_{(m',n')}\\
&\qquad-\frac{1}{2}\mu\delta_{(m_1+m_2,n_1+n_2),(0,0)}\bigg\}\\
&\qquad+q^{n_1m_2}\bigg\{-\sum_{(m,n)\in\mathbb{Z}^2} q^{n(m_1+m_2)} Q_{(m+m_1+m_2,n+n_1+n_2)} P_{(m,n)}\\
&\qquad-\frac{1}{2}\mu\delta_{(m_1+m_2,n_1+n_2),(0,0)}\bigg\}\\
&\quad=-q^{n_2m_1}e_{11}(m_1+m_2,n_1+n_2)+q^{n_1m_2}e_{11}(m_1+m_2,n_1+n_2).
\end{aligned}$$

Similarly to the above case, one can check that

$$\big[e_{22}(m_1,n_1),e_{22}(m_2,n_2)\big]=q^{n_1m_2}e_{22}(m_1+m_2,n_1+n_2)-q^{n_2m_1}e_{22}(m_1+m_2,n_1+n_2),$$

$$\begin{aligned}
&\big[e_{11}(m_1,n_1),e_{12}(m_2,n_2)\big]\\
&\quad=\mu q^{-m_2n_2}\sum_{(m,n)\in\mathbb{Z}^2} q^{nm_1}[Q_{(m+m_1,n+n_1)}P_{(m,n)},P_{(-m_2,-n_2)}]\\
&\qquad+\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m',n')\in\mathbb{Z}^2\\(m'',n'')\in\mathbb{Z}^2}} q^{nm_1+n_2m''+n'm_2+n'm''}\\
&\qquad\times[Q_{(m+m_1,n+n_1)}P_{(m,n)},Q_{(m'+m''+m_2,n'+n''+n_2)}P_{(m',n')}P_{(m'',n'')}]\\
&\quad=-\mu q^{-m_2n_2+(-n_1-n_2)m_1}P_{(-m_1-m_2,-n_1-n_2)}\\
&\qquad+\sum_{\substack{(m',n')\in\mathbb{Z}^2\\(m'',n'')\in\mathbb{Z}^2}} q^{(n'+n''+n_2)m_1+n_2m''+n'm_2+n'm''}
\end{aligned}$$

$$\times Q_{(m'+m''+m_2+m_1,n'+n''+n_2+n_1)}P_{(m',n')}P_{(m'',n'')}$$
$$-\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m'',n'')\in\mathbb{Z}^2}} q^{nm_1+n_2m''+(n+n_1)m_2+(n+n_1)m''}$$
$$\times Q_{(m+m''+m_1+m_2,n+n''+n_1+n_2)}P_{(m,n)}P_{(m'',n'')}$$
$$-\sum_{\substack{(m',n')\in\mathbb{Z}^2\\(m,n)\in\mathbb{Z}^2}} q^{nm_1+n_2(m+m_1)+n'm_2+n'(m+m_1)}Q_{(m'+m+m_1+m_2,n'+n+n_1+n_2)}P_{(m',n')}P_{(m,n)}$$
$$=-\mu q^{-m_2n_2+(-n_1-n_2)m_1}P_{(-m_1-m_2,-n_1-n_2)}$$
$$-\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m'',n'')\in\mathbb{Z}^2}} q^{nm_1+n_2m''+(n+n_1)m_2+(n+n_1)m''}$$
$$\times Q_{(m+m''+m_1+m_2,n+n''+n_1+n_2)}P_{(m,n)}P_{(m'',n'')}$$

(the second and the fourth terms are negative to each other)

$$=q^{n_1m_2}\Big(-\mu q^{-(m_1+m_2)(n_1+n_2)}P_{(-m_1-m_2,-n_1-n_2)}$$
$$-\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m'',n'')\in\mathbb{Z}^2}} q^{nm_1+n_2m''+nm_2+(n+n_1)m''}Q_{(m+m''+m_1+m_2,n+n''+n_1+n_2)}P_{(m,n)}P_{(m'',n'')}\Big)$$
$$=q^{n_1m_2}e_{12}(m_1+m_2,n_1+n_2).$$

In a similar way, one may obtain that

$$\big[e_{22}(m_1,n_1),e_{12}(m_2,n_2)\big]=-q^{n_2m_1}e_{21}(m_1+m_2,n_1+n_2).$$

$$\big[e_{12}(m_1,n_1),e_{21}(m_2,n_2)\big]$$
$$=-q^{-m_1n_1}\mu\big[P_{(-m_1,-n_1)},Q_{(m_2,n_2)}\big]$$
$$-\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m',n')\in\mathbb{Z}^2}} q^{n_1m'+nm_1+nm'}[Q_{(m+m'+m_1,n+n'+n_1)}P_{(m,n)}P_{(m',n')},Q_{(m_2,n_2)}]$$
$$=-\delta_{(-m_1,-n_1),(m_2,n_2)}q^{-m_1n_1}\mu-\sum_{(m,n)\in\mathbb{Z}^2} q^{n_1m_2+nm_1+nm_2}Q_{(m+m_2+m_1,n+n_2+n_1)}P_{(m,n)}$$
$$-\sum_{(m',n')\in\mathbb{Z}^2} q^{n_1m'+n_2m_1+n_2m'}Q_{(m'+m_2+m_1,n'+n_2+n_1)}P_{(m',n')}$$
$$=q^{n_1m_2}\Big(-\sum_{(m,n)\in\mathbb{Z}^2} q^{(m_1+m_2)n}Q_{(m+m_2+m_1,n+n_2+n_1)}P_{(m,n)}-\frac{1}{2}\mu\delta_{(-m_1,-n_1),(m_2,n_2)}\Big)$$
$$-q^{n_2m_1}\Big(\sum_{(m',n')\in\mathbb{Z}^2} q^{(n_1+n_2)m'}Q_{(m'+m_2+m_1,n'+n_2+n_1)}P_{(m',n')}+\frac{1}{2}\mu\delta_{(-m_1,-n_1),(m_2,n_2)}\Big)$$

$$= q^{n_1 m_2} e_{11}(m_1+m_2, n_1+n_2) - q^{n_2 m_1} e_{22}(m_1+m_2, n_1+n_2).$$

Next we shall handle the most complicated situation:

$$\begin{aligned}
&\big[e_{12}(m_1,n_1), e_{12}(m_2,n_2)\big] \\
&\quad = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m',n')\in\mathbb{Z}^2\\(\bar m,\bar n)\in\mathbb{Z}^2\\(\bar m',\bar n')\in\mathbb{Z}^2}} q^{n_1 m' + n m_1 + n m' + n_2 \bar m' + \bar n m_2 + \bar n \bar m'} \\
&\qquad \times \big[Q_{(m+m'+m_1, n+n'+n_1)} P_{(m,n)} P_{(m',n')}, Q_{(\bar m+\bar m'+m_2, \bar n+\bar n'+n_2)} P_{(\bar m,\bar n)} P_{(\bar m',\bar n')}\big] \\
&\qquad + q^{(-m_1 n_1)} \mu \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m',n')\in\mathbb{Z}^2}} q^{n_2 m' + n m_2 + n m'} \\
&\qquad \times \big[P_{(-m_1,-n_1)}, Q_{(m+m'+m_2, n+n'+n_2)} P_{(m,n)} P_{(m',n')}\big] \\
&\qquad + q^{(-m_2 n_2)} \mu \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m',n')\in\mathbb{Z}^2}} q^{n_2 m' + n m_2 + n m'} \\
&\qquad \times \big[Q_{(m+m'+m_1, n+n'+n_1)} P_{(m,n)} P_{(m',n')}, P_{(-m_2,-n_2)}\big] \\
&\quad = J_1 + J_2 - J_3 - J_4 + J_5 + J_6,
\end{aligned}$$

where

$$J_1 = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(\bar m,\bar n)\in\mathbb{Z}^2\\(\bar m',\bar n')\in\mathbb{Z}^2}} q^{n_1(\bar m+\bar m'+m_2) + n m_1 + n(\bar m+\bar m'+m_2) + n_2 \bar m' + \bar n m_2 + \bar n \bar m'} \times Q_{(m+\bar m+\bar m'+m_2+m_1, n+\bar n+\bar n'+n_2+n_1)} P_{(m,n)} P_{(\bar m,\bar n)} P_{(\bar m',\bar n')},$$

$$J_2 = \sum_{\substack{(m',n')\in\mathbb{Z}^2\\(\bar m,\bar n)\in\mathbb{Z}^2\\(\bar m',\bar n')\in\mathbb{Z}^2}} q^{n_1 m' + (\bar n+\bar n'+n_2) m_1 + (\bar n+\bar n'+n_2) m' + n_2 \bar m' + \bar n m_2 + \bar n \bar m'} \times Q_{(\bar m+\bar m'+m'+m_2+m_1, \bar n+\bar n'+n'+n_2+n_1)} P_{(m',n')} P_{(\bar m,\bar n)} P_{(\bar m',\bar n')},$$

$$J_3 = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(\bar m,\bar n)\in\mathbb{Z}^2\\(m',n')\in\mathbb{Z}^2}} q^{n_1 m' + n m_1 + n m' + n_2(m+m'+m_1) + \bar n m_2 + \bar n(m+m'+m_1)} \times Q_{(\bar m+m+m'+m_2+m_1, \bar n+n+n'+n_2+n_1)} P_{(\bar m,\bar n)} P_{(m,n)} P_{(m',n')},$$

$$J_4 = \sum_{\substack{(m',n')\in\mathbb{Z}^2\\(m,n)\in\mathbb{Z}^2\\(\bar{m}',\bar{n}')\in\mathbb{Z}^2}} q^{n_1m'+nm_1+nm'+n_2\bar{m}'+(n+n'+n_1)m_2+(n+n'+n_1)\bar{m}'} \times Q_{(m+m'+\bar{m}'+m_2+m_1,n+n'+\bar{n}'+n_2+n_1)}P_{(\bar{m}',\bar{n}')}P_{(m,n)}P_{(m',n')},$$

$$J_5 = q^{-m_1n_1}\mu\sum_{(m,n)\in\mathbb{Z}^2} q^{n_2(-m_1-m_2-m+nm_2+n(-m_1-m_2-m))}P_{(-m_1-m_2-m,-n_1-n_2-n)}P_{(m,n)},$$

$$J_6 = -q^{-m_2n_2}\mu\sum_{(m',n')\in\mathbb{Z}^2} q^{n_1m'+(-n_1-n_2-n')m_1+(-n_1-n_2-n')m'}P_{(-m_1-m_2-m',-n_1-n_2-n')}P_{(m',n')}.$$

Note that $J_1 = J_4$, $J_2 = J_3$ and $J_5 = -J_6$. Thus

$$\big[e_{12}(m_1,n_1), e_{12}(m_2,n_2)\big] = 0.$$

It is clear that $[e_{21}(m_1,n_1), e_{21}(m_2,n_2)] = 0$. Next we check the identities involving D_1 and D_2.

It is obvious that the following identities hold:

$$[D_1, D_2] = 0, \qquad \big[D_1, e_{21}(m_1,n_1)\big] = m_1e_{21}(m_1,n_1).$$

$$\begin{aligned}
&\big[D_1, e_{12}(m_1,n_1)\big]\\
&\quad= -q^{-m_1n_1}\mu\sum_{(m,n)\in\mathbb{Z}^2} m[Q_{(m,n)}P_{(m,n)}, P_{(-m_1,-n_1)}]\\
&\qquad- \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m',n')\in\mathbb{Z}^2\\(m'',n'')\in\mathbb{Z}^2}} mq^{n_1m''+n'm_1+n'm''}[Q_{(m,n)}P_{(m,n)}, Q_{(m'+m''+m_1,n'+n''+n_1)}P_{(m',n')}P_{(m'',n'')}]\\
&\quad= q^{-m_1n_1}\mu(-m_1)P_{(-m_1,-n_1)}\\
&\qquad- \sum_{\substack{(m',n')\in\mathbb{Z}^2\\(m'',n'')\in\mathbb{Z}^2}} (m'+m''+m_1)q^{n_1m''+n'm_1+n'm''}Q_{(m'+m''+m_1,n'+n''+n_1)}P_{(m',n')}P_{(m'',n'')}\\
&\qquad+ \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m'',n'')\in\mathbb{Z}^2}} mq^{n_1m''+nm_1+nm''}Q_{(m+m''+m_1,n+n''+n_1)}P_{(m,n)}P_{(m'',n'')}\\
&\qquad+ \sum_{\substack{(m',n')\in\mathbb{Z}^2\\(m,n)\in\mathbb{Z}^2}} mq^{n_1m+n'm_1+n'm}Q_{(m'+m+m_1,n'+n+n_1)}P_{(m',n')}P_{(m,n)}\\
&\quad= -m_1q^{-m_1n_1}\mu P_{(-m_1,-n_1)} - m_1\sum_{\substack{(m',n')\in\mathbb{Z}^2\\(m,n)\in\mathbb{Z}^2}} q^{n_1m+n'm_1+n'm}Q_{(m'+m+m_1,n'+n+n_1)}P_{(m',n')}P_{(m,n)}\\
&\quad= m_1e_{12}(m_1,n_1).
\end{aligned}$$

$$
\begin{aligned}
&\big[D_1, e_{11}(m_1,n_1)\big] \\
&\quad = -\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m',n')\in\mathbb{Z}^2}} mq^{n'm_1}\big[Q_{(m,n)}P_{(m,n)}, Q_{(m'+m_1,n'+n_1)}P_{(m',n')}\big] \\
&\quad = -\sum_{(m',n')\in\mathbb{Z}^2} (m'+m_1)q^{n'm_1}Q_{(m'+m_1,n'+n_1)}P_{(m',n')} + \sum_{(m,n)\in\mathbb{Z}^2} mq^{nm_1}Q_{(m+m_1,n+n_1)}P_{(m,n)} \\
&\quad = m_1\bigg(-\sum_{(m,n)\in\mathbb{Z}^2} mq^{nm_1}Q_{(m+m_1,n+n_1)}P_{(m,n)} - \frac{1}{2}\mu\delta_{(m_1,n_1),(0,0)}\bigg) \\
&\quad = m_1e_{11}(m_1,n_1).
\end{aligned}
$$

Similarly to the above case, one has

$$\big[D_1, e_{22}(m_1,n_1)\big] = m_1e_{22}(m_1,n_1).$$

Replacing D_1 by D_2 in the above proof, one can show that

$$
\begin{aligned}
&\big[D_2, e_{21}(m_1,n_1)\big] = n_1e_{21}(m_1,n_1), \qquad &\big[D_2, e_{11}(m_1,n_1)\big] = n_1e_{11}(m_1,n_1), \\
&\big[D_2, e_{22}(m_1,n_1)\big] = n_1e_{22}(m_1,n_1), \qquad &\big[D_2, e_{12}(m_1,n_1)\big] = n_1e_{12}(m_1,n_1).
\end{aligned}
$$

Therefore by comparing with (1.11) we see that the linear map $\pi_{X,\mu}$ is indeed a Lie algebra homomorphism. □

3. Hermitian forms

Here we shall unify the hermitian forms independently studied by Wakimoto [19] and Jakobsen–Kac [15].

Define $\omega : \widetilde{\mathfrak{gl}_2(\mathbb{C}_q)} \mapsto \widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$ a $\mathbb{R}$-linear map as the following:

$$\omega(\lambda x) = \bar{\lambda}\omega(x), \quad \forall\lambda \in \mathbb{C},\ x \in \widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}, \tag{3.1}$$

$$\omega\big(E_{ij}(a)\big) = (-1)^{i+j}E_{ji}(\bar{a}), \quad a \in \mathbb{C}_q, \tag{3.2}$$

$$\omega(d_s) = d_s, \qquad \omega(d_t) = d_t, \qquad \omega(c_s) = c_s, \qquad \omega(c_t) = c_t, \tag{3.3}$$

where $\mathbb{R}$-linear map $\bar{\ }: \mathbb{C}_q \to \mathbb{C}_q$ is defined as $\overline{\lambda s^m t^n} = \bar{\lambda}t^{-n}s^{-m} = \bar{\lambda}q^{mn}s^{-m}t^{-n}$, and $\bar{\lambda}$ is the complex conjugate, for any $\lambda \in \mathbb{C}$, and $m, n \in \mathbb{Z}$.

In the following sections, we always assume that $q\bar{q} = 1$ (or $|q| = 1$). This assumption will guarantee that the map $\bar{\ }$ is of order two.

Lemma 3.4. *ω is an anti-linear anti-involution of $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$.*

Proof. Since

$$\omega\big(E_{ij}\big(s^m t^n\big)\big) = (-1)^{i+j}q^{mn}E_{ji}\big(s^{-m}t^{-n}\big), \tag{3.5}$$

we have

$$\begin{aligned}\omega^2\big(E_{ij}\big(s^m t^n\big)\big) &= \omega\big((-1)^{i+j} q^{mn} E_{ji}\big(s^{-m} t^{-n}\big)\big)\\ &= (-1)^{i+j}\bar{q}^{mn}(-1)^{i+j} q^{mn} E_{ij}\big(s^m t^n\big)\\ &= E_{ij}\big(s^m t^n\big),\end{aligned}$$

so $\omega^2 = \mathrm{id}$. We only need to check $\omega([a, b]) = [\omega(b), \omega(a)]$, for any $a, b \in \widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$.

$$\begin{aligned}&\big[\omega\big(E_{ij}\big(s^{m_1} t^{n_1}\big)\big), \omega\big(E_{kl}\big(s^{m_2} t^{n_2}\big)\big)\big]\\ &\quad = (-1)^{j+k} q^{m_1n_1+m_2n_2+m_2n_1}\delta_{il} E_{jk}\big(s^{-(m_1+m_2)} t^{-(n_1+n_2)}\big)\\ &\qquad - (-1)^{l+i} q^{m_1n_1+m_2n_2+m_1n_2}\delta_{jk} E_{li}\big(s^{-(m_1+m_2)} t^{-(n_1+n_2)}\big)\\ &\qquad - q^{m_1n_1+m_2n_2+m_2n_1} m_1\delta_{jk}\delta_{il}\delta_{n_1+n_2,0}\delta_{m_1+m_2,0} c_s\\ &\qquad - q^{m_1n_1+m_2n_2+m_2n_1} n_1\delta_{jk}\delta_{il}\delta_{n_1+n_2,0}\delta_{m_1+m_2,0} c_t.\end{aligned}$$

Thus

$$\begin{aligned}&\omega\big(\big[E_{ij}\big(s^{m_1} t^{n_1}\big), E_{kl}\big(s^{m_2} t^{n_2}\big)\big]\big)\\ &\quad = \bar{q}^{m_2n_1}\delta_{jk}\omega\big(E_{il}\big(s^{m_1+m_2} t^{n_1+n_2}\big)\big) - \bar{q}^{m_1n_2}\delta_{il}\omega\big(E_{kj}\big(s^{m_1+m_2} t^{n_1+n_2}\big)\big)\\ &\qquad + m_1\bar{q}^{m_2n_1}\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\delta_{n_1+n_2,0}\omega(c_s) + n_1\bar{q}^{m_2n_1}\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\delta_{n_1+n_2,0}\omega(c_t)\\ &\quad = (-1)^{l+i} q^{m_1n_1+m_2n_2+m_1n_2}\delta_{jk} E_{li}\big(s^{-(m_1+m_2)} t^{-(n_1+n_2)}\big)\\ &\qquad - (-1)^{j+k} q^{m_1n_1+m_2n_2+m_2n_1}\delta_{il} E_{jk}\big(s^{-(m_1+m_2)} t^{-(n_1+n_2)}\big)\\ &\qquad + m_1 q^{m_1n_1+m_2n_2+m_2n_1}\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\delta_{n_1+n_2,0} c_s\\ &\qquad + n_1 q^{m_1n_1+m_2n_2+m_2n_1}\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\delta_{n_1+n_2,0} c_t\\ &\quad = -\big[\omega\big(E_{ij}\big(s^{m_1} t^{n_1}\big)\big), \omega\big(E_{kl}\big(s^{m_2} t^{n_2}\big)\big)\big]\\ &\quad = \big[\omega\big(E_{kl}\big(s^{m_2} t^{n_2}\big)\big), \omega\big(E_{ij}\big(s^{m_1} t^{n_1}\big)\big)\big].\end{aligned}$$

As for identities involving d_s and d_t, we have

$$\begin{aligned}\big[\omega(d_s), \omega\big(E_{ij}\big(s^m t^n\big)\big)\big] &= \big[d_s, (-1)^{j+i} q^{mn} E_{ji}\big(s^{-m} t^{-n}\big)\big]\\ &= -(-1)^{j+i} m q^{mn} E_{ji}\big(s^{-m} t^{-n}\big),\end{aligned}$$

$$\omega\big(\big[d_s, E_{ij}\big(s^m t^n\big)\big]\big) = \omega\big(m E_{ij}\big(s^m t^n\big)\big) = (-1)^{j+i} m q^{mn} E_{ji}\big(s^{-m} t^{-n}\big).$$

Hence we get $[\omega(d_s), \omega(E_{ij}(s^m t^n))] = \omega(-[d_s, E_{ij}(s^m t^n)])$. Similarly,

$$\big[\omega(d_t), \omega\big(E_{ij}\big(s^m t^n\big)\big)\big] = \omega\big(-\big[d_t, E_{ij}\big(s^m t^n\big)\big]\big).$$

The other cases are trivial and so the proof is completed. □

Theorem 3.6. *Assume that μ is a real number. Then there exists a contravariant, with respect to $\pi_{X,\mu}$ and ω, hermitian form $(\cdot,\cdot)$ on V so that*

$$\big(\pi_{X,\mu}(a).f, g\big) = \big(f, \pi_{X,\mu}\big(\omega(a)\big).g\big) \tag{3.7}$$

for every $f, g \in V$, $a \in \widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$.

Proof. Since V is a polynomial algebra, it is sufficient to define the form on a pair of monomials in the variables $x_{(m,n)}$, $(m,n) \in \mathbb{Z}^2$. Given a monomial

$$f = \prod_{(m,n)\in\mathbb{Z}^2} x_{(m,n)}^{\boldsymbol{A}_{(m,n)}}, \tag{3.8}$$

whose degree in $x_{(m,n)}$ is positive, where $\boldsymbol{A}_{(m,n)} \in \mathbb{Z}_+ \cup \{0\}$ and only finitely many $\boldsymbol{A}_{(m,n)} \neq 0$, denote by $f_{\widehat{(m,n)}}$ the unique monomial such that

$$f = x_{(m,n)} f_{\widehat{(m,n)}}. \tag{3.9}$$

Denote by $\deg f$ the total degree of f.

Now we define a hermitian form (f, g) on V inductively on the degree of f. Since a hermitian form requires $(f, g) = \overline{(g, f)}$, we only need to define (f, g) with $\deg g \leqq \deg f$.

We set

$$(1,1) = 1, \tag{3.10}$$

$$(x_{(m,n)}, 1) = 0, \tag{3.11}$$

$$(x_{(m,n)}, x_{(l,k)}) = \mu a_{(m,n)} \overline{a_{(m,n)}} \delta_{(m,n),(l,k)}. \tag{3.12}$$

Fix a positive integer N and assume that the form is defined for all monomials f, g such that $\deg g, \deg f \leqq N-1$ and satisfies

$$\big(\pi_{X,\mu}\big(E_{21}\big(s^m t^n\big)\big).f, g\big) = \big(f, \pi_{X,\mu}\big(\omega\big(E_{21}\big(s^m t^n\big)\big)\big).g\big) \tag{3.13}$$

with $\deg g \leqq N-2$, $\deg f \leqq N-1$. It is easy to see that (3.13) holds true when $\deg f$, $\deg g \leqslant 1$. Take f with $\deg f = N$, and choose $(m,n) \in \mathbb{Z}^2$ such that the degree of f in $x_{(m,n)} \geqslant 1$.

Observe that

$$f = x_{(m,n)} f_{\widehat{(m,n)}} = a_{(m,n)} Q_{(m,n)} f_{\widehat{(m,n)}} - c_{(m,n)} P_{(m,n)} f_{\widehat{(m,n)}}$$

by (2.4). Since $Q_{(m,n)} = \pi_{X,\mu}(E_{21}(s^m t^n))$ this can be written as

$$f = a_{(m,n)} \pi_{X,\mu}\big(E_{21}\big(s^m t^n\big)\big) f_{\widehat{(m,n)}} - c_{(m,n)} P_{(m,n)} f_{\widehat{(m,n)}}. \tag{3.14}$$

Suppose first that $\deg g < N$. Then set

$$(f, g) := a_{(m,n)}\big(f_{\widehat{(m,n)}}, \pi_{X,\mu}\big(\omega\big(E_{21}\big(s^m t^n\big)\big)g\big) - c_{(m,n)}\big(P_{(m,n)} f_{\widehat{(m,n)}}, g\big)\big). \tag{3.15}$$

Note that all terms here are defined by induction and (3.13) holds.

Suppose now that $\deg g = N$. The first term in (3.15) still makes sense as

$$\deg \pi_{X,\mu}\big(\omega\big(E_{21}\big(s^m t^n\big)\big)g\big) < \deg g.$$

On the other hand, $(g, P_{(m,n)} f_{\widehat{(m,n)}})$ is defined by (3.15) as

$$\deg P_{(m,n)} f_{\widehat{(m,n)}} < \deg f_{\widehat{(m,n)}} < N.$$

Then the last term is also defined by applying the same formula to g and $P_{(m,n)} f_{\widehat{(m,n)}}$ and using the fact that the form is hermitian.

We have to show (3.15) is well defined, which means that the right-hand side of (3.15) is independent of the choice of (m, n). Namely, we need to show that if $\boldsymbol{A}_{(m,n)} \geqslant 1$, $\boldsymbol{A}_{(l,k)} \geqslant 1$ and $(m, n) \neq (l, k)$, we have

$$\begin{aligned} &a_{(m,n)}\big(f_{\widehat{(m,n)}}, -q^{mn} e_{12}(-m, -n).g\big) - c_{(m,n)}(P_{(m,n)} f_{\widehat{(m,n)}}, g) \\ &\quad = a_{(l,k)}\big(f_{\widehat{(l,k)}}, -q^{lk} e_{12}(-l, -k).g\big) - c_{(l,k)}(P_{(l,k)} f_{\widehat{(l,k)}}, g). \end{aligned} \tag{3.16}$$

Since

$$f_{\widehat{(m,n)}} = a_{(l,k)} e_{21}(l, k).f_{\widehat{(m,n)}\widehat{(l,k)}} - c_{(l,k)} P_{(l,k)} f_{\widehat{(m,n)}\widehat{(l,k)}}, \tag{3.17}$$

substituting to the left-hand side of (3.16), we obtain

$$\begin{aligned} &\text{LHS of (3.16)} \\ &\quad = a_{(m,n)}\big(a_{(l,k)} e_{21}(l, k).f_{\widehat{(m,n)}\widehat{(l,k)}} - c_{(l,k)} P_{(l,k)} f_{\widehat{(m,n)}\widehat{(l,k)}}, -q^{mn} e_{12}(-m, -n).g\big) \\ &\qquad - c_{(m,n)}\big(P_{(m,n)}\big(a_{(l,k)} e_{21}(l, k).f_{\widehat{(m,n)}\widehat{(l,k)}} - c_{(l,k)} P_{(l,k)} f_{\widehat{(m,n)}\widehat{(l,k)}}\big), g\big) \\ &\quad = a_{(m,n)} a_{(l,k)}\big(e_{21}(l, k).f_{\widehat{(m,n)}\widehat{(l,k)}}, -q^{mn} e_{12}(-m, -n).g\big) \\ &\qquad - a_{(m,n)} c_{(l,k)}\big(P_{(l,k)} f_{\widehat{(m,n)}\widehat{(l,k)}}, -q^{mn} e_{12}(-m, -n).g\big) \\ &\qquad - c_{(m,n)} a_{(l,k)}\big(P_{(m,n)} e_{21}(l, k).f_{\widehat{(m,n)}\widehat{(l,k)}}, g\big) + c_{(m,n)} c_{(l,k)}(P_{(m,n)} P_{(l,k)} f_{\widehat{(m,n)}\widehat{(l,k)}}, g) \\ &\quad = a_{(m,n)} a_{(l,k)}\big(f_{\widehat{(m,n)}\widehat{(l,k)}}, (-1)q^{lk} e_{12}(-l, -k).(-1)q^{mn} e_{12}(-m, -n).g\big) \\ &\qquad - a_{(m,n)} c_{(l,k)}\big(e_{21}(m, n) P_{(l,k)} f_{\widehat{(m,n)}\widehat{(l,k)}}, g\big) - c_{(m,n)} a_{(l,k)}\big(P_{(m,n)} e_{21}(l, k).f_{\widehat{(m,n)}\widehat{(l,k)}}, g\big) \\ &\qquad + c_{(m,n)} c_{(l,k)}(P_{(m,n)} P_{(l,k)} f_{\widehat{(m,n)}\widehat{(l,k)}}, g). \end{aligned} \tag{3.18}$$

Exchanging (m, n) and (l, k) in (3.18) and noting that $f_{\widehat{(m,n)}\widehat{(l,k)}} = f_{\widehat{(l,k)}\widehat{(m,n)}}$, we get the right-hand side of (3.16):

$$\begin{aligned}\text{RHS of (3.16)} &= a_{(l,k)}a_{(m,n)}\big(f_{\widehat{(m,n)}\,\widehat{(l,k)}}, q^{mn}e_{12}(-m,-n).q^{lk}e_{12}(-p,-q).g\big)\\ &\quad - a_{(l,k)}c_{(m,n)}\big(e_{21}(l,k)P_{(m,n)}f_{\widehat{(m,n)}\,\widehat{(l,k)}}, g\big)\\ &\quad - c_{(l,k)}a_{(m,n)}\big(P_{(l,k)}e_{21}(m,n).f_{\widehat{(m,n)}\,\widehat{(l,k)}}, g\big)\\ &\quad + c_{(l,k)}c_{(m,n)}(P_{(l,k)}P_{(m,n)}f_{\widehat{(m,n)}\,\widehat{(l,k)}}, g).\end{aligned} \tag{3.19}$$

Since $[e_{12}(-m,-n), e_{12}(-l,-k)] = [P_{(m,n)}, P_{(l,k)}] = 0$, subtracting (3.18) from (3.19) we have

$$\begin{aligned}(3.19) - (3.18) &= -a_{(l,k)}c_{(m,n)}\big([Q_{(l,k)}, P_{(m,n)}].f_{\widehat{(m,n)}\,\widehat{(l,k)}}, g\big)\\ &\quad - c_{(l,k)}a_{(m,n)}\big([P_{(l,k)}, Q_{(m,n)}].f_{\widehat{(m,n)}\,\widehat{(l,k)}}, g\big)\\ &= \delta_{(l,k),(m,n)}(a_{(l,k)}c_{(m,n)} - c_{(l,k)}a_{(m,n)})(f_{\widehat{(m,n)}\widehat{(l,k)}}, g) = 0.\end{aligned}$$

Hence (3.16) holds true and (3.15) is well defined. So we obtained a form on V, and the form satisfies (3.13) for any $f, g \in V$.

Since (3.13) holds and $E_{22}(x)$ is a linear combination of $E_{11}(x)$ and $[E_{12}(x'), E_{21}(x'')]$, in order to prove that the form we defined is contravariant it remains to check

$$\big(\pi_{X,\mu}(d_s).f, g\big) = \big(f, \pi_{X,\mu}\big(\omega(d_s)\big).g\big), \tag{3.20}$$

$$\big(\pi_{X,\mu}(d_t).f, g\big) = \big(f, \pi_{X,\mu}\big(\omega(d_t)\big).g\big), \tag{3.21}$$

$$\big(\pi_{X,\mu}\big(E_{11}\big(s^m t^n\big)\big).f, g\big) = \big(f, \pi_{X,\mu}\big(\omega\big(E_{11}\big(s^m t^n\big)\big)\big).g\big). \tag{3.22}$$

We do this by using induction on the degree of f and g.

First we have

$$\big(\pi_{X,\mu}(d_s).1, 1\big) = \big(1, \pi_{X,\mu}\big(\omega(d_s)\big).1\big) = 0,$$
$$\big(\pi_{X,\mu}(d_s).x_{(m,n)}, 1\big) = 0 = \big(x_{(m,n)}, \pi_{X,\mu}(d_s).1\big),$$
$$\big(\pi_{X,\mu}(d_s).x_{(m,n)}, x_{(l,k)}\big) = m(x_{(m,n)}, x_{(l,k)}) = l(x_{(m,n)}, x_{(l,k)}) = \big(x_{(m,n)}, \pi_{X,\mu}\big(\omega(d_s)\big).x_{(l,k)}\big)$$

for $(m,n), (l,k) \in \mathbb{Z}^2$.

Assuming for any $\deg g \leqq \deg f \leqq N-1$, we have

$$\big(\pi_{X,\mu}(d_s).f, g\big) = \big(f, \pi_{X,\mu}\big(\omega(d_s)\big).g\big).$$

According to (3.14) or (2.4),

$$f = a_{(m,n)}e_{21}(m,n).h - c_{(m,n)}P_{(m,n)}h, \tag{3.23}$$

where $h = f_{\widehat{(m,n)}}$ and $\deg h = N-1$, together with the assumption, then

$$\begin{aligned}
&\big(\pi_{X,\mu}(d_s).f, g\big)\\
&\quad= a_{(m,n)}\big(\big(D_1 e_{21}(m,n)\big).h, g\big) - c_{(m,n)}(D_1 P_{(m,n)}h, g)\\
&\quad= a_{(m,n)}\big(e_{21}(m,n)D_1.h + \big[D_1, e_{21}(m,n)\big].h, g\big) - c_{(m,n)}(P_{(m,n)}h, D_1.g)\\
&\quad= a_{(m,n)}\big(D_1.h, -q^{mn}e_{12}(-m,-n).g\big) + a_{(m,n)}\big(me_{21}(m,n).h, g\big)\\
&\qquad - c_{(m,n)}(P_{(m,n)}.h, D_1.g)\\
&\quad= a_{(m,n)}\big(h, -q^{mn}D_1 e_{12}(-m,-n).g\big) + a_{(m,n)}m\big(h, -q^{(mn)}e_{12}(-m,-n).g\big)\\
&\qquad - c_{(m,n)}(P_{(m,n)}.h, D_1.g)\\
&\quad= a_{(m,n)}\big(h, -q^{mn}\big(e_{12}(-m,-n)D_1 + \big[D_1, e_{12}(-m,-n)\big]\big).g\big)\\
&\qquad + a_{(m,n)}m\big(h, -q^{(mn)}e_{12}(-m,-n).g\big) - c_{(m,n)}(P_{(m,n)}.h, D_1.g)\\
&\quad= a_{(m,n)}\big(h, -q^{mn}e_{12}(-m,-n)D_1.g\big) + a_{(m,n)}\big(h, +mq^{mn}e_{12}(-m,-n).g\big)\\
&\qquad + a_{(m,n)}m\big(h, -q^{(mn)}e_{12}(-m,-n).g\big) - c_{(m,n)}(P_{(m,n)}.h, D_1.g)\\
&\quad= a_{(m,n)}\big(e_{21}(m,n).h, D_1.g\big) - c_{(m,n)}(P_{(m,n)}.h, D_1.g)\\
&\quad= (f, D_1.g) = \big(f, \pi_{X,\mu}(d_s).g\big).
\end{aligned}$$

Therefore (3.20) holds true and so does (3.21).

As for (3.22) we first have

$$\big(\pi_{X,\mu}\big(E_{11}\big(s^l t^k\big)\big).1, 1\big) = -\frac{1}{2}\mu\delta_{(l,k),(0,0)} = \big(1, \pi_{X,\mu}\big(\omega\big(E_{11}\big(s^l t^k\big)\big)\big).1\big),$$
$$\big(\pi_{X,\mu}\big(E_{11}\big(s^l t^k\big)\big).x_{m,n}, 1\big) = 0 = \big(x_{m,n}, \pi_{X,\mu}\big(\omega\big(E_{11}\big(s^l t^k\big)\big)\big).1\big).$$

Secondly,

$$\begin{aligned}
&\big(\pi_{X,\mu}\big(E_{11}\big(s^l t^k\big)\big).x_{(m_1,n_1)}, x_{(m_2,n_2)}\big)\\
&\quad= -q^{ln_1}a_{(m_1,n_1)}d_{(m_1+l,n_1+k)}(x_{(m_1+l,n_1+k)}, x_{(m_2,n_2)}) - \frac{1}{2}\mu\delta_{(l,k),(0,0)}(x_{(m_1,n_1)}, x_{(m_2,n_2)})\\
&\quad= -q^{ln_1}a_{(m_1,n_1)}\overline{a_{(m_2,n_2)}}\mu\delta_{(m_1+l,n_1+k),(m_2,n_2)} - \frac{1}{2}|a_{(m_2,n_2)}|^2\mu^2\delta_{(l,k),(0,0)}\delta_{(m_1,n_1),(m_2,n_2)}
\end{aligned}$$

and

$$\begin{aligned}
&\big(x_{(m_1,n_1)}, \pi_{X,\mu}\big(\omega\big(E_{11}\big(s^l t^k\big)\big)\big).x_{(m_2,n_2)}\big)\\
&\quad= -\overline{q^{lk-ln_2}}\,\overline{a_{(m_2,n_2)}}\,\overline{d_{(m_2-l,n_2-k)}}(x_{(m_1,n_1)}, x_{(m_2-l,n_2-k)})\\
&\qquad - \overline{q^{lk}}\frac{1}{2}\mu\delta_{(-l,-k),(0,0)}(x_{(m_1,n_1)}, x_{(m_2,n_2)})\\
&\quad= -q^{ln_1}a_{(m_1,n_1)}\overline{a_{(m_2,n_2)}}\mu\delta_{(m_1+l,n_1+k),(m_2,n_2)} - \frac{1}{2}|a_{(m_2,n_2)}|^2\mu^2\delta_{(l,k),(0,0)}\delta_{(m_1,n_1),(m_2,n_2)}
\end{aligned}$$

yields

$$\big(\pi_{X,\mu}\big(E_{11}\big(s^l t^k\big)\big).x_{(m_1,n_1)}, x_{(m_2,n_2)}\big) = \big(x_{(m_1,n_1)}, \pi_{X,\mu}\big(\omega\big(E_{11}\big(s^l t^k\big)\big)\big).x_{(m_2,n_2)}\big).$$

Now assume for any $\deg g \leqq \deg f \leqq N-1$, we have

$$\big(\pi_{X,\mu}\big(E_{11}\big(s^m t^n\big)\big).f, g\big) = \big(f, \pi_{X,\mu}\big(\omega\big(E_{11}\big(s^m t^n\big)\big)\big).g\big).$$

According to (3.23), we have

$$\begin{aligned}
&\big(\pi_{X,\mu}\big(E_{11}\big(s^l t^k\big)\big).f, g\big)\\
&\quad = a_{(m,n)}\big(e_{11}(l,k)e_{21}(m,n).h, g\big) - c_{(m,n)}\big(e_{11}(l,k)P_{(m,n)}h, g\big)\\
&\quad = a_{(m,n)}\big(e_{21}(m,n)e_{11}(l,k).h, g\big) + a_{(m,n)}\big(\big[e_{11}(l,k), e_{21}(m,n)\big].h, g\big)\\
&\qquad - c_{(m,n)}\big(P_{(m,n)}.h, q^{lk}e_{11}(-l,-k).g\big)\\
&\quad = a_{(m,n)}\big(e_{11}(l,k).h, q^{mn}e_{12}(-m,-n)g\big) + a_{(m,n)}\big(-q^{nl}e_{21}(m+l,n+k).h, g\big)\\
&\qquad - c_{(m,n)}\big(P_{(m,n)}.h, q^{lk}e_{11}(-l,-k).g\big)\\
&\quad = a_{(m,n)}\big(h, q^{mn}q^{lk}e_{11}(-l,-k)e_{12}(-m,-n).g\big)\\
&\qquad + a_{(m,n)}\big(h, -\bar{q}^{nl}q^{(m+l)(n+k)}e_{12}(-m-l,-n-k).g\big)\\
&\qquad - c_{(m,n)}\big(P_{(m,n)}.h, q^{lk}e_{11}(-l,-k).g\big)\\
&\quad = a_{(m,n)}\big(h, q^{mn+lk}e_{12}(-m,-n)e_{11}(-l,-k).g\big)\\
&\qquad + a_{(m,n)}\big(h, q^{mn+lk}\big[e_{11}(-l,-k), e_{12}(-m,-n)\big].g\big)\\
&\qquad + a_{(m,n)}\big(h, -q^{-nl}q^{(m+l)(n+k)}e_{12}(-m-l,-n-k).g\big)\\
&\qquad - c_{(m,n)}\big(P_{(m,n)}.h, q^{lk}e_{11}(-l,-k)g\big)\\
&\quad = a_{(m,n)}\big(e_{21}(m,n).h, q^{lk}e_{11}(-l,-k).g\big)\\
&\qquad + a_{(m,n)}\big(h, q^{mn+lk}q^{km}e_{12}(-m-l,-n-k).g\big)\\
&\qquad + a_{(m,n)}\big(h, -q^{mn+lk+mk}e_{12}(-m-l,-n-k).g\big) - c_{(m,n)}\big(P_{(m,n)}h, q^{lk}e_{11}(-l,-k)g\big)\\
&\quad = a_{(m,n)}\big(e_{21}(m,n).h, q^{lk}e_{11}(-l,-k).g\big) - c_{(m,n)}\big(P_{(m,n)}.h, q^{lk}e_{11}(-l,-k)g\big)\\
&\quad = \big(f, q^{lk}e_{11}(-l,-k)g\big) = \big(f, \pi_{X,\mu}\big(\omega\big(E_{11}\big(s^l t^k\big)\big)\big).g\big).
\end{aligned}$$

Hence (3.22) is also true and the form is indeed a contravariant hermitian form on V. □

4. Conditions for unitarity

It is important to have the contravariant hermitian form on V to be positive definite so that the underlying module is unitarizable.

From the definition of our contravariant form in Theorem 3.6, we see that

$$\big(x^2_{(m,n)}, 1\big) = a_{(m,n)}\big(x_{(m,n)}, q^{mn}e_{12}(-m,-n).1\big) - c_{(m,n)}(P_{(m,n)}.x_{(m,n)}, 1) = -a_{(m,n)}c_{(m,n)}.$$

Thus the hermitian form on different degrees can be nonzero. It is therefore difficult to determine when the hermitian form is positive definite. For this we use another base of V as in [15] rather than the natural monomial base of V. We further work out the necessary and sufficient conditions for the unitarity. We shall follow the approach in [15].

Definition 4.1. If the hermitian form is positive definite, $\pi_{X,\mu}$ is said to be unitarizable (w.r.t. ω).

Here we simplify $\pi_{X,\mu}(E_{ij}(r)).v$ as $E_{ij}(r).v$, for any $v \in V$, $r \in \mathbb{C}_q$.

Lemma 4.2. *The elements* $E_{21}(r_1)E_{21}(r_2)\dots E_{21}(r_k).1$, *where* $k \in \mathbb{Z}_+ \cup \{0\}$, $r_i = s^{m_i}t^{n_i}$, $i = 1,\dots,n$, $m_i, n_i \in \mathbb{Z}$ *forms a basis for* V. *Moreover, if* f *is a monomial of degree* N, *then* f *can be written as a linear combination of* $E_{21}(r_1)E_{21}(r_2)\dots E_{21}(r_k).1$ *with* $k \leqslant N$.

Proof. Prove by induction on the degree of f. It is obvious true for $\deg f = 0$, i.e. $f = 1$.

If $\deg f = 1$, $f = x_{(m,n)} = E_{21}(s^m t^n).1$. Now we assume that f is a monomial whose degree in $x_{(m,n)}$ is positive, then

$$f = x_{(m,n)} f_{\widehat{(m,n)}} = a_{(m,n)} E_{21}\left(s^m t^n\right) f_{\widehat{(m,n)}} - c_{(m,n)} P_{(m,n)} f_{\widehat{(m,n)}},$$

here $\deg f_{\widehat{(m,n)}} = N - 1$. The induction proves our claim.

Hence $E_{21}(r_1)E_{21}(r_2)\dots E_{21}(r_k).1$, $k \in \mathbb{R}_+ \cup \{0\}$, $r_i = s^{m_i}t^{n_i}$, $i = 1,\dots,n$, $m_i, n_i \in \mathbb{Z}$, spans V over $\mathbb{C}$. Note that the elements $E_{21}(r_1)E_{21}(r_2)\dots E_{21}(r_k).1$ are independent of the order in which the operators are applied.

Since the leading term of $E_{21}(r_1)E_{21}(r_2)\dots E_{21}(r_k).1$ is $\prod_{i=1}^{k} x_{(m_i,n_i)}$, we know that

$$E_{21}(r_1)E_{21}(r_2)\dots E_{21}(r_k).1$$

form a base for V with k ranges in $\{0,1,2,3,\dots\}$ and r_i ranges in $\{s^m t^n\colon m,n \in \mathbb{Z}\}$. □

It immediately follows from Lemma 4.2 that V is generated as a $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$-module by 1, and

$$\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}.1 = -\frac{1}{2}\mu\kappa(a_1).1 + \frac{1}{2}\mu\kappa(a_3).1 \tag{4.3}$$

for any $a_1, a_2, a_3 \in \mathbb{C}_q$, here $\kappa(a)$ is defined as in (1.4). The subalgebra

$$\mathcal{B} = \left\{\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}\colon a_1, a_2, a_3 \in \mathbb{C}_q\right\} \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t \oplus \mathbb{C}d_s \oplus \mathbb{C}d_t$$

is a Borel subalgebra of $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$ in the sense of (0.1).

Hence we have

Proposition 4.4. V *is a highest weight module of highest weight* $\lambda : \mathcal{B} \to \mathbb{C}$, *where* λ *is defined as follows*

$$\lambda\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} = -\frac{1}{2}\mu\kappa(a_1) + \frac{1}{2}\mu\kappa(a_3),$$
$$\lambda(c_s) = \lambda(c_t) = \lambda(d_s) = \lambda(d_t) = 0$$

and 1 *is the highest weight vector.*

Let $i \in \mathbb{N}$, $\gamma = (\gamma_1, \dots, \gamma_s)$ be the s-partition of i. Denote by $Par_s(i)$ the set of all partitions $\gamma = (\gamma_1, \dots, \gamma_s)$ of i with s parts.

Given $\gamma \in Par_s(N)$, we say that $\pi_1' \times \pi_2' \in S_N \times S_N$ is equivalent to $\pi_1 \times \pi_2 \in S_N \times S_N$, here S_N is the permutation group of N letters, if for all $z_1, \dots, z_N, w_1, \dots, w_N \in \mathbb{C}_q$,

$$\kappa(z_{\pi_1'(1)} w_{\pi_2'(1)} \dots z_{\pi_1'(\gamma_1)} w_{\pi_2'(\gamma_1)}) \dots \kappa(z_{\pi_1'(\gamma_1+\dots+\gamma_{s-1}+1)} w_{\pi_2'(\gamma_1+\dots+\gamma_{s-1}+1)} \dots z_{\pi_1'(N)} w_{\pi_2'(N)})$$

can be obtained from the analogous expression for $\pi_1 \times \pi_2$ by a permutation of the s factors $\kappa(\cdots)$ and/or by cyclic permutation of the variables (e.g., $\kappa(z_1w_1z_2w_2z_3w_3) = \kappa(z_3w_3z_1w_1z_2w_2)$).

The set of equivalence classes is denoted by $[S_N \times S_N](\gamma)$. The following result was due to Jakobsen–Kac [15].

Lemma 4.5. *Let* $z_1, z_2, \dots, z_N, w_1, w_2, \dots, w_N \in \mathbb{C}_q[s^{\pm 1}, t^{\pm 1}]$

$$\begin{aligned}
&\begin{pmatrix} 0 & z_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & z_2 \\ 0 & 0 \end{pmatrix}\cdots\begin{pmatrix} 0 & z_N \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w_2 & 0 \end{pmatrix}\cdots\begin{pmatrix} 0 & 0 \\ w_N & 0 \end{pmatrix}.1 \\
&= \sum_{s=1}^{N} \sum_{\gamma \in Par_s(N)} \sum_{[\pi_1 \times \pi_2] \in (S_N \times S_N)(\gamma)} (-1)^{\gamma_1 - 1}(-\mu)\kappa(z_{\pi_1(1)} w_{\pi_2(1)} \dots z_{\pi_1(\gamma_1)} w_{\pi_2(\gamma_1)}) \\
&\quad .(-1)^{\gamma_2 - 1}(-\mu)\kappa(z_{\pi_1(\gamma_1+1)} w_{\pi_2(\gamma_1+1)} \dots z_{\pi_1(\gamma_2)} w_{\pi_2(\gamma_2)}) \\
&\quad \dots (-1)^{\gamma_s - 1}(-\mu)\kappa(z_{\pi_1(\gamma_1+\dots+\gamma_{s-1}+1)} w_{\pi_2(\gamma_1+\dots+\gamma_{s-1}+1)} \dots z_{\pi_1(N)} w_{\pi_2(N)}).1. \qquad (4.6)
\end{aligned}$$

We shall call k the level of the element $E_{21}(r_1) \dots E_{21}(r_k).1 \in V$, where $k \in \mathbb{Z}_+ \cup \{0\}$, $r_i = s^{m_i} t^{n_i}$, $i = 1, \dots, n$, $m_i, n_i \in \mathbb{Z}$.

Proposition 4.7.

(i) *The hermitian form on different level is* 0.
(ii) *Let h be an element of level n. Then (h, h) is a polynomial in μ with the leading term $c(h)\mu^n$ for some constant $c(h) > 0$.*

Proof. Since

$$\begin{aligned}
&\left(1, \begin{pmatrix} 0 & 0 \\ z_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ z_2 & 0 \end{pmatrix}\cdots\begin{pmatrix} 0 & 0 \\ z_r & 0 \end{pmatrix}.1\right) \quad r \geqq 1 \\
&= \left(\begin{pmatrix} 0 & \overline{z_1} \\ 0 & 0 \end{pmatrix}.1, \begin{pmatrix} 0 & 0 \\ z_2 & 0 \end{pmatrix}\cdots\begin{pmatrix} 0 & 0 \\ z_r & 0 \end{pmatrix}.1\right) = 0
\end{aligned}$$

then WLOG, assume $s > t$,

$$\left(\begin{pmatrix}0&0\\z_1&0\end{pmatrix}\begin{pmatrix}0&0\\z_2&0\end{pmatrix}\cdots\begin{pmatrix}0&0\\z_s&0\end{pmatrix}.1,\begin{pmatrix}0&0\\r_1&0\end{pmatrix}\begin{pmatrix}0&0\\r_2&0\end{pmatrix}\cdots\begin{pmatrix}0&0\\r_t&0\end{pmatrix}.1\right)$$
$$=\left(1,(-1)^s\begin{pmatrix}0&\overline{z_s}\\0&0\end{pmatrix}\begin{pmatrix}0&\overline{z_{s-1}}\\0&0\end{pmatrix}\cdots\begin{pmatrix}0&\overline{z_1}\\0&0\end{pmatrix}\begin{pmatrix}0&0\\r_1&0\end{pmatrix}\begin{pmatrix}0&0\\r_2&0\end{pmatrix}\cdots\begin{pmatrix}0&0\\r_t&0\end{pmatrix}.1\right)$$
$$=\left(1,(-1)^s\begin{pmatrix}0&\overline{z_s}\\0&0\end{pmatrix}\begin{pmatrix}0&\overline{z_{s-1}}\\0&0\end{pmatrix}\cdots\begin{pmatrix}0&\overline{z_{s-r}}\\0&0\end{pmatrix}c.1\right)$$
(according to Lemma 4.5 here $c\in\mathbb{C}$)
$$=0.$$

This proves (i).

Let $h=E_{21}(\overline{z_N})E_{21}(\overline{z_{N-1}})\ldots E_{21}(\overline{z_1})$, $h'=E_{21}(w_1)E_{21}(w_2)\ldots E_{21}(w_N)$, where $\overline{z_i}=s^{m_i}t^{n_i}$, $w_i=s^{l_i}t^{r_i}$, then according to Lemma 4.5,

$$\overline{(h,h')}=(-1)^N.\sum_{s=1}^{N}\sum_{\gamma\in Par_s(N)}\sum_{[\pi_1\times\pi_2]\in(S_N\times S_N)(\gamma)}(-1)^{\gamma_1-1}(-\mu)$$
$$\times\kappa(z_{\pi_1(1)}w_{\pi_2(1)}\ldots z_{\pi_1(\gamma_1)}w_{\pi_2(\gamma_1)})$$
$$.(-1)^{\gamma_2-1}(-\mu)\kappa(z_{\pi_1(\gamma_1+1)}w_{\pi_2(\gamma_1+1)}\ldots z_{\pi_1(\gamma_2)}w_{\pi_2(\gamma_2)}).$$
$$\ldots(-1)^{\gamma_s-1}(-\mu)\kappa(z_{\pi_1(\gamma_1+\cdots+\gamma_{s-1}+1)}w_{\pi_2(\gamma_1+\cdots+\gamma_{s-1}+1)}\ldots z_{\pi_1(N)}w_{\pi_2(N)})$$

is a polynomial P of μ, whose coefficients depends on h and h'.

If $\deg P=N$, then there exists at least a $\pi\in S_N$, such that $\kappa(z_iw_{\pi(i)})\neq 0$, that is $\kappa(t^{-n_i}s^{-m_i}s^{l_{\pi(i)}}t^{r_{\pi(i)}})=q^{(l_{\pi(i)}-m_i)(-n_i)}\delta_{(l_{\pi(i)}-m_i,r_{\pi(i)}-n_i),(0,0)}\neq 0$, hence $\overline{z_i}=w_{\pi(i)}$ for any $1\leqslant i\leqslant N$. So if $h=h'$, the coefficient of μ^n is the number of such elements π, otherwise it equals zero. Hence with Lemma 4.2, we proved (ii). □

Next we prove the unitarity of the hermitian form.

Theorem 4.8. *$(\pi_{X,\mu},V)$ is unitarizable if and only if $\mu>0$.*

Proof. Since

$$\left(\begin{pmatrix}0&0\\s^mt^n&0\end{pmatrix}.1,\begin{pmatrix}0&0\\s^lt^k&0\end{pmatrix}.1\right)=\mu\delta_{m-l,0}\delta_{n-k,0},$$

for any $m,n\in N$, then if $(\pi_{X,\mu},V)$ is unitarizable, $\mu>0$.

Let $w_i=s^{m_i}t^{n_i}$, $z_j=t^{-l_j}s^{-k_j}$, for $i,j=1,\ldots,N$, then

$$\kappa(z_1w_1z_2w_2\ldots z_rw_r)=\kappa\left(t^{-l_1}s^{-k_1}s^{m_1}t^{n_1}t^{-l_2}s^{-k_2}s^{m_2}t^{n_2}\ldots t^{-l_r}s^{-k_r}s^{m_r}t^{n_r}\right)$$
$$=q^{\alpha}\delta_{-k_1+m_1-k_2+m_2-\cdots-k_r+m_r,0}\delta_{-l_1+n_1-l_2+n_2-\cdots-l_r+n_r,0},$$

where

$$\alpha=(-k_1+m_1)(-l_1)+(-k_2+m_2)(-l_1+n_1-l_2)+\cdots$$
$$+(-k_r+m_r)(-l_1+n_1-l_2+n_2-\cdots-l_{r-1}+n_{r-1}-l_r).$$

Consider the linear transformation $T_{a,b}$ of $\mathbb{C}_q$ determined by

$$T_{a,b}\big(s^{c_1}t^{d_1}s^{c_2}t^{d_2}\dots s^{c_k}t^{d_k}\big)=s^{c_1+a}t^{d_1+b}s^{c_2+a}t^{d_2+b}\dots s^{c_k+a}t^{d_k+b}$$

$(a,b\in\mathbb{Z})$. Extend this operator to a linear operator $\widetilde{T_{a,b}}$ on V by

$$\begin{aligned}\widetilde{T_{a,b}}&\left[\begin{pmatrix}0&0\\r_1&0\end{pmatrix}\begin{pmatrix}0&0\\r_2&0\end{pmatrix}\cdots\begin{pmatrix}0&0\\r_t&0\end{pmatrix}.1\right]\\&=\begin{pmatrix}0&0\\T_{a,b}r_1&0\end{pmatrix}\begin{pmatrix}0&0\\T_{a,b}r_2&0\end{pmatrix}\cdots\begin{pmatrix}0&0\\T_{a,b}r_t&0\end{pmatrix}.1,\end{aligned}$$

for $r_1,\dots,r_t\in\mathbb{C}_q[s^{\pm1},t^{\pm1}]$.

Let $\widetilde{z_i}=T_{-a,-b}(z_i)$, $\widetilde{w_j}=T_{a,b}(w_j)$, then

$$\begin{aligned}&\kappa(\widetilde{z_1}\widetilde{w_1}\widetilde{z_2}\widetilde{w_2}\dots\widetilde{z_r}\widetilde{w_r})\\&\quad=\kappa\big(t^{-l_1-b}s^{-k_1-a}s^{m_1+a}t^{n_1+b}t^{-l_2-b}s^{-k_2-a}s^{m_2+a}t^{n_2+b}\dots t^{-l_r-b}s^{-k_r-a}s^{m_r+a}t^{n_r+b}\big)\\&\quad=q^{\widetilde{\alpha}}\delta_{-k_1+m_1-k_2+m_2-\cdots-k_r+m_r,0}\delta_{-l_1+n_1-l_2+n_2-\cdots-l_r+n_r,0},\end{aligned}$$

where

$$\begin{aligned}\widetilde{\alpha}&=(-k_1-a+m_1+a)(-l_1-b)+(-k_2-a+m_2+a)(-l_1-b+n_1+b-l_2-b)+\cdots\\&\quad+(-k_r-a+m_r+a)\\&\quad\times(-l_1-b+n_1+b-l_2-b+n_2+b-\cdots-l_{r-1}-b+n_{r-1}+b-l_r-b)\\&=(-k_1+m_1)(-l_1)+(-k_2+m_2)(-l_1+n_1-l_2)+\cdots\\&\quad+(-k_r+m_r)(-l_1+n_1-l_2+n_2-\cdots-l_{r-1}+n_{r-1}-l_r)\\&\quad-b(-k_1+m_1-k_2+m_2-\cdots-k_r+m_r),\end{aligned}$$

so $\kappa(\widetilde{z_1}\widetilde{w_1}\widetilde{z_2}\widetilde{w_2}\dots\widetilde{z_r}\widetilde{w_r})=\kappa(z_1w_1z_2w_2\dots z_rw_r)$. It then follows from Lemma 4.2 that $\widetilde{T_{a,b}}$ preserves the hermitian form on V.

We need to prove positivity at all levels as the hermitian form on different levels are zero.

Since $\widetilde{T_{a,b}}$ preserves the hermitian form on V, we may then assume that h_r in level r only involves elements $s^{m_i}t^{n_i}$ with $m_i\geqq 0$, $n_i\geqq 0$. Denote

$$\begin{aligned}L_r^+(M,N)=Span\Bigg\{&\begin{pmatrix}0&0\\s^{m_1}t^{n_1}&0\end{pmatrix}\begin{pmatrix}0&0\\s^{m_2}t^{n_2}&0\end{pmatrix}\cdots\begin{pmatrix}0&0\\s^{m_r}t^{n_r}&0\end{pmatrix}.1\;\Bigg|\;m_i\geqq 0,\ n_i\geqq 0,\\&\sum_{i=1}^r m_i\leqq M,\ \sum_{i=1}^r n_i\leqq N\Bigg\}.\end{aligned}$$

From the above discussions, we know that the hermitian form restricted to every level should be positive definite for μ big enough. Assume that the form is not positive definite for some (possible all) $\mu>0$. Let s_0 be the lowest level at which there is nonunitarity. It is clear that

$s_0 > 1$. So there exist M, N such that the form restricted to $L_{s_0}^+(M,N)$ is not positive definite. Following (4.7), the form on $L_{s_0}^+(M,N)$ varies smoothly with μ, then we can find a μ_0 (we can think it is the first place going from ∞ towards 0) at which the form is not positive definite, while for all $\mu > \mu_0$, the form is positive definite. We write $(\cdot,\cdot)_\mu$ to be the hermitian form at μ.

Claim. *The radical of the form is nontrivial at μ_0.*

At first for all $h' \in L_{s_0}^+(M,N)$, $(h',h')_{\mu_0} \geqslant 0$. Otherwise, there exists $h \in L_{s_0}^+(M,N)$ such that $(h,h)_{\mu_0} < 0$. From (4.7), the form varies smoothly with μ, and $(h,h)_\mu > 0$ if $\mu \to \infty$, then there exist $\mu' > \mu_0$ such that $(h,h)_{\mu'} = 0$, this contradicts with the fact that for all $\mu > \mu_0$, the form is positive definite.

Since the form is positive semi-definite but not positive definite at μ_0, the radical of the form must be nontrivial. Thus,

$$\exists 0 \neq \widetilde{h} \in L_{s_0}^+(M,N),\ \forall h \in L_{s_0}^+(M,N) \quad \text{such that} \quad (\widetilde{h}, h)_{\mu_0} = 0.$$

Let h_{s_0-1} be an arbitrary element of $L_{s_0-1}^+(M,N)$, and let $c \in \mathbb{C}$, then

$$\left(\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\widetilde{h}, h_{s_0-1}\right)_{\mu_0} = 0.$$

From the assumption of s_0, we have $\left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)\widetilde{h} = 0$, for any $c \in \mathbb{C}$. Replacing $\widetilde{h}$ by $\widetilde{T_{-m,-n}}(\widetilde{h})$ if necessary, we can write

$$\widetilde{h} = \sum_{i=1}^{s_0} a_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^i x_i.1,$$

where $x_i = \sum \prod_{j=1}^{s_0-i} \left(\begin{smallmatrix} 0 & 0 \\ v_{i,j} & 0 \end{smallmatrix}\right)$ (here is the finite sum), and each $v_{i,j}$ is the form of $s^l t^k$ (here l, k cannot both be 0).

Let i_0 be the smallest i, $1 \leqslant i \leqslant s_0$, such that $a_{i_0} \neq 0$. It follows that

$$\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\widetilde{h} = \beta a_{i_0} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{i_0-1} x_{i_0}.1 + R = 0,$$

where R contains a power of $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ greater than $i_0 - 1$. Observe that

$$\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{i_0} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{i_0}\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} + i_0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{i_0-1}\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$$
$$+ (-2c)\frac{i_0(i_0-1)}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{i_0-1}$$

(this can be easily proved by induction on i_0).

Since

$$\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} x_{i_0}.1 = x_{i_0} \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}.1 + (-2c)(s_0 - i_0)x_{i_0},$$

we have

$$\beta = c\big(-i_0\mu_0 + i_0(-2)(s_0 - i_0)\big) + (-c)i_0(i_0 - 1) = ci_0\big(-\mu_0 - (s_0 - i_0) - (s_0 - 1)\big).$$

Since $s_0 \geqq i_0 \geqq 1$ and $\mu_0 > 0$, $\beta \neq 0$ which contradicts with $\left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)\widetilde{h} = 0$.
So for any $\mu > 0$, the hermitian form is positive definite. □

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References

[1] B.N. Allison, S. Azam, S. Berman, Y. Gao, A. Pianzola, Extended affine Lie algebras and their root systems, Mem. Amer. Math. Soc. 126 (605) (1997).
[2] B.N. Allison, S. Berman, Y. Gao, A. Pianzola, A characterization of affine Kac–Moody Lie algebras, Comm. Math. Phys. 185 (1997) 671–688.
[3] S. Berman, Y. Gao, Y. Krylyuk, Quantum tori and the structure of elliptic quasi-simple Lie algebras, J. Funct. Anal. 135 (1996) 339–389.
[4] S. Berman, Y. Gao, S. Tan, A unified view of some vertex operator constructions, Israel J. Math. 134 (2003) 29–60.
[5] S. Berman, J. Szmigielski, Principal realization for extended affine Lie algebra of type sl_2 with coordinates in a simple quantum torus with two variables, in: Contemp. Math., vol. 248, 1999, pp. 39–67.
[6] S. Eswara Rao, Unitary modules for EALAs coordinatized by a quantum torus, Comm. Algebra 31 (2003) 2245–2256.
[7] P. Etingof, I. Frenkel, A. Kirillov, Lectures on Representation Theory and Knizhnik–Zamolodchikov Equations, Math. Surveys Monogr., vol. 58, 1998.
[8] B. Feigin, E. Frenkel, Affine Kac–Moody algebras and semi-infinite flag manifolds, Comm. Math. Phys. 128 (1990) 161–189.
[9] Y. Gao, Representations of extended affine Lie algebras coordinatized by certain quantum tori, Compos. Math. 123 (2000) 1–25.
[10] Y. Gao, Vertex operators arising from the homogeneous realization for $\widehat{gl}_N$, Comm. Math. Phys. 211 (2000) 745–777.
[11] Y. Gao, Fermionic and bosonic representations of the extended affine Lie algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$, Canad. Math. Bull. 45 (2002) 623–633.
[12] M. Golenishcheva-Kutuzova, V.G. Kac, Γ-conformal algebras, J. Math. Phys. 39 (1998) 2290–2305.
[13] R. Høegh-Krohn, B. Torresani, Classification and construction of quasi-simple Lie algebras, J. Funct. Anal. 89 (1990) 106–136.
[14] H.P. Jakobsen, V.G. Kac, A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras, in: Lecture Notes in Phys., vol. 226, 1985, pp. 1–20.
[15] H.P. Jakobsen, V.G. Kac, A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras. II, J. Funct. Anal. 82 (1989) 69–90.
[16] Y.I. Manin, Topics in Noncommutative Geometry, Princeton Univ. Press, 1991.
[17] K. Saito, Extended affine root systems 1 (Coxeter transformations), Publ. Res. Inst. Math. Sci. 21 (1985) 75–179.
[18] M. Varagnolo, E. Vasserot, Double-loop algebras and the Fock space, Invent. Math. 133 (1998) 133–159.
[19] M. Wakimoto, Extended affine Lie algebras and a certain series of Hermitian representations, preprint, 1985.
[20] M. Wakimoto, Representations of the affine Lie algebra $A_1^{(1)}$, Comm. Math. Phys. 104 (1986) 605–609.