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Reciprocity Theorems for Dedekind Sums and Generalizations

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1. INTRODUCTION

Let

$$((x)) = x - [x] - \tfrac{1}{2}, \qquad \text{if } x \text{ is not an integer},$$
$$= 0, \qquad \text{otherwise}.$$

If h and k are integers, the classical Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{j \pmod{k}} ((hj/k))((j/k)).$$

The most fundamental result in the theory of Dedekind sums is the reciprocity theorem. If $(h, k) = 1$ and h and k are positive, then

$$s(h, k) + s(k, h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk}\right). \tag{1.1}$$

For several proofs of (1.1), see [28].

Historically, Dedekind sums first arose in the transformation formulas of $\log \eta(z)$, where $\eta(z)$ denotes the Dedekind eta-function. In the past couple decades, several generalizations of Dedekind sums have been defined. Many of these arise in the transformation formulas of functions similar to $\log \eta(z)$. These generalizations of $s(h, k)$ possess reciprocity theorems as well. In those instances where the generalized Dedekind sums appear in transformation formulas, the latter can be used to derive the reciprocity theorems.

In this paper we consider generalizations of ordinary Dedekind sums

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that involve the first Bernoulli function $((x))$. There are many generalizations of Dedekind sums that involve higher order Bernoulli functions, and the reader should consult [28] for references. Our goal is to prove reciprocity theorems, or generalizations thereof, for various types of Dedekind sums. Our methods, which are all analytic, are of three types. The first method uses contour integration and was first employed by Rademacher [24] to prove (1.1). Iseki [20] and Grosswald [17] have used essentially the same method to prove (1.1). Hardy [19] employed a different technique in contour integration to establish (1.1); we have commented on that method in [7]. The second method used here was also invented by Rademacher [25] to prove (1.1), and uses Riemann–Stieltjes integrals. The third method uses the Poisson summation formula or the periodic Poisson summation formula which has been developed by Schoenfeld and the author [8]. A proof of (1.1) using the ordinary Poisson summation formula has been given by the author [5]. An advantage of the Poisson or periodic Poisson summation formula is that it requires little prior intuition concerning the shape of the reciprocity theorem. The calculation of the integrals involved presents the only possible impediment.

Our first goal is to give two new, short proofs of the three-term relation for ordinary Dedekind sums first established by Rademacher [26].

In the next section, we consider Dedekind–Rademacher sums, first defined in complete generality by Rademacher [27]. We first give what we consider to be the shortest proof of the reciprocity theorem for Dedekind–Rademacher sums. We then prove a three-term relation for Dedekind–Rademacher sums that generalizes a result of Carlitz [14].

Next, we consider some sums similar to those defined by Carlitz [9]. Essentially, the same sums have also been recently studied by Zagier [29]. These sums involve the product of several Bernoulli functions. We prove a reciprocity theorem for such sums that is simpler than that of Carlitz.

In the sixth section, we consider modified Dedekind sums. The sums involve roots of unity and first arose in a large class of transformation formulas found by the author [4]. We prove a reciprocity theorem for modified Dedekind sums.

In the last two sections, we consider two further generalizations of Dedekind sums. These involve periodic coefficients and periodic Bernoulli functions defined by Schoenfeld and the author [8]. Three-term relations and reciprocity theorems are established for these sums. In the case when the periodic coefficient is a primitive character and the periodic Bernoulli function is a generalized Bernoulli function, these Dedekind sums first arose in transformation formulas of character analogs of $\log \eta(z)$ [3, 6].

2. Preliminary Results

First, recall the ordinary Poisson summation formula. If f is of bounded variation on $[\alpha, \beta]$, then

$$\tfrac{1}{2} \sum_{n=\alpha}^{\beta}{}' \{f(n+0)+f(n-0)\} = \int_\alpha^\beta f(x)\,dx + 2\sum_{n=1}^{\infty} \int_\alpha^\beta f(x)\cos(2\pi n x)\,dx, \tag{2.1}$$

where the prime on the summation sign at the left indicates that if $n = \alpha$ or $n = \beta$, only $f(\alpha + 0)$ or $f(\beta - 0)$, respectively, is counted.

In Section 7 we shall need the periodic Poisson summation formula and some results on "periodic Bernoulli numbers" and "periodic Bernoulli functions." All results quoted below are proved in [8].

Let $A = \{a_n\}$, $-\infty < n < \infty$, be a sequence of complex numbers with period k, i.e., $a_n = a_{n+k}$ for every integer n. Define the complementary sequence $B = \{b_n\}$, $-\infty < n < \infty$, by

$$b_n = (1/k) \sum_{j=0}^{k-1} a_j e^{-2\pi i j n/k}. \tag{2.2}$$

It is easily shown that (2.2) holds if and only if

$$a_n = \sum_{j=0}^{k-1} b_j e^{2\pi i j n/k}, \qquad -\infty < n < \infty. \tag{2.3}$$

We now state the periodic Poisson summation formula. If f is of bounded variation on $[\alpha, \beta]$,

$$\tfrac{1}{2} \sum_{n=\alpha}^{\beta}{}' a_n\{f(n+0)+f(n-0)\}$$

$$= b_0 \int_\alpha^\beta f(x)\,dx + \sum_{n=1}^{\infty} \int_\alpha^\beta (b_n e^{2\pi i n x/k} + b_{-n} e^{-2\pi i n x/k}) f(x)\,dx. \tag{2.4}$$

The periodic Bernoulli numbers $B_n(A)$, $0 \leqslant n < \infty$, and periodic

Bernoulli functions $\mathscr{B}_n(x, A)$, $0 \leqslant n < \infty$, are defined recursively as follows. Let

$$\mathscr{B}_0(x, A) = B_0(A) = (1/k) \sum_{j=0}^{k-1} a_j ,$$

$$B_1(A) = (1/k) \sum_{j=0}^{k-1} (j - \tfrac{1}{2}k)\, a_j , \tag{2.5}$$

and for $x \geqslant 0$,

$$\mathscr{B}_1(x, A) = B_0(A)\, x - B_1(A) - \sum_{0 \leqslant j \leqslant x}{}' a_j , \tag{2.6}$$

where the prime indicates that if $j = x$, the last term a_x is to be halved. For $n \geqslant 2$ and $x \geqslant 0$, let

$$\mathscr{B}_n(x, A) = n \int_0^x \mathscr{B}_{n-1}(u, A)\, du + (-1)^n B_n(A),$$

where

$$B_n(A) = ((-1)^{n+1} n/k) \int_0^k (k - u)\, \mathscr{B}_{n-1}(u, A)\, du.$$

It can be shown that $\mathscr{B}_n(x, A)$ has period k. The definition of $\mathscr{B}_n(x, A)$ is then extended to $-\infty < x < \infty$ by periodicity. If $A = I \equiv \{1\}$, $B_n(I) = B_n$ and $\mathscr{B}_n(x, I) = \mathscr{B}_n(x)$, where B_n and $\mathscr{B}_n(x)$ denote the ordinary Bernoulli numbers and functions, respectively.

In the sequel, we shall need the fact that

$$\begin{aligned}\mathscr{B}_n(x, A) &= k^{n-1} \sum_{j=0}^{k-1} a_j \mathscr{B}_n((x - j)/k) \\ &= k^{n-1} \sum_{j=0}^{k-1} a_{-j} \mathscr{B}_n((x + j)/k).\end{aligned} \tag{2.7}$$

The Fourier series representation of $\mathscr{B}_n(x, A)$, $n \geqslant 1$, will also be utilized. For $-\infty < x < \infty$,

$$\mathscr{B}_n(x, A) = -n! \sum_{j=1}^{\infty} (k/2\pi ij)^n \{b_j e^{2\pi ijx/k} + (-1)^n\, b_{-j} e^{-2\pi ijx/k}\}. \tag{2.8}$$

3. The Three-Term Relation for Dedekind Sums

In [5] we showed how to derive the reciprocity theorem for Dedekind sums from the ordinary Poisson summation formula. A modification in that proof will enable us to prove the three-term relation for Dedekind sums. We shall give a second very easy proof of the three-term relation by modifying slightly a proof of the reciprocity theorem given by Rademacher [24] and Grosswald [17]. See also [28, pp. 21, 22].

Lemma 3.1. *For* $(h, k) = 1$,

$$\int_0^1 ((hx))((kx))\, dx = 1/12hk.$$

For a proof of Lemma 3.1, see [28, pp. 24, 25].

Theorem 3.2 (Three-term relation). *Let* $a, b,$ *and* c *be positive integers such that* $(a, b) = (a, c) = (b, c) = 1$. *Let* $a', b',$ *and* c' *be integers chosen so that* $aa' \equiv 1 \pmod{b}$, $bb' \equiv 1 \pmod{c}$, *and* $cc' \equiv 1 \pmod{a}$. *Then,*

$$s(bc', a) + s(ca', b) + s(ab', c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right). \tag{3.1}$$

The relation (3.1) was first proven by Rademacher [26]. Generalizations of (3.1) have been given by Carlitz [9, 10, 13, 14]. The hypotheses on a', b', and c' are less restrictive than those of Rademacher and Carlitz, for they require $aa' \equiv 1 \pmod{bc}$, $bb' \equiv 1 \pmod{ca}$, and $cc' \equiv 1 \pmod{ab}$. Dieter [15] has given a proof of (3.1) under the same hypotheses on a', b', and c' as we have. His proof uses the transformation formula of $\eta(z)$. If we let $c = 1$, (3.1) reduces to the reciprocity theorem (1.1).

First proof. In (2.1), let $\alpha = 0$, $\beta = a$, and $f(x) = ((bx/a))((cx/a))$. We find that

$$\tfrac{1}{4} + \sum_{n=1}^{a-1} ((bn/a))((cn/a))$$

$$= \int_0^a ((bx/a))((cx/a))\, dx + 2 \sum_{n=1}^{\infty} \int_0^a ((bx/a))((cx/a)) \cos(2\pi nx)\, dx.$$

Upon using Lemma 3.1 and noting that $(a, c') = 1$ and $cc' \equiv 1 \pmod{a}$, we find that the above may be written as

$$\tfrac{1}{4} + s(bc', a) = a/12bc + 2a \sum_{n=1}^{\infty} I(a, b, c, n), \tag{3.2}$$

where

$$\begin{aligned} I(a, b, c, n) &\\ &= \int_0^1 ((bx))((cx)) \cos(2\pi nax)\, dx \\ &= \int_0^1 (bx - \tfrac{1}{2})((cx)) \cos(2\pi nax)\, dx - \sum_{\mu=0}^{b-1} \mu \int_{\mu/b}^{(\mu+1)/b} ((cx)) \cos(2\pi nax)\, dx \\ &= \int_0^1 (bx - \tfrac{1}{2})(cx - \tfrac{1}{2}) \cos(2\pi nax)\, dx - \sum_{\nu=0}^{c-1} \nu \int_{\nu/c}^{(\nu+1)/c} (bx - \tfrac{1}{2}) \cos(2\pi nax) dx \\ &\quad - \sum_{\mu=0}^{b-1} \mu \int_{\mu/b}^{(\mu+1)/b} (cx - \tfrac{1}{2}) \cos(2\pi nax)\, dx + \int_0^1 [bx][cx] \cos(2\pi nax)\, dx \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{3.3}$$

say.

Two integrations by parts easily give

$$I_1 = 2bc/(2\pi na)^2. \tag{3.4}$$

Secondly, write

$$\begin{aligned} I_2 &= -\sum_{r=1}^{c-1} \int_{r/c}^{1} (bx - \tfrac{1}{2}) \cos(2\pi nax)\, dx \\ &= \frac{1}{2\pi na} \sum_{r=1}^{c-1} (br/c - \tfrac{1}{2}) \sin(2\pi nar/c) - \frac{b}{(2\pi na)^2} \sum_{r=1}^{c-1} \{1 - \cos(2\pi nar/c)\} \\ &= \frac{1}{2\pi na} \sum_{r=1}^{c-1} (br/c - \tfrac{1}{2}) \sin(2\pi nar/c) - bc/(2\pi na)^2 + b\, \delta(n, c)/(2\pi na)^2, \end{aligned} \tag{3.5}$$

where $\delta(n, c) = c$ if $n \equiv 0 \pmod{c}$ and $\delta(n, c) = 0$, otherwise. Similarly,

$$I_3 = (1/2\pi na) \sum_{r=1}^{b-1} (cr/b - \tfrac{1}{2}) \sin(2\pi nar/b) - bc/(2\pi na)^2 + c\delta(n, b)/(2\pi na)^2. \tag{3.6}$$

Lastly,

$$I_4 = \sum_{j=0}^{bc-1} [bj/bc][cj/bc] \int_{j/bc}^{(j+1)/bc} \cos(2\pi nax)\, dx$$

$$= (1/2\pi na) \sum_{j=1}^{bc} [(j-1)/c][(j-1)/b] \sin(2\pi naj/bc)$$

$$- (1/2\pi na) \sum_{j=0}^{bc-1} [j/c][j/b] \sin(2\pi naj/bc)$$

$$= -(1/2\pi na) \sum_{r=1}^{b-1} [cr/b] \sin(2\pi nar/b)$$

$$-(1/2\pi na) \sum_{r=1}^{c-1} [br/c] \sin(2\pi nar/c), \tag{3.7}$$

since $(b, c) = 1$. Putting (3.4)–(3.7) into (3.3), we find that

$$I(a, b, c, n) = (1/2\pi na) \sum_{r=1}^{c-1} ((br/c)) \sin(2\pi nar/c)$$

$$+ (1/2\pi na) \sum_{r=1}^{b-1} ((cr/b)) \sin(2\pi nar/b) + \{b\delta(n, c) + c\delta(n, b)\}/(2\pi na)^2.$$

Putting the above in (3.2) and then using the well-known Fourier series

$$((x)) = - \sum_{n=1}^{\infty} ((\sin(2\pi nx))/\pi n), \tag{3.8}$$

we deduce that

$$\tfrac{1}{4} + s(bc', a) - a/12bc$$

$$= - \sum_{r=1}^{c-1} ((br/c))((ar/c))$$

$$- \sum_{r=1}^{b-1} ((cr/b))((ar/b)) + (2abc/(2\pi ca)^2) \sum_{j=1}^{\infty} j^{-2} + (2abc/(2\pi ba)^2) \sum_{j=1}^{\infty} j^{-2}$$

$$= -s(ab', c) - s(a'c, b) + b/12ac + c/12ab,$$

since $(b', c) = (a', b) = 1$, $bb' \equiv 1 \pmod{c}$, and $aa' \equiv 1 \pmod{b}$. This completes the proof.

Second proof. We shall give just a brief sketch as the details are similar to those in [17] or [28].

Let C be the positively oriented, indented rectangle with vertices at $\pm iM$ and $1 \pm iM$, $M > 0$, where the indentations are small semicircles of the same radius centered at 0 and 1 and to the left of 0 and 1, respectively. Let $F(z) = \cot(\pi az)\cot(\pi bz)\cot(\pi cz)$. Letting M tend to ∞, we find, as in [17], that

$$S \equiv (1/2\pi i)\int_C F(z)\,dz = -1/\pi. \tag{3.9}$$

On the interior of C, $F(z)$ has simple poles at $z = l/a$, $1 \leqslant l \leqslant a-1$, $z = m/b$, $1 \leqslant m \leqslant b-1$, and $z = n/c$, $1 \leqslant n \leqslant c-1$. Furthermore, $F(z)$ has a triple pole at $z = 0$. Hence, by the residue theorem,

$$\begin{aligned} S = {} & \frac{1}{\pi a}\sum_{l=1}^{a-1}\cot(\pi bl/a)\cot(\pi cl/a) \\ & + \frac{1}{\pi b}\sum_{m=1}^{b-1}\cot(\pi am/b)\cot(\pi cm/b) + \frac{1}{\pi c}\sum_{n=1}^{c-1}\cot(\pi an/c)\cot(\pi bn/c) \\ & - \frac{1}{3\pi}\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right). \end{aligned} \tag{3.10}$$

Now for $(h, k) = 1$, $k > 0$ [17, p. 641; 28, p. 18],

$$s(h, k) = (1/4k)\sum_{r=1}^{k-1}\cot(\pi r/k)\cot(\pi hr/k).$$

Thus, (3.10) becomes

$$S = \frac{4}{\pi}\{s(bc', a) + s(ca', b) + s(ab', c)\} - \frac{1}{3\pi}\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right).$$

Combining the above with (3.9), we arrive at (3.1).

4. Dedekind–Rademacher Sums

Let h and k be integers and let x and y be arbitrary real numbers. Then the Dedekind–Rademacher sum $s(h, k; x, y)$ is defined by

$$s(h, k; x, y) = \sum_{j \pmod k}\left(\left(h\frac{j+y}{k} + x\right)\right)\left(\left(\frac{j+y}{k}\right)\right).$$

THEOREM 4.1 (Reciprocity theorem). *If h and k are coprime, positive integers, and if x and y are not both integers,*

$$s(h, k; x, y) + s(k, h; y, x)$$
$$= ((x))((y)) + \frac{1}{2}\left(\frac{h}{k}\mathscr{B}_2(y) + \frac{k}{h}\mathscr{B}_2(x) + \frac{1}{hk}\mathscr{B}_2(hy + kx)\right), \qquad (4.1)$$

where $\mathscr{B}_2(z)$ denotes the second Bernoulli function.

If x and y are integers, $s(h, k; x, y) = s(h, k)$ whose reciprocity theorem was proved in the last section.

The reciprocity theorem for $s(h, k; x, y)$ was first proven by Rademacher [27]. More recent proofs have been given by Grosswald [18] and the author [2]. In cases where x and y are certain rational numbers, previous proofs of the reciprocity theorem were given by Meyer [22, 23] and Dieter [16]. Reciprocity theorems for generalizations of $s(h, k; x, y)$ have been given by Carlitz [11, 12]. The new proof we give below is at least as short as any of the previous proofs.

LEMMA 4.2. *Let h and k be positive integers with $(h, k) = g$. Let x and y be arbitrary real numbers. Then,*

$$I \equiv \int_0^1 ((ht + x))((kt + y))\, dt = (g^2/2hk)\,\mathscr{B}_2(hy/g - kx/g). \qquad (4.2)$$

Proof. First, assume that $g = 1$. Write

$$I = \sum_{n=0}^{h-1} \int_{n/h}^{(n+1)/h} ((ht + x))((kt + y))\, dt,$$

and put $t = (u + n)/h$. Then,

$$I = \frac{1}{h}\sum_{n=0}^{h-1} \int_0^1 ((u + x))((k(u + n)/h + y))\, du$$
$$= \frac{1}{h}\int_0^1 ((u + x)) \sum_{n=0}^{h-1} \left(\left(\frac{ku + hy + kn}{h}\right)\right) du$$
$$= \frac{1}{h}\int_0^1 ((u + x))((ku + hy))\, du,$$

where we have used the familiar property [28, p. 4],

$$\sum_{n(\mathrm{mod}\,h)} (((a+n)/h)) = ((a)). \tag{4.3}$$

By repeating the above calculation with h replaced by k, we arrive at

$$I = (1/hk) \int_0^1 ((t+kx))((t+hy))\,dt = (1/hk)L, \tag{4.4}$$

say.

Observe that

$$L = \int_0^1 ((t+a))((t+b))\,dt,$$

where $a = kx - [kx]$ and $b = hy - [hy]$. Without loss of generality, assume that $a \leqslant b$. Then,

$$\begin{aligned}
L &= \int_0^{1-b} (t+a-\tfrac{1}{2})(t+b-\tfrac{1}{2})\,dt + \int_{1-b}^{1-a} (t+a-\tfrac{1}{2})(t+b-\tfrac{3}{2})\,dt \\
&\quad + \int_{1-a}^{1} (t+a-\tfrac{3}{2})(t+b-\tfrac{3}{2})\,dt \\
&= \int_0^1 (t+a-\tfrac{1}{2})(t+b-\tfrac{1}{2})\,dt - \int_{1-b}^{1} (t+a-\tfrac{1}{2})\,dt - \int_{1-a}^{1} (t+b-\tfrac{3}{2})dt \\
&= \tfrac{1}{2}\{(b-a)^2 - (b-a) + \tfrac{1}{6}\} \\
&= \tfrac{1}{2} B_2(b-a) \\
&= \tfrac{1}{2} \mathscr{B}_2(hy-kx).
\end{aligned}$$

Putting the above in (4.4), we arrive at (4.2) for $g = 1$.

In the more general case, set $h = gr$ and $k = gs$. Then,

$$\begin{aligned}
I &= \int_0^1 ((grt+x))((gst+y))\,dt \\
&= (1/g) \int_0^g ((ru+x))((su+y))\,du \\
&= \int_0^1 ((ru+x))((su+y))\,du \\
&= (1/2rs)\,\mathscr{B}_2(ry-sx) = (g^2/2hk)\,\mathscr{B}_2(hy/g - kx/g),
\end{aligned}$$

where we have employed the result from the first part of our proof.

Proof of Theorem 4.1. Without loss of generality, assume that $0 \leqslant x, y < 1$. Apply (2.1) with $\alpha = -y$, $\beta = k - y$, and $f(u) = ((h(u+y)/k + x))(((u+y)/k))$. We find that

$$s(h, k; x, y) = \int_{-y}^{k-y} \left(\left(h\frac{u+y}{k} + x\right)\right)\left(\left(\frac{u+y}{k}\right)\right) du + 2\sum_{n=1}^{\infty} I(h, k, x, y, n), \quad (4.5)$$

where

$$I(h, k, x, y, n) = \int_{-y}^{k-y} \left(\left(h\frac{u+y}{k} + x\right)\right)\left(\left(\frac{u+y}{k}\right)\right) \cos(2\pi nu)\, du.$$

By letting $u + y = kt$ and using Lemma 4.2 in the first instance, we easily deduce that

$$\int_{-y}^{k-y} \left(\left(h\frac{u+y}{k} + x\right)\right)\left(\left(\frac{u+y}{k}\right)\right) du = \frac{k}{2h}\mathscr{B}_2(x) \quad (4.6)$$

and

$$\begin{aligned} I(h, k, x, y, n) &= k\int_0^1 ((ht + x))((t)) \cos(2\pi n(kt - y))\, dt \\ &= k\int_0^1 (ht + x - \tfrac{1}{2})(t - \tfrac{1}{2}) \cos(2\pi n(kt - y))\, dt \\ &\quad - k\sum_{r=1}^{h-1} r\int_{(r-x)/h}^{(r+1-x)/h} (t - \tfrac{1}{2}) \cos(2\pi n(kt - y))\, dt \\ &\quad - hk\int_{1-x/h}^{1} (t - \tfrac{1}{2}) \cos(2\pi n(kt - y))\, dt \\ &= I_1 + I_2 + I_3, \end{aligned} \quad (4.7)$$

say.

Firstly, we find that after two integrations by parts,

$$I_1 = -(1/2\pi n)(x - \tfrac{1}{2} + \tfrac{1}{2}h) \sin(2\pi ny) + (2hk/(2\pi nk)^2) \cos(2\pi ny). \quad (4.8)$$

Secondly, write

$$I_2 + I_3 = -k\sum_{\mu=1}^{h} \int_{(\mu-x)/h}^{1} (t - \tfrac{1}{2}) \cos(2\pi n(kt - y))\, dt,$$

and integrate by parts to get

$$I_2 + I_3 = \sum_{\mu=1}^{h} \frac{1}{2\pi n} \left\{ \frac{1}{2} \sin(2\pi ny) + \left(\frac{\mu - x}{h} - \frac{1}{2}\right) \sin\left(2\pi n\left(k\frac{\mu - x}{h} - y\right)\right) + \int_{(\mu-x)/h}^{1} \sin(2\pi n(kt-y))dt \right\}$$

$$= \frac{h}{4\pi n} \sin(2\pi ny) + \frac{1}{2\pi n} \sum_{\mu=1}^{h} \left(\frac{\mu - x}{h} - \frac{1}{2}\right) \sin\left(2\pi n\left(k\frac{\mu - x}{h} - y\right)\right) - \frac{hk}{(2\pi nk)^2} \cos(2\pi ny) + \frac{k}{(2\pi nk)^2} \sum_{\mu=1}^{h} \cos\left(2\pi n\left(k\frac{\mu - x}{h} - y\right)\right). \tag{4.9}$$

Using (4.8) and (4.9) in (4.7), we find that

$$I(h, k, x, y, n) = -\frac{1}{2\pi n}\left(x - \frac{1}{2}\right) \sin(2\pi ny) + \frac{hk}{(2\pi nk)^2} \cos(2\pi ny) + \frac{1}{2\pi n} \sum_{\mu=1}^{h} \left(\frac{\mu - x}{h} - \frac{1}{2}\right) \sin\left(2\pi n\left(k\frac{\mu - x}{h} - y\right)\right) + \frac{k}{(2\pi nk)^2} \sum_{\mu=1}^{h} \cos\left(2\pi n\left(k\frac{\mu - x}{h} - y\right)\right).$$

The last sum on the right side above has the value 0 unless $n = jh$, where j is a positive integer, $1 \leqslant j < \infty$, in which case the sum has value $h \cos(2\pi j(hy + kx))$. Hence,

$$2 \sum_{n=1}^{\infty} I(h, k, x, y, n) = -\left(x - \frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi ny)}{\pi n} + \frac{h}{2\pi^2 k} \sum_{n=1}^{\infty} \frac{\cos(2\pi ny)}{n^2} + \sum_{\mu=1}^{h} \left(\frac{\mu - x}{h} - \frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi n(k(\mu - x)/h - y))}{\pi n} + \frac{1}{2\pi^2 hk} \sum_{j=1}^{\infty} \frac{\cos(2\pi j(hy + kx))}{j^2}$$

$$= \left(x - \frac{1}{2}\right)((y)) + \frac{h}{2k}\mathscr{B}_2(y) - \sum_{\mu=1}^{h} \left(\frac{\mu - x}{h} - \frac{1}{2}\right)\left(\left(k\frac{\mu - x}{h} - y\right)\right) + \frac{1}{2hk}\mathscr{B}_2(hy + kx),$$

by (2.7) and (3.8). Now,

$$\frac{\mu - x}{h} - \frac{1}{2} = \left(\left(\frac{\mu - x}{h}\right)\right) + \frac{1}{2}, \qquad \text{if } \mu = h \quad \text{and} \quad x = 0,$$

$$= \left(\left(\frac{\mu - x}{h}\right)\right), \qquad \text{otherwise.}$$

Hence, we may write the above as

$$2 \sum_{n=1}^{\infty} I(h, k, x, y, n) = ((x))((y)) + (h/2k)\,\mathscr{B}_2(y) - s(k, h; -y, -x) + (1/2hk)\,\mathscr{B}_2(hy + kx). \tag{4.10}$$

By letting the index of summation $j = h - \mu$, we readily see that $s(k, h; -y, -x) = s(k, h; y, x)$. Putting (4.6) and (4.10) in (4.5), we arrive at (4.1) forthwith.

We now derive a three-term relation for Dedekind–Rademacher sums. Let a, b, and c be nonzero integers and let x, y, and z be real. Define

$$S(a, b, c; x, y, z) = \sum_{j(\mathrm{mod}\, c)} \left(\left(a\frac{j + z}{c} - x\right)\right)\left(\left(b\frac{j + z}{c} - y\right)\right).$$

Observe that $S(a, 1, c; -x, 0, z) = s(a, c; x, z)$. But, in fact, the sums $S(a, b, c; x, y, z)$ are no more general than the Dedekind–Rademacher sums. If we replace $((u))$ in the above by $B_1(u - [u])$, we obtain the sum $-s(a, b, c; x, y, z)$ first defined by Carlitz [14] who derived a three-term relation for $s(a, b, c; x, y, z)$ under the hypotheses $(a, b) = (b, c) = (c, a) = 1$. These hypotheses are removed in the theorem we prove below. Our three-term relation for $S(a, b, c; x, y, z)$ contains as special cases generalizations of Theorems 3.2 and 4.1. The method that we use is an extension of an idea of Rademacher [25].

THEOREM 4.3 (Three-term relation). *Let $(a, b) = f$, $(b, c) = g$, and $(c, a) = h$, where a, b, and c are positive integers. Then*

$$S(a, b, c; x, y, z) + S(c, a, b; z, x, y) + S(b, c, a; y, z, x) = -N/4 + (cf^2/2ab)\,\mathscr{B}_2(ay/f - bx/f) + (ag^2/2bc)\,\mathscr{B}_2(bz/g - cy/g) + (bh^2/2ac)\,\mathscr{B}_2(cx/h - az/h), \tag{4.11}$$

where N is the number of distinct triples r, s, t such that

$$0 \leqslant (r + x)/a = (s + y)/b = (t + z)/c < 1.$$

Proof. Let $\epsilon > 0$ be chosen so that $((at - x))$, $((bt - y))$, and $((ct - z))$ have no discontinuities on $(0, \epsilon]$ and $[1 - \epsilon, 1)$. Let $t_1, \ldots, t_m$ be those points (if any) on $(0, 1)$ where any two or three of the functions $((at - x))$, $((bt - y))$, and $((ct - z))$ have common discontinuities. Let $I_j = (t_j - \epsilon_j, t_j + \epsilon_j)$, where $0 < 2\epsilon_j < \inf(1/a, 1/b, 1/c)$, $1 \leqslant j \leqslant m$. Let S be the complement in $[\epsilon, 1 - \epsilon]$ of $\bigcup_{j=1}^{m} I_j$. Let

$$S'(a, b, c; x, y, z) = \sum_{j(\mathrm{mod}\, c)}{}' \left(\left(a\frac{j+z}{c} - x\right)\right)\left(\left(b\frac{j+z}{c} - y\right)\right),$$

where the prime on the summation sign means that if z is an integer, the term corresponding to $j + z \equiv 0 \pmod c$ is omitted from the summation.

Now [25; 28, p. 22],

$$\begin{aligned} I_\epsilon &\equiv \int_S ((at - x))\, d(((bt - y))((ct - z))) \\ &= \int_S ((at - x))((bt - y))\, d(((ct - z))) \\ &\quad + \int_S ((at - x))((ct - z))\, d(((bt - y))) \\ &= -\sum_{j(\mathrm{mod}\, c)}{}' \left(\left(a\frac{j+z}{c} - x\right)\right)\left(\left(b\frac{j+z}{c} - y\right)\right) + c\int_S ((at - x))((bt - y))\, dt \\ &\quad - \sum_{j(\mathrm{mod}\, b)}{}' \left(\left(a\frac{j+y}{b} - x\right)\right)\left(\left(c\frac{j+y}{b} - z\right)\right) + b\int_S ((at - x))((ct - z))\, dt. \end{aligned} \tag{4.12}$$

In the two sums on the right side of (4.12), we have added the terms arising from $t_1, \ldots, t_m$. However, each of these terms has value zero. Letting $\epsilon, \epsilon_1, \ldots, \epsilon_m$ tend to 0 in (4.12), we find with the help of Lemma 4.2 that

$$\begin{aligned} \lim_{\epsilon \to 0} I_\epsilon &= -S'(a, b, c; x, y, z) - S'(c, a, b; z, x, y) \\ &\quad + (cf^2/2ab)\, \mathscr{B}_2(ay/f - bx/f) + (bh^2/2ac)\, \mathscr{B}_2(cx/h - az/h). \end{aligned} \tag{4.13}$$

On the other hand, putting $F(t; x, y, z) = ((at-x))((bt-y))((ct-z))$, we find upon integrating by parts that

$$I_\epsilon = F(t; x, y, z)\Big|_{\epsilon}^{x_1-\epsilon} + \sum_{j=1}^{m-1} F(t; x, y, z)\Big|_{x_j+\epsilon_j}^{x_{j+1}-\epsilon_{j+1}} + F(t; x, y, z)\Big|_{x_m+\epsilon_m}^{1-\epsilon}$$

$$- \int_S ((bt-y))((ct-z))\, d(((at-x))). \tag{4.14}$$

Let $G(x, y, z)$ denote the limit as $\epsilon, \epsilon_1, \ldots, \epsilon_m$ tend to 0 of the sum of the first three expressions on the right side of (4.14). Letting $\epsilon, \epsilon_1, \ldots, \epsilon_m$ tend to 0, we find with the aid of Lemma 4.2 that

$$\lim_{\epsilon\to 0} I_\epsilon = G(x, y, z) + S'(b, c, a; y, z, x)$$
$$-(ag^2/2bc)\, \mathscr{B}_2(bz/g - cy/g). \tag{4.15}$$

We next calculate $G(x, y, z)$. We find that $G(x, y, z) = N/4$ in each of the following three cases: x, y, and z are all integers; exactly two of the parameters x, y, and z are integers; and none of the parameters x, y, and z is an integer. Lastly, if exactly one of the parameters x, y, and z is an integer,

$$\begin{aligned} G(x, y, z) &= ((y))((z)) + N/4, &&\text{if } x \text{ is an integer,}\\ &= ((x))((z)) + N/4, &&\text{if } y \text{ is an integer,}\\ &= ((x))((y)) + N/4, &&\text{if } z \text{ is an integer.}\end{aligned}$$

Observe that if z is an integer,

$$S'(a, b, c; x, y, z) + ((x))((y)) = S(a, b, c; x, y, z). \tag{4.16}$$

Similar relations hold if either x or y is an integer. If we combine (4.15) with (4.13) and use (4.16) and its analogs, we arrive at (4.11), and the proof is complete.

COROLLARY 4.4. *Let $(a, b) = f$, $(b, c) = g$, and $(c, a) = h$, where a, b, and c are positive integers. Let $S(a, b, c) = S(a, b, c; 0, 0, 0)$. Then,*

$$S(a, b, c) + S(c, a, b) + S(b, c, a) = -\frac{N}{4} + \frac{1}{12}\left(\frac{cf^2}{ab} + \frac{ag^2}{bc} + \frac{bh^2}{ac}\right).$$

Proof. Apply Theorem 4.3 with $x = y = z = 0$. Recall that $\mathscr{B}_2(0) = B_2 = \frac{1}{6}$.

If $f = g = h = 1$, Corollary 4.4 reduces to Theorem 3.2.

COROLLARY 4.5. *Let $(a, b) = f$, where a and b are positive integers. Then,*

$$s(a, b; x, y) + s(b, a; y, x) = -N/4 + ((x))((y)) + \frac{f^2}{2ab}\mathscr{B}_2(ay/f + bx/f) + \frac{a}{2b}\mathscr{B}_2(y) + \frac{b}{2a}\mathscr{B}_2(x),$$

where $N = 1$ if x and y are both integers and $N = 0$ otherwise.

Proof. Apply Theorem 4.3 with $z = 0$, $c = 1$, and x replaced by $-x$. Observe that $S(a, b, 1; -x, y, 0) = -((x))((y))$.

If $f = 1$ and x and y are integers, Corollary 4.5 reduces to the reciprocity theorem (1.1) for $s(a, b)$; if $f = 1$ and x and y are not both integers, Corollary 4.5 reduces to Theorem 4.1.

5. THE GENERALIZED DEDEKIND SUMS OF CARLITZ

Let $h_1, \dots, h_n$ and k be nonzero integers. Define

$$s(h_1, \dots, h_n; k) = \sum_{j_1, \dots, j_n (\mathrm{mod}\, k)} ((\{h_1 j_1 + \cdots + h_n j_n\}/k))((j_1/k)) \cdots ((j_n/k)).$$

These sums are very similar to sums first defined by Carlitz [9], who derived $(n + 1)$ and $(n + 2)$-term relations for his sums. We shall use contour integration to derive $(n + 1)$ and $(n + 2)$-term relations for $s(h_1, \dots, h_n; k)$. Our results take simpler forms than those of Carlitz. Observe that if n is even, $s(h_1, \dots, h_n; k) = 0$.

LEMMA 5.1. *We have for $n \geqslant 1$ and $k > 1$,*

$$s(h_1, \dots, h_n; k) = -(-\tfrac{1}{2}i)^{n+1}(1/k) \sum_{\mu=1}^{k-1} \cot(\pi\mu/k)\cot(\pi h_1\mu/k) \cdots \cot(\pi h_n \mu/k).$$

Proof. The finite Fourier series for $((j/k))$ may be written as [28, p. 14; 17, p. 641]

$$((j/k)) = (i/2k) \sum_{\mu=1}^{k-1} \cot(\pi\mu/k)\, e^{2\pi i j\mu/k}. \tag{5.1}$$

Thus,

$$s(h_1, \ldots, h_n; k)$$
$$= (i/2k)^{n+1} \sum_{\mu=1}^{k-1} \sum_{\mu_1=1}^{k-1} \cdots \sum_{\mu_n=1}^{k-1} \cot(\pi\mu/k) \cot(\pi\mu_1/k)$$
$$\cdots \cot(\pi\mu_n/k) \sum_{j_1,\ldots,j_n (\mathrm{mod}\, k)} \exp(2\pi i(j_1(h_1\mu + \mu_1) + \cdots + j_n(h_n\mu + \mu_n))/k).$$

The sum on j_r, $1 \leqslant r \leqslant n$, is equal to 0 unless $h_r\mu + \mu_r \equiv 0 \pmod{k}$ in which case the sum is equal to k. Thus,

$$s(h_1, \ldots, h_n; k) = (\tfrac{1}{2} i)^{n+1}(1/k) \sum_{\mu=1}^{k-1} \cot(\pi\mu/k) \cot(-\pi h_1\mu/k) \cdots \cot(-\pi h_n\mu/k),$$

and the result follows.

Let $\tilde{h}_{jk} = (h_1, \ldots, h_{j-1}, h_{j+1}, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{n+2})$. Thus, $\tilde{h}_{jk}$ is an n-vector; it is obtained from an $(n+2)$-vector by removing the jth and kth components, $j \neq k$. In the following, it will be convenient to write

$$s(\tilde{h}_{jk}; h) = s(h_1, \ldots, h_{j-1}, h_{j+1}, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{n+2}; h).$$

THEOREM 5.2. *Let $n \geqslant 1$ be odd. Let $h_1, \ldots, h_{n+2}$ be positive integers that are relatively prime in pairs. Pick $k = k(j) \neq j$, $1 \leqslant j \leqslant n+2$, and define h_k^{-1} by $h_k^{-1}h_k \equiv 1 \pmod{h_j}$, $1 \leqslant j \leqslant n+2$. (It makes no difference how we choose $k \neq j$.) Define*

$$T(h) = T(h_1, \ldots, h_{n+2})$$
$$= \frac{1}{h_1 \cdots h_{n+2}} \sum_{2(j_1+\cdots+j_{n+2})=n+1} \frac{B_{2j_1} \cdots B_{2j_{n+2}} h_1^{2j_1} \cdots h_{n+2}^{2j_{n+2}}}{(2j_1)! \cdots (2j_{n+2})!}.$$

Then,

$$\sum_{j=1}^{n+2} s(h_k^{-1}\tilde{h}_{jk}; h_j) = -2^{-n-1} + T(h). \tag{5.2}$$

Proof. Let C be the same contour as in the second proof of Theorem 3.2. Let

$$F(z) = \prod_{j=1}^{n+2} \cot(\pi h_j z).$$

On the interior of C, $F(z)$ has simple poles at $z = \mu_j/h_j$, $1 \leqslant \mu_j \leqslant h_j - 1$, $1 \leqslant j \leqslant n+2$. The poles are simple since $h_1, \ldots, h_{n+2}$ are relatively prime in pairs. Furthermore, $F(z)$ has a pole of order $n+2$ at $z = 0$. The residue at $z = \mu_j/h_j$ is

$$\frac{1}{\pi h_j} \prod_{\substack{r=1 \\ r \neq j}}^{n+2} \cot(\pi h_r \mu_j / h_j), \qquad 1 \leqslant j \leqslant n+2.$$

Since [21, p. 204],

$$\cot z = \frac{1}{z} + \sum_{j=1}^{\infty} \frac{(-1)^j\, 2^{2j} B_{2j} z^{2j-1}}{(2j)!},$$

we may write

$$F(z) = \frac{1}{(\pi z)^{n+2}\, h_1 \cdots h_{n+2}} \prod_{r=1}^{n+2} \left\{ 1 + \sum_{j=1}^{\infty} \frac{B_{2j} (2\pi i h_r z)^{2j}}{(2j)!} \right\}.$$

Hence, the residue of $F(z)$ at $z = 0$ is

$$\frac{(2\pi i)^{n+1}}{\pi^{n+2} h_1 \cdots h_{n+2}} \sum_{2(j_1 + \cdots + j_{n+2}) = n+1} \frac{B_{2j_1} \cdots B_{2j_{n+2}} h_1^{2j_1} \cdots h_{n+2}^{2j_{n+2}}}{(2j_1)! \cdots (2j_{n+2})!} = (2i)^{n+1}\, T(h)/\pi.$$

The contributions to

$$S \equiv (1/2\pi i) \int_C F(z)\, dz$$

along the indented vertical sides of C cancel since $F(z) = F(z+1)$. As M tends to $\pm\infty$, $\cot(x + iM)$ tends to $\mp i$. Since n is odd, we easily deduce, as in [17, p. 643] or [28, p. 21], that $S = i^{n+1}/\pi$. Combining this with the value of S obtained from the residue theorem, we find with the aid of Lemma 5.1 that

$$\begin{aligned} S = i^{n+1}/\pi &= \frac{1}{\pi} \sum_{j=1}^{n+2} \frac{1}{h_j} \sum_{\mu_j=1}^{h_j-1} \prod_{\substack{r=1 \\ r \neq j}}^{n+2} \cot(\pi h_r \mu_j / h_j) + (2i)^{n+1}\, T(h)/\pi \\ &= -\frac{1}{\pi} (2i)^{n+1} \sum_{j=1}^{n+2} s(h_k^{-1} \tilde{h}_{jk}\, ; h_j) + (2i)^{n+1}\, T(h)/\pi, \end{aligned}$$

where we have used the facts that $h_k^{-1}k_k \equiv 1 \pmod{h_j}$ and $(h_k^{-1}, h_j) = 1$, $1 \leqslant j \leqslant n + 2$. A rearrangement of the above yields (5.2).

If $n = 1$, (5.2) reduces to the three-term relation (3.1). The next result gives an $(n + 1)$-term relation which includes (1.1) as a special case.

COROLLARY 5.3. *Let $h_1, \ldots, h_{n+1}$, $n \geqslant 1$, be positive integers that are relatively prime in pairs. Put $\tilde{h}_{j(n+2)} = \tilde{h}_j$. Then in the notation of Theorem* 5.2,

$$\sum_{j=1}^{n+1} s(\tilde{h}_j\,; h_j)$$

$$= -2^{-n-1} + \frac{1}{h_1 \cdots h_{n+1}} \sum_{2(j_1+\cdots+j_{n+2})=n+1} \frac{B_{2j_1} \cdots B_{2j_{n+2}} h_1^{2j_1} \cdots h_{n+1}^{2j_{n+1}}}{(2j_1)! \cdots (2j_{n+2})!}.$$

Proof. In Theorem 5.2, let $h_{n+2} = 1$. For $j \neq n + 2$, pick $k = n + 2$ in all cases. Then we can take $h_{n+2}^{-1} = 1$ in all such cases. For $j = n + 2$, we see from the definition of $s(h_1, \ldots, h_n\,; k)$ that $s(h_k^{-1}\tilde{h}_{(n+2)k}\,; 1) = 0$. With these substitutions, Theorem 5.2 reduces to Corollary 5.3.

6. MODIFIED DEDEKIND SUMS

In recent work of the author [4] on transformation formulas of certain generalized Eisenstein series, certain modified Dedekind sums have arisen. Suppose that α, β, h, and k are positive integers with $(h, k) = 1$. Define h^{-1} by

$$h^{-1}h \equiv 1 \pmod{k}.$$

Then the modified Dedekind sum $s_{\alpha,\beta}(h, k)$ is defined by

$$s_{\alpha,\beta}(h, k) = \sum_{\mu \pmod{hk}} e(\mu\alpha/h + \mu\beta/k)((\mu/hk))((\mu h^{-1}/k)),$$

where $e(x) = e^{2\pi i x}$. It is not difficult to show that $s_{0,0}(h, k) = s(h, k)$ [4]. In [4], we used transformation formulas to derive a reciprocity theorem for $s_{\alpha,\beta}(h, k)$. Here, we give a direct proof by contour integration.

LEMMA 6.1. *We have for $k > 1$,*

$$s_{\alpha,\beta}(h, k) = (1/4k) \sum_{j=1}^{k-1} \cot(\pi(j/k + \{\alpha k + \beta h\}/hk)) \cot(\pi hj/k). \quad (6.1)$$

Proof. From (5.1),

$$s_{\alpha,\beta}(h,k) = -(1/4hk^2)\sum_{r=1}^{hk-1}\sum_{j=1}^{k-1}\cot(\pi r/hk)\cot(\pi j/k)$$

$$\cdot \sum_{\mu(\bmod hk)} \exp(2\pi i\mu(\alpha k+\beta h+r+jhh^{-1})/hk).$$

The inner sum is zero except when $\alpha k+\beta h+r+jhh^{-1}\equiv 0 \pmod{hk}$, in which case the sum has value hk. Hence,

$$s_{\alpha,\beta}(h,k) = (1/4k)\sum_{j=1}^{k-1}\cot(\pi(\alpha k+\beta h+jhh^{-1})/hk)\cot(\pi j/k).$$

As j runs through a complete residue system modulo k, hj does as well since $(h, k) = 1$. Replacing j by hj in the above, we obtain (6.1) at once.

THEOREM 6.2 (Reciprocity theorem for modified Dedekind sums). *Let* α, β, h, *and* k *be positive integers with* $(h,k)=(\alpha,h)=(\beta,k)=1$ *and* $h, k \geqslant 2$. *Then,*

$$s_{\alpha,\beta}(h,k)+s_{\beta,\alpha}(k,h)$$
$$= (1/4hk)\csc^2(\pi(\alpha k+\beta h)/hk) - \tfrac{1}{4}(1+\cot(\pi\alpha k/h)\cot(\pi\beta h/k)). \quad (6.2)$$

Proof. Let C be the contour in the second proof of Theorem 3.2. Let

$$F(z) = \cot(\pi(z+(\alpha k+\beta h)/hk))\cot(\pi hz)\cot(\pi kz).$$

First, we observe that $(\alpha k+\beta h)/hk$ is not an integer since $(h,k)=(\alpha,h)=(\beta,k)=1$. Secondly, since $(h,k)=1$, we see that $(\alpha k+\beta h)/hk$ is not an integral multiple of $1/h$ or of $1/k$. Hence, on the interior of C, $F(z)$ has a simple pole at $z_0 \neq 0$, where $z_0 \equiv (\alpha k+\beta h)/hk \pmod 1$. Next, $F(z)$ has poles at $z = j/h$, $1 \leqslant j \leqslant h-1$, and at $z = r/k$, $1 \leqslant r \leqslant k-1$. All of these poles are simple since $(h,k)=1$. Lastly, $F(z)$ has a double pole at $z=0$. The residue of $F(z)$ at $z=z_0$ is

$$(1/\pi)\cot(\pi\beta h/k)\cot(\pi\alpha k/h).$$

The residues of $F(z)$ at $z = j/h$ and at $z = r/k$ are, respectively,

$$(1/\pi h)\cot(\pi(j/h+(\alpha k+\beta h)/hk))\cot(\pi kj/h)$$

and

$$(1/\pi k)\cot(\pi(r/k+(\alpha k+\beta h)/hk))\cot(\pi hr/k).$$

Now,

$$F(z)=\{\cot(\pi(\alpha k+\beta h)/hk)-\pi\csc^2(\pi(\alpha k+\beta h)/hk)\,z+\cdots\}$$
$$\cdot\{1/\pi hz-\pi hz/3+\cdots\}\{1/\pi kz-\pi kz/3+\cdots\},$$

and so the residue of $F(z)$ at $z=0$ is

$$-(1/\pi hk)\csc^2(\pi(\alpha k+\beta h)/hk).$$

Hence, by the residue theorem and Lemma 6.1, we find that

$$S\equiv(1/2\pi i)\int_C F(z)\,dz=(4/\pi)\,s_{\beta,\alpha}(k,h)+(4/\pi)\,s_{\alpha,\beta}(h,k)$$
$$+(1/\pi)\cot(\pi\alpha k/h)\cot(\pi\beta h/k)-(1/\pi hk)\csc^2(\pi(\alpha k+\beta h)/hk). \qquad (6.3)$$

As before, the integrals of $F(z)$ over the indented, vertical sides of C cancel. As in [17, p. 643] or [28, p. 21], $F(x+iM)$ tends to $\pm i$ as M tends to $\pm\infty$. Hence, we find that

$$S=-1/\pi. \qquad (6.4)$$

Equations (6.3) and (6.4) taken together yield (6.2), and we are done.

7. Dedekind Sums with Periodic Coefficients

Let $A=\{a_n\}$ and $C=\{c_n\}$, $-\infty<n<\infty$, each be sequences of period k, and let a and c be nonzero integers. Then the Dedekind sum $s(a,c;A,C)$ associated with the sequences A and C is defined by

$$s(a,c;A,C)=\sum_{j(\operatorname{mod} ck)} a_j\mathscr{B}_1(aj/c,C)\,\mathscr{B}_1(j/ck).$$

For brevity, we call $s(a,c;A,C)$ a periodic Dedekind sum.

When $A=\chi\equiv\{\chi(n)\}$ and $C=\bar\chi$, where $\chi(n)$ is a primitive character of modulus k, $s(a,c;\chi,\bar\chi)$ is what we called in [3] a Dedekind character sum. These sums arose in the transformation formulas of natural character analogs of $\log\eta(z)$ [3]. Using our transformation formulas, we derived a reciprocity theorem for $s(a,c;\chi,\bar\chi)$. The transformation formulas developed in [3] can be generalized for general Dirichlet characters and for a very limited class of other periodic sequences. However, the multiplicativity of $\chi(n)$ was an essential ingredient of the method. (We emphasize that our notation here conflicts with that in [3]. What we called $\mathscr{B}_n(x, \chi)$ in [3] is denoted by $\chi(-1)\mathscr{B}_n(x, \bar{\chi})$ here.)

We develop a three-term relation for periodic Dedekind sums. From this theorem a reciprocity theorem is easily deduced. Then, several special cases will be examined. We first need a lemma.

LEMMA 7.1. *Let a, b, and c be positive integers with $(ak, b) = 1$. Define b' by $bb' \equiv 1 \pmod{k}$ and let A' be the sequence defined by $A' = \{a_{nb'}\}$. Then*

$$\int_0^{ck} \mathscr{B}_1(ax/c, A)\, \mathscr{B}_1(bx/ck)\, dx = (c/2ab)\, B_2(A').$$

Proof. Upon the use of two elementary changes of variable, we have

$$\begin{aligned}
&\int_0^{ck} \mathscr{B}_1(ax/c, A)\, \mathscr{B}_1(bx/ck)\, dx \\
&= ck \sum_{n=0}^{a-1} \int_{n/a}^{(n+1)/a} \mathscr{B}_1(aky, A)\, \mathscr{B}_1(by)\, dy \\
&= (ck/a) \int_0^1 \sum_{n=0}^{a-1} \mathscr{B}_1(k(n+x), A)\, \mathscr{B}_1(b(n+x)/a)\, dx \\
&= (ck/a) \int_0^1 \mathscr{B}_1(kx, A) \sum_{n=0}^{a-1} \mathscr{B}_1((n+bx)/a)\, dx,
\end{aligned}$$

since $\mathscr{B}_1(x, A)$ has period k and since $(a, b) = 1$. If we use (4.3) and then (2.7), this last expression becomes

$$\begin{aligned}
(ck/a) \sum_{j=0}^{k-1} a_j \int_0^1 ((x - j/k))((bx))\, dx &= (ck/2ab) \sum_{j=0}^{k-1} a_j \mathscr{B}_2(bj/k) \\
&= (ck/2ab) \sum_{n=0}^{k-1} a_{nb'} \mathscr{B}_2(n/k) \\
&= (c/2ab)\, B_2(A'),
\end{aligned}$$

where we have employed Lemma 4.2 and lastly used (2.7) again. This completes the proof.

THEOREM 7.2 (Three-term relation). *Let A and C be sequences with period k. Let a, b, and c be positive integers such that $(a, b) = (a, c) = (b, c) = (b, k) = 1$. Determine b' by $bb' \equiv 1 \pmod{k}$, and let $A' = \{a_{nb'}\}$ and $C' = \{c_{nb'}\}$. Then,*

$$\sum_{n(\mathrm{mod}\, ck)} a_n \mathscr{B}_1(an/c, C)\, \mathscr{B}_1(bn/ck)$$

$$+ \sum_{n(\mathrm{mod}\, ak)} c_n \mathscr{B}_1(cn/a, A)\, \mathscr{B}_1(bn/ak) + \sum_{n(\mathrm{mod}\, b)} \mathscr{B}_1(ckn/b, A)\, \mathscr{B}_1(akn/b, C)$$

$$= -\tfrac{1}{4} a_0 c_0 + (c/2ab) B_0(A) B_2(C') + (a/2bc)\, B_0(C) B_2(A') + (b/2ac) E(a,c;A,C), \tag{7.1}$$

where

$$E(a, c; A, C) = \sum_{r,j=0}^{k-1} a_r c_j \mathscr{B}_2((ar - cj)/k). \tag{7.2}$$

Proof. In the periodic Poisson summation formula (2.4), let $\alpha = 0$, $\beta = ck$, and $f(x) = \mathscr{B}_1(ax/c, C)\, \mathscr{B}_1(bx/ck)$. Because of the hypotheses $(a, c) = (b, k) = 1$, $\mathscr{B}_1(ax/c, C)$ and $\mathscr{B}_1(bx/ck)$ have no common discontinuities on $(0, ck)$. Using (2.6) and the fact that $\mathscr{B}_1(x, C)$ has period k, we find that

$$\lim_{x \to 0+} f(x) = \tfrac{1}{2} \{B_1(C) + c_0\}, \qquad \lim_{x \to ck-} f(x) = -\tfrac{1}{2} B_1(C).$$

Hence, (2.4) yields, with the aid of Lemma 7.1,

$$\tfrac{1}{4} a_0 c_0 + \sum_{n=1}^{ck-1} a_n \mathscr{B}_1(an/c, C)\, \mathscr{B}_1(bn/ck) = (c/2ab)\, B_0(A)\, B_2(C')$$

$$+ \sum_{n=1}^{\infty} \int_0^{ck} (b_n e^{2\pi i n x/k} + b_{-n} e^{-2\pi i n x/k})\, \mathscr{B}_1(ax/c, C)\, \mathscr{B}_1(bx/ck)\, dx, \tag{7.3}$$

where $\{b_n\}$ is defined by (2.2). For each integer $n \neq 0$, let

$$I_n = \int_0^1 e^{2\pi i n c y} \mathscr{B}_1(aky, C)\, \mathscr{B}_1(by)\, dy$$

$$= \sum_{j=0}^{k-1} c_{-j} K_j(n), \tag{7.4}$$

by (2.7), where

$$K_j(n) = \int_0^1 e^{2\pi i n c y} ((ay + j/k))((by))\, dy. \tag{7.5}$$

To evaluate the integrals $K_j(n)$, $0 \leqslant j \leqslant k-1$, we integrate by parts and then use a generalization of the integration by parts formula [28, p. 22]. Since $(b, ak) = 1$, $((by))$ and $((ay + j/k))$, $0 \leqslant j \leqslant k-1$, have no discontinuities in common on $(0, 1)$. If $j = 0$, $((by))$ and $((ay + j/k))$ do have common discontinuities at $y = 0$ and $y = 1$. In this case, integrate over $[\epsilon, 1-\delta]$, where $\epsilon, \delta > 0$, and then let ϵ and δ tend to 0. At each discontinuity of $((by))$ or $((ay + j/k))$, but not both, there is a "jump" of (-1). Keeping all of the above considerations in mind, we find that

$$\begin{aligned} K_j(n) &= -\frac{1}{2\pi inc}\int_0^1 e^{2\pi incy}\, d\{((ay + j/k))((by))\} \\ &= -\frac{1}{2\pi inc}\int_0^1 e^{2\pi incy}((ay + j/k))\, d((by)) \\ &\quad -\frac{1}{2\pi inc}\int_0^1 e^{2\pi incy}((by))\, d((ay + j/k)) \\ &= \frac{1}{2\pi inc}\sum_{r=0}^{b-1} e^{2\pi incr/b}((ar/b + j/k)) - \frac{b}{2\pi inc}\int_0^1 e^{2\pi incy}((ay + j/k))\, dy \\ &\quad + \frac{1}{2\pi inc}\sum_{r=1}^{a} e^{2\pi inc(r-j/k)/a}((b(r - j/k)/a)) - \frac{a}{2\pi inc}\int_0^1 e^{2\pi incy}((by))\, dy. \end{aligned} \tag{7.6}$$

The two integrals on the far right side of (7.6) are each evaluated by a further integration by parts. Thus, since $(b, c) = 1$,

$$\begin{aligned} -\frac{a}{2\pi inc}\int_0^1 e^{2\pi incy}((by))\, dy &= \frac{a}{(2\pi inc)^2}\int_0^1 e^{2\pi incy}\, d((by)) \\ &= -\frac{a}{(2\pi inc)^2}\sum_{r=0}^{b-1} e^{2\pi incr/b} = -\frac{a\,\delta(n, b)}{(2\pi inc)^2}, \end{aligned} \tag{7.7}$$

where $\delta(n, b) = b$ if $n \equiv 0 \pmod{b}$ and $\delta(n, b) = 0$, otherwise. Since $(a, c) = 1$, a similar calculation gives

$$-\frac{b}{2\pi inc}\int_0^1 e^{2\pi incy}((ay + j/k))\, dy = -\frac{b\,\delta(n, a)\, e^{-2\pi incj/ak}}{(2\pi inc)^2}. \tag{7.8}$$

Putting (7.7) and (7.8) into (7.6), we find that

$$K_j(n) = \frac{1}{2\pi inc} \sum_{r=0}^{b-1} e^{2\pi incr/b}((ar/b + j/k))$$

$$+ \frac{1}{2\pi inc} \sum_{r=1}^{a} e^{2\pi inc(r-j/k)/a}((b(r - j/k)/a))$$

$$- \frac{b\,\delta(n, a)\, e^{-2\pi incj/ak}}{(2\pi inc)^2} - \frac{a\,\delta(n, b)}{(2\pi inc)^2}, \tag{7.9}$$

and then putting (7.9) into (7.4) and using (2.7), we deduce that

$$I_n = \frac{1}{2\pi inc} \sum_{r(\mathrm{mod}\, b)} e^{2\pi incr/b} \mathscr{B}_1(akr/b, C)$$

$$+ \frac{1}{2\pi inc} \sum_{m(\mathrm{mod}\, ak)} c_m e^{2\pi incm/ak}((bm/ak))$$

$$- \frac{b\,\delta(n, a)}{(2\pi inc)^2} \sum_{j=0}^{k-1} c_{-j} e^{-2\pi incj/ak} - \frac{ak\,\delta(n, b)}{(2\pi inc)^2} B_0(C). \tag{7.10}$$

Substituting (7.10) into (7.3) and utilizing (2.8), we see that

$$\tfrac{1}{4} a_0 c_0 + \sum_{n(\mathrm{mod}\, ck)} a_n \mathscr{B}_1(an/c, C)\, \mathscr{B}_1(bn/ck)$$

$$= (c/2ab)\, B_0(A) B_2(C') - \sum_{r(\mathrm{mod}\, b)} \mathscr{B}_1(ckr/b, A)\, \mathscr{B}_1(akr/b, C)$$

$$- \sum_{m(\mathrm{mod}\, ak)} c_m \mathscr{B}_1(cm/a, A)\, \mathscr{B}_1(bm/ak)$$

$$- (bk/ac) \sum_{j=0}^{k-1} c_{-j} \sum_{n=1}^{\infty} (1/2\pi in)^2 \{b_{na} e^{-2\pi incj/k} + b_{-na} e^{2\pi incj/k}\}$$

$$- (a/bc)\, B_0(C) \sum_{n=1}^{\infty} (k/2\pi in)^2 \{b_{nb} + b_{-nb}\}. \tag{7.11}$$

Now, from (2.2) or (2.3) we observe that $\{b_{nb}\}$ is the complementary sequence of $\{a_{nb'}\}$, where $bb' \equiv 1 \pmod{k}$. Thus,

$$-(a/bc)\, B_0(C) \sum_{n=1}^{\infty} (k/2\pi in)^2 \{b_{nb} + b_{-nb}\} = (a/2bc)\, B_0(C)\, B_2(A'). \tag{7.12}$$

Next, substitute (2.2) for b_{na} and b_{-na} and employ (2.8) again. Accordingly, we find that

$$
\begin{aligned}
&-(bk/ac)\sum_{j=0}^{k-1} c_{-j}\sum_{n=1}^{\infty}(1/2\pi in)^2\{b_{na}e^{-2\pi incj/k}+b_{-na}e^{2\pi incj/k}\}\\
&\quad=-(b/ac)\sum_{j=0}^{k-1} c_{-j}\sum_{r=0}^{k-1} a_r\sum_{n=1}^{\infty}(1/2\pi in)^2\{e^{-2\pi in(cj+ar)/k}+e^{2\pi in(cj+ar)/k}\}\\
&\quad=(b/2ac)\sum_{j=0}^{k-1} c_{-j}\sum_{r=0}^{k-1} a_r\mathscr{B}_2((cj+ar)/k)\\
&\quad=(b/2ac)E(a,c;A,C), \qquad (7.13)
\end{aligned}
$$

by (7.2). If we substitute (7.12) and (7.13) into (7.11), we arrive at (7.1).

THEOREM 7.3 (Reciprocity theorem). *Let a and c be coprime, positive integers, and suppose that A and C are arbitrary sequences, each with period k. Then,*

$$
\begin{aligned}
&s(a,c;A,C)+s(c,a;C,A)\\
&\quad=-\tfrac{1}{4}a_0c_0-\mathscr{B}_1(0,A)\,\mathscr{B}_1(0,C)+(c/2a)\,B_0(A)\,B_2(C)\\
&\qquad+(a/2c)\,B_0(C)\,B_2(A)+(1/2ac)\,E(a,c;A,C), \qquad (7.14)
\end{aligned}
$$

where $E(a, c; A, C)$ is defined by (7.2).

Proof. In Theorem 7.2, let $b = 1$. Observe that $A' = A$ and $C' = C$. Equation (7.14) is now then immediate from (7.1).

PROPOSITION 7.4. *Let a and c be coprime, positive integers with $c \equiv 0 \pmod{k}$. Define a' by $aa' \equiv 1 \pmod{k}$, and let $A'' = \{a_{na'}\}$. Then,*

$$E(a,c;A,C)=B_0(C)\,B_2(A''). \qquad (7.15)$$

Proof. Since $(a, c) = 1$ and $c \equiv 0 \pmod{k}$, it follows that $(a, k) = 1$. Thus, a' exists. Hence,

$$
\begin{aligned}
E(a,c;A,C)&=\sum_{j=0}^{k-1} c_j\sum_{r=0}^{k-1} a_r\mathscr{B}_2(ar/k)\\
&=kB_0(C)\sum_{r=0}^{k-1} a_{ra'}\mathscr{B}_2(r/k)\\
&=B_0(C)\,B_2(A''),
\end{aligned}
$$

by (2.7), and the proof is complete.

If the hypothesis $c \equiv 0 \pmod{k}$ is replaced by $a \equiv 0 \pmod{k}$ in Proposition 7.4, then,

$$E(a, c; A, C) = B_0(A)\, B_2(C'''), \tag{7.16}$$

where $C''' = \{c_{nc'}\}$ with $cc' \equiv 1 \pmod{k}$. If either $a \equiv 0 \pmod{k}$ or $c \equiv 0 \pmod{k}$, then (7.15), or (7.16), enables us to simplify Theorems 7.2 and 7.3.

If $A = C = I$, then $s(a, c; I, I) = s(a, c)$, $B_0(I) = 1$, $B_2(I) = \frac{1}{6}$, $E(a, c; I, I) = \frac{1}{6}$, and $\mathscr{B}_1(0, I) = 0$. Hence, (7.1) reduces to the three-term relation (3.1), and (7.14) reduces to the reciprocity theorem (1.1) for ordinary Dedekind sums.

A sequence $A = \{a_n\}$ is said to be *even* if $a_n = a_{-n}$ for every integer n; A is said to be *odd* if $a_n = -a_{-n}$ for every integer n. If A is even, then by (2.8) it is clear that $\mathscr{B}_1(0, A) = 0$. Hence, if either A or C is even, Theorem 7.3 simplifies slightly. If A is odd, it follows from (2.5) that $B_0(A) = 0 = B_0(A')$, since A' is odd. Furthermore, from (2.8) it is obvious that $B_2(A) = 0 = B_2(A')$. Hence, Theorems 7.2 and 7.3 greatly simplify if A or C is odd.

COROLLARY 7.5 (Reciprocity theorem for Dedekind character sums). *Let χ denote a nonprincipal character of modulus k. Let $a, c > 0$ with $(a, c) = 1$ and either a or $c \equiv 0 \pmod{k}$. Then,*

$$s(a, c; X, \bar{X}) + s(c, a; \bar{X}, X) = -B_1(X)\, B_1(\bar{X}).$$

Proof. Since χ is nonprincipal, $a_0 = c_0 = B_0(\chi) = B_0(\bar{\chi}) = 0$. With the use of (7.15) and (7.16), the result now follows at once from Theorem 7.3.

In the case when χ is primitive, Corollary 7.5 was first proved by the author in [3].

8. TWO FURTHER DEDEKIND SUMS INVOLVING PERIODIC BERNOULLI FUNCTIONS

As before, let $A = \{a_n\}$ denote a sequence with period k, and let a and c be integers. Define

$$s_1(a, c; A) = \sum_{j \pmod{ck}} a_j \mathscr{B}_1(aj/c)\, \mathscr{B}_1(j/ck)$$

and

$$s_2(a, c; A) = \sum_{j(\operatorname{mod} ck)} \mathscr{B}_1(aj/c, A)\,\mathscr{B}_1(j/ck).$$

In the case $A = \chi$, where χ is primitive, the sums $s_1(a, c; \chi)$ and $s_2(a, c; \chi)$ arose in the transformation formulas of certain other character analogs of $\log \eta(z)$ [6]. In [6], we used these transformation formulas to deduce a type of reciprocity relation for $s_1(a, c; \chi)$ and $s_2(c, a; \chi)$. In a manner similar to that of Section 7, we first prove a three-term relation from which we deduce a reciprocity theorem for $s_1(a, c; A)$ and $s_2(c, a; A)$.

THEOREM 8.1 (Three-term relation). *Let a, b, and c be positive integers relatively prime in pairs with $(ab, k) = 1$. Let A' be defined as in Theorem 7.2. Then,*

$$\sum_{n(\operatorname{mod} ak)} a_n \mathscr{B}_1(cn/a)\,\mathscr{B}_1(bn/ak) + \sum_{n(\operatorname{mod} ck)} \mathscr{B}_1(an/c, A)\,\mathscr{B}_1(bn/ck) + \sum_{n(\operatorname{mod} b)} \mathscr{B}_1(akn/b, A)\,\mathscr{B}_1(ckn/b)$$
$$= -\tfrac{1}{4} a_0 + (a/12bc)\,B_0(A) + (b/12ac)\,B_0(A) + (c/2ab)\,B_2(A'). \tag{8.1}$$

Proof. We apply the ordinary Poisson summation formula (2.1) with $\alpha = 0$, $\beta = ck$, and $f(x) = \mathscr{B}_1(ax/c, A)\,\mathscr{B}_1(bx/ck)$. Proceeding in precisely the same fashion as in the proof of Theorem 7.2, we have

$$\tfrac{1}{4} a_0 + \sum_{n=1}^{ck-1} \mathscr{B}_1(an/c, A)\,\mathscr{B}_1(bn/ck)$$
$$= (c/2ab)B_2(A') + 2 \sum_{n=1}^{\infty} \int_0^{ck} \mathscr{B}_1(ax/c, A)\,\mathscr{B}_1(bx/ck) \cos(2\pi nx)\, dx$$
$$= (c/2ab)\,B_2(A') + 2ck \sum_{n=1}^{\infty} \sum_{j=0}^{k-1} a_{-j} L_j(n), \tag{8.2}$$

where

$$L_j(n) = \int_0^1 ((ay + j/k))((by)) \cos(2\pi ncky)\, dy.$$

The integrals $L_j(n)$ are calculated in exactly the same fashion as the integrals $K_j(n)$ defined by (7.5). Accordingly, we get

$$L_j(n) = \frac{1}{2\pi nck}\sum_{r=0}^{b-1}\sin(2\pi nckr/b)((ar/b + j/k))$$

$$+ \frac{1}{2\pi nck}\sum_{r=1}^{a}\sin(2\pi nck(r - j/k)/a)((b(r - j/k)/a))$$

$$+ \frac{b\,\delta(n, a)\cos(2\pi ncj/a)}{(2\pi nck)^2} + \frac{a\,\delta(n, b)}{(2\pi nck)^2}.$$

The hypothesis $(a, k) = 1$, not needed for Theorem 7.2, is needed in the above calculation. Thus, (8.2) becomes

$$\frac{1}{4}a_0 + \sum_{n=1}^{ck-1}\mathscr{B}_1(an/c, A)\,\mathscr{B}_1(bn/ck)$$

$$= \frac{c}{2ab}B_2(A') + \sum_{r=0}^{b-1}\sum_{j=0}^{k-1}a_{-j}((ar/b + j/k))\sum_{n=1}^{\infty}\frac{\sin(2\pi nckr/b)}{\pi n}$$

$$+ \sum_{r=1}^{a}\sum_{j=0}^{k-1}a_{-j}((b(r - j/k)/a))\sum_{n=1}^{\infty}\frac{\sin(2\pi nck(r - j/k)/a)}{\pi n}$$

$$+ \frac{b}{2\pi^2 ack}\sum_{j=0}^{k-1}a_{-j}\sum_{n=1}^{\infty}\frac{\cos(2\pi ncj)}{n^2} + \frac{a}{2\pi^2 bck}\sum_{j=0}^{k-1}a_{-j}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= -\sum_{r(\mathrm{mod}\,b)}\mathscr{B}_1(akr/b, A)\,\mathscr{B}_1(ckr/b) - \sum_{n(\mathrm{mod}\,ak)}a_n\mathscr{B}_1(bn/ak)\,\mathscr{B}_1(cn/a)$$

$$+ \frac{c}{2ab}B_2(A') + \frac{b}{12ac}B_0(A) + \frac{a}{12bc}B_0(A), \tag{8.3}$$

where we have employed (2.7) and (2.8). Obviously, (8.3) is equivalent to (8.1), and the proof is complete.

THEOREM 8.2 (Reciprocity theorem). *Let a and c be coprime, positive integers with* $(a, k) = 1$. *Then*

$$s_1(c, a; A) + s_2(a, c; A) = -\frac{1}{4}a_0 + \frac{a}{12c}B_0(A) + \frac{1}{12ac}B_0(A) + \frac{c}{2a}B_2(A).$$

Proof. Set $b = 1$ in Theorem 8.1.

The previous two results were proved under the hypothesis $(a, k) = 1$. We now suppose that $a \equiv 0 \pmod{k}$. Except for one slight change, the proof of the next result is like that of Theorem 8.1.

THEOREM 8.3 (Three-term relation). *Let a, b, and c be positive integers relatively prime in pairs with $(b, k) = 1$ and $a \equiv 0 \pmod{k}$. Let A' be defined as in Theorem 7.2. Let c' be defined by $cc' \equiv 1 \pmod{k}$, and let $A''' = \{a_{nc'}\}$. Then,*

$$\sum_{n(\mathrm{mod}\, ak)} a_n \mathscr{B}_1(cn/a)\, \mathscr{B}_1(bn/ak) + \sum_{n(\mathrm{mod}\, ck)} \mathscr{B}_1(an/c, A)\, \mathscr{B}_1(bn/ck)$$
$$+ \sum_{n(\mathrm{mod}\, b)} \mathscr{B}_1(akn/b, A)\, \mathscr{B}_1(ckn/b)$$
$$= -\frac{1}{4} a_0 + \frac{a}{12bc} B_0(A) + \frac{b}{2ac} B_2(A''') + \frac{c}{2ab} B_2(A').$$

THEOREM 8.4 (Reciprocity theorem). *Let a and c be coprime, positive integers with $a \equiv 0 \pmod{k}$. Let A''' be defined as in Theorem 8.3. Then,*

$$s_1(c, a; A) + s_2(a, c; A) = -\frac{1}{4} a_0 + \frac{a}{12c} B_0(A) + \frac{1}{2ac} B_2(A''') + \frac{c}{2a} B_2(A).$$

If $A = I$, Theorems 8.1 and 8.3 both reduce to the three-term relation (3.1), and Theorems 8.2 and 8.4 both reduce to the reciprocity theorem (1.1) for ordinary Dedekind sums. If A is odd, the above theorems reduce to trivialities. We shall examine the results when $A = \chi$, where χ is an even character.

COROLLARY 8.5. *Let a, b, and c be positive integers relatively prime in pairs with $(b, k) = 1$. Let χ be an even, nonprincipal character modulo k. Then,*

$$\sum_{n(\mathrm{mod}\, ak)} \chi(n)\, \mathscr{B}_1(cn/a)\, \mathscr{B}_1(bn/ak) + \sum_{n(\mathrm{mod}\, ck)} \mathscr{B}_1(an/c, \chi)\, \mathscr{B}_1(bn/ck)$$
$$+ \sum_{n(\mathrm{mod}\, b)} \mathscr{B}_1(akn/b, \chi)\, \mathscr{B}_1(ckn/b)$$
$$= (c/2ab)\, \bar\chi(b)\, B_2(\chi), \qquad \text{if } (a, k) = 1,$$
$$= (c/2ab)\, \bar\chi(b)\, B_2(\chi) + (b/2ac)\, \bar\chi(c)\, B_2(\chi), \qquad \text{if } a \equiv 0 \pmod{k}.$$

Proof. For $(a, k) = 1$, apply Theorem 8.1. Since $A' = \{\chi(nb')\} = \chi(b')\{\chi(n)\} = \bar{\chi}(b)\{\chi(n)\}$, the first part of the theorem follows. For $a \equiv 0 \pmod{k}$, $A''' = \{\chi(nc')\} = \bar{\chi}(c)\{\chi(n)\}$, and so the second part of the theorem follows from Theorem 8.3.

The next result was proved by the use of transformation formulas in [6] under slightly more restrictive hypotheses.

COROLLARY 8.6. *Let a and c be coprime, positive integers, and let χ be an even, nonprincipal character modulo k. Then,*

$$\begin{aligned} s_1(c, a; \chi) &+ s_2(a, c; \chi) \\ &= (c/2a)\, B_2(\chi), && \text{if } (a, k) = 1, \\ &= (c/2a)\, B_2(\chi) + (1/2ac)\, \bar{\chi}(c)\, B_2(\chi), && \text{if } a \equiv 0 \pmod{k}. \end{aligned}$$

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