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Maximal B-regular boundary value problems with parameters ☆

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Abstract

This study focuses on non-local boundary value problems (BVP) for elliptic differential-operator equations (DOE) defined in Banach-valued Besov (B) spaces. Here equations and boundary conditions contain certain parameters. This study found some conditions that guarantee the maximal regularity and fredholmness in Banach-valued B-spaces uniformly with respect to these parameters. These results are applied to non-local boundary value problems for a regular elliptic partial differential equation with parameters on a cylindrical domain to obtain algebraic conditions that guarantee the same properties.

Keywords: Boundary value problems (BVP); Differential-operator equations (DOE); Maximal regular BVP; Banach-valued Besov spaces; Operator-valued multipliers; Interpolation of Banach spaces

1. Introduction and notations

BVPs for DOE have been elaborated in detail by authors of [5,18,23,34,43,45]. The solvability and spectrum of BVPs for elliptic DOE have been studied also in [4,7,8,12, 16,22,33,35–39,44]. A comprehensive introduction to the differential-operator equations

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and historical references may be found in [22,45]. In these works Hilbert-valued function spaces essentially have been considered. The maximal regular initial and BVPs in Banach-valued function spaces have been investigated, e.g., in [4,13,38,42] . The main objective of the present paper is to discuss non-local BVPs for DOE with parameters in Banach-valued B-spaces. In this work:

(1) The BVPs are considered in Banach-valued B spaces.
(2) DOE and BVP contain certain parameters in principal party.
(3) Boundary conditions are, generally, non-local and non-homogeneous.
(4) Operators occurring in equations and boundary conditions are, in general, unbounded.

The uniformly maximal regularity and fredholmness of these problems with respect to these parameters are proved. These results are applied to non-local boundary value problems for elliptic, quasi-elliptic partial differential equations with parameters, and their finite or infinite systems on cylindrical domains.

Let E be a Banach space and $x = (x_1, x_2, \ldots, x_n) \in \Omega \subset R^n$. Let $L_p(\Omega; E)$ denote a space of strongly measurable E-valued Bochner functions on Ω with the norm

$$\|f\|_{L_p(\Omega;E)} = \left(\int \|f(x)\|_E^p\,dx\right)^{1/p}, \quad 1 \leqslant p < \infty.$$

By $L_{\mathbf{p}}(\Omega)$ and $B^l_{\mathbf{p},q}(\Omega)$, $\mathbf{p} = (p_1, p_2)$, will be denoted a scalar-valued $\mathbf{p}$-summable function space and Besov space with mixed norm, respectively (see, e.g., [9]). Let $S(R^n; E)$ denote a Schwartz class, i.e. a space of E-valued rapidly decreasing smooth functions φ on R^n and $S'(R^n; E)$ denotes a E-valued tempered distributions. Let $y \in R$, $m \in N$, and let e_i, $i = 1, 2, \ldots, n$, be standard unit vectors in R^n. Let

$$\begin{aligned}
&\Delta_i(y)f(x) = f(x + ye_i) - f(x),\\
&\quad\vdots\\
&\Delta_i^m(y)f(x) = \Delta_i(y)\left[\Delta_i^{m-1}(y)f(x)\right] = \sum_{k=0}^{m}(-1)^{m+k}C_m^k f(x + kye_i),\\
&\Delta_i(y) = \Delta_i(\Omega, y) = \begin{cases} \Delta_i(y)f(x), & \text{for } [x, x + mye_i] \subset \Omega,\\ 0 & \text{for } [x, x + mye_i] \notin \Omega.\end{cases}
\end{aligned}$$

Let m_i be positive integers, k_i be non-negative integers, s_i be positive numbers and $m_i > s_i - k_i > 0$, $i = 1, 2, \ldots, n$, $s = (s_1, s_2, \ldots, s_n)$, $1 \leqslant p \leqslant \infty$, $1 \leqslant \theta \leqslant \infty$, $0 < y_0 < \infty$. Let F denote Fourier transform. The Banach-valued Besov spaces $B^s_{p,\theta}(\Omega; E)$ are defined as

$$\begin{aligned}
B^s_{p,\theta}(\Omega; E) = \Bigg\{ f:\ & f \in L_p(\Omega; E),\ \|f\|_{B^s_{p,\theta}(\Omega;E)} = \|f\|_{L_p(\Omega;E)}\\
&+ \sum_{i=1}^{n}\left(\int_0^{h_0} h^{-[(s_i-k_i)q+1]}\left\|\Delta_i^{m_i}(h,\Omega)D_i^{k_i}f(x)\right\|^{\theta}_{L_p(\Omega;E)}\,dy\right)^{1/\theta} < \infty\\
&\text{for } 1 \leqslant \theta < \infty\Bigg\},
\end{aligned}$$

$$\|f\|_{B^s_{p,q}(\Omega;E)} = \sum_{i=1}^{n} \sup_{0<h<h_0} \frac{\Delta_i^{m_i}(h,\Omega) D_i^{k_i} f(x)_{L_p(\Omega;E)}}{h^{s_i-k_i}} \quad \text{for } \theta = \infty.$$

A Banach space E is said to be ζ-convex (see [10,11,13,30]) if there exists a symmetric real-valued function $\zeta(u,v)$ on $E \times E$ which is convex with respect to each of the variables and satisfies the conditions

$$\zeta(0,0) > 0, \qquad \zeta(u,v) \leqslant \|u+v\| \quad \text{for } \|u\| = \|v\| = 1.$$

A ζ-convex Banach space E is often called a UMD-space and written as $E \in \mathrm{UMD}$. It is shown [10] that a Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy$$

is bounded in $L_p(R;E)$, $p \in (1,\infty)$, for those and only those spaces E which possess the property of UMD spaces. UMD spaces contain, e.g., L_p, l_p and Lorentz spaces L_{pq}, $p,q \in (1,\infty)$.

Let $\mathbf{C}$ be a set of complex numbers and

$$S_\varphi = \{\lambda\colon\ \lambda \in \mathbf{C},\ |\arg \lambda - \pi| \leqslant \pi - \varphi\} \cup \{0\}, \quad 0 < \varphi \leqslant \pi.$$

A linear operator A is said to be φ-positive in a Banach space E, with bound $M > 0$ if:

- $D(A)$ is dense on E and $\|(A-\lambda I)^{-1}\|_{L(E)} \leqslant M(1+|\lambda|)^{-1}$ with $\lambda \in S_\varphi$, $\varphi \in (0,\pi]$,
- I is an identity operator in E,
- $L(E)$ is a space of bounded linear operators acting in E.

Sometimes, instead of $A + \lambda I$, it will be written $A + \lambda$ or denoted by A_λ. It is known [41, Section 1.15.1] that there exist fractional powers A^θ of the positive operator A. Let $E(A^\theta)$ denote a space $D(A^\theta)$ with graph norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \|A^\theta u\|^p\right)^{1/p}, \quad 1 \leqslant p < \infty,\ -\infty < \theta < \infty.$$

Let E_1 and E_2 be two Banach spaces. By $(E_1,E_2)_{\theta,p}$, $0<\theta<1$, $1 \leqslant p < \infty$, will be denoted interpolation spaces defined by real method (see, e.g., [26] or [41, Sections 1.3–1.8]). Let $\Omega \in R^n$, $l = (l_1, l_2, \ldots, l_n)$, $D_k^{l_k} = \partial^{l_k}/\partial x_k^{l_k}$. Let E_0 and E be two Banach spaces with E_0 continuously and densely embedded into E. Let us introduce Besov–Lions type spaces $B^{l,s}_{p,q}(\Omega;E_0,E)$ which are collections of functions $u \in B^s_{p,q}(\Omega;E_0)$ having the generalized derivatives $D_k^{l_k} u \in B^s_{p,q}(\Omega;E)$, $k = 1,2,\ldots,n$, with norm

$$\|u\|_{B^{l,s}_{p,q}(\Omega;E_0,E)} = \|u\|_{B^s_{p,q}(\Omega;E_0)} + \sum_{k=1}^{n} \left\| D_k^{l_k} u \right\|_{B^s_{p,q}(\Omega;E)} < \infty.$$

Let $l = (l_1, l_2, \ldots, l_n)$, $s = (s_1, s_2, \ldots, s_n)$ and $l_i \geqslant s_i > 0$. We set $B^{l+s}_{p,q}(\Omega;E_0,E) = B^s_{p,q}(\Omega;E_0) \cap B^l_{p,q}(\Omega;E)$ with norm

$$\|u\|_{B^l_{p,q}(\Omega;E_0,E)} = \|u\|_{B^s_{p,q}(\Omega;E_0)} + \|u\|_{B^l_{p,q}(\Omega;E)}.$$

Let m be integer and $(a,b)\subset R$. Then for $n=1$, $\Omega=(a,b)$, $l_1=l_2=\cdots=l_n=m$ and for $s_1=s_2=\cdots=s_n=s$, spaces $B^{l,s}_{p,q}(\Omega;E)$, $B^{l,s}_{p,q}(\Omega;E_0,E)$ and $B^{l+s}_{p,q}(\Omega;E_0,E)$ will be denoted by $B^{m,s}_{p,q}(a,b;E)$, $B^{m,s}_{p,q}(a,b;E_0,E)$, and $B^{l+s}_{p,q}(a,b;E_0,E)$, respectively.

Let $t=(t_1,t_2,\dots,t_n)$, where t_j are positive parameters. We define in $B^{l,s}_{p,q}(\Omega;E_0,E)$ parameterized norm

$$\|u\|_{B^{l,s}_{p,q,t}(\Omega;E_0,E)}=\|u\|_{B^s_{p,q}(\Omega;E_0)}+\sum_{k=1}^n\big\|t_kD_k^{l_k}u\big\|_{B^s_{p,q}(\Omega;E)}.$$

A function $\Psi\in C^{(l)}(R^n;L(E_1,E_2))$ is called a Fourier multiplier from $B^s_{p,q}(R^n;E_1)$ to $B^s_{p,q}(R^n;E_2)$ if there exists a constant $C>0$ such that

$$\big\|F^{-1}\Psi(\xi)Fu\big\|_{B^s_{p,q}(R^n;E_2)}\leqslant C\|u\|_{B^s_{p,q}(R^n;E_1)}$$

for all $u\in B^s_{p,q}(R^n;E_1)$, where F and F^{-1} are the Fourier and the inverse Fourier transforms, respectively. The set of all multipliers from $B^s_{p,q}(R^n;E_1)$ to $B^s_{p,q}(R^n;E_2)$ will be denoted by $M^s_{p,q}(E_1,E_2)$. For the case $E_1=E_2=E$ it will be denoted by $M^s_{p,q}(E)$. Let

$$H_k=\big\{\Psi_h\in M_{p,q}(E_1,E_2):\ h=(h_1,h_2,\dots,h_L)\in Q\big\}$$

be a collection of multipliers in $M^s_{p,q}(E_1,E_2)$. We say that $\Psi_h=\Psi_h(\xi)$ is a uniformly bounded multiplier with respect to h if there exists a constant $C>0$, independent of $h\in K(h)$ and such that

$$\big\|F^{-1}\Psi_hFu\big\|_{B^s_{p,q}(R^n,E_2)}\leqslant C\|u\|_{B^s_{p,q}(R^n,E_1)}$$

for all $h\in K(h)$ and $u\in B^s_{p,q}(R^n;E_1)$.

The exposition of the theory of L_p-multipliers of the Fourier transformation and some related references can be found in [41, Sections 2.2.1–2.2.4]. On the other hand, in vector-valued function spaces, Fourier multipliers have been studied by [4,11,13,19,27,42]. Let

$$\beta=(\beta_1,\beta_2,\dots,\beta_n),$$
$$V_n=\big\{\xi=(\xi_1,\xi_2,\dots,\xi_n)\in R^n:\ \xi_i\neq0,\ i=1,2,\dots,n\big\}.$$

Definition 1. A Banach space E satisfies a B-multiplier condition with respect to $p\in(1,\infty)$ and $q\in[1,\infty]$, when $\Psi\in C^n(R^n;B(E))$, $|\beta|\leqslant n$, $\xi\in V_n$, if the estimate

$$|\xi_1|^{\beta_1}|\xi_2|^{\beta_2}\cdots|\xi_n|^{\beta_n}\big\|D^\beta\Psi(\xi)\big\|_{L(E)}\leqslant C\quad\text{implies}\quad\Psi\in M^s_{p,q}(E).$$

Remark. By virtue of [4] or [19], all ζ-convex spaces satisfies B-multiplier condition. In a similar way as in [4, Theorems 7.1, 7.3] or [19, Corollary 4.11], it is shown that if $\Psi_h=\Psi_h(\xi)\in C^n(R^n;B(E))$ satisfies the above estimate uniformly with respect to h then Ψ_h is a uniformly bounded multiplier in $B^s_{p,q}(R^n;E)$ with respect to h.

It is well known (e.g., see [4,19]) that any Hilbert space satisfies the B-multiplier condition with respect to any $p\in(1,\infty)$ and $q\in[1,\infty]$. There are, however, Banach spaces that are not Hilbert spaces but satisfy the B-multiplier condition, e.g., UMD spaces (see, e.g., [4,42]). However, for UMD spaces additional conditions are needed for operator-valued multipliers to be satisfied by the L_p-multiplier condition (e.g., R-boundedness [11,13,42]).

An operator $A(t)$ is said to be φ-positive in a Banach space E uniformly with respect to t if $D(A(t))$ is independent of t, $D(A(t))$ is dense in E, and

$$\left\|\left(A(t)-\lambda I\right)^{-1}\right\| \leqslant \frac{M}{1+|\lambda|} \quad \text{for all } \lambda \in S(\varphi),\ \varphi \in (0,\pi).$$

Let $\sigma_\infty(E)$ denote a space of compact operators acting in E.

2. Background material

Embedding theorems of vector-valued Besov spaces play important role in the present investigation. Such theorems have been studied, e.g., in [4,6,39,40]. Let

$$s=(s_1,s_2,\ldots,s_n), \quad l=(l_1,l_2,\ldots,l_n), \quad s_k>0,\ l_k>0.$$

By using a similar technique as in [9, Section 18] and [41, Section 14] we obtain the following.

Lemma A₁. *Let E be a Banach space satisfying the multiplier condition with respect to $p \in (1,\infty)$ and $q \in [1,\infty]$. Let $\Omega \in R^n$ be a region such that there exists a bounded linear extension operator acting from $B^s_{p,q}(\Omega;E)$ to $B^s_{p,q}(R^n;E)$ and also from $B^{l+s}_{p,q}(R^n;E)$ to $B^{l+s}_{p,q}(R^n;E)$. Then*

$$B^{l,s}_{p,q}(\Omega;E)=B^{l+s}_{p,q}(\Omega;E).$$

By using techniques similar to [36–39], we obtain the following.

Theorem A₁. *Let the following conditions be satisfied*:

(1) *E is a Banach space satisfying the multiplier condition with respect to $p \in (1,\infty)$ and $q \in [1,\infty]$.*
(2) *$\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$, $l=(l_1,l_2,\ldots,l_n)$ are n-tuples of non-negative integer numbers such that*

$$\varkappa=\sum_{k=1}^{n}\frac{\alpha_k}{l_k}\leqslant 1, \qquad 0\leqslant\mu\leqslant 1-\varkappa.$$

(3) *A is a φ-positive operator in E for $0<\varphi\leqslant\pi$ and $0<h\leqslant h_0<\infty$, $0<t_k\leqslant T<\infty$.*
(4) *$\Omega \in R^n$ is a region such that there exists a bounded linear extension operator acting from $B^s_{p,q}(\Omega;E)$ to $B^s_{p,q}(R^n;E)$ and from $B^{l,s}_{p,q}(\Omega;E(A),E)$ to $B^{l,s}_{p,q}(R^n;E(A),E)$.*

Then an embedding

$$D^\alpha B^{l,s}_{p,q}\big(\Omega;E(A),E\big)\subset B^s_{p,q}\big(\Omega;E\big(A^{1-\varkappa-\mu}\big)\big)$$

is continuous and there exists a positive constant C_μ such that

$$\prod_{k=1}^{n} t_k^{\alpha_k/l_k} \|D^\alpha u\|_{B^s_{p,q}(\Omega;E(A^{1-\varkappa-\mu}))} \leqslant C_\mu\big[h^\mu \|u\|_{B^{l,s}_{p,q}(\Omega;E(A),E)} + h^{-(1-\mu)}\|u\|_{B^s_{p,q}(\Omega;E)}\big] \tag{1}$$

for all $u \in B^{l,s}_{p,q}(\Omega; E(A), E)$.

Proof. The theorem, at first, is proved for the case $\Omega = R^n$. In this case inequality (1) is equivalent to

$$\prod_{k=1}^{n} t_k^{\alpha_k/l^k} \big\|F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu}\hat{u}\big\|_{B^s_{p,q}(R^n;E)} \leqslant C_\mu\left\{h^\mu\left(\|F^{-1}A\hat{u}\|_{B^s_{p,q}(R^n;E)} + \sum_{k=1}^{n}\big\|t_k F^{-1}\big((i\xi_k)^{l_k}\hat{u}\big)\big\|_{B^s_{p,q}(R^n;E)}\right) + h^{-(1-\mu)}\|F^{-1}\hat{u}\|_{B^s_{p,q}(R^n;E)}\right\}.$$

So it is sufficient to show that the operator-function

$$\Psi_t(\xi) = \Psi_{t,h,\mu}(\xi) = \prod_{k=1}^{n} t_k^{\alpha_k/l_k}(i\xi)^\alpha A^{1-\varkappa-\mu}h^{-\mu}\left[A + \sum_{k=1}^{n} t_k\delta^{l_k}(\xi_k) + h^{-1}\right]$$

is a multiplier in $B^s_{p,q}(R^n; E)$, where $\delta \in C^\infty(R)$ with $\delta(y) \geqslant 0$ for all $y \geqslant 0$, $\delta(y) = 0$ for $|y| \leqslant 1/2$, $\delta(y) = 1$ for $|y| \geqslant 1$ and $\delta(-y) = -\delta(y)$ for all y. Indeed as in [36,37], using the Moment inequality for powers of positive operators and the Young inequality, we obtain that the operator function $\Psi_t(\xi)$ for all with $|\beta| \leqslant n$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in R^n$, $\xi_i \neq 0$, satisfies the estimate

$$\big\|D^\beta \Psi_t(\xi)\big\|_{L(E)} \leqslant C|\xi|^{-\beta}$$

uniformly with respect to $\xi \in R^n$ and parameters h, t. Therefore, by Definition 1, $\Psi_t(\xi)$ is multiplier in $B^s_{p,q}(R^n; E)$. Thus, the theorem for the case of $\Omega = R^n$ is proved. Then, by using the extension operator in spaces $B^{l,s}_{p,q}(\Omega; E(A), E)$ and $B^s_{p,q}(\Omega; E)$, we obtain Theorem A_1 for the general case of Ω. □

Theorem A_2. *Suppose all conditions of Theorem* A_1 *are satisfied and suppose* Ω *is a bounded region in* R^n, $A^{-1} \in \sigma_\infty(E)$. *Then, for* $0 < \mu \leqslant 1 - \varkappa$, *an embedding*

$$D^\alpha B^{l,s}_{p,q}\big(\Omega; E(A), E\big) \subset B^s_{p,q}\big(\Omega; E(A^{1-\varkappa-\mu})\big)$$

is compact.

Proof. By putting $h = \|u\|_{B^s_{p,q}(\Omega;E)}/\|u\|_{B^{l,s}_{p,q,t}(\Omega;E(A),E)}$ in (1) we obtain multiplicative inequality

$$\prod_{k=1}^{n} t_k^{\alpha_k/l_k} \|D^\alpha u\|_{B^s_{p,q}(\Omega;E(A^{1-\varkappa-\mu}))} \leqslant C_\mu \|u\|^\mu_{B^s_{p,q}(\Omega;E)} \|u\|^{1-\mu}_{B^{l,s}_{p,q,t}(\Omega;E(A),E)}. \tag{2}$$

By virtue of [6], an embedding

$$B^{l+s}_{p,q}\big(\Omega; E(A), E\big) \subset B^{s}_{p,q}(\Omega; E)$$

is compact. Then, by virtue of Lemma A_1 and by estimate (2), we obtain the assertion of Theorem A_2. □

Theorem A_3. *Suppose all conditions of Theorem* A_1 *are satisfies for* $\varphi \in (0, \pi/2)$ *and* $0 < \mu < 1 - \varkappa$.

Then an embedding

$$D^{\alpha} B^{l,s}_{p,q}\big(\Omega; E(A), E\big) \subset B^{s}_{p,q}\big(\Omega; \big(E(A), E\big)_{\varkappa,1}\big)$$

is continuous and there exists a positive constant C_μ *such that*

$$\prod_{k=1}^{n} t_k^{\alpha_k/l_k} \|D^{\alpha} u\|_{B^{s}_{p,q}(\Omega;(E(A),E)_{\varkappa+\mu,1})} \leqslant C_\mu\big[h^{\mu}\|u\|_{B^{l,s}_{p,q,t}(\Omega;E(A),E)} + h^{-(1-\mu)}\|u\|_{B^{s}_{p,q}(\Omega;E)}\big]$$

for all $u \in B^{l,s}_{p,q}(\Omega; E(A), E)$.

By reasoning as in Theorem A_2, we obtain

Theorem A_4. *Suppose all the conditions of Theorem* A_3*s are satisfied and suppose* Ω *is a bounded region in* R^n, $A^{-1} \in \sigma_\infty(E)$. *Then, for* $0 < \mu < 1 - \varkappa$, *an embedding*

$$D^{\alpha} B^{l,s}_{p,q}\big(\Omega; E(A), E\big) \subset B^{s}_{p,q}\big(\Omega; \big(E(A), E\big)_{\varkappa+\mu,1}\big)$$

is compact.

By using of estimates of semigroups generated by positive operators in $B^{s}_{p,q}(0, b; E)$ and by reasoning as [12], we obtain:

Theorem A_5. *Let* A *be a positive operator of type* φ *with a bound* M *in a Banach space* E *and* $0 < s < 1$. *Moreover, let* m *be a positive integer,*

$$\eta = \eta(s, m, \alpha) = \frac{\alpha}{m} + \frac{s}{2m} - \frac{1}{2mp},$$

$1 \leqslant p < \infty$, $1 \leqslant q < \infty$ *and* $\frac{1}{2p} - s < \alpha < m + \frac{1}{2p} - s$.

Then for $\lambda \in S(\varphi)$ an operator $-A_\lambda^{1/2}$ generates a semigroup $e^{-A_\lambda^{1/2}x}$ which is holomorphic for $x > 0$ and strongly continuous for $x \geqslant 0$. Moreover, there exists a constant $C > 0$ (depending only on $M, \varphi, m, \alpha, s, p, q$) such that

$$\big\|A_\lambda^{\alpha} e^{-xA_\lambda^{1/2}} u\big\|_{B^{s}_{p,q}(0,b;E)} \leqslant C\big(\|u\|_{(E,E(A^m))_{\eta,p}} + |\lambda|^{\eta/m}\|u\|_E\big)$$

for every $u \in (E, E(A^m))_{\eta,p}$ and $\lambda \in S(\varphi)$.

By using the similar technique as in [41, Section 1.8.2] and [9, Section 18], we obtain:

Theorem A$_6$. *Let the following conditions be satisfied*:

(1) $0 < s < 1$, *l is a positive integers and* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$;
(2) $\theta_j = (j + 1/p)/(s + l + 1/q - 1/p)$, $0 < t \leqslant t_0 < \infty$, $0 \leqslant j \leqslant l - 1$, $0 < h \leqslant h_0$, $0 < \mu \leqslant 1 - \theta_j$.

Then transformations $u \to u^{(j)}(0)$ are bounded and linear from $B^{s+l}_{p,q}(0, b; E_0, E)$ to $(E_0, E)_{\theta_j,q}$, and for $u \in B^{s+l}_{p,q}(0, 1; E_0, E)$ the following inequalities hold:

$$t^{\theta_j} \left\| u^{(j)}(0) \right\|_{(E_0,E)_{\theta_j+\mu,q}} \leqslant h^{\mu} \|u\|_{B^{s+l}_{p,q,t}(0,1;E_0,E)} + h^{-(1-\mu)} \|u\|_{B^{s}_{p,q}(0,1;E)}. \tag{3}$$

3. Statement of the problem

Consider the following non-local boundary value problem:

$$L(t)u = -tu''(x) + Au(x) + t^{\frac{1}{2}} B_1(x)u'(x) + B_2(x)u(x) = f(x), \quad x \in (0, 1), \tag{4}$$

$$\begin{aligned} L_k u = t^{\theta_k} \left[\alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{j=1}^{N_k} \delta_{k_j} u^{(m_k)}(x_{k,j}) \right] + \sum_{j=1}^{M_k} t^{\gamma} T_{kj} u(x_{kj0}) \\ = f_k, \quad k = 1, 2, \end{aligned} \tag{5}$$

in $B^{s}_{p,q}(0, 1; E)$, where

$$\theta_k = \frac{(m_k + 1/p)}{s + 2 + |1/p - 1/q|}, \qquad \gamma = \gamma(s) = \frac{1}{p(s + 2 + |1/p - 1/q|)},$$

for $0 < s < 1$, $m_k \in \{0, 1\}$; α_k, β_k, δ_{kj} are complex numbers, and t is a parameter, $0 < t \leqslant t_0 < \infty$; moreover, A, $B_k(x)$ for $x \in [0, 1]$, and T_{kj} are, in general, unbounded operators in E, also $f_k \in E_k = (E(A), E)_{\theta_k,q}$, $x_{kj} \in (0, 1)$, $x_{kj0} \in [0, 1]$ for $k = 1, 2$.

A function belonging to space $B^{2+s}_{p,q}(0, 1; E(A), E)$ and satisfying Eq. (4) a.e. on $(0, 1)$ is said to be a solution of Eq. (4) on $(0, 1)$.

Let

$$B^{2+s}_{p,q}\big(0, 1; E(A), E, L_k\big) = \big\{u\colon\ u \in B^{2+s}_{p,q}\big(0, 1; E(A), E\big),\ L_k u = 0,\ k = 1, 2\big\}.$$

4. Homogeneous equations

Let us first consider the following non-local boundary value problem:

$$\big|L_0(t) + \lambda\big| u = -tu''(x) + (A + \lambda)u(x) = 0, \tag{6}$$

$$L_{k0} u = t^{\theta_k} \left[\alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{j=1}^{N_k} \delta_{kj} u^{(m_k)}(x_{kj}) \right] = f_k, \quad k = 1, 2, \tag{7}$$

where λ is a complex parameter, $m_k \in \{0, 1\}$; α_k, β_k, δ_{kj} are complex numbers, A is, in general, an unbounded operator in E, and $D = \frac{d}{dx}$.

Condition 1. Assume the following conditions hold:

(1) $\sqrt{|1/p-1/q|} \leqslant s < 1-|1/p-1/q|$, $1<p<\infty$, $1\leqslant q\leqslant\infty$, $0<t\leqslant t_0<\infty$;
(2) $\delta=(-1)^{m_1}\alpha_1\beta_2-(-1)^{m_2}\alpha_2\beta_1\neq 0$;
(3) A is a φ-positive operator in a Banach space E for $0<\varphi\leqslant\pi$.

Theorem 1. *Let Condition* 1 *be satisfied. Then problem* (6)–(7) *for* $f_k\in E_k$ *and* $|\arg\lambda|\leqslant\pi-\varphi$, *with sufficiently large* $|\lambda|$ *has a unique solution that belongs to the space* $B^{2+s}_{p,q}(0,1;E(A),E)$ *and coercive uniformity which is defined by*

$$\sum_{i=0}^{2} t^{i/2}|\lambda|^{1-i/2}\|u^{(i)}\|_{B^s_{p,q}(0,1;E)}+\|Au\|_{B^s_{p,q}(0,1;E)} \leqslant M\sum_{k=1}^{2}\big(\|f_k\|_{E_k}+|\lambda|^{\theta_k}\|f_k\|_E\big) \tag{8}$$

with respect to parameters t and λ the estimate holds for the solution of problem (6)–(7).

Proof. Consider a boundary value problem

$$\big[L_0(t)+\lambda\big]u=-u''(x)+t^{-1}(A+\lambda)u(x)=0, \tag{9}$$

$$L_{k0}u=0,\quad k=1,2, \tag{10}$$

which is equivalent to (6)–(7). By definition of positive operators for $0<t\leqslant t_0<\infty$, an operator $\frac{A}{t}$ is positive uniformly with respect to the parameter $t>0$ and, for all $\lambda\in S_\varphi$, we have an estimate

$$\left\|\left(\frac{A}{t}-\lambda I\right)^{-1}\right\|\leqslant M\frac{t}{1+t|\lambda|}. \tag{11}$$

By virtue of estimate (11) and by using the similar technique of [45, Lemma 5.4.2/6], we obtain that there exists a holomorphic for $x>0$ and strongly continuous for $x\geqslant 0$ semigroup $e^{-x(t^{-1}A_\lambda)^{1/2}}$ and an arbitrary solution of Eq. (9), for $|\arg\lambda|\leqslant\pi-\varphi$, belonging to space $B^{2+s}_{p,q}(0,1;E(A),E)$ has a form

$$u(x)=e^{-xt^{-1/2}A_\lambda^{1/2}}g_1+e^{-(1-x)t^{-1/2}A_\lambda^{1/2}}g_2, \tag{12}$$

where

$$g_k\in\big(E(A),E\big)_{\gamma,p},\quad k=1,2.$$

Now, taking into account boundary conditions (10), we obtain an algebraic linear equations with respect to g_1, g_2;

$$(-1)^{m_k}e^{-xt^{-1/2}A_\lambda^{1/2}}A_\lambda^{m_k/2}\left\{\left[\alpha_k+\beta_k e^{-t^{-1/2}A_\lambda^{1/2}}+\sum_{j=1}^{N_k}\delta_{kj}e^{-x_{kj}t^{-1/2}A_\lambda^{1/2}}\right]g_1 + \left[\alpha_k e^{-t^{-1/2}A_\lambda^{1/2}}+\beta_k+\sum_{j=1}^{N_k}\delta_{kj}e^{-(1-x_{kj})t^{-1/2}A_\lambda^{1/2}}\right]g_2\right\}=f_k,\quad k=1,2. \tag{13}$$

By using properties of positive operators and holomorphic semigroups [41, Section 1.14.5], we get $\|D(\lambda,t)\|_{B(E^2)} \to 0$ for $|\lambda| \to \infty$ uniformly with respect to t, where $D(\lambda,t)$ is a determinant-operator of system (13). Then by condition (3) for $|\arg\lambda| \leqslant \pi - \varphi$, $\lambda \to \infty$ the operator-matrix $Q(\lambda,t) = [\delta + D(\lambda,t)]^{-1}$ is invertible and is bounded uniformly with respect to parameters λ and t. By solving system (13), we obtain

$$g_1 = \left|Q(\lambda,t)\right| \left\{ \left[\alpha_2 e^{-t^{-1/2}A_\lambda^{1/2}} + \beta_2 + \sum_{j=1}^{N_k} \delta_{2j} e^{-(1-x_{2j})t^{-1/2}A_\lambda^{1/2}} \right] A_\lambda^{-m_1/2} f_1 - \left[\alpha_1 e^{-t^{-1/2}A_\lambda^{1/2}} + \beta_1 + \sum_{j=1}^{N_k} \delta_{1j} e^{-(1-x_{1j})t^{-1/2}A_\lambda^{1/2}} \right] A_\lambda^{-m_2/2} f_2 \right\},$$

$$g_2 = \left|Q(\lambda,t)\right| \left\{ (-1)^{m_1} \left[\alpha_1 + \beta_1 e^{-t^{-1/2}A_\lambda^{1/2}} + \sum_{j=1}^{N_k} \delta_{1j} e^{-x_{1j}t^{-1/2}A_\lambda^{1/2}} \right] A_\lambda^{-m_2/2} f_2 - (-1)^{m_2} \left[\alpha_2 + \beta_2 e^{-t^{-1/2}A_\lambda^{1/2}} + \sum_{j=1}^{N_k} \delta_{2j} e^{-x_{2j}t^{-1/2}A_\lambda^{1/2}} \right] A_\lambda^{-m_1/2} f_1 \right\}, \tag{14}$$

where $|Q(\lambda,t)|$ denote a determinant-operator of matrix-operator $Q(\lambda,t)$. From the representation of the operators $D(\lambda,t)$ and $|Q(\lambda,t)|$, it follows that these operators are bounded uniformly with respect to t and λ and the operators contained in equality (14) commute with any powers of operators $A_\lambda^{1/2}$. Consequently, substituting (14) into (12), we obtain a representation of the solution of problem (9)–(10). By part (1) of Condition 1 we get

$$\theta_k \leqslant \eta_0 = \frac{p(1-s)+1}{2p}.$$

Then, by using Lemma A_1 and Theorem A_5, we obtain the assertion. □

5. Non-homogeneous equations

Now consider a non-local boundary value problems for a non-homogeneous equation

$$\left|L_0(t) + \lambda\right| u = -tu''(x) + (A + \lambda I)u(x) = f(x), \quad x \in (0,1), \tag{15}$$

$$L_{0k}u = t^{\theta_k} \left[\alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{j=1}^{N_k} \delta_{kj} u^{(m_k)}(x_{kj}) \right] = f_k, \quad k = 1,2. \tag{16}$$

Theorem 2. *Let Condition* 1 *be satisfied and let E be a Banach space satisfying the multiplier condition with respect to p and q. Then an operator $u \to [D_0(t) + \lambda]u = \{[L_0(t) + \lambda]u, L_{10}u, L_{20}u\}$, for $|\arg\lambda| \leqslant \pi - \varphi$, $0 < \varphi \leqslant \pi$, and sufficiently large $|\lambda|$, is an isomorphism from $B_{p,q}^{2+s}(0,1; E(A), E)$ onto $B_{p,q}^{s}(0,1; E) + E_1 + E_2$. Moreover, coercive uniformity which is defined by*

$$\sum_{j=0}^{2} t^{j/2}|\lambda|^{1-j/2}\|u^{(j)}\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} \leqslant C\left[\|f\|_{B^s_{p,q}} + \sum_{k=1}^{2}\|f_k\|_{E_k} + |\lambda|^{\theta_k}\|f_k\|_E\right] \tag{17}$$

with respect to parameters λ and t, the estimate holds for the solution of problem (15)–(16).

Proof. By the definition of the space $B^{2+s}_{p,q}(0,1;E(A),E)$, by Theorem A_6 and Lemma A_1, we obtain that an operator $u \to [D_0(t)+\lambda]u$ is bounded linear from $B^{2+s}_{p,q}(0,1;E(A),E)$ to $B^s_{p,q}(0,1;E)+E_1+E_2$. By virtue of the Banach theorem, it suffices to prove that it is algebraic isomorphism. To see this, we have to prove uniqueness of the solution of problem (15)–(16) in Theorem 1. Let us define

$$\tilde f(x) = \begin{cases} f(x) & \text{if } x \in [0,1], \\ 0 & \text{if } x \notin [0,1]. \end{cases}$$

We show that a solution of problem (15)–(16) which belong to space $B^{2+s}_{p,q}(0,1;E(A),E)$ can be represented as a sum $v(x) = u_1(x) + u_2(x)$, where u_1 is a restriction on $[0,1]$ of a solution u of the equation

$$\left[L_0(t,\xi)+\lambda\right]u = \tilde f(x), \quad x \in R = (-\infty,\infty), \tag{18}$$

u_2 is a solution of the problem

$$\left[L_0(t,\xi)+\lambda\right]u = 0, \qquad L_{k0}u = f_k - L_{k0}u_1. \tag{19}$$

The solution of Eq. (18) is given by the formula

$$u(x) = F^{-1}\left[L_0(t,\xi)+\lambda\right]^{-1}F\tilde f(x),$$

where $F\tilde f(x)$ is the Fourier transform of the function $\tilde f(x)$, and $[L_0(t,\xi)+\lambda]$ is a characteristic operator pencil of Eq. (18), i.e.

$$\left[L_0(t,\xi)+\lambda\right] = (t\xi^2+\lambda)I + A.$$

Let us show that operator-functions

$$\Psi_{t,\lambda}(\xi) = A\left[L_0(t,\xi)+\lambda\right]^{-1},$$
$$\Psi_{t,\lambda,j}(\xi) = t^{j/2}|\lambda|^{1-j/2}\xi^j\left[L_0(t,\xi)+\lambda\right]^{-1}, \quad j=0,1,2,$$

are Fourier multipliers in $B^s_{p,q}(R;E)$ uniformly with respect to parameters λ and t. Due to positivity of A for $|\arg\lambda| \leqslant \pi-\varphi$ and $\xi \in (-\infty,\infty)$, we have

$$\left\|\frac{d^i}{d\xi^i}\Psi_{t,\lambda}(\xi)\right\| \leqslant C|\xi|^{-i}, \qquad \left\|\frac{d^i}{d\xi^i}\Psi_{t,\lambda,j}(\xi)\right\| \leqslant C|\xi|^{-i}, \quad i=0,1,\ j=0,1,2, \tag{20}$$

uniformly with respect to parameters t and λ. Therefore, by virtue of Definition 1 and by Remark in Section 1, from (20) we obtain that operator functions $\Psi_{t,\lambda}(\xi)$ and $\Psi_{t,\lambda,j}(\xi)$ are uniformly bounded multipliers in $B^s_{p,q}(R;E)$ with respect to t and λ. This implies then that problem (18) for all $\tilde f(x) \in B^s_{p,q}(R)$ has a solution $u \in B^{2+s}_{p,q}(R;E(A),E)$ and

$$\sum_{j=0}^{2} t^{j/2}|\lambda|^{1-j/2}\|u^{(j)}\|_{B^s_{p,q}(R;E)} + \|Au\|_{B^s_{p,q}(R;E)} \leqslant C\|\tilde f(x)\|_{B^s_{p,q}(R;E)}. \tag{21}$$

Then we obtain that $u_1 \in B^{2+s}_{p,q}(0,1;E(A),E)$ is solution of Eq. (18) on $(0,1)$. By virtue of Theorem A_6, we get that $u_1^{(m_k)}(x_0) \in E_k$, $k = 1,2$, for all $x_0 \in [0,1]$. Hence, $L_{0k}u_1 \in E_k$. Thus, by virtue of Theorem 1, problem (19) has the unique solution $u_2(x)$ that belongs to space $B^{2+s}_{p,q}(0,1;E(A),E)$ for $|\arg\lambda| \leqslant \pi - \varphi$ and for sufficiently large $|\lambda|$. Moreover, we have

$$\begin{aligned}
&\sum_{j=0}^{2} t^{j/2}|\lambda|^{1-j/2}\big\|u_2^{(j)}\big\|_{B^s_{p,q}(R;E)} + \|Au_2\|_{B^s_{p,q}(R;E)} \\
&\quad\leqslant C\sum_{k=1}^{2}\Big[\|f_k\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|_E + |\lambda|^{1-\theta_k}\|L_{0k}u_1\|_E + \big\|u_1^{(m_k)}\big\|_{C([0,1];E_k)} \\
&\qquad + |\lambda|^{1-\theta_k}\big\|u_1^{(m_k)}\big\|_{C([0,1];E)}\Big].
\end{aligned} \tag{22}$$

From (21) for $|\arg\lambda| \leqslant \pi - \varphi$ we obtain

$$\sum_{j=0}^{2} t^{j/2}|\lambda|^{1-j/2}\big\|u_1^{(j)}\big\|_{B^s_{p,q}(R;E)} + \|Au_1\|_{B^s_{p,q}(R;E)} \leqslant C\|\tilde f(x)\|_{B^s_{p,q}(R;E)}. \tag{23}$$

Therefore, by Theorem A_6 and by virtue of estimate (23), we obtain

$$t^{\theta_k}\big\|u_1^{(m_k)}(x_0)\big\|_{E_k} \leqslant C\|u_1\|_{B^{2+s}_{p,q,t}(0,1;E(A),E)} \leqslant C\|f\|_{B^s_{p,q}(0,1;E)}, \quad k = 1,2. \tag{24}$$

In a similar way, we have

$$|\lambda|^{1-\theta_k}t^{\theta_k}\big\|u_1^{(m_k)}(x_0)\big\|_E \leqslant C\|f\|_{B^s_{p,q}}, \quad j = 1,2,\ k = 1,2, \tag{25}$$

uniformly with respect to parameters t and λ. Hence from (22) and (24)–(25) we obtain

$$\begin{aligned}
&\sum_{j=0}^{2} t^{j/2}|\lambda|^{1-j/2}\big\|u_2^{(j)}\big\|_{B^s_{p,q}} + \|Au_2\|_{B^s_{p,q}} \\
&\quad\leqslant C\left(\|f\|_{B^s_{p,q}} + \sum_{k=1}^{2}\big(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|_E\big)\right).
\end{aligned} \tag{26}$$

Then estimates (23) and (26) imply (17). □

6. Coerciveness on the space variable and fredholmness

Consider problem (4)–(5).

Theorem 3. *Let Condition* 1 *be satisfied for* $\varphi \in (0, \pi/2)$. *Moreover, assume the following conditions hold*:

(1) $A^{-1} \in \sigma_\infty(E)$, $0 < t \leqslant t_0 < \infty$.
(2) *For any* $\varepsilon > 0$ *and for almost all* $x \in [0,1]$,

$$\left\| B_1(x)u \right\| \leqslant \varepsilon \|u\|_{(E(A),E)_{1/2,1}} + C(\varepsilon)\|u\|, \quad u \in \bigl(E(A), E\bigr)_{1/2,1},$$
$$\left\| B_2(x)u \right\| \leqslant \varepsilon \|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A),$$

the functions $B_1(x)u$, $B_2(x)u$ *for* $u \in (E(A),E)_{1/2,1}$ *and* $u \in D(A)$, *respectively, are measurable on* $[0,1]$ *in* E.
(3) *If* $m_k = 0$, *then* $T_{kj} = 0$, *if* $m_k = 1$ *then for* $\varepsilon > 0$ *and* $u \in E_\gamma = (E(A),E)_{\gamma,g}$, $\gamma = 1/(p(s+2+|1/p-1/q|))$,

$$\|T_{kj}u\|_{E_k} \leqslant \varepsilon \|u\|_{E_\gamma} + C(\varepsilon)\|u\|.$$

Then for all $u \in B^{2+s}_{p,q}(0,1;E(A),E)$ *coercive uniformity which is defined by*

$$\sum_{i=0}^{2} t^{i/2}|\lambda|^{1-i/2}\|u^{(i)}\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} \leqslant C\left[\|Lu\|_{B^s_{p,q}} + \sum_{k=1}^{2}\|L_k u\|_{E_k} + \|u\|_{B^s_{p,q}}\right] \tag{27}$$

with respect to parameter t *and* λ, *the coercive estimate holds for the solution of problem* (4)–(5).

Proof. The general case is reduced to the latter if operators $A + \lambda I$ and $B_2(x) - \lambda I$ for some sufficiently large $|\lambda|$, are considered instead of operators A and B_2. Let $u \in B^{2+s}_{p,q}(0,1;E(A),E)$ be a solution of problem (4)–(5). Then $u(x)$ is a solution of a problem

$$-t\frac{d^2}{dx^2}u(x) + (A+\lambda I)u(x) = f(x) + \lambda u(x) - t^{1/2}B_1(x)\frac{d}{dx}u(x) - B_2(x)u(x),$$
$$L_{k0}u = f_k - \sum_{j=1}^{M_k} t^\gamma T_{kj}u(x_{kj0}), \quad k = 1,2,$$

where L_{k0} are defined by Eq. (7). By Theorem 2, for sufficiently large $\lambda_0 > 0$ we have

$$\sum_{i=0}^{2} t^{i/2}|\lambda|^{1-i/2}\|u^{(i)}\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} \leqslant C\left[\left\|f + \lambda_0 u - t^{1/2}B_1 u^{(1)} - B_2 u\right\|_{B^s_{p,q}} + \sum_{k=1}^{2}\left\| f_k - \sum_{j=1}^{M_k} t^\gamma T_{kj}u(x_{kj0})\right\|_{E_k}\right]. \tag{28}$$

By condition (2) and by virtue of Theorem A$_3$, for all $u \in B^{2+s}_{p,q}(0,1;E(A),E)$ we have

$$t^{1/2}\|B_1u^{(1)}\|_{B^s_{p,q}} \leqslant \varepsilon\|u^{(1)}\|_{B^{1+s}_{p,q,t}(0,1;(E(A),E)_{\frac{1}{2},1}E)} + C(\varepsilon)\|t^{1/2}u^{(1)}\|_{B^s_{p,q}},$$
$$\|B_2u\|_{B^s_{p,q}} \leqslant \varepsilon\|u\|_{B^{1+s}_{p,q,t}(0,1;E(A),E)} + C(\varepsilon)\|u\|_{B^s_{p,q}}, \quad \varepsilon > 0. \tag{29}$$

Moreover, from Theorem A_3 we obtain that an operator $u \mapsto du/dx$ is bounded from $B^{2+s}_{p,q}(0,1;E)$ into $B^{1+s}_{p,q}(0,1;E)$ and for $u \in B^{2+s}_{p,q}(0,1;E)$

$$\exists\varepsilon > 0\colon \quad \|t^{1/2}u^{(1)}\|_{B^s_{p,q}} \leqslant \varepsilon\|u\|_{B^{2+s}_{p,q,t}(0,1;E)} + C(\varepsilon)\|u\|_{B^s_{p,q}}. \tag{30}$$

Hence, in view of inequalities (29) and (30), it follows:

$$\|t^{1/2}B_1u^{(1)}\|_{B^s_{p,q}} \leqslant \varepsilon\|u\|_{B^{2+s}_{p,q,t}(0,1;E(A),E)} + C(\varepsilon)\|u\|_{B^s_{p,q}},$$
$$\|B_2u\|_{B^s_{p,q}} \leqslant \varepsilon\|u\|_{B^{2+s}_{p,q,t}(0,1;E(A),E)} + C(\varepsilon)\|u\|_{B^s_{p,q}} \tag{31}$$

for $u \in B^{2+s}_{p,q}(0,1;E(A),E)$. By virtue of Theorem A$_6$, an operator $u \to u(x_0)$ from $B^{2+s}_{p,q}(0,1;E(A),E)$ into E_γ is bounded and for $u \in B^{2+s}_{p,q}(0,1;E(A),E)$ we have

$$t^\gamma\|u(x_0)\|_{E_\gamma} \leqslant \varepsilon\|u\|_{B^{2+s}_{p,q,t}(0,1;E(A),E)} + C(\varepsilon)\|u\|_{B^s_{p,q}}.$$

Consequently, from condition (3) and by the above estimate for all $u \in B^{2+s}_{p,q}(0,1;E(A),E)$, it follows:

$$t^\gamma\|T_{kj}u(x_{kj0})\|_{E_k} \leqslant \varepsilon\|u\|_{B^{2+s}_{p,q,t}(0,1;E(A),E)} + C(\varepsilon)\|u\|_{B^s_{p,q}}. \tag{32}$$

Substituting (31) and (32) into (28), we get (27). □

Theorem 4. *Let all conditions of Theorem* 3 *be satisfied. Then an operator* $u \to D(t)u = \{L(t)u, L_1u, L_2u\}$ *is Fredholm from* $B^{2+s}_{p,q}(0,1;E(A),E)$ *into* $B^s_{p,q}(0,1;E) + E_1 + E_2$.

Proof. Let $D(t) = [D_0(t) + \lambda] + (D_1(t) - \lambda]$, where

$$[D_0(t) + \lambda]u = \{[D_0(t) + \lambda]u, L_{10}, L_{20}\},$$
$$[D_1(t) - \lambda]u = \left\{-\lambda u + t^{1/2}B_1(x)u^{(1)}(x) + B_2(x)u(x), \sum_{j=1}^{M_1} t^\gamma T_{1j}u(x_{1j}), \sum_{j=1}^{M_2} t^\gamma T_{2j}u(x_{2j})\right\}$$

and L_0, L_{10}, L_{20} are defined by Eqs. (15), (16). By Theorem 3 we obtain that operator $D_0(t) + \lambda$ has a bounded inverse from $B^{2+s}_{p,q}(0,1;E(A),E)$ onto $B^s_{p,q}(0,1;E) + E_1 + E_2$. From estimate (27) and in view of Theorem A$_2$, it follows that operator D_1 from $B^{2+s}_{p,q}(0,1;E(A),E)$ into $B^s_{p,q}(0,1;E) + E_1 + E_2$ is compact. Then in view of Theorem 3 and by the perturbation theory of linear operators [24], it follows that operator $D(t)$ from $B^{2+s}_{p,q}(0,1;E(A),E)$ into $B^s_{p,q}(0,1;E) + E_1 + E_2$ is Fredholm. □

7. Non-local boundary value problems for elliptic equations with parameters

Fredholm property of boundary value problems for elliptic equations with parameters in smooth domains was studied in [1–3,28]; for non-smooth domains it was treated in [17,25,31,32]. In [14,15,32] and [20,21] the non-local BVPs were studied. In this section, by applying Theorems 3, 4, the coercive estimate and fredholmness of non-local elliptic boundary value problem with parameters in Besov spaces are obtained. Let $\Omega \subset R^m$, $m \geqslant 2$, be a bounded domain with an $(m-1)$-dimensional boundary $\partial\Omega$ which locally admits rectification. Let $G = [0,1] \times \Omega$. Let us consider a non-local boundary value problem for an elliptic differential equation of second order:

$$\begin{aligned} \big[L(t)+\lambda\big] &= -tD_x^2 u(x,y) - \sum_{k,j=1}^{m} a_{kj}(y) D_k D_j u(x,y) + t^{1/2} a(x,y) D_x u(x,y) \\ &\quad + \sum_{j=1}^{m} a_j(x,y) D_j u(x,y) + a_0(x,y) u(x,y) + \lambda u(x,y) \\ &= f(x,y), \quad (x,y) \in G, \end{aligned} \tag{33}$$

$$\begin{aligned} L_k u &= t^{\theta_k}\left(\alpha_k \frac{\partial^{m_k}}{\partial x^{m_k}} u(0,y) + \beta_k \frac{\partial^{m_k}}{\partial x^{m_k}} u(1,y)\right) + \sum_{j=1}^{N_k} t^{\theta_k} \delta_{kj} \frac{\partial^{m_k}}{\partial x^{m_k}} u(x_{kj},y) \\ &\quad + \sum_{j=1}^{M_k} t^{\gamma} T_{kj} u(x_{kj0},y) \\ &= f_k(y), \quad k=1,2,\ y \in \Omega, \end{aligned} \tag{34}$$

$$L_0 u = \sum_{j=1}^{m} c_j(y') \frac{\partial}{\partial y_j} u(x,y') + c_0(y') u(x,y') = 0, \quad x \in (0,1),\ y' \in \partial\Omega, \tag{35}$$

where $0 < t \leqslant t_0 < \infty$, $D_x = \partial/\partial x$, $D_j = -i\partial/\partial y_j$, $D_y = (D_1,\dots,D_m)$, $m_k \in \{0,1\}$,

$$\theta_k = \frac{m_k + 1/p}{2+s+|1/p-1/q|}, \qquad \gamma = \frac{1}{p(s+2)+|1/p-1/q|}$$

and α_k, β_k are complex numbers, $y = (y_1,\dots,y_m)$, $x_{kj} \in (0,1)$, $x_{kj0} \in [0,1]$, T_{kj} are, generally speaking, unbounded operators in $B^s_{p,q}(\Omega)$. Let

$$r = \operatorname{ord} L_0, \qquad s_k = 2+s-\frac{2(m_k+1/p)}{2+s+|1/p-1/q|},$$
$$s_0 = 2+s-\frac{1}{p(2+s)+p|1/p-1/q|}.$$

Remark 1. By using [41, Sections 2.9.3, 3.6.4, 4.3.3] and the localization technique for $r < s_k$, the following equalities are obtained:

$$B_k = \big(B^{2+s}_{p_0,q}(\Omega; L_0 u = 0), B^s_{p_0,q}(\Omega)\big)_{\theta_k,q} = B^{s_k}_{p_0,q}(\Omega; L_0 u = 0),$$
$$B_{k_0} = \big(B^{2+s}_{p_0,q}(\Omega; L_0 u = 0), B^s_{p_0,q}(\Omega)\big)_{\gamma,q} = B^{s_0}_{p_0,q}(\Omega; L_0 u = 0).$$

Remark 2. Let $\mathbf{p} = (p_0, p)$. We set

$$B = B^s_{p,q}\big(0, 1; B^s_{p_0,q}(\Omega)\big), \qquad B_0 = B^{2+s}_{p,q}\big(0, 1; B^{2+s}_{p_0,q}(\Omega), B^s_{p_0,q}(\Omega)\big).$$

It is clear to see that

$$B \subset B^s_{\mathbf{p},q}(G), \qquad B_0 \subset B^{2+s}_{\mathbf{p},q}(G).$$

Theorem 5. *Let the following conditions be satisfied*:

(1) $a_{kj} \in C(\Omega)$, $a, a_j, a_0 \in C^1(\Omega)$, $c_j \in C^1(\Omega)$, $c_0 \in C(\Omega)$; $\partial\Omega \in C^2$, $0 < t \leqslant t_0 < \infty$;
(2) $\sum_{j=1}^m c_j(y')\sigma_j \neq 0$, $y' \in \partial\Omega$, $\sigma \in R^m$, *where* $\sigma \in R^m$ *is normal to* ∂G *and* $c_0 \in C^1(\overline{G})$ *for* $r = 1$; $c_0(y') \neq 0$, $y' \in \partial G$ *for* $r = 0$;
(3) *for* $y \in \Omega$, $\sigma \in R^m$, $\arg\lambda = \pi$, $|\sigma| + |\lambda| \neq 0$,

$$\lambda + \sum_{k,j=1}^m a_{kj}(y)\sigma_k\sigma_j \neq 0;$$

(4) *for the tangent vector* σ *and a normal vector* σ *to* $\partial\Omega$ *at the point* $y' \in \partial\Omega$, *the following boundary value problem*:

$$\left[\lambda + \sum_{k,j=1}^m a_{kj}(y')\left(\sigma'_k - i\sigma_j\frac{d}{dt}\right)\left(\sigma'_k - i\sigma_j\frac{d}{dt}\right)\right]u(\xi) = 0,$$

$$\xi > 0,\ \lambda \leqslant 0,\ \sum_{j=1}^m c_j(y')\left(\sigma'_k - i\sigma_j\frac{d}{dt}\right)u(\xi)\bigg|_{\xi=0} = h \tag{36}$$

for $r = 1$ (*and for* $r = 0$ *the problem generated by Eqs.* (36) *with* $u(0) = h$) *has one and only one solution, including all its derivatives, tending to zero as* $\xi \to \infty$ *for any numbers* $h \in C^1$;
(5) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$;
(6) *if* $m_k = 0$ *then* $T_{kj} = 0$, *if* $m_k = 1$ *then for* $\varepsilon > 0$ *and* $u \in B^{s_k}_{p_0,q}(\Omega; L_0u = 0)$, $r < s_k$,

$$\|T_{kj}u\|_{B^{s_k}_{p_0,q}(\Omega)} \leqslant \varepsilon\|u\|_{B^{s_0}_{p_0,q}(\Omega)} + c(\varepsilon)\|u\|_{L_p(\Omega)}.$$

Then:

(a) *Problem* (33)–(35) *for all* $f \in B$, $1 < p_0, p < \infty$, $1 \leqslant q \leqslant \infty$, $f_k \in B_k$, *for* $|\arg\lambda| = \pi$ *and sufficiently large* $|\lambda|$, *has a unique solution that belongs to the space* B_0 *and coercive uniformity which is defined by*

$$\|u\|_{B_0} \leqslant C\left[\big\|(L+\lambda)u\big\|_B + \sum_{k=1}^2 \|L_k u\|_{B_k} + \|u\|_B\right]$$

with respect to parameters t *and* λ, *estimate holds for the solution of problem* (33)–(35);
(b) *An operator* $u \to Q(t)u = \{L(t)u, L_1u, L_2u\}$ *from* B_0 *into* $B \times \prod_{k=1}^2 B_k$ *is Fredholm.*

Proof. Let $E = B^s_{p_0,q}(\Omega)$. Consider an operator A which is defined by the equalities

$$D(A) = B^{2+s}_{p_0,q}(\Omega; L_0 u = 0), \quad Au = -\sum_{k,j=1}^{m} a_{kj}(y) D_k D_j u(y).$$

For $x \in [0, 1]$ also consider operators

$$B_1(x)u = a(x, y)u(y), \qquad B_2(x)u = \sum_{j=1}^{m} a_j(x, y) D_j u(y) + a_0(x, y)u(x, y).$$

Then problem (33)–(35) can be rewritten in the form:

$$\begin{aligned}
&-t\frac{d^2u(x)}{dx^2} + Au(x) + t^{1/2}B_1(x)\frac{du(x)}{dx} + B_2(x)u(x) + \lambda u(x) = f(x),\\
&\quad x \in (0, 1),\\
&\alpha_k t^{\theta_k}\left[u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{j=1}^{N_k} \delta_{kj} u^{(m_k)}(x_{kj})\right] + \sum_{j=1}^{M_k} t^{\gamma} T_{kj} u(x_{kj0}) = f_k,\\
&\quad k = 1, 2,
\end{aligned} \tag{37}$$

where $u(x) = u(x, \cdot)$, $f(x) = f(x, \cdot)$ are functions with values in space $E = B^s_{p_0,q}(\Omega)$ and $f_k = f_k(\cdot)$. Let us apply Theorem 3 to problem (37). By virtue of [29], a problem

$$\lambda u(y) - \sum_{k,j=1}^{m} a_{kj}(y) D_k D_j u(y) = f(y), \quad y \in \Omega,$$

$$\sum_{j=1}^{m} c_j(y') D_j u(y') + c_0(y')u(y') = 0, \quad y' \in \partial\Omega,$$

for $\arg\lambda = \pi$, $|\lambda| \to \infty$ and $f \in B^s_{p_0,q}(\Omega)$ has a unique solution u and

$$|\lambda| \, \|u\|_{B^s_{p_0,q}(\Omega)} \leqslant C\|f\|_{B^s_{p_0,q}(\Omega)}.$$

Consequently, $\|(A - \lambda I)^{-1}\| \leqslant C|\lambda|^{-1}$ for $\arg\lambda = \pi$, $|\lambda| \to \infty$, i.e. part (3) of Condition 1 is fulfilled. By Theorem A_2 an embedding $B^{2+s}_{p_0,q}(\Omega) \subset B^s_{p_0,q}(\Omega)$ is compact. Consequently, condition (1) of Theorem 3 is fulfilled too. Conditions (5) and (6) coincide with part (2) of Condition 1 and condition (3) of Theorem 3, respectively. Using interpolation properties of Besov spaces (see, e.g., [9, Section 18], [41, Section 4]) and Remark 1, it is easy to see that other conditions of Theorems 3, 4 are fulfilled. □

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Further reading

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