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Double-super-connected digraphs☆

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ABSTRACT

A strongly connected digraph D is said to be super-connected if every minimum vertex-cut is the out-neighbor or in-neighbor set of a vertex. A strongly connected digraph D is said to be double-super-connected if every minimum vertex-cut is both the out-neighbor set of a vertex and the in-neighbor set of a vertex. In this paper, we characterize the double-super-connected line digraphs, Cartesian product and lexicographic product of two digraphs. Furthermore, we study double-super-connected Abelian Cayley digraphs and illustrate that there exist double-super-connected digraphs for any given order and minimum degree.

1. Introduction

By a simple digraph $D = (V(D), A(D))$, we mean a directed graph without loops and multiple arcs. Let $D = (V, A)$ be a strongly connected digraph and let x and y be two distinct vertices of D. For a vertex $x \in V$, we use $N_D^+(x)$ and $N_D^-(x)$, or simply $N^+(x)$ and $N^-(x)$, to denote the out-neighbor set and in-neighbor set of x in D, respectively. Set $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$. As usual, $\delta^+(D)$ and $\delta^-(D)$ denote the minimum out-degree and minimum in-degree of D, respectively. $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ denotes the minimum degree. If $d^+(x) = d^-(x) = d$ for each vertex $x \in V$, then D is a d-regular digraph. The *reverse digraph* of D is the digraph $D^{(r)} = (V, \{(x, y)|(y, x) \in A\})$; D is a *symmetric digraph* if $A = A^{(r)}$.

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs, where $V_1 = \{x_1, x_2, \dots, x_{n_1}\}$ and $V_2 = \{y_1, y_2, \dots, y_{n_2}\}$. The *line digraph* of D_1, denoted by $L(D_1)$, is the digraph with vertex set $V(L(D_1)) = \{a_{ij}|a_{ij} = (x_i, x_j) \in A_1\}$, and a vertex a_{ij} is adjacent to a vertex a_{st} in $L(D_1)$ if and only if $x_j = x_s$ in D_1. The *Cartesian product* $D_1 \times D_2$ of D_1 and D_2 has vertex set $V_1 \times V_2$ and $((x_1, y_1), (x_2, y_2)) \in A(D_1 \times D_2)$ if and only if either $(x_1, x_2) \in A_1$ and $y_1 = y_2$, or $x_1 = x_2$ and $(y_1, y_2) \in A_2$. The *lexicographic product* $D_1[D_2]$ of D_1 and D_2 has vertex set $V_1 \times V_2$ and $((x_1, y_1), (x_2, y_2)) \in A(D_1[D_2])$ if and only if either $(x_1, x_2) \in A_1$, or $x_1 = x_2$ and $(y_1, y_2) \in A_2$. Let $S_1, S_2, \dots, S_{n_1-1}$ and S_{n_1} be n_1 digraphs. The digraph $D_1[S_1, S_2, \dots, S_{n_1}]$ is the digraph obtained from D_1 by replacing the ith vertex of D_1 by a copy of the digraph S_i in such a way that for every arc (x_i, x_j) in D_1, $D_1[S_1, S_2, \dots, S_{n_1}]$ contains all possible arcs from $V(S_i)$ to $V(S_j)$. Furthermore, all the original arcs of S_i are also in $D_1[S_1, S_2, \dots, S_{n_1}]$. Clearly, if $S_1 \cong S_2 \cong \cdots \cong S_{n_1} \cong D_2$, then $D_1[S_1, S_2, \dots, S_{n_1}] \cong D_1[D_2]$.

A digraph D is said to be *vertex-transitive* if the automorphism group $\mathrm{Aut}(D)$ acts transitively on V, and is *arc-transitive* if $\mathrm{Aut}(D)$ acts transitively on A. For a group G and a subset $S \subset G \setminus \{1\}$, the *Cayley digraph* $Cay(G, S)$ is the digraph with vertex

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set G and arc set $\{(g, gs)|g \in G, s \in S\}$. In particular, if G is Abelian, then $Cay(G, S)$ is an *Abelian Cayley digraph*; if $G = Z_n$, then $Cay(Z_n, S)$ is a *circulant digraph*. It is well known that a Cayley digraph is vertex-transitive.

The *vertex-connectivity* $\kappa(D)$ (*arc-connectivity* $\lambda(D)$) is the minimum cardinality of all vertex-cuts (arc-cuts) of digraph D. We call a digraph D maximally connected, or max-κ for short, if $\kappa(D) = \delta(D)$. A strongly connected digraph D is said to be *super-connected*, or *super-κ* for short, if there exists a vertex x such that $U = N^+(x)$ or $N^-(x)$ for any minimum vertex-cut U. It is *hyper-connected*, or *hyper-κ* for short, if the removal of any minimum vertex-cut results in exactly two strongly connected components one of which is a singleton. A hyper-connected digraph is clearly super-connected. Now we give a new definition:

Definition 1.1. A strongly connected digraph D is said to be *double-super-connected* if there exist two vertices x and y such that $U = N^+(x) = N^-(y)$ for every minimum vertex-cut U.

Double-super-connected digraphs are super-connected and super-connected symmetric digraphs are double-super-connected. If D is a double-super-connected digraph, then $\delta^+(D) = \delta^-(D)$.

Hamidoune and Tindell [2,3,8] studied super-connected Abelian Cayley digraphs. In [5,10,11], the authors studied the connectivity of line graphs (digraphs). Shieh [7] studied the super-connected and super-edge-connected Cartesian product of two regular graphs. Liu and Meng [4] studied the super-connected and super-arc-connected Cartesian product of digraphs. Meng and Zhang [6] characterized the super-connected arc-transitive digraphs. In this paper, we will characterize the double-super-connected line digraph, Cartesian product and lexicographic product of two digraphs. Furthermore, we will study double-super-connected Abelian Cayley digraphs and illustrate that there is a double-super-connected digraph for any given order and minimum degree.

All digraphs in this paper are finite. Notation and definitions not given here can be found in [1,9].

2. Operations on digraphs

Firstly, we give the characterization of the double-super-connected line digraphs.

Lemma 2.1 (*[9]*). *Let D be a digraph; then $\lambda(D) = \kappa(L(D))$.*

Theorem 2.2. *Let $D = (V, A)$ be a simple digraph; then $L(D)$ is double-super-connected if and only if $\lambda(D) = 1$ and any cut-arc $(x_i, x_j) \in A$ satisfies that $d^+(x_i) = d^-(x_j) = 1$ in D.*

Proof. If $\lambda(D) = 1$, then $\kappa(L(D)) = 1$ by Lemma 2.1. For each cut-vertex a_{ij} in $L(D)$, $a_{ij} = (x_i, x_j) \in A(D)$ is a cut-arc in D; if it satisfies that $d^+(x_i) = d^-(x_j) = 1$, then there exist two vertices $a_{si} = (x_s, x_i), a_{jt} = (x_j, x_t) \in V(L(D))$ such that $N^+(a_{si}) = N^-(a_{jt}) = \{a_{ij}\}$ in $L(D)$. Therefore $L(D)$ is double-super-connected.

On the other hand, let $L(D)$ be double-super-connected and U be a minimum vertex-cut of $L(D)$. Thus there exist two vertices $a_{ij} = (x_i, x_j), a_{st} = (x_s, x_t) \in V(L(D))$ such that $N^+(a_{ij}) = U = N^-(a_{st})$. By the definition of a line digraph, we know that there are $|U|$ parallel arcs from x_j to x_s in D. Since D is simple, we have $|U| = 1$. Thus $\kappa(L(D)) = 1$. Therefore $\lambda(D) = 1$ by Lemma 2.1. If $d^+(x_i) \neq 1$ or $d^-(x_j) \neq 1$ for any cut-arc $(x_i, x_j) \in A(D)$, then there is no vertex a_{si} such that $N^+(a_{si}) = \{a_{ij}\}$ or vertex a_{jt} such that $N^+(a_{jt}) = \{a_{ij}\}$, a contradiction. □

Next, we characterize the double-super-connected Cartesian product $D_1 \times D_2$ of two digraphs D_1 and D_2. In the following part of this section, we will assume that $\delta(D) = \delta^+(D) = \delta^-(D)$ for digraph D. For convenience, we use the symbols n_i, δ_i, κ_i to denote the order, the minimum degree and the connectivity of digraph D_i, respectively, for $i = 1, 2$.

By the definition of "double-super-connected", we know that if $D_1 \times D_2$ is super-κ and there exists a vertex $(x, y) \in V(D_1 \times D_2)$ such that $U = N^+((x, y)) = N^-((x, y))$ for each minimum vertex-cut U of $D_1 \times D_2$, then $D_1 \times D_2$ is double-super-connected. Thus, in the following theorem, we will consider that there exists a minimum vertex-cut U of $D_1 \times D_2$ such that $U = N^+((x, y)) = N^-((x, y))$ does not hold for any vertex $(x, y) \in V(D_1 \times D_2)$.

The following theorem in [4] for $\kappa_i = \delta_i = 1$ is useful in our proof.

Theorem 2.3 (*[4]*). *Let D_1 and D_2 be two simple strongly connected digraphs and let $\delta_i^+ = \delta_i^- = \delta_i$ for $i = 1, 2$. If $\delta_i = \kappa_i$, then $D_1 \times D_2$ is super-κ if and only if $D_1 \times D_2 \ncong D \times \overrightarrow{K_n}$ ($D \times \overrightarrow{K_n} \ncong \overrightarrow{K_2} \times \overrightarrow{K_2}, \overrightarrow{K_2} \times \overrightarrow{K_3}$), where $\kappa(D) = \delta(D) = 1, n \geq 2$.*

Therefore, if $\kappa_i = \delta_i = 1$ for $i = 1, 2$, then $D_1 \times D_2$ is super-κ if and only if $D_1 \times D_2 \ncong D \times \overrightarrow{K_2}$, where $D \ncong \overrightarrow{K_2}$ and $\kappa(D) = \delta(D) = 1$.

Theorem 2.4. *Let D_1 and D_2 be two strongly connected digraphs; then $D_1 \times D_2$ is double-super-connected if and only if the following conditions hold:*

(i) $\kappa_i = \delta_i = 1$ *for* $i = 1, 2$,
(ii) $D_1 \times D_2 \ncong D \times \overrightarrow{K_2}$, *where* $D \ncong \overrightarrow{K_2}$ *and* $\kappa(D) = \delta(D) = 1$,
(iii) $N^-_{D_i}(N^+_{D_i}(x)) = \{x\}$ *for any* $x \in V_i$ *with* $d^+(x) = 1$, *and* $N^+_{D_i}(N^-_{D_i}(x)) = \{x\}$ *for any* $x \in V_i$ *with* $d^-(x) = 1$ *for* $i = 1, 2$.

Proof. If (i) and (ii) hold, then $D_1 \times D_2$ is super-κ by Theorem 2.3. By (i) and (iii), $U = N^+((x, y)) = \{(x, y_1), (x_1, y)\} = N^-((x_1, y_1))$ for each minimum vertex-cut U; thus $D_1 \times D_2$ is double-super-connected.

On the other hand, if $D_1 \times D_2$ is double-super-connected, then $D_1 \times D_2$ is super-κ. We first prove (i). Without loss of generality, suppose that $\delta_1 \geq 2$. We assume that U is a minimum vertex-cut of $D_1 \times D_2$. Since $D_1 \times D_2$ is double-super-connected, there are two vertices (x, y) and $(x', y')((x, y) \neq (x', y'))$ with $N^+((x, y)) = U = N^-((x', y'))$. Set $N^+_{D_1}(x) = \{x_1, \dots, x_{\delta_1}\}, N^+_{D_2}(y) = \{y_1, \dots, y_{\delta_2}\}, N^-_{D_1}(x') = \{x'_1, \dots, x'_{\delta_1}\}, N^-_{D_2}(y') = \{y'_1, \dots, y'_{\delta_2}\}$; then

$$N^+((x, y)) = \{(x_1, y), \dots, (x_{\delta_1}, y), (x, y_1), \dots, (x, y_{\delta_2})\},$$
$$N^-((x', y')) = \{(x'_1, y'), \dots, (x'_{\delta_1}, y'), (x', y'_1), \dots, (x', y'_{\delta_2})\}.$$

Since $\delta_1 \geq 2$ and $\delta_2 \geq 1$, we have $N^+((x, y)) \neq N^-((x', y'))$, a contradiction, so (i) holds. By Theorem 2.3, if (i) holds and $D_1 \times D_2$ is super-κ, then (ii) holds. Finally, we prove (iii). Without loss of generality, suppose that there exists a vertex $x \in V_1$ with $d^+(x) = 1$ such that $N^-_{D_1}(N^+_{D_1}(x)) = N^-_{D_1}(x') = \{x, x_1, \dots\}$ where $N^+_{D_1}(x) = x'$. For any vertex $y \in V_2$ with $d^+(y) = 1$, let $N^+_{D_2}(y) = y'$; then $U = N^+((x, y)) = \{(x', y), (x, y')\} \subsetneq N^-((x', y'))$ is a minimum vertex-cut of $D_1 \times D_2$, and there is no $(x'', y'') \in V(D_1 \times D_2)$ such that $U = N^-((x'', y''))$, a contradiction. □

Lemma 2.5. *Let $D = (V, A)$ be a strongly connected d-regular digraph. If there exists a vertex $y \in V$ such that $U = N^+_D(y) = N^-_D(y)$ for any vertex-cut $U = N^+_D(x)$ (or $N^-_D(x)$), then D is a symmetric digraph.*

Proof. Suppose that D is not a symmetric digraph; then there is an arc $(x, y) \in A$ and $(y, x) \notin A$. Since D is strongly connected regular digraph, there exists a vertex $z \in V$ such that $(z, x) \in A$ and $(x, z) \notin A$. Let $N^+(x) = \{x_1, x_2, \dots, x_{d-1}, y\} = U$, $N^-(x) = \{x'_1, x'_2, \dots, x'_{d-1}, z\} = U'$; there exist two distinct vertices $s, t \in V$ such that $U = N^+_D(s) = N^-_D(s)$, $U' = N^+_D(t) = N^-_D(t)$. Let $N^-(y) = \{y_1, y_2, \dots, y_{d-2}, s, x\} = U''$; there exists a vertex $x_i \in U (1 \leq i \leq d-1)$ such that $U'' = N^+_D(x_i) = N^-_D(x_i)$. Since $t \notin U''$ and $(x_i, t), (t, x_i) \in A$, we have $N^+_D(x_i) = N^-_D(x_i) = \{y_1, y_2, \dots, y_{d-2}, s, x, t\}$, a contradiction. □

Theorem 2.6. *Let D_1 and D_2 be two strongly connected regular digraphs. Then $D_1 \times D_2$ is double-super-connected if and only if one of the following conditions holds:*

(i) *D_1 and D_2 are symmetric digraphs and $D_1 \times D_2$ is super-κ.*
(ii) *D_1 and D_2 are directed cycles except for $D_1 \times D_2 \cong \overrightarrow{C_2} \times \overrightarrow{C_k} (k \geq 3)$, where $\overrightarrow{C_k}$ denotes the directed cycle of length k.*

Proof. If (i) holds, then $D_1 \times D_2$ is a super-κ and symmetric digraph; thus, $D_1 \times D_2$ is double-super-connected. If (ii) holds, then $D_1 \times D_2$ is double-super-connected by Theorem 2.4.

On the other hand, if $D_1 \times D_2$ is double-super-connected, then $D_1 \times D_2$ is super-κ; we consider two cases:

Case 1. If there exists a vertex $(x, y) \in V(D_1 \times D_2)$ such that $U = N^+((x, y)) = N^-((x, y))$ for any minimum vertex-cut U of $D_1 \times D_2$, then there exists a vertex $x \in V_i$ such that $U' = N^+(x) = N^-(x)$ for any vertex-cut $U' = N^+(z)$ (or $N^-(z)$) of D_i for $i = 1, 2$. By Lemma 2.5, D_1 and D_2 are symmetric digraphs, (i) holds.

Case 2. If there exists a minimum vertex-cut U of $D_1 \times D_2$ such that $U = N^+((x, y)) = N^-((x, y))$ does not hold for any vertex $(x, y) \in V(D_1 \times D_2)$, then $\delta_1 = \delta_2 = 1$ by Theorem 2.4. Since D_1 and D_2 are strongly connected regular digraphs, we have that D_1 and D_2 are directed cycles, so (ii) holds. □

Finally, we characterize the double-super-connected lexicographic product of two digraphs.

Proposition 2.7. *Let $D = \overrightarrow{C_n}[S_1, S_2, \dots, S_n]$, where $\overrightarrow{C_n}$ denotes a directed cycle of length n, and let $|V(S_i)|$ be minimum for some $i \in \{1, 2, \dots, n\}$. If there exist two vertices $x \in V(S_{i-1}), y \in V(S_{i+1})$ such that $d^+_{S_{i-1}}(x) = d^-_{S_{i+1}}(y) = 0$, then D is double-super-connected.*

Proof. If $D = \overrightarrow{C_n}[S_1, S_2, \dots, S_n]$ and $|V(S_i)|$ is minimum for some $i \in \{1, 2, \dots, n\}$, then the vertex set of S_i is a minimum vertex-cut of D. If there exist two vertices $x \in V(S_{i-1}), y \in V(S_{i+1})$ such that $d^+_{S_{i-1}}(x) = d^-_{S_{i+1}}(y) = 0$, then $N^+(x) = V(S_i)$ and $N^+(y) = V(S_i)$; thus D is double-super-connected. □

The subdigraph $D_2^{x_i}$ is the digraph with vertex set $\{(x_i, y_j)|j = 1, 2, \dots, n_2\}$ and arc set $\{((x_i, y_j), (x_i, y_{j'}))|(y_j, y_{j'}) \in A_2\}$. Clearly, $D_2^{x_i}$ is isomorphic to digraph D_2 for $i = 1, 2, \dots, n_1$. The out-degree of the vertex (x, y) is $d^+_{D_1[D_2]}((x, y)) = d^+_{D_1}(x)n_2 + d^+_{D_2}(y)$ and the minimum degree of the digraph $D_1[D_2]$ is $\delta_1 n_2 + \delta_2$. From the definition of a lexicographic product, it is easy to see that $D_1[D_2]$ can be obtained from D_1 by replacing each vertex of D_1 with a copy of D_2 in such a way that every arc (x_i, x_j) in D_1 contains all possible arcs from $D_2^{x_i}$ to $D_2^{x_j}$.

It is clear that if D_1 is an isolated vertex, then $D_1[D_2] \cong D_2$, and if D_2 is an isolated vertex, then $D_1[D_2] \cong D_1$. In the following, we always assume that D_1 and D_2 are strongly connected digraphs with at least two vertices.

Theorem 2.8. *$D_1[D_2]$ is max-κ if and only if D_1 is a complete graph and D_2 is max-κ.*

Proof. Suppose that D_1 is not a complete graph. Let x_i be a vertex of D_1 with minimum degree and $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{\delta_1}}\}$ be the out-neighbor (or in-neighbor) set of x_i. Then the vertex set of $\cup_{j=1}^{\delta_1} D_2^{x_{ij}}$ is a vertex-cut with cardinality $\delta_1 n_2$. So we have $\delta_1 n_2 < \delta_1 n_2 + \delta_2$ (note that $\delta_2 > 0$). Thus $D_1[D_2]$ is not max-κ. Furthermore, D_2 must be max-κ, since otherwise, $D_1[D_2]$ cannot be max-κ. On the other hand, since D_1 is a complete graph and D_2 is max-κ, $\delta(D_1[D_2]) = n_2(n_1 - 1) + \delta_2$. Assume U is a minimum vertex-cut, and let $j \in \{1, 2, \ldots, n_1\}$ and $U' = \cup_{i=1, i\neq j}^{n_1} D_2^{x_i}$; then $U' \subseteq U$. Otherwise, $D_1[D_2] - U$ is strongly connected. Thus $(D_1[D_2] - U') \cong D_2^{x_i}$. Since D_2 is max-κ, $D_1[D_2]$ is max-κ. □

Similarly, we can give some necessary and sufficient conditions for a digraph to be super-κ, hyper-κ, and double-super-connected. By Theorem 2.8, if D_1 is a complete graph, then the connected properties of $D_1[D_2]$ are similar to those of D_2; thus the following theorems can be obtained easily.

Theorem 2.9. *$D_1[D_2]$ is super-κ if and only if D_1 is a complete graph and D_2 is super-κ.*

Theorem 2.10. *$D_1[D_2]$ is hyper-κ if and only if D_1 is a complete graph and D_2 is hyper-κ.*

Theorem 2.11. *$D_1[D_2]$ is double-super-connected if and only if D_1 is a complete graph and D_2 is double-super-connected.*

3. Double-super-connected Abelian Cayley digraphs

Let G be a finite Abelian group, and $S \subset G \setminus \{0\}$; then $X = Cay(G, S)$ is an Abelian Cayley digraph. If $|S| = |G| - 1$, then X is a complete graph. Now we consider $|S| \leq |G| - 2$.

Lemma 3.1. *Let $X = Cay(G, S)$ be an Abelian Cayley digraph, and $S = \{s_1, s_2, \ldots, s_k\}$; then $X = Cay(G, S)$ is double-super-connected if and only if X is super-connected and there exists an ordering $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ of S such that there is an element g of G satisfying*

$$s_{i_{j+1}} + s_{i_{k-j}} = g \quad (j = 0, 1, 2, \ldots, k - 1).$$

Proof. If X is a double-super-connected digraph, then there are two vertices $x, y \in V(X)$ such that $N^+(x) = N^-(y)$; without loss of generality, let

$$N^+(x) = \{x + s_1, x + s_2, \ldots, x + s_k\},$$
$$N^-(y) = \{y - s_1, y - s_2, \ldots, y - s_k\}.$$

For $x + s_1 \in N^+(x)$, there exists $s_j \in S$ such that $x + s_1 = y - s_j$; thus $y = x + s_1 + s_j$. For $y - s_1 \in N^-(y)$, there exists $s_i \in S$ such that $y - s_1 = x + s_i$; thus $y = x + s_1 + s_i$. Therefore, we have $s_i = s_j$.

Hence, there exists an ordering $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ of S, such that

$$s_{i_1} + s_{i_k} = s_{i_2} + s_{i_{k-1}} = \cdots = s_{i_{j+1}} + s_{i_{k-j}} = \cdots = s_{i_k} + s_{i_1} \quad (j = 0, 1, 2, \ldots k - 1).$$

Let $s_{i_{j+1}} + s_{i_{k-j}} = g \in G$; then $y = x + g$.

On the other hand, if X is super-κ and there exists an ordering $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ of S such that there is an element g of G satisfying $s_{i_{j+1}} + s_{i_{k-j}} = g$ $(j = 0, 1, 2, \ldots, k - 1)$, then there exists a vertex x such that $U = N^+(x)$ or $N^-(x)$ for any minimum vertex-cut U, say $U = N^+(x)$; thus $U = N^+(x) = \{x + s_{i_1}, x + s_{i_2}, \ldots, x + s_{i_k}\}$. Since $s_{i_{j+1}} + s_{i_{k-j}} = g$, this implies that $U = \{x + g - s_{i_k}, x + g - s_{i_{k-1}}, \ldots, x + g - s_{i_1}\} = N^-(x + g)$. Therefore, $X = Cay(G, S)$ is double-super-connected. The proof is completed. □

A subset $P \subset G$ is said to be an arithmetic progression with difference d when $P = \{a, a + d, \ldots, a + sd\}$ for some $a \in G$, $s \in N$. When $G = Z_n$, and $d = 1$, we say that P is consecutive. A subset S is said to be a semi-progression when $B = S \cup \{0\}$ is an arithmetic progression with difference d and $\{-d, d\} \subset S$.

Lemma 3.2 ([2]). *Suppose $S \cup \{0\}$ is an arithmetic progression. Then $X = Cay(G, S)$ is super-connected if and only if S is not a semi-progression.*

Theorem 3.3. *Suppose $S \cup \{0\}$ is an arithmetic progression. Then $Cay(G, S)$ is double-super-connected if and only if S is not a semi-progression.*

Proof. If $S \cup \{0\}$ is an arithmetic progression, then S is an arithmetic progression with difference d and $S = \{s_1, s_2, \cdots, s_k\} = \{s_1, s_1 + d, \ldots, s_1 + (k-1)d\}$, where $s_i = s_1 + (i-1)d$. Therefore $s_i + s_{k-i+1} = s_1 + (i-1)d + s_1 + (k-i)d = 2s_1 + (k-1)d \in G$, $1 \leq i \leq k$. By Lemmas 3.1 and 3.2, we know that $Cay(G, S)$ is double-super-connected if and only if S is not a semi-progression. □

Corollary 3.4. *The Harary digraph $Cay(Z_n, \{1, 2, \ldots, s\})$ is double-super-connected for each $1 \leq s < n - 1$.*

Proof. Let $S = \{1, 2, \ldots, s\}$; since $S \cup \{0\} = \{0, 1, 2, \ldots, s\}$ and $1 \leq s < n - 1$, $S \cup \{0\}$ is an arithmetic progression and $-1 \notin S$, we have that S is not a semi-progression. Thus $Cay(Z_n, S)$ is double-super-connected by Theorem 3.3. □

For a digraph D of order n, if $\delta(D) = n - 1$, then D is a complete digraph.

Corollary 3.5. *There exist double-super-connected digraphs for any given order and minimum degree.*

4. Conclusions

This paper introduces the notion of a double-super-connected digraph and characterizes being double-super-connected for some particular digraphs. Lastly, we prove that there are double-super-connected digraphs of any given order and maximum degree. In the future, readers could offer others criteria for determining whether certain digraphs are double-super-connected or not. Furthermore, readers could characterize some particular digraphs being double-super-connected, such as vertex-transitive digraphs, arc-transitive digraphs and so on.

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