More on Representation Theory for Default Logic[1]

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In this paper, we investigate the representability of a family of theories as the set of extensions of a default theory. First, we present both new necessary conditions and sufficient ones for the representability by means of general default theories, which improves on similar results known before. Second, we show that one always obtains representable families by eliminating countably many theories from a representable family. Finally, we construct two examples of denumerable, representable families; one is not supercompactly nonincluding, and the other consists of mutually inconsistent theories but fails to be represented by a normal default theory. © 2001 Academic Press

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1. INTRODUCTION

Reiter's default logic [12] is one of the most prominent formalizations of nonmonotonic reasoning, and is extensively investigated by the community of logical foundations of artificial intelligence [1, 2, 4, 5, 7, 15, 16]. A default theory Δ describes a collection of formula sets, namely the family of all extensions of Δ, which represents a family of belief sets of a reasoning agent. An important issue is, then, to characterize those families of sets that can be represented as the set of extensions for a certain default theory. This issue is called the representation theory for default logic.

Several significant contributions have been made to the representation theory for default logic (see, for example, [8–11, 14]). In particular, the case for default theories with a finite set of defaults and the case for normal default theories were addressed successfully. However, the infinite, nonnormal case is considered a rather hard open problem [11]. This paper aims at tackling this hard problem.

By $ext(\Delta)$ we denote the family of all extensions of a default theory Δ. There is a well-known constraint on the family $ext(\Delta)$ obtained first by Reiter [12] that such a family must be *nonincluding*. Moreover, Marek *et al.* [11] showed that if a finite family is nonincluding, then it can be represented by a default theory. Unfortunately, this is not true for infinite families as they actually constructed a denumerable family of nonincluding theories [11] that is not representable. In this paper, we present a new necessary condition for those families that can be represented by default theories. In addition, we present sufficient conditions for representability for infinite families of theories, which are much weaker than those in [11]. We further address the representability problem with respect to subfamilies of a representable family.

The family of extensions of a *normal* default theory is not only nonincluding, but all its members are pairwise inconsistent [12]. Again, this stronger constraint could not fully characterize those families of theories representable by normal default theories. Marek and Truszczyński [9] raised an interesting question: Is the family $ext(\Delta)$ of all extensions of a default theory Δ representable by a normal default theory under the conditions that $ext(\Delta)$ is not empty with all its members mutually inconsistent?[2] Marek *et al.* [11] constructed an example of a denumerable family of pairwise inconsistent theories that are not representable by normal default theories. However, this example is not representable by any default theory and thus does not answer the question above. In this paper, we construct another denumerable family of pairwise inconsistent theories, which is representable by a default theory but not by any normal default theory, and hence we answer this question negatively.

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[2] In personal correspondence by e-mail, Marek told the author of this paper that the answer to the question is negative.

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2. PRELIMINARIES

In this paper, by $\mathcal{L}$ we denote a language of propositional logic with a denumerable set of atoms At. We denote *propositional provability* by $\vdash$ and the corresponding consequence operator by Cn. By a *theory* we *always* mean a subset of $\mathcal{L}$ *closed under propositional provability*. Let B be a set of standard monotone inference rules. As in [9], we obtain a formal proof system (denoted by $PC+B$) by extending propositional calculus with the rules from B. The derivations in the system $PC+B$ are built by means of propositional provability and rules in B. More formally, by a *proof* in the system $PC+B$, we mean any sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that for every $i = 1, \ldots, n$,

1. φ_i stands for a tautology, or is obtained from formulas φ_j and φ_k, with $j, k < i$, by means of modus ponens, or
2. there is a rule $\frac{\alpha}{\beta}$ in B such that $\alpha = \varphi_j$, for some $j < i$ and $\beta = \varphi_i$.

The corresponding provability operator is analogously denoted by $\vdash_B$ and the consequence operator by $Cn^B(\cdot)$.

A default is an expression d of the form $\frac{\alpha:\Gamma}{\beta}$, where α and β are formulas from $\mathcal{L}$ and Γ is a *finite* list of formulas from $\mathcal{L}$. The formula α is called the *prerequisite* of d ($p(d)$, in symbols) and β is called the *consequent* of d ($c(d)$, in symbols). The set of formulas Γ is called the *justification* set of d and is denoted by $j(d)$. This terminology is naturally extended to a set of defaults D. Namely, the *prerequisite, consequent*, and justification set of D, in symbols $p(D)$, $c(D)$, and $j(D)$, are defined by

$$p(D) = \bigcup_{d\in D}\{p(d)\}, \quad c(D) = \bigcup_{d\in D}\{c(d)\}, \quad j(D) = \bigcup_{d\in D} j(d).$$

If $p(d)$ is a tautology, d is called *prerequisite-free*. In such case, $p(d)$ is usually omitted from the notation of d. If $j(d) = \{c(d)\}$, d is called *normal*. If $j(d) = \{c(d) \wedge \gamma\}$ for some formula γ, d is called *seminormal*.

By a *default theory* we mean a pair $\Delta = (D, W)$, where D is a set of defaults and W a set of formulas. The set W is called the *objective* part of (D, W). We say that (D, W) is a *prerequisite-free* if all defaults in D are prerequisite-free, *normal* if they are normal, and *seminormal* if they are seminormal.

Following [9], we give an alternative definition of extension, which is proved equivalent to the original one by Reiter. First, we need the following concepts.

For a default d and a set of defaults D, define

$$Mon(d) = \frac{p(d)}{c(d)} \quad \text{and} \quad Mon(D) = \left\{ \frac{p(d')}{c(d')} \,\middle|\, d' \in D \right\}.$$

Given a set of formulas S, those defaults d such that $S \not\vdash \neg\gamma$ for every $\gamma \in j(d)$ are called *S-applicable*. A set D of defaults is *S-applicable* if all its members are S-applicable.

Remark that for every set S of formulas, those defaults with the empty justification set are S-applicable (even if S is inconsistent), and if a finite set D of defaults is S-applicable, then those default with the justification set $j(D)$ are S-applicable also.

We define

$$D_S = Mon(\{d \in D : S \not\vdash \neg\gamma \text{ for every } \gamma \in j(d)\}).$$

A theory S is an *extension* of a default theory (D, W), if and only if

$$S = Cn^{D_S}(W).$$

The family of all extensions of (D, W) is denoted by $ext(D, W)$.

We now define some key concepts concerning the representability issue for default logic.

DEFINITION 2.1. Two default theories Δ and Δ' are said to be equivalent to each other (denoted by $\Delta \approx \Delta'$), if they have the same extensions, i.e., $ext(\Delta) = ext(\Delta')$. A default theory is said to be representable in a class of default theories if the default theory is equivalent to some default theory in the class.

DEFINITION 2.2. Let $\mathcal{T}$ be a family of theories contained in $\mathcal{L}$. The family is representable by a default theory Δ if $ext(\Delta) = \mathcal{T}$. A family (of theories) is called *representable* if it is representable by a default theory.

3. REPRESENTABILITY BY GENERAL DEFAULT THEORIES

We get a necessary condition for extensions of a default theory, which is stronger than the well-known necessary condition obtained first by Reiter [12] that extensions be pairwise nonincluding. We then present some sufficient conditions improving on similar conditions known previously (see, for example, [11, Proposition 3.3, Theorem 4.2]).

To prove these results, we need a lemma allowing us to replace any default theory with an equivalent default theory in which all defaults are prerequisite-free. This result was obtained independently by Schaub [17], Bonatti and Eiter [3], and Marek *et al.* [11]. However, our argument shows that the equivalent theory can be simpler, in the *inference-free* form defined as follows.

DEFINITION 3.1. A default theory (D, W) is called *inference-free* if

1. The default theory is prerequisite-free.
2. The objective part W is empty.
3. For prerequisite-free defaults d, d', if $d \in D$ and d' has the same justification set as d and $c(d')$ is logically inferred from $c(d)$, then $d' \in D$ also.
4. For every two defaults d, d' in D, the default: $j(d) \cup j(d')/c(d) \wedge c(d')$ is also in D.

LEMMA 3.1. *For every default theory Δ there is an inference-free default theory Δ' equivalent to Δ.*

Proof. Given a default theory $\Delta = (D, W)$, we define another default theory $\Delta' = (D', \emptyset)$[3] as

$$D' = \left\{ \frac{:j(D'')}{\varphi} \;\middle|\; \varphi \in \mathcal{L},\ D'' \text{ is a finite subset of } D \text{ and } W \vdash_{Mon(D'')} \varphi \right\}.$$

It is trivial to check that the default theory Δ' is inference-free. We now prove that Δ' is equivalent to Δ; i.e., they have exactly the same extensions. Recalling the definition of an extension, it suffices to show that, for all sets of formulas S,

$$Cn^{D_S}(W) = Cn^{D'_S}(\emptyset)$$

or that for every formula φ,

$$W \vdash_{D_S} \varphi \text{ iff } \vdash_{D'_S} \varphi.$$

Assume first that $W \vdash_{D_S} \varphi$. Then there is a finite subset B of D_S such that $W \vdash_B \varphi$. There must be, by the definition of D_S, a set D'' of defaults such that $Mon(D'') = B$ and D'' is S-applicable. Hence, the default $:j(D'')/\varphi$ is in D' and is S-applicable. Since $Mon(:j(D'')/\varphi) = \top/\varphi$, where $\top$ is a tautology, we have that the rule $\top/\varphi$ is in D''_S, which leads to $\vdash_{D'_S} \varphi$.

To prove the converse implication, assume that $\vdash_{D'_S} \varphi$. Then there must be a finite subset D^* of D' such that all defaults in D^* are S-applicable and $\vdash_{Mon(D^*)} \varphi$. By the definition of D', we can assume that

$$D^* = \left\{ \frac{:j(D''_i)}{\varphi_i} \;\middle|\; i < n \right\},$$

[3] By $\emptyset$ we denote the empty set.

where n is a natural number. Observe that, for each i,

$$W \vdash_{Mon(D''_i)} \varphi_i.$$

Moreover, since $:j(D''_i)/\varphi_i$ is S-applicable, those defaults in D''_i are also S-applicable. Observe that $\vdash_{Mon(D^*)} \varphi$ is equivalent to $\{\varphi_i \mid i < n\} \vdash \varphi$. Hence,

$$W \vdash_{Mon\left(\bigcup\limits_{i<n} D''_i\right)} \varphi.$$

It follows that $W \vdash_{D_S} \varphi$. ■

We now introduce three notions of a *nonincluding family* of sets.

DEFINITION 3.2. Let $\mathcal{F}$ be a family of theories. Then we say that $\mathcal{F}$ is *nonincluding* if $\mathcal{F}$ is an antichain (that is, for $T, T' \in \mathcal{F}$, if $T \subseteq T'$ then $T = T'$); *compactly nonincluding* if for every theory $E \in \mathcal{F}$ and every formula $\alpha \in E$, there exists a finite list of families $\mathcal{F}_0, \ldots, \mathcal{F}_{k-1}$ such that

1. $\bigcup_{i<k} \mathcal{F}_i = \{T \in \mathcal{F} : \alpha \notin T\}$
2. for every $i < k$, $(\cap\mathcal{F}_i - E) \neq \emptyset$;

and *supercompactly nonincluding* if for every theory $E \in \mathcal{F}$, there exists a finite list of families $\mathcal{F}_0, \ldots, \mathcal{F}_{k-1}$ such that

1. $\bigcup_{i<k} \mathcal{F}_i = \mathcal{F} - \{E\}$
2. for every $i < k$, $(\cap\mathcal{F}_i - E) \neq \emptyset$.

It is important to remark that the list of families $\mathcal{F}_0, \ldots, \mathcal{F}_{k-1}$ in the last two parts of Definition 3.2 could be empty. In this case, we have that $k = 0$, and by mathematical conventions, $\bigcup_{i<k} \mathcal{F}_i = \emptyset$ and it is trivially true that "for every $i < k$, $(\cap\mathcal{F}_i - E) \neq \emptyset$." Accordingly, the family of a single inconsistent theory is both compactly nonincluding and supercompactly nonincluding.

Clearly, if a family of theories $\mathcal{F}$ is supercompactly nonincluding, then it is compactly nonincluding, but not vice versa (see the example in the proof of Proposition 5.1). In addition, we have the following result.

PROPOSITION 3.1. *If a family* (*of theories*) *is compactly nonincluding, then it is nonincluding.*

Proof. Suppose that a family $\mathcal{F}$ is compactly nonincluding. We must show it is nonincluding also. Assume conversely that $F_1 \subseteq F_2$ hold for some distinct theories $F_1, F_2 \in \mathcal{F}$, and hence there exists a formula $\alpha \in F_2 - F_1$. By the assumption that $\mathcal{F}$ is compactly nonincluding, there exists a list of families $\mathcal{F}_0, \ldots, \mathcal{F}_{k-1}$ such that

1. $\bigcup_{i<k} \mathcal{F}_i = \{T \in \mathcal{F} : \alpha \notin T\}$
2. for every $i < k$, $(\cap\mathcal{F}_i - F_2) \neq \emptyset$.

As $\alpha \notin F_1$, there is a $j < k$ such that $F_1 \in \mathcal{F}_j$ and hence $\cap\mathcal{F}_j \subseteq F_1$. So, by the assumption that $F_1 \subseteq F_2$, we have that $\cap\mathcal{F}_j - F_2$ is an empty set, contradicting the latter assertion about $\mathcal{F}_i$s. ■

The converse of the above proposition does not necessarily hold. We demonstrate this with an example. Let $\{p_0, p_1 \ldots\}$ be a set of propositional atoms. Define $T_i = \{p_i\}$, $i = 0, 1, \ldots,$ and $\mathcal{T} = \{T_i : i = 0, 1, \ldots\}$. It follows immediately that this family $\mathcal{T}$ is nonincluding but not compactly nonincluding.

To gain more intuitive impression on these three notions, we restate their definitions using the concept of a *hitting set*, which was widely used in the literature (see, for example, [6, 12]). A set H is a *hitting set* for a family $\mathcal{F}$ of theories if $H \cap F \neq \emptyset$ holds for every $F \in \mathcal{F}$.

PROPOSITION 3.2. *Let $\mathcal{F}$ be a family of theories.*

(a) *$\mathcal{F}$ is nonincluding iff, for each $F \in \mathcal{F}$, there is a hitting set for $\mathcal{F} - \{F\}$ that is not a hitting set for $\mathcal{F}$.*

(b) *$\mathcal{F}$ is compactly nonincluding iff, for each $F \in \mathcal{F}$ and $\alpha \in F$, there is a* finite *hitting set for $\{T \in \mathcal{F} : \alpha \notin T\}$ that is not a hitting set for $\{T \in \mathcal{F} : \alpha \notin T\} \cup \{F\}$.*

(c) *$\mathcal{F}$ is supercompactly nonincluding iff, for each $F \in \mathcal{F}$, there is a* finite *hitting set for $\mathcal{F} - \{F\}$ that is not a hitting set for $\mathcal{F}$.*

Proof. The result of (a) follows immediately by the definition.

To prove the "if" part of (b), we can associate each formula φ_j in the hitting set being considered with the family of those theories that contain the formula. For the "only if" part of (b), given $F \in \mathcal{F}$ and $\alpha \in F$, we then get the families $\mathcal{F}_j$ as in the definition of compact nonincluding. The result follows if we select a formula α_j in $\cap\mathcal{F}_j - F$ for each j and consider the set of those α_js. This completes the proof of (b).

The proof of (c) is similar to that of (b). ■

The next proposition says that the three notions above are equivalent to each other for finite families.

PROPOSITION 3.3. *Let $\mathcal{F}$ be a finite family of theories. Then the following three propositions are equivalent to each other*:

(a) *$\mathcal{F}$ is nonincluding.*

(b) *$\mathcal{F}$ is compactly nonincluding.*

(c) *$\mathcal{F}$ is supercompactly nonincluding.*

Proof. As we have demonstrated already that (a) implies (b) and (b) implies (c), it suffices to prove (c) implies (a), but this is trivial by Proposition 3.2 and the finiteness of $\mathcal{F}$. ■

What follows is a new necessary for families representable by default theories.

THEOREM 3.1. *If $\mathcal{F}$ is representable by a default theory, then it is compactly nonincluding.*

Proof. Suppose that $\mathcal{F}$ is representable by a default theory Δ, E is a theory in $\mathcal{F}$ and α is a formula in E. By Lemma 3.1, we can assume that Δ is inference-free. Let $\Delta = (D, \emptyset)$. As the extensions of the default theory $(D, \emptyset)$ are exactly those theories in $\mathcal{F}$, E is an extension of $(D, \emptyset)$. By the definition of extension, we then have that $E = Cn^{D_E}(\emptyset)$, and hence $\alpha \in Cn^{D_E}(\emptyset)$. It follows that there exists a finite subset D' of D such that each default in D' is E-applicable and $\vdash_{Mon(D')} \alpha$ holds, i.e., $c(D') \vdash \alpha$ since the default theory $(D, \emptyset)$ is inference-free. We now have that the default

$$d_\alpha = \frac{:j(D')}{\alpha} \in D.$$

Suppose that $j(D')$ consists of α_js (for $j < k$). For each α_j, let

$$\mathcal{F}_j = \{T \in \mathcal{F} \mid \neg\alpha_j \in T \text{ and } \alpha \notin T\}.$$

Then $\mathcal{F}_j \subseteq \{T \in \mathcal{F} \mid \alpha \notin T\}$. On the other hand, for those $T \in \mathcal{F}$ such that $\alpha \notin T$, we have that $\alpha \notin Cn^{D_T}(\emptyset)$, which yields that d_α is not T-applicable. Thus there is an α_j such that $\neg\alpha_j \in T$, and we have that $T \in \mathcal{F}_j$. So we have proved that

$$\bigcup_{i<k} \mathcal{F}_i = \{T \in \mathcal{F} \mid \alpha \notin T\}.$$

Now we need only to show that, for each $j < k$, $\cap\mathcal{F}_j - E$ is nonempty, but $\neg\alpha_j \in \cap\mathcal{F}_j$ and $\neg\alpha_j \notin E$ (recall that $\alpha_j \in j(D')$ and each default in D' is E-applicable). ■

Since a family being compactly nonincluding implies the family nonincluding, we get the following well-known result [9, 12] as a corollary.

COROLLARY 3.1. *If $\mathcal{F}$ is representable by a default theory then it is nonincluding.*

The converse of Theorem 3.1 does not necessarily hold. In other words, Theorem 3.1 gives a necessary condition for representable families, but this conditions is not a sufficient one.

PROPOSITION 3.4. *There exists a family that is compactly nonincluding but not representable.*

Proof. The result follows from a cardinality argument. There are continuum-many default theories in the given (denumerable) language, while there are more than continuum-many families that are compactly nonincluding since there is a default theory Δ with continuum-many extensions and hence all subsets of $ext(\Delta)$ are compactly nonincluding. ■

We wonder whether there exists a denumerable family that is compactly nonincluding family but not representable.

The next result completely characterizes subfamilies of representable families.

THEOREM 3.2. *A family $\mathcal{F}$ is a subset of a representable family if and only if it is compactly non-including.*

Proof. The "only if" part follows immediately from Theorem 3.1 and the fact that if a family is compactly nonincluding, so are its subfamilies. As for the "if" part, assume that a family of theories $\mathcal{F}$ is compactly nonincluding. Given $E \in \mathcal{F}$ and $\alpha \in E$, we obtain, by the assumption above, a finite list of families $\mathcal{F}_0, \ldots, \mathcal{F}_{k-1}$ such that

$$\bigcup_{j<k} \mathcal{F}_j = \{T \in \mathcal{F} : \alpha \notin T\}$$

and for each $j < k$, there is an α_j such that

$$\alpha_j \in (\cap \mathcal{F}_j - E).$$

By $d_{E,\alpha}$, we denote the default $:\{\neg\alpha_0, \ldots, \neg\alpha_{k-1}\}/\alpha$. Now consider the default theory $(D, \emptyset)$, where

$$D = \{d_{T,\varphi} \mid T \in \mathcal{F} \text{ and } \varphi \in T\}.$$

We want to prove that $T \subseteq ext(D, \emptyset)$, i.e., for each $T \in \mathcal{F}$,

$$T = Cn^{D_T}(\emptyset).$$

On one hand, for each $\varphi \in T$, $d_{T,\varphi}$ is T-applicable, and $\varphi \in Cn^{D_T}(\emptyset)$. On the other hand, for all T-applicable defaults $d_{T',\alpha'}$, we must prove that $\alpha' \in T$. By the construction of $d_{T',\alpha'}$, we can assume that $j(d_{T',\alpha'}) = \{\alpha'_0, \ldots, \alpha'_{k'-1}\}$, and there are families $\mathcal{F}'_0, \ldots, \mathcal{F}'_{k'-1}$ such that

$$\bigcup_{j<k'} \mathcal{F}'_j = \{S \in \mathcal{F} : \alpha' \notin S\}$$

and for every $j < k'$,

$$\neg\alpha'_j \in \cap\mathcal{F}'_j.$$

Suppose that $\alpha' \notin T$. Then we would have that $T \in \mathcal{F}'_j$ for some $j < k'$. Thus, $\neg\alpha'_j \in T$, contradicting the assumption that the default $d_{T',\alpha'}$ is T-applicable. ■

We now propose a sufficient condition for the representablity of a family.

THEOREM 3.3. *If a family $\mathcal{F}$ is supercompactly nonincluding, then it is representable by a default theory.*

Proof. Assume that a family of theories $\mathcal{F}$ is supercompactly nonincluding. Since the empty family is representable, we assume further that $\mathcal{F}$ is a nonempty family. Given an arbitrary $E \in \mathcal{F}$, we obtain, by the assumption above, a finite list of families $\mathcal{F}_0, \ldots, \mathcal{F}_{k-1}$ such that

$$\bigcup_{j<k} \mathcal{F}_j = \mathcal{F} - \{E\}$$

and for each $j < k$, there is an α_j such that

$$\alpha_j \in (\cap \mathcal{F}_j - E).$$

For each $E \in \mathcal{F}$, we fix one such finite set of $\neg\alpha_j$s, denoted by J_E. As in the proof of Theorem 3.2, by $d_{E,\alpha}$, we denote the default $:J_E/\alpha$. Now consider the default theory $(D, \emptyset)$, where

$$D = \{d_{T,\varphi} \mid T \in \mathcal{F} \text{ and } \varphi \in T\}.$$

We shall prove that $\mathcal{F}$ is representable by the default theory $(D, \emptyset)$. First, we show that $\mathcal{F} \subseteq ext(D, \emptyset)$. Given $T \in \mathcal{F}$, we must prove $T = Cn^{D_T}(\emptyset)$. On one hand, for each $\varphi \in T$, $d_{T,\varphi}$ is T-applicable, and hence $\varphi \in Cn^{D_T}(\emptyset)$. On the other hand, for every T-applicable default $d_{T',\alpha'}$ in D, we prove $\alpha' \in T$ as follows. By the construction of $d_{T',\alpha'}$, we can assume that $j(d_{T',\alpha'}) = \{\neg\alpha'_0, \ldots, \neg\alpha'_{k'-1}\}$, and there are families $\mathcal{F}'_0, \ldots, \mathcal{F}'_{k'-1}$ such that

$$\bigcup_{j<k'} \mathcal{F}'_j = \mathcal{F} - \{T'\}$$

and for every $j < k'$,

$$\alpha'_j \in \cap \mathcal{F}'_j.$$

Suppose that $\alpha' \notin T$. Then we would have that $T \neq T'$ for $\alpha' \in T'$, and hence $T \in \mathcal{F}'_j$ for some $j < k'$. Thus, $\alpha'_j \in T$, contradicting the assumption that the default $d_{T',\alpha'}$ is T-applicable.

Now, we show that $\mathcal{F} \supseteq ext(D, \emptyset)$. Given an $E \in ext(D, \emptyset)$, we must prove $E \in \mathcal{F}$.

Without loss of generality, we assume that $E \neq Cn(\emptyset)$; otherwise, all extensions of $(D, \emptyset)$ are $Cn(\emptyset)$, and the result follows since $\mathcal{F} \subseteq ext(D, \emptyset)$ and $\mathcal{F} \neq \emptyset$. As $E = Cn^{D_E}(\emptyset) \neq Cn(\emptyset)$, there is an E-applicable default $d_{T',\alpha'}$ in D, where $T' \in \mathcal{F}$. Hence, for all $\varphi' \in T'$, $d_{T',\varphi'}$ is also E-applicable. This leads to $\varphi' \in Cn^{D_E}(\emptyset) = E$. Thus, $T' \subseteq E$. As a result, $T' = E$ since we have proved that T' is an extension of $(D, \emptyset)$. This completes the proof. ■

Again, the converse of Theorem 3.3 does not hold. In Section 5, we will actually construct a denumerable family that is representable by a default theory but is not supercompactly nonincluding.

We say that a family $\mathcal{F}$ of theories has a *strong system of distinct representatives* (SSDR) if for every $T \in \mathcal{F}$ there is a formula $\varphi_T \in T$ that does not belong to any other theory in $\mathcal{F}$ (see [11]).

PROPOSITION 3.5. *If a family $\mathcal{F}$ of theories is compactly nonincluding and has an SSDR then it is supercompactly nonincluding.*

Proof. Suppose that $\mathcal{F}$ is a compactly nonincluding family that has an SSDR. By the definition, for each E and $\alpha \in E$, there exists a finite list of families $\mathcal{F}_0, \ldots, \mathcal{F}_{k-1}$ such that

$$\bigcup_{j<k} \mathcal{F}_j = \{T \in \mathcal{F} : \alpha \notin T\},$$

and for each $j < k$,

$$(\cap \mathcal{F}_j - E) \neq \emptyset.$$

Consider the case where α is the formula $\varphi_T \in T$, which does not belong to any other theory in $\mathcal{F}$. It follows immediately that $\mathcal{F}$ is supercompactly nonincluding, since we have that

$$\{T \in \mathcal{F} : \varphi_T \notin T\} = \mathcal{F} - \{E\}. \qquad \blacksquare$$

As a corollary of Theorem 3.3 and the proposition above, we obtain another sufficient condition for the representablity of a family.

COROLLARY 3.2. *If a family $\mathcal{F}$ is representable by a default theory, then every family $\mathcal{G} \subseteq \mathcal{F}$ that has an SSDR is representable by a default theory.*

Corollary 3.2 improves a closely related result in [11].

COROLLARY 3.3 [11, THEOREM 4.2]. *If a family $\mathcal{F}$ is representable by a default theory and has an SSDR, then every family $\mathcal{G} \subseteq \mathcal{F}$ is representable by a default theory.*

Proof. The result follows immediately from the fact that if a family $\mathcal{F}$ has an SSDR, so does every subfamily of $\mathcal{F}$. $\blacksquare$

The result of [11, Theorem 4.2] essentially says that a family that has an SSDR is representable under the condition that it is a subfamily of a representable family that has an SSDR, while Corrollary 3.2 weakens the condition by that it is a subfamily of a representable family. In the next section, we shall improve the result of [11, Theorem 4.2] in another direction.

4. ELIMINATING COUNTABLY MANY EXTENSIONS

In this section, we address the representablity problem with respect to subfamilies of a representable family. Marek *et al.* [11] showed that if $\mathcal{F}$ is representable by a default theory and has an SSDR, then every subfamily of $\mathcal{F}$ is also representable by a default theory. Note that every family that has an SSDR must be countable. We surprisingly show that the above result of Marek *et al.* still holds if we replace the condition that $\mathcal{F}$ has an SSDR simply by that $\mathcal{F}$ is a countable family.

THEOREM 4.1. *Let $\mathcal{F}$ be representable by a default theory, and let $\mathcal{F}' \subseteq \mathcal{F}$ be countable. Then $\mathcal{G} = \mathcal{F} - \mathcal{F}'$ is also representable by a default theory.*

Proof. Suppose that $\mathcal{F}$ is a representable family and $E_0, \ldots, E_j, \ldots$ are theories in $\mathcal{F}$. We must prove that $\mathcal{F} - \{E_0, \ldots, E_j, \ldots\}$ is representable.

First, we can assume, by Lemma 3.1, that $\mathcal{F}$ is representable by an inference-free default theory $(D, \emptyset)$.

Since the theories in $\mathcal{F}$ are nonincluding, for every number j and every $i < j$, there is a formula $\varphi_{i,j}$ such that $\varphi_{i,j} \in E_j$ and $\varphi_{i,j} \notin E_i$. Thus, the formula $\varphi_{0,j} \wedge \cdots \wedge \varphi_{j-1,j}$, denoted by φ_j, is in E_j, but not in E_i for each $i < j$. For convenience, let φ_0 be a fixed tautological formula.

To construct a default theory such that all its extensions are exactly those in $\mathcal{F} - \{E_0, \ldots, E_j, \ldots\}$, we introduce some notations. Let g be a fixed encoding function for defaults in D; thus, for each default $d \in D$, $g(d)$ is the unique encoding number of d. For a finite subset D' of D, by $m(D')$ we denote the default in D' with the largest encoding number.

For a default $d \in D$, by N_d, we denote the set of those natural numbers k such that there is a finite subset D' of D satisfying that

1. every default in D' is E_k-applicable,
2. $c(D') \vdash \varphi_k$,
3. $d = m(D')$.

Note that for each finite subset D' of D, there is at most one k satisfying the first two properties. To prove this, suppose that for another k', every default in D' is $E_{k'}$-applicable and $c(D') \vdash \varphi_{k'}$. Then both $c(D') \subseteq Cn^{D_{E_{k'}}}(\emptyset) = E_{k'}$ and $c(D') \subseteq Cn^{D_{E_k}}(\emptyset) = E_k$ hold, and hence both $E_{k'} \vdash \varphi_k$ and $E_k \vdash \varphi_{k'}$.

This is impossible for different k and k'. On the other hand, for a fixed d, there are only finitely many finite subsets D' of D such that $d = m(D')$. Thus, we obtain that N_d is a finite set of numbers.

For a finite set of natural numbers N, let $\prod_{i\in N} E_i$ be the Cartesian product of the family $E_i : i \in N$. Thus, every f in $\prod_{i\in N} E_i$ is a function on N and for each $i \in N$, we have that $f(i) \in E_i$.

Now, we define

$$N \otimes d = \left\{ \frac{:j(d) \cup \{\neg(\varphi_n \wedge f(n) : n \in N\}}{c(d)} \;\middle|\; f \in \prod_{i\in N} E_i \right\}.$$

So, given a sequence of formulas α_n $(n \in N)$ such that for every $n \in N$, $\alpha_n \in E_n$, we have that

$$\frac{:j(d) \cup \{\neg(\varphi_n \wedge \alpha_n : n \in N\}}{c(d)} \in N \otimes d.$$

We have the following observation immediately from the fact that for every natural number $j, \varphi_j \in E_j$.

Claim 0. If $j \in N$ then each default in $N \otimes d$ is not E_j-applicable.

Now, consider the default theory $(D^*, \emptyset)$, where D^* is obtained by replacing every default $d \in D$ by all defaults in $N_d \otimes d$; i.e.,

$$D^* = \bigcup_{d\in D} N_d \otimes d.$$

It is easy to see the following two claims holding.

Claim 1. For each $d' \in D^*$, there exists a $d \in D$ such that $j(d) \subseteq j(d')$ and $c(d) = c(d')$.

Claim 2. For an arbitrary theory S, if for every k, $E_k \not\subseteq S$, then for each S-applicable default $d \in D$, there is an S-applicable default $d' \in D^*$ such that $c(d) = c(d')$.

Accordingly, we have that, for an arbitrary theory S,

$$Cn^{D^*_S}(\emptyset) \subseteq Cn^{D_S}(\emptyset),$$

and if $E_j \not\subseteq S$ for all js then

$$Cn^{D^*_S}(\emptyset) = Cn^{D_S}(\emptyset).$$

Claim 1 is trivially true. To prove Claim 2, suppose $E_k \not\subseteq S$ for all ks. We must show that for an arbitrary S-applicable default $d \in D$ there is S-applicable default $d' \in D^*$ with $c(d') = c(d)$. Let $\beta_k \in E_k - S$, and consider the default

$$d' = \frac{:j(d) \cup \{\neg(\varphi_n \wedge \beta_n) : n \in N\}}{c(d)}.$$

Then $d' \in N_d \otimes d \subseteq D^*$ by $\beta_n \in E_n$, and d' is S-applicable by $\beta_n \notin S$; therefore, we have proved Claim 2.

To complete the whole proof, we want to show that $ext(D^*, \emptyset) = \mathcal{F} - \{E_j : j \in \omega\}$. Since $\mathcal{F}$ is nonincluding, it follows immediately that each $T \in F - \{E_0, \ldots, E_j, \ldots\}$ is an extension of $(D^*, \emptyset)$ as $Cn^{D^*_T}(\emptyset) = Cn^{D_T}(\emptyset) = T$. Therefore, it suffices to show the following claims:

Claim 3. If $E = Cn^{D^*_E}(\emptyset)$ then $E \in \mathcal{F}$.

Claim 4. $E_j \neq Cn^{D^*_{E_j}}(\emptyset)$

To prove Claim 3, suppose $E = Cn^{D^*_E}(\emptyset)$. If $E_j \not\subseteq E$ for all js, then $Cn^{D^*_E}(\emptyset) = Cn^{D_E}(\emptyset)$ and hence E is also an extension of $(D, \emptyset)$. Thus, we get $E \in \mathcal{F}$. Otherwise, for some j, $E_j \subseteq E$. This leads to

$Cn^{D_E}(\emptyset) \subseteq Cn^{D_{E_j}}(\emptyset)$. As $Cn^{D_S^*}(\emptyset) \subseteq Cn^{D_S}(\emptyset)$, for an arbitrary theory S, we have that

$$E_j \subseteq E = Cn^{D_E^*}(\emptyset) \subseteq Cn^{D_E}(\emptyset) \subseteq Cn^{D_{E_j}}(\emptyset) = E_j.$$

Accordingly, $E = Cn^{D_E}(\emptyset)$.

As for Claim 4, it suffices to show $\varphi_j \notin Cn^{D^*_{E_j}}(\emptyset)$. Suppose not. Then there is a finite subset D' of D^* such that every default in D' is E_j-applicable and $c(D') \vdash \varphi_j$.

By the definition of D^*, for each default $d' \in D^*$ there is a default in D, denoted by $Ker(d')$, such that d' is of the form

$$d' = \frac{:j(Ker(d') \cup \Gamma)}{c(Ker(d'))}$$

and for each $k \in N_{Ker(d')}$, d' is not E_k-applicable. Note also that for an arbitrary set of formulas S, if $Ker(d')$ is S-applicable, so is the default d'.

Let $D'' = \{Ker(d') : d' \in D'\}$. Then all defaults in D'' are E_j-applicable since all defaults $d' \in D'$ are E_j-applicable. As $c(D'') = c(D') \vdash \varphi_j$, we have that $j \in N_{m(D'')}$. It follows that there exists a default $d^* \in D'$ such $Ker(d^*) = m(D'')$, and hence d^* is not E_j-applicable. This is a contradiction. ■

COROLLARY 4.1. *Let $\mathcal{F}$ be a denumerable family that is representable by a default theory, and let $\mathcal{G} \subseteq \mathcal{F}$. Then $\mathcal{G}$ is also representable by a default theory.*

We conclude this section with an example of a denumerable family that is supercompactly non-including (and hence representable) but has not any SSDR; Corollary 4.1 is thus an improvement of [11, Theorem 4.2].

EXAMPLE 4.1. Let $P = \{p_0, p_1, \ldots\}$ be a set of propositional atoms. Define $T_i = Cn(P - \{p_i\})$, $i = 0, 1, \ldots$, and $\mathcal{T} = \{T_i : i = 0, 1, \ldots\}$.

It is clear that $\mathcal{T}$ is countable and supercompactly nonincluding as for each T_i, $\{p_i\}$ is a hitting set for $\mathcal{T} - \{T_i\}$ but not for $\mathcal{T}$. Moreover, we show that $\mathcal{T}$ has not any SSDR as follows. It suffices to prove that, for an arbitrary formulas φ, if $T_i \vdash \varphi$ holds for some i, then there is another natural number j such that $T_j \vdash \varphi$. By the condition that $T_i \vdash \varphi$ and $T_i \subseteq Cn(P)$, we have a finite set $S \subseteq P$ such that $S \vdash \varphi$. The result follows since there are infinitely many T_js such that $S \subseteq T_j$.

5. TWO MORE EXAMPLES

In this section, we actually construct two denumerable, representable families. The first family is not supercompactly nonincluding. The second one consists of mutually inconsistent theories but is not representable by any normal default theory.

PROPOSITION 5.1. *There is a denumerable family that is representable but not supercompactly non-including.*

Proof. Let us construct an example of denumerable, representable families that are not supercompactly nonincluding. Let $P = \{p_i : i \in \omega\}$ and $Q = \{q_i : i \in \omega\}$[4] be two disjoint sets of propositional atoms. Define, for every $j \in \omega$,

$$S_j = \{p_i : i < j\} \cup \{q_i : i \geq j\}$$

and

$$\mathcal{F} = \{Cn(S_j) : j \in \omega\} \cup \{Cn(P)\}.$$

[4] By ω, we denote the set of all natural numbers.

We prove that $\mathcal{F}$ is not supercompactly nonincluding as follows. Assuming H is a finite hitting set for $\{Cn(S_j) : j \in \omega\}$, we want to show that it is a hitting set for $\mathcal{F}$ also. By the finiteness of H, there is a formula φ such that $\varphi \in Cn(S_j)$ for infinitely many S_js. Let k be a natural number such that those atoms q_i, where $i \geq k$, do not occur in φ. Let $k' \geq k$ be such that $\varphi \in Cn(S_{k'})$, i.e., $S_{k'} \vdash \varphi$. Replacing q_is, $i \geq k'$ in $S_{k'} \vdash \varphi$ by a tautological formula $\top$, we obtain that $\{p_i : i < k'\} \vdash \varphi$ (see the following remark for the justification of this technique). Hence $\varphi \in Cn(P)$ and H is a hitting set for $\mathcal{F}$.

To complete the proof, we must show that $\mathcal{F}$ is representable. Denoting the set $\{\neg q_i : i < j\} \cup \{\neg p_j\}$ by Γ_j, we define

$$D = \left\{ \frac{\neg q_i}{p_i} : i \in \omega \right\} \bigcup \left\{ \frac{\Gamma_i}{\alpha} : i \in \omega \text{ and } \alpha \in S_i \right\}.$$

We now consider the default theory $(D, \emptyset)$. It is easy to check that $Cn(P)$ and $Cn(S_i)$s are extensions of this default theory, that is, $\mathcal{F} \subseteq ext(D, \emptyset)$. On the other hand, we shall prove that $\mathcal{F} \supseteq ext(D, \emptyset)$, whence $\mathcal{F}$ is representable by $(D, \emptyset)$. Suppose $S \in ext(D, \emptyset)$, and we want to show $S \in \mathcal{F}$. Here are two cases:

Case 1. If $q_i \notin S$ for all i's, then all defaults $\neg q_i / a_i$ in D are S-applicable; hence $p_i \in Cn^{D_S}(\emptyset) = S$. Accordingly, $Cn(P) \subseteq S$, which leads to $Cn(P) = S$ since extensions of a default theory are nonincluding.

Case 2. If there exists an i such that $q_i \in S$, then let n be the least such one. Since $\neg q_n \in \Gamma_i$ for all $i > n$, we have that all S-applicable defaults in D fall into the set

$$D' = \left\{ \frac{\neg q_i}{p_i} : i \neq n \right\} \bigcup \left\{ \frac{\Gamma_i}{\alpha} : i \leq n \text{ and } \alpha \in S_i \right\}.$$

Thus $S = Cn^{D_S}(\emptyset) \subseteq Cn(c(D'))$. Clearly $c(D') \not\vdash p_n$ and hence $S \not\vdash p_n$. Thus the defaults Γ_n/α, where $\alpha \in S_n$, are S-applicable, which leads to that $S = Cn^{D_S}(\emptyset) \supseteq S_n$. It follows that $S = S_n$.

In both cases, S is in $\mathcal{F}$, so we have proved the result of (b). ■

Remark. In the proof above, there is a maneuver of replacing p_j by $\top$ (*true*) and then inferring something. The justification for this technique is as follows:

Given an arbitrary set S of propositional formulas, propositional formulas φ and α, and primitive formula p, by $\varphi(\frac{p}{\alpha})$ we denote the formula obtained from φ by replacing each occurrence of p with α. By $S(\frac{p}{\alpha})$ we denote the set $\{\phi(\frac{p}{\alpha}) \mid \phi \in S\}$. "Replacing p in the provability relation $S \models \varphi$ with α" results in the provability relation $S(\frac{p}{\alpha}) \models \varphi(\frac{p}{\alpha})$. What we want to show here is that $S \models \varphi$ implies that $S(\frac{p}{\alpha}) \models \varphi(\frac{p}{\alpha})$.

By the replacement theorem for propositional calculus, $\models \varphi$ implies that $\models \varphi(\frac{p}{\alpha})$, and $S \models \varphi$ iff there are $\alpha_1, \ldots \alpha_n \in S$ such that $\models (\alpha_1 \wedge, \ldots, \wedge \alpha_n) \to \varphi$, which by the same reason implies that $\models ((\alpha_1 \wedge, \ldots, \wedge \alpha_n) \to \varphi)(\frac{p}{\alpha})$; that is, $\models (\alpha_1(\frac{p}{\alpha}) \wedge, \ldots, \wedge \alpha_n(\frac{p}{\alpha})) \to \varphi(\frac{p}{\alpha})$, implying that $S(\frac{p}{\alpha}) \models \varphi(\frac{p}{\alpha})$.

PROPOSITION 5.2. *There is a denumerable family of mutually inconsistent theories $\mathcal{F}$ such that $\mathcal{F}$ is representable by a default theory but not by any normal default theory.*

This proposition is significant in that it guarantees the existence of such a default theory that has exactly infinitely many, mutually inconsistent extensions but fails to be representable in the class of normal default theories, which gives a negative answer to the question of Marek and Truszczyński as to whether the following proposition holds without the assumption that the extensions are finitely generated [9, p. 130].

PROPOSITION 5.3 [9, COROLLARY 5.10]. *If a default theory (D, W) has at least one extension and all the extensions for (D, W) are finitely generated and pairwise inconsistent, then the default theory is representable in the class of normal default theories.*

As mentioned in the introduction, Marek *et al.* [11] constructed an example of a denumerable family of pairwise inconsistent theories that are not representable by normal default theories. However, their example is not representable by any default theory and thus does not answer the question above.

To prove Propositon 5.2, we present our counterexample as follows:

EXAMPLE 5.1.[5] Let q_is, p_js, and a_{ij}s $(i \leq j)$ be different atoms, and let

$$W_0 = \{q_i \to \neg p_j \mid i \neq j\}$$

and for $i < j$,

$$a_{ji} = \neg a_{ij}.$$

Then, we define

$$E_j = Cn(W_0 \cup \{p_j\} \cup \{a_{ij} \mid i \in \omega\}),$$

for all $j \in \omega$, and

$$\mathcal{E} = \{E_j \mid j \in \omega\}.$$

It is clear that $\mathcal{E}$ is denumerable and consists of mutually inconsistent theories. In addition, $\mathcal{E}$ is supercompactly nonincluding and hence representable, because for each E_j in $\mathcal{E}$, $\{\neg q_j\}$ is a hitting set for $\mathcal{E} - \{E_j\}$ but not for $\mathcal{E}$. It remains to show that $\mathcal{E}$ is not representable by any normal default theory. For this purpose, we must get some essential properties to distinguish the family $\mathcal{E}$ from those families representable by normal default theories. Fortunately, the next lemma gives an important feature of those families that are representable by normal default theories, and Lemma 5.2 guarantees that our family $\mathcal{E}$ does not share this very feature.

LEMMA 5.1. *Let (D, W) be a normal default theory, then for all extensions E, E' of (D, W) such that $E \neq E'$ there is a sentence α such that $\alpha \in E$ and $E' \cup \{\alpha\}$ is inconsistent, and for every extension E^* of (D, W), either $\alpha \in E^*$ or $E^* \cup \{\alpha\}$ is inconsistent.*

The proof of this lemma is omitted here. It can be inferred easily from the complete representability result for normal default theories obtained by Marek *et al.* [11] that a family $\mathcal{T}$ of theories in $\mathcal{L}$ is representable by a normal default theory if and only if $\mathcal{T} = \mathcal{L}$ or there is a set of formulas Ψ such that $T = \{Cn(\Phi) : \Phi \subseteq \Psi$ is maximal so that Φ is consistent$\}$.

Lemma 5.1, which gives a new essential feature of normal default theories, is closely related to Reiter's corresponding result [12, Theorem 3.3], i.e., the inconsistency of any two extensions of normal default theory, which says that there is a sentence α in one extension but the negation of the sentence is in the other. Somewhat stronger than Reiter's theorem, our theorem says that this sentence α can be such one as if $\frac{:\alpha}{\alpha}$ were a default of the default theory being considered; in other words, for every extension E^*, the consistency of $E^* \cup \{\alpha\}$ implies $\alpha \in E^*$.

LEMMA 5.2. *Let E_js be as in Example* 5.1, *γ an arbitrary sentence such that $\gamma \in E_0$, and $\neg\gamma \in E_1$. Then there is an E_i such that $\gamma \notin E_i$ and $\neg\gamma \notin E_i$.*

The proof of this lemma is omitted also.

6. CONCLUSION AND OPEN QUESTIONS

The representability theory is of key importance for default logic. Representability by certain classes of default theories provides insights into the expressive power of the corresponding branches of default logic. This kind of study will be helpful for users to find simpler representation for their default theories.

The usefulness of Reiter's default logic for specifying multiple belief sets of an agent was investigated by Marek *et al.* [11]. However, they did not find a complete characterization of families of theories that are representable by general default theories. This paper has given a new necessary condition for the

[5] This example was previously presented as a default theory in [14].

representability by means of general default theories and presented several sufficient conditions that are weaker than similar ones known previously. Moreover, this paper has improved some interesting results of [11]. For example, Marek *et al*. showed that every subfamily of a representable family that has an SSDR that is representable, whereas we have obtained that every subfamily of a denumerable representable family is representable. Our result is more satisfying in that an arbitrary family that has an SSDR must be denumerable.

It is well known that there exists an extension and two different extensions are inconsistent for every normal default theory (i.e., the existence and orthogonality of extensions, see [12, Theorems 3.1 and 3.3]). In this paper, we have given a new feature of extensions of a normal default theory, by which we show that our example of a denumerable, representable family of mutually inconsistent theories is not presentable by any normal default theory.

However, we have not found some essential and distinguishing features for other important classes of default theories, such as the class of semi-normal default theories [7], and the class of unitary default theories (i.e., the class of those default theories in which each default has exactly one justification). The following problems seem more difficult and remain open:

Problem 1: Is each default theory equivalent to a semi-normal default theory?

Problem 2: Is each default theory equivalent to a unitary default theory?

Problem 3: Is each unitary default theory equivalent to a semi-normal default theory?

We note that if Problem 1 or Problem 2 has an affirmative answer, then, from a representation viewpoint, the user is allowed to characterize families of belief sets for his/her agents by using only semi-normal default theories or unitary default theories instead of general ones.

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