On the different formulations in linear bifurcation analysis of laminated cylindrical shells

Y. Goldfeld *,1, E.A. Ejgenberg 2

Technion—Israel Institute of Technology, Faculty of Civil and Environmental Engineering, Technion City, 32000 Haifa, Israel

Received 15 November 2006; received in revised form 24 June 2007; accepted 26 June 2007
Available online 4 July 2007

Abstract

The stability pattern of shells is governed by a set of nonlinear partial differential equations. The solution procedure can be simplified, and fast and accurate predictions of the critical buckling load obtained, with the aid of a multilevel approach. Under this approach the lower levels are implemented by means of the perturbation technique, with the nonlinear prebuckling deformation disregarded, and a linear set of equations solved for each state. It turns out, however, that in these circumstances the prediction may differ depending on the chosen formulation. In an attempt to find the reasons for these differences, the linear bifurcation buckling behavior of laminated cylindrical shells was examined via two well-known formulations, with \( u-v-w \) and \( w-F \) as the unknowns. A third, mixed formulation, was found the most reliable in predicting the buckling behavior.

© 2007 Elsevier Ltd. All rights reserved.

Keywords: Buckling; Composite shells; Linear bifurcation analysis; Alternative formulations

1. Introduction

Composite laminated shells are widely used in industrial applications. Their stability behavior is a vital safety consideration, and improvement of the accuracy of the predicted behavior is thus essential for reliable design.

The nonlinear behavior of thin-walled structures is generally characterized by a limit point rather than by a bifurcation point (classical eigenvalue problems), and a nonlinear equilibrium state of the path is naturally involved. However, bifurcation analysis is still useful as a guideline and a basic procedure for examining the imperfection sensitivity by which the limit point is characterized, Koiter (1945), Arbocz and Hol (1989, 1990). Also, the analytical buckling loads can serve as the design load in conjunction with a suitable “knockdown” factor.
Arbocz and Starnes (2002, 2003) developed, for predicting the critical buckling load of compression-loaded thin-walled cylindrical shells, a hierarchical procedure comprising three levels of fidelity, which are defined in brief as follows. Level-1 assumes membrane prebuckling and neglects the edge restraint, and Level-2 incorporates the effect of nonlinear prebuckling; in both cases the buckling state is calculated as an eigenvalue problem. Finally, Level-3 analysis calculates the full nonlinear behavior. In theory, the prediction accuracy is supposed to improve the higher the level number, but when only the linear prebuckling deformation is considered the result may differ depending on the chosen formulation. The main objective of this paper is examination of the accuracy of different formulations of Level-2.

Extensive research on the stability behavior of laminated cylindrical shells is reported in literature. Most of them were based on Donnell’s theory, thus permitting recourse to the Airy stress function \( F \), whereby the number of unknown functions is reduced from the three displacement components (axial \( u \), circumferential \( v \), and normal \( w \)) to two \( (w, F) \). Sheinman and Goldfeld (2001) compared the linear bifurcation buckling loads obtained by the two formulations, and found that for the special cases of angle-ply shells, the buckling loads calculated by the \( u-v-w \) formulation were lower (by 30\% and even more) than those obtained by its \( w-F \) counterpart. Since such was the case throughout that study, these lower values were mistakenly accepted as the correct ones; they are certainly on the safe side.

In the present work, the reasons for these discrepancies were sought. It was found that in reality the \( w-F \) formulation yields more reliable results than the other, as exclusion of the cubic operators in the third equilibrium buckling equation leads to inconsistency in satisfying the other in-plane equations, and not satisfying the natural boundary conditions, both for \( u-v-w \) formulation. As a result, in certain cases (in particular, angle-ply configurations with high stretching-bending coupling) the \( u-v-w \) formulation can yield a completely different characteristic buckling pattern. As a way out, a third—alternative is presented—the mixed formulation, whose main advantages are that it satisfies the natural boundary conditions, and only linear and quadratic operators are involved.

The Level 2 procedure is based here on expansion of the variables in Fourier series in the circumferential direction and their presentation as finite differences in the axial direction. The Galerkin method is used to minimize the errors caused by the truncation. Parametric study of an angle-ply laminated cylindrical shell shows that the \( u-v-w \) may fail to yield the accurate buckling load, but the \( w-F \) and mixed formulations do yield it.

2. Governing equation

The solution procedure has been used before by Sheinman and Goldfeld (2001), and is briefly repeated here for the sake of completeness.

2.1. Kinematics relation—Donnell’s Shell Theory (1933)

The strain–displacement relation can be written as:

\[
\begin{cases}
{e}_{xx} \\
{e}_{	heta	heta} \\
\gamma_{x	heta}
\end{cases} = \begin{bmatrix} e_{xx}^0 & e_{x	heta}^0 \\ e_{	heta	heta}^0 & e_{	heta	heta}^0 \\ \gamma_{x	heta}^0 & \gamma_{x	heta}^0 \\ \end{bmatrix} + z \begin{bmatrix} Z_{xx} \\ Z_{	heta	heta} \end{bmatrix} = \begin{bmatrix} w_x + \frac{1}{2} w_x^2 \\ w_{\theta} + \frac{1}{2} w_{\theta}^2 \\ \frac{w_x}{R} + \frac{w_{\theta} w_x}{R} \end{bmatrix} + z \begin{bmatrix} -w_{xx} \\ -w_{\theta	heta} \\ -w_{x\theta\theta} \end{bmatrix}
\]

where \( \{e^0\} \) and \( \{Z\} \) are, respectively, the strain and change-of-curvature vectors of the reference surface; \( (\_)_x \) and \( (\_)_\theta \) denote the derivatives with respect to the axial (\( x \)) and circumferential (\( \theta \)) coordinates, respectively; \( R \) is the radius (see Fig. 1).

2.2. Constitutive equations—laminated cylindrical shells

Under the classical laminate theory, Jones (1975) and Whitney (1987), the constitutive equation reads:

\[
\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} e^0 \\ Z \end{bmatrix}
\]
where \( \{N\}^T = \{N_{xx}, N_{00}, N_{x0}\} \) and \( \{M\}^T = \{M_{xx}, M_{00}, M_{x0}\} \) are the force and moment resultant vectors, \( \{e_0^0\}^T = \{e_{xx}^0, e_{00}^0, e_{x0}^0\} \) and \( \{z\}^T = \{z_{xx}, z_{00}, z_{x0}\} \) are the strain at the reference surface and the change of curvature and twist of the middle surface, respectively. The coefficients of the elastic stiffness matrix are given by:

\[
\{A, B, D\} = \left\{ \sum_{k=1}^n [\mathbb{Q}]_k (t_k - t_{k-1}), \sum_{k=1}^n [\mathbb{Q}]_k (\tilde{t}_k^2 - \tilde{t}_{k-1}^2), \sum_{k=1}^n [\mathbb{Q}]_k (\tilde{t}_k^3 - \tilde{t}_{k-1}^3) \right\}
\]

(3)

\( A, B, \) and \( D \) are, respectively, the membrane, coupling and flexural rigidities, and \( \mathbb{Q} \) the laminate transformed reduced stiffness matrix.

For the \( w-F \) formulation, the relevant constitutive relations are:

\[
\begin{bmatrix}
\{e_0^0\} \\
\{M\}
\end{bmatrix} =
\begin{bmatrix}
a & -b \\
b^T & d
\end{bmatrix}
\begin{bmatrix}
\{N\} \\
\{Z\}
\end{bmatrix}
\]

(4)

where the elastic matrices are defined as:

\[
a = A^{-1} \quad b = A^{-1} B \quad d = D - BA^{-1} B.
\]

(5)

### 2.3. Equilibrium equations

The nonlinear equilibrium equations and the appropriate boundary conditions are derived on the basis of the potential-energy approach, with the energy \( \Pi \) reading as follows:

\[
\Pi = \frac{1}{2} \int_0^1 \int_x \left[ N_{xx} e_{xx}^0 + N_{00} e_{00}^0 + N_{x0} e_{x0}^0 + M_{xx} z_{xx} + M_{00} z_{00} + 2 M_{x0} z_{x0} \right] \, dx \, d\theta \\
- \int_0^1 \int_x \left( q_{xx} u + q_{x0} v + q_{00} w \right) \, dx \, d\theta - \int_0^1 \left( \mathbb{N}_{xx} u + \mathbb{N}_{x0} v + \mathbb{Q} w + \mathbb{M}_{xx} w_x \right) \, d\theta |_{x=1}^{x=0}
\]

\[
\frac{\partial}{\partial z_x} \left( \frac{1}{2} \int_0^1 \int_x \left[ N_{xx} e_{xx}^0 + N_{00} e_{00}^0 + N_{x0} e_{x0}^0 + M_{xx} z_{xx} + M_{00} z_{00} + 2 M_{x0} z_{x0} \right] \, dx \, d\theta \right)
\]

(6)

and \( q_{xx}, q_{x0}, \) and \( q_{00} \) being the external distributed loading in the axial, circumferential, and normal directions, respectively. \( \mathbb{N}_{xx}, \mathbb{N}_{x0}, \mathbb{Q}, \) and \( \mathbb{M}_{xx} \) are, respectively, the axial, torsional, shearing forces, and the bending moment applied at the boundaries.

For equilibrium the potential energy must be stationary. Its first variation \( \delta \Pi \) yields the following three nonlinear equilibrium equations:
N_{xx'} + \frac{N_{00,xx}}{R} + q_u = 0 \quad (7a)
N_{x0} + \frac{N_{00,x0}}{R} + q_v = 0 \quad (7b)
M_{xx'} + \frac{M_{00,xx}}{R^2} + \frac{2M_{00,x0}}{R} - \frac{N_{00}}{R} + N_{xx}w_x + \frac{2N_{00}w_{xx}}{R} + \frac{N_{00}w_{00}}{R^2} + \left( N_{xx'} + \frac{N_{00,xx}}{R} \right)w_x + \left( N_{x0'} + \frac{N_{00,x0}}{R} \right)w_0 + q_w = 0 \quad (7c)

with the following four boundary conditions:
\begin{align*}
N_{xx} &= \overline{N}_{xx} \quad \text{or} \quad u = \overline{u} \\
N_{x0} &= \overline{N}_{x0} \quad \text{or} \quad v = \overline{v} \\
M_{xx'} + \frac{2M_{00,xx}}{R} + N_{xx}w_x + \frac{N_{00}w_0}{R} = \overline{Q} \quad \text{or} \quad w = \overline{w} \\
M_{xx} &= \overline{M}_{xx} \quad \text{or} \quad w_{xx} = \overline{w}_{xx}.
\end{align*}

Note that the first and second equations are incorporated in the third. For \( q_u = q_v = 0 \), Eqs. (7a) and (7b) reduce to \( N_{xx} + \frac{1}{R}N_{00,xx} = 0 \) and \( N_{x0} + \frac{1}{R}N_{00,x0} = 0 \), and Eq. (7c) reduces to:
\begin{equation}
M_{xx'} + \frac{2M_{00,xx}}{R^2} + \frac{2M_{00,x0}}{R} - \frac{N_{00}}{R} + N_{xx}w_x + \frac{2N_{00}w_{xx}}{R} + \frac{N_{00}w_{00}}{R^2} = 0.
\end{equation}

Introducing the transverse shear force \( Q \), Eq. (7c) can be split in two, namely
\begin{align*}
Q &= M_{xx'} + \frac{2M_{00,xx}}{R^2} + N_{xx}w_x + \frac{N_{00}w_0}{R} \\
Q_x &= \frac{M_{00,xx}}{R^2} - \frac{N_{00}}{R} + \frac{1}{R} \left[ N_{00}w_t + \frac{N_{00}w_0}{R} \right] + q_w = 0
\end{align*}

and the set of three nonlinear equilibrium equations is now reduced by one of four (Eqs. (7a), (7b), (10) and (11)).

2.4. \( u-v-w \) formulation

Substitution of the kinematic relations, Eq. (1), and the constitutive equation (2), in the equilibrium equations, Eqs. (7)—yields the following differential equations in terms of the unknown displacement functions \( u(x, \theta), v(x, \theta), \) and \( w(x, \theta) \):
\begin{align*}
L_h^1[w] + L_q^1[v] + L_e^1[u] + LL_e^2[w, w] + q_u &= 0 \\
L_h^2[w] + L_q^2[v] + L_e^2[u] + LL_e^2[w, w] + q_0 &= 0 \\
L_h^3[w] + L_q^3[v] + L_e^3[u] + LL_e^3[w, w] + LL_v^3[u, v] + LLL_v^3[w, w, w] + q_w &= 0
\end{align*}

where \( L_e, LL_e, \) and \( LLL_e, \) \( e = 1, 2, 3, \) are, respectively, linear, quadratic, and cubic differential operators with variable coefficients, see Sheinman and Goldfeld (2001), where the subscripts are explained.

2.5. \( w-F \) formulation

The \( w-F \) formulation can be found in Tennyson and Muggeridge (1969) and later in Sheinman et al. (1983) and Sheinman and Firer (1994).

By introducing the Airy stress function \( F(x, \theta) \), the in-plane equilibrium equations (7a) and (7b) are identically satisfied, and the relevant equations are those of compatibility:
\begin{equation}
\frac{\epsilon_{xx,00}}{R^2} + \frac{\gamma_{x0,xx}}{R} - \frac{\gamma_{00,xx}}{R} = \frac{w_{xx}w_{00}}{R^2} + \frac{w_{xx}}{R}
\end{equation}
and the third equilibrium equation, Eq. (7c). Substituting the kinematic relations, Eq. (1), and the constitutive equations, Eq. (4), in Eqs. (7c) and (13)—the following two equations are obtained in terms of the unknown functions $w(x, \theta)$ and $F(x, \theta)$:

\[
L_k^{[1]}(w) + L_\theta^{[1]}(w) + LL^{[1]}(w, F) - \frac{F_{xx} + N_{\theta\theta}}{R} + q_w = 0
\]

\[
L_k^{[2]}(w) + L_\theta^{[2]}(w) + \frac{1}{2} LL^{[2]}(w, w) - \frac{w_{xx}}{R} = 0
\]  \hspace{1cm} (14)

where $L^{[c]}$ and $LL^{[c]}$ are differential operators given by Sheinman and Frostig (1988). The displacements $u$ and $v$ in the boundary conditions are replaced by conditions on their derivatives; see Budiansky (1967), Arbocz and Hol (1989), Sheinman and Simitses (1977).

In addition one must make sure that the circumferential periodicity condition $\int_0^{2\pi} \frac{q_w}{R} d\theta = 0$ is satisfied, Arbocz and Hol (1989).

The advantage of the $w-F$ formulation over its $u-v-w$ counterpart lies mainly in reducing the number of unknown functions from three to two, and in dispensing with the cubic operator of the $u-v-w$. Besides, it always yields the correct answer. On the downside, it is restricted to Donnell-type equations.

2.6. Mixed formulation

The mixed formulation involves a set of eight unknown functions: the displacements—$u(x, \theta)$, $v(x, \theta)$, $w(x, \theta)$, and $w_{z}(x, \theta)$, and the resultant forces and moment—$N_{xx}(x, \theta)$, $N_{\theta\theta}(x, \theta)$, $Q(x, \theta)$, and $M_{xx}(x, \theta)$. Given the four equilibrium equations: Eqs. (7a), (7b), (10), and (11), the three constitutive equations for $N_{xx}(x, \theta)$, $N_{\theta\theta}(x, \theta)$, $M_{xx}(x, \theta)$, and the derivative of $w$ in the axial direction, the following eight governing equations can be written in operator form as:

\[
N_{xx} + \frac{N_{\theta\theta}}{R} + q_u = 0
\]

\[
N_{xx}^{[2]}(u) + N_{xx}^{[2]}(v) + N_{xx}^{[2]}(w) + LL_{xx}^{[2]}(w, w) + LL_{xx}^{[2]}(w, w) + LL_{xx}^{[2]}(w, w) + LL_{xx}^{[2]}(w, w) + N_{xx} + q_u = 0
\]

\[
N_{xx}^{[3]}(u) + N_{xx}^{[3]}(v) + N_{xx}^{[3]}(w) + LL_{xx}^{[3]}(w, w) + LL_{xx}^{[3]}(w, w) + LL_{xx}^{[3]}(w, w) + LL_{xx}^{[3]}(w, w) + N_{xx} + q_u = 0
\]

\[
N_{xx}^{[4]}(u) + N_{xx}^{[4]}(v) + N_{xx}^{[4]}(w) + LL_{xx}^{[4]}(w, w) + LL_{xx}^{[4]}(w, w) + LL_{xx}^{[4]}(w, w) + LL_{xx}^{[4]}(w, w) + N_{xx} + q_u = 0
\]

\[
N_{xx}^{[5]}(u) + N_{xx}^{[5]}(v) + N_{xx}^{[5]}(w) + LL_{xx}^{[5]}(w, w) + LL_{xx}^{[5]}(w, w) + LL_{xx}^{[5]}(w, w) + LL_{xx}^{[5]}(w, w) + N_{xx} + q_u = 0
\]

\[
N_{xx}^{[6]}(u) + N_{xx}^{[6]}(v) + N_{xx}^{[6]}(w) + LL_{xx}^{[6]}(w, w) + LL_{xx}^{[6]}(w, w) + LL_{xx}^{[6]}(w, w) + LL_{xx}^{[6]}(w, w) + N_{xx} + q_u = 0
\]

\[
N_{xx}^{[7]}(u) + N_{xx}^{[7]}(v) + N_{xx}^{[7]}(w) + LL_{xx}^{[7]}(w, w) + LL_{xx}^{[7]}(w, w) + LL_{xx}^{[7]}(w, w) + LL_{xx}^{[7]}(w, w) + M_{xx} = 0
\]

\[
N_{xx}^{[8]}(w) - w_{xx} = 0
\]  \hspace{1cm} (15)

The main advantage of this formulation lies in satisfying directly both the natural $(N_{xx}, N_{\theta\theta}, Q, M_{xx})$ and essential $(u, v, w, w_{z})$ boundary conditions. A further advantage is that the first derivative in the $x$-direction is the highest, whereas in the $u-v-w$ it is the fourth. All equations except the third comprise only linear and quadratic operators; the third has also cubic operators that vanish in the case of an axisymmetric prebuckling state (which involves derivatives in the $\theta$-direction). On the downside it has eight unknowns, with the resulting increase in the computational resources needed.

3. Prebuckling and buckling equations

The buckling equations are straightforward and derived with the aid of the perturbation technique:

\[
\{S\} = \{S^{(0)}\} + \xi \{S^{(1)}\} + \cdots
\]  \hspace{1cm} (16)
where $\xi$ is the perturbation parameter, which represents the normalized amplitude of the buckling mode. The superscripts $^{(0)}$ and $^{(1)}$ denote the prebuckling and buckling state, respectively; $S$ represents the unknown function.

Applying Eq. (16), perturbation of the differential operators yields:

$$L(S) = L(S^{(0)}) + \xi L(S^{(1)}) + O(\xi^2)$$  \hspace{1cm} (17)

$$LL(S, T) = LL(S^{(0)}, T^{(0)}) + \xi [LL(S^{(0)}, T^{(1)}) + LL(S^{(1)}, T^{(0)})] + O(\xi^2)$$  \hspace{1cm} (18)

$$LLL(S, T, S) = LLL(S^{(0)}, T^{(0)}, S^{(0)}) + \xi [LL(S^{(0)}, T^{(0)}, S^{(1)}) + LLL(S^{(0)}, T^{(1)}, S^{(0)}) + LLL(S^{(0)}, T^{(0)}, S^{(1)})] + O(\xi^2).$$  \hspace{1cm} (19)

Substitution of the above operators in the set of equilibrium equations yields the partial differential equations of the prebuckling and buckling states.

The zero-order terms yield the following partial differential equations of the prebuckling state

$$L^{[e]}(S^{(0)}) + LL^{[e]}(S^{(0)}, T^{(0)}) + LLL^{[e]}(S^{(0)}, T^{(1)}, S^{(0)}) = P^{[e]}$$  \hspace{1cm} (20)

where the applied loading $P^{[e]}$ consists of axial compression, external pressure, and clockwise or counterclockwise torsion. The first-order terms yield the following partial differential equations of the buckling state

$$L^{[e]}(S^{(1)}) + [LL^{[e]}(S^{(0)}, T^{(1)}) + LL^{[e]}(S^{(1)}, T^{(0)})] + [LLL^{[e]}(S^{(1)}, T^{(0)}, S^{(0)}) + LLL^{[e]}(S^{(0)}, T^{(1)}, S^{(0)}) + LLL^{[e]}(S^{(0)}, T^{(0)}, S^{(1)})] = 0$$  \hspace{1cm} (21)

where $n = 3, 2,$ and $8$ for the $u-v-w, w-F$ and mixed formulation, respectively.

Because of our concentration here on the linear prebuckling state, Eq. (20) reduce to $L^{[e]}(S^{(0)}) = P^{[e]}$. For the buckling state, an eigenvalue problem for the critical load $\lambda_c$ is thus obtained, and the governing equations reduce to:

$$L^{[e]}(S^{(1)}) + \lambda_c [LL^{[e]}(S^{(0)}, T^{(1)}) + LLL^{[e]}(S^{(1)}, T^{(0)})] = 0$$  \hspace{1cm} (22)

where $\lambda_c$ corresponds to the bifurcation buckling load. As this contingency requires that the cubic operator in the $u-v-w$ formulation be excluded, the first and second equilibrium equations are not satisfied identically, which leads to problems in Eq. (7c), represented by the terms multiplied by $\delta_1$ in the following equation:

$$M^{(1)}_{xx} + \frac{M^{(1)}_{00,00}}{R^2} + \frac{2M^{(1)}_{00,00}}{R} - \frac{N^{(1)}_{00}}{R} + \delta_1 N^{(1)}_{xx} w^{(0)} + N^{(0)}_{00} w^{(1)} + \frac{2N^{(1)}_{00} w^{(0)}}{R} + \frac{2N^{(0)}_{00} w^{(1)}}{R} + \frac{N^{(1)}_{00} w^{(0)}}{R^2} + \frac{N^{(0)}_{00} w^{(1)}}{R^2} + \delta_1 \left( N^{(1)}_{xx} w^{(0)} + N^{(0)}_{00} w^{(1)} \right) + \delta_2 \left( N^{(1)}_{xx} w^{(0)} + N^{(0)}_{00} w^{(1)} \right) = 0$$  \hspace{1cm} (23)

For $q_u = q_t = 0$, the first and second in-plane prebuckling equilibrium equations (last terms in Eq. (23)) are satisfied under linear prebuckling.

One more problem arising from the $u-v-w$ formulation is that the natural boundary conditions in the buckling state are not fully satisfied. For example, in the case of axial compressive load, one applied $N_{xx} = N^{(0)}_{xx} + \xi N^{(1)}_{xx} = 1.0$ at the boundaries. Therefore, at prebuckling there should be: $N^{(0)}_{xx} = 1.0$, and at buckling $N^{(1)}_{xx} = 0$. Substituting the constitutive and kinematic relations, the in-plane axial force is written in terms of the unknown displacement. However, on recourse to the Fourier series, the equilibrium at buckling is not satisfied and $N^{(1)}_{xx}$ does not vanish at the boundaries, nor does the term containing $N^{(1)}_{xx}$ in the third equation (the term multiplied by $\delta_2$ in Eq. (23)) vanish. Its contribution is especially pronounced in moderate and short shells. Both effects lead to discrepancy in the predicted buckling load.

To overcome these problems, in some commercial computer codes such as STAGS-A (Almroth et al. (1973)) and ABAQUS (2006), which also use the displacement functions as the unknowns, the first and second equilibrium equations are satisfied in the third by eliminating their multipliers. In linear bifurcation analysis, STAGS-A excludes the prebuckling rotations in calculating the buckling state (that is, $w^{(0)}_{xx} = w^{(0)}_{00} = w^{(0)}_{00} = \cdots = 0$), while ABAQUS assumes a membrane prebuckling state (with the derivatives of $w^{(0)}$ again zero). Both codes have the option for nonlinear bifurcation analysis, where all terms are retained.
4. Numerical solution procedure

The solution of each set of equations admits separable solutions of the form:

\[ S(x, \theta) = \sum_{m=0}^{2N_S} S_m(x) g_m(\theta) \]  

(24)

where \( 2N_S \) is the number of retained terms in the relevant truncated Fourier series, and:

\[ g_m(\theta) = \begin{cases} 
\cos jm\theta & m = 0, 1, 2, 3, \ldots N_S \\
\sin jm\theta & m = N_S + 1, \ldots 2N_S 
\end{cases} \]

(25)

\( j \) denoting the characteristic circumferential wave number, and \( N_S = N_{u\nu}, N_{v\nu}, N_{w\nu}, N_{F\nu}, N_{Q\nu}, N_{N_{xx\nu}}, N_{N_{x\theta\nu}}, N_{M_{xx\nu}} \) or \( N_{w_{xx\nu}} \) according to the equation number.

The \( \theta \)-dependence is eliminated by applying Galerkin’s procedure. A central finite difference scheme is used to reduce the ordinary differential equations to the following algebraic ones:

For the prebuckling state,

\[ [K]\{Z^{(0)}\} = \{P\} \]

(26)

and for the buckling state,

\[ \{[K] + \lambda[G]\}\{Z^{(1)}\} = 0 \]

(27)

where \( K \) and \( G \) are the stiffness and geometry matrices, respectively, and \( Z^{(0)} \) and \( Z^{(1)} \) unknown vectors for the prebuckling and buckling state, respectively, namely: \([u, v, w]\) for the \( u-v-w \); \([w, F]\) for the \( w-F \); \([u, v, w, w_{xx}, N_{xx}, N_{x\theta}, Q, M_{xx}]\) for the mixed formulation, Eq. (27) is an eigenproblem in which \( \lambda \) represents the buckling load parameter and \( Z^{(1)} \) the buckling mode.

5. Results and discussion

For the procedure outlined a special computer code was written, for each formulation, covering the prebuckling and buckling behavior of laminated cylindrical shell under arbitrary loads and boundary conditions.

The parametric study of the characteristic buckling behavior had two main goals: (1) establish the relative accuracies of the three formulations, (2) find the terms that affect these inaccuracies—with a view to the necessary corrections. To this end, an angle-ply (±\( \beta \)) graphite/epoxy cylindrical shell under axial compression was examined. The material properties were (Sheinman and Firer (1994)) as follows: 2-ply laminate with a thickness \( t = 0.0127 \) m (\( R_1/t = 100 \)).

The buckling loads according to the three formulations are compared in Fig. 2, for SS1 at one end and SS4 at the other. Representative results are summarized in Table 1. For the linear prebuckling calculation, they show significant discrepancies (up to 50% and even more) between the formulations in the 15° < \( \beta \) < 75° range, the \( u-v-w \) yielding lower buckling loads than the other two. As was mentioned before, the reasons for this are: (a) inconsistency in satisfying the first and second equilibrium buckling equations in the third, (b) non-compliance with the in-plane natural boundary conditions. Under the membrane assumption there are no discrepancies.

In Fig. 3a, the axial buckling mode deformations are given for the three formulations for ±\( \beta = 45^\circ \), with the same circumferential wave number in all three cases, see Table 1. However, in the axial direction the buckling mode deformations according to the \( w-F \) and the mixed formulation are identical, but that according to the \( u-v-w \) formulation is slightly different. In Fig. 3b, the in-plane axial force is given for the three formula-
It is seen that for the $u-v-w$ formulation, the in-plane axial force does not vanish at $x=0$, which significantly modifies the whole result.

In order to identify the most dominant reason, the parametric study was repeated with the $d_1$ and the $d_2$ terms (associated the first and second reasons, respectively) artificially eliminated from the third equation. Results are given in Fig. 4 and in Table 1. As was expected, the results now coincided with their "membrane" counterparts. In this case, each of the above terms contributes evenly in correcting the results. Comparison with the linear bifurcation analysis obtained with the commercial computer codes STAGS-A and ABAQUS is given in Table 2; these results further validate those of the $w-F$ and mixed formulations.

For the case $\pm \beta = 40^\circ$, the one with the maximum discrepancy, the buckling load was also calculated by nonlinear bifurcation analysis (STAGS-A) as follows: $N_{xx} = 836$ kN/m with $n = 3$, the same as obtained in the linear case. It is thus seen that the lower levels yield reliable predictions which further validates the accuracy of the $w-F$ and mixed formulations.

For the SS3–SS4 boundary conditions, results are given in Fig. 5 and Table 3. Here, the significant discrepancy occurred in the $25^\circ < \pm \beta < 60^\circ$ range, and again, the $u-v-w$ yields much lower buckling loads than the others, the maximum discrepancy being 40%. The buckling patterns, and the internal axial force calculated by...
the different formulations, are shown in Fig. 6 for $\pm \beta = 45^\circ$, and the patterns are seen to differ widely: the one obtained with the $u-v-w$ is characterized by boundary layers, while that obtained with the other formulations is distributed evenly along the shell, which is commonly the case in axisymmetric buckling behavior; moreover the axial in-plane load, which must in that case vanish along the shell, does not vanish at both boundaries with the $u-v-w$, see Fig. 6b.

To correct the $u-v-w$ results the $\delta_1$ and $\delta_2$ terms were again eliminated, see Fig. 7. In this case, the first term (associated with the first and second equilibrium equations) is the dominant one, owing to the essential boundary condition $v = 0$, which ensures $N_x^{(1)} = 0$ at the boundary.
Table 2
Axial compressive buckling load ($N_{xx}$ [kN/m]) for $L/R = 2$, $R/t = 100$, SS1–SS4, calculated by ABAQUS and STAGS-A

<table>
<thead>
<tr>
<th>$\pm \beta$</th>
<th>ABAQUS</th>
<th>STAGS-A</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1228 (7)$^a$</td>
<td>1150 (7)</td>
</tr>
<tr>
<td>10</td>
<td>1140 (6)</td>
<td>1078 (6)</td>
</tr>
<tr>
<td>20</td>
<td>1045 (4)</td>
<td>1011 (5)</td>
</tr>
<tr>
<td>30</td>
<td>931 (4)</td>
<td>913 (4)</td>
</tr>
<tr>
<td>40</td>
<td>845 (3)</td>
<td>836 (3)</td>
</tr>
<tr>
<td>45</td>
<td>834 (3)</td>
<td>828 (3)</td>
</tr>
<tr>
<td>50</td>
<td>861 (3)</td>
<td>857 (3)</td>
</tr>
<tr>
<td>60</td>
<td>1014 (3)</td>
<td>1021 (3)</td>
</tr>
<tr>
<td>70</td>
<td>1228 (3)</td>
<td>1255 (3)</td>
</tr>
<tr>
<td>80</td>
<td>1278 (8)</td>
<td>1159 (8)</td>
</tr>
<tr>
<td>90</td>
<td>1208 (6)</td>
<td>1178 (6)</td>
</tr>
</tbody>
</table>

$^a$ In parentheses—circumferential wave number.

---

Table 3
Axial compressive buckling load ($N_{xx}$ [kN/m]) for $L/R = 2$, $R/t = 100$, SS3–SS4

<table>
<thead>
<tr>
<th>$\pm \beta$</th>
<th>$u$–$v$–$w$ formulation</th>
<th>$w$–$F$ formulation</th>
<th>Mixed formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Membrane</td>
<td>Linear</td>
<td>Linear $\delta_1 = 0$</td>
</tr>
<tr>
<td>0</td>
<td>1341 (8)$^a$</td>
<td>1348 (8)</td>
<td>1344 (8)</td>
</tr>
<tr>
<td>10</td>
<td>1189 (8)</td>
<td>1197 (8)</td>
<td>1197 (8)</td>
</tr>
<tr>
<td>20</td>
<td>1326 (6)</td>
<td>1336 (6)</td>
<td>1336 (6)</td>
</tr>
<tr>
<td>30</td>
<td>1335 (3)</td>
<td>1336 (3)</td>
<td>1338 (3)</td>
</tr>
<tr>
<td>40</td>
<td>1284 (0)</td>
<td>1236 (0)</td>
<td>1248 (0)</td>
</tr>
<tr>
<td>45</td>
<td>1293 (0)</td>
<td>907 (0)</td>
<td>1264 (0)</td>
</tr>
<tr>
<td>50</td>
<td>1399 (0)</td>
<td>991 (0)</td>
<td>1337 (0)</td>
</tr>
<tr>
<td>60</td>
<td>1735 (3)</td>
<td>1729 (0)</td>
<td>1738 (0)</td>
</tr>
<tr>
<td>70</td>
<td>1468 (8)</td>
<td>1471 (8)</td>
<td>1472 (8)</td>
</tr>
<tr>
<td>80</td>
<td>1206 (8)</td>
<td>1199 (8)</td>
<td>1199 (8)</td>
</tr>
<tr>
<td>90</td>
<td>1326 (8)</td>
<td>1326 (8)</td>
<td>1326 (8)</td>
</tr>
</tbody>
</table>

$^a$ In parentheses—circumferential wave number.

---

Fig. 5. Buckling load vs. angle-ply ($\pm \beta$) for SS3–SS4 boundary condition, three formulations.
For the CC1–CC4 and CC3–CC4 boundary conditions results are given in Fig. 8 and Table 4. Here, the discrepancy range for CC1–CC4 is $35^\circ < \pm \beta < 55^\circ$, while for CC3–CC4 the results coincide. It can be concluded that the more restrictive the boundary conditions, the less decisive the role of the chosen formulation.

The clamped boundary condition ($w = w_x = 0$) ultimately satisfies the terms associated with $d_2$ and the first term associated with $d_1$ in the third equation.

The influence of the $L/R$ ratio on the results is presented in Fig. 9 and Table 5, which show differences between the formulations throughout the $L/R$ range. For shorter shells the discrepancy is larger because of the greater weight of the boundary conditions.
A linear bifurcation analysis (Level-2) based on Donnell’s theory is presented for three alternatives: the $u$–$v$–$w$, $w$–$F$, and mixed formulations. It was found that the $u$–$v$–$w$ yields an inaccurate result—unlike the $w$–$F$ and mixed formulations—especially in cases of a composite material with high stretching-bending coupling. The reasons for that were found to be:

1. Inconsistency in satisfying the first and second (in-plane) equilibrium equations for the buckling state. In order to achieve an eigenvalue problem, the solution procedure excludes the cubic operator, as a consequence of which the first two equations are not satisfied in the third (out-of-plane) equation.
2. The boundary conditions: the $u$–$v$–$w$ satisfies exactly only the essential boundary conditions ($u$, $v$), the $w$–$F$—only the natural boundary conditions ($N_{xx}$, $N_{x0}$), and the mixed formulation—both types. The mixed formulation seems to be adequate for Level-2 prediction of the buckling behavior.

### Table 4
Axial compressive buckling load ($N_{xx}$ [kN/m]) for $L/R = 2$, $R/t = 100$, CC

<table>
<thead>
<tr>
<th>$\pm \beta$</th>
<th>CC1–CC4 $u$–$v$–$w$ formulation</th>
<th>$w$–$F$ formulation</th>
<th>Mixed formulation</th>
<th>CC3–CC4 $u$–$v$–$w$ formulation</th>
<th>$w$–$F$ formulation</th>
<th>Mixed formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1430 (8)</td>
<td>1449 (8)</td>
<td>1429 (8)</td>
<td>1430 (8)</td>
<td>1430 (8)</td>
<td>1429 (8)</td>
</tr>
<tr>
<td>10</td>
<td>1225 (8)</td>
<td>1222 (8)</td>
<td>1227 (8)</td>
<td>1229 (8)</td>
<td>1227 (8)</td>
<td>1226 (8)</td>
</tr>
<tr>
<td>20</td>
<td>1349 (6)</td>
<td>1341 (6)</td>
<td>1332 (3)</td>
<td>1350 (6)</td>
<td>1350 (6)</td>
<td>1347 (6)</td>
</tr>
<tr>
<td>30</td>
<td>1304 (3)</td>
<td>1325 (3)</td>
<td>1332 (3)</td>
<td>1332 (3)</td>
<td>1331 (3)</td>
<td>1330 (3)</td>
</tr>
<tr>
<td>40</td>
<td>1134 (3)</td>
<td>1297 (0)</td>
<td>1285 (0)</td>
<td>1285 (0)</td>
<td>1297 (0)</td>
<td>1296 (0)</td>
</tr>
<tr>
<td>45</td>
<td>1133 (3)</td>
<td>1314 (0)</td>
<td>1302 (0)</td>
<td>1302 (0)</td>
<td>1315 (0)</td>
<td>1314 (0)</td>
</tr>
<tr>
<td>50</td>
<td>1215 (3)</td>
<td>1397 (0)</td>
<td>1394 (0)</td>
<td>1394 (0)</td>
<td>1398 (0)</td>
<td>1397 (0)</td>
</tr>
<tr>
<td>60</td>
<td>1668 (3)</td>
<td>1718 (3)</td>
<td>1729 (2)</td>
<td>1729 (2)</td>
<td>1716 (2)</td>
<td>1716 (2)</td>
</tr>
<tr>
<td>70</td>
<td>1455 (8)</td>
<td>1453 (8)</td>
<td>1460 (8)</td>
<td>1460 (8)</td>
<td>1458 (8)</td>
<td>1457 (8)</td>
</tr>
<tr>
<td>80</td>
<td>1200 (8)</td>
<td>1198 (8)</td>
<td>1200 (8)</td>
<td>1200 (8)</td>
<td>1200 (8)</td>
<td>1200 (8)</td>
</tr>
<tr>
<td>90</td>
<td>1333 (8)</td>
<td>1332 (8)</td>
<td>1334 (8)</td>
<td>1332 (8)</td>
<td>1332 (8)</td>
<td>1332 (8)</td>
</tr>
</tbody>
</table>

* In parentheses—circumferential wave number.

### 6. Conclusion

A linear bifurcation analysis (Level-2) based on Donnell’s theory is presented for three alternatives: the $u$–$v$–$w$, $w$–$F$, and mixed formulations. It was found that the $u$–$v$–$w$ yields an inaccurate result—unlike the $w$–$F$ and mixed formulations—especially in cases of a composite material with high stretching-bending coupling. The reasons for that were found to be:

1. Inconsistency in satisfying the first and second (in-plane) equilibrium equations for the buckling state. In order to achieve an eigenvalue problem, the solution procedure excludes the cubic operator, as a consequence of which the first two equations are not satisfied in the third (out-of-plane) equation.
2. The boundary conditions: the $u$–$v$–$w$ satisfies exactly only the essential boundary conditions ($u$, $v$), the $w$–$F$—only the natural boundary conditions ($N_{xx}$, $N_{x0}$), and the mixed formulation—both types. The mixed formulation seems to be adequate for Level-2 prediction of the buckling behavior.

Fig. 8. Buckling load vs. angle-ply ($\pm \beta$) for CC boundary condition, three formulations.
The \( u - v - w \) solution can be corrected by assumption of membrane prebuckling or by exclusion of the pre-buckling rotations in the buckling state.

\section*{Acknowledgment}

The authors are indebted to E. Goldberg for editorial assistance.

\section*{References}


