Oscillation Criteria for Second Order Nonlinear Perturbed Differential Equations*  

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We present new oscillation criteria for the second order nonlinear perturbed differential equations. These criteria are of a high degree of generality and they extend and unify a number of existing results.

INTRODUCTION

In this paper we study the oscillation of solutions of the nonlinear second order perturbed differential equation

\[ [r(t)x'(t)]' + p(t)x'(t) + Q(t, x(t)) = P(t, x(t), x'(t)). \]  

We recall that a function \( x: [t_0, t_1) \to (\mathbb{R}, \mathbb{R}) \) is called a solution of Eq. (1) if \( x(t) \) satisfies Eq. (1) for all \( t \in [t_0, t_1) \). In the sequel it will be always assumed that solutions of Eq. (1) exist for any \( t_0 \geq 0 \). Further, a solution \( x(t) \) of Eq. (1) is called regular (infinitely continuable) if \( x(t) \) exists for all \( t \geq t_0 \). A regular solution \( x(t) \) of Eq. (1) is called oscillatory if it has arbitrarily large zeroes, otherwise it is called nonoscillatory.

The oscillation problem for Eq. (1) and for the less general equations such as the linear equation

\[ x''(t) + q(t)x(t) = 0, \]  

the nonlinear equation

\[ [r(t)x'(t)]' + q(t)f(x(t)) = 0, \]  

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and the nonlinear equation with damping term

\[
[r(t)x'(t)]' + p(t)x'(t) + q(t)f(x(t)) = 0
\] (4)

has been discussed by numerous authors and by different methods (see, for example, the papers [1–19] and the references quoted therein).

An important tool in the study of oscillatory behaviour of solutions for Eqs. (1)–(4) is the averaging technique which goes back as far as the classical results of Wintner [16] and Hartman [9] giving a sufficient condition for oscillation of Eq. (2). The result of Wintner was improved by Kamenev [10], and further extensions of Kamenev’s criterion have been obtained for Eq. (2) by Philos [13] and for Eq. (3) with \( f(x) = x \) by Li [12] (see also corrections to the latter paper in Rogovchenko [14]); while for Eq. (4) the extensions of the mentioned above results of Philos have been obtained by Grace [4] and they have been further refined in the recent paper by Rogovchenko [15].

Following Philos [13], let us introduce now the class of functions \( \mathcal{P} \) which will be extensively used in the sequel. Let \( D_0 = \{(t, s): t > s \geq t_0 \} \) and \( D = \{(t, s): t \geq s \geq t_0 \} \). The function \( H \in C(D; (-\infty, \infty)) \) is said to belong to the class \( \mathcal{P} \) if

1. \( H(t, t) = 0 \) for \( t \geq t_0 \), \( H(t, s) > 0 \) on \( D_0 \);
2. \( H \) has a continuous and nonpositive partial derivative on \( D_0 \) with respect to the second variable.

**Theorem A** [15, Theorem 1]. Suppose that the derivative \( f'(x) \) exists and satisfies the inequality \( f'(x) \geq K > 0 \) with some constant \( K \) for all \( x \neq 0 \). Suppose further that the functions \( h, H \in C(D, (-\infty, \infty)) \) are such that \( H \) belongs to the class \( \mathcal{P} \) and \( -\partial H/\partial s(t, s) = h(t, s)\sqrt{H(t, s)} \) for all \( (t, s) \in D_0 \).

If there exists a function \( g \in C^1([t_0, \infty); (0, \infty)) \) such that

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[ H(t, s)\psi(s) - \frac{a(s)r(s)}{4K} \right] \times \left( h(t, s) + \frac{p(s)}{r(s)} \sqrt{H(t, s)} \right)^2 ds = \infty,
\]

where \( \psi(s) = K^{-1}(Kg(s) - p(s)g(s) - [r(s)g(s)]' + r(s)g^2(s)) \) and \( a(s) = \exp(-2\int g(u) du) \), then any regular solution of Eq. (4) is oscillatory.
Recently Yeh [19] has obtained extensions of the result of Grace and Lalli [5] for Eq. (1) and has proved the following

**Theorem B** [19, Theorem 1]. Suppose that

(a) $r, p \in C(I = [t_0, \infty), R), r(t) > 0$ for all $t \geq t_0 > 0$;

(b) $Q \in C(I \times R, R)$ and $P \in C(I \times R^2, R)$ satisfy

\[ xQ(t, x) \geq xq(t)f_1(x) \quad \text{and} \quad xP(t, x, x') \leq xh(t)g(x')f_2(x) \]

for all $x \neq 0$, where $g, h \in C(I, R)$ and $g, f_1, f_2 \in C(R, R)$ are such that

(i) $xf_1(x) > 0$ and $xf_2(x) \geq 0$ for $x \neq 0$;

(ii) $h(t) > 0$ for $t \geq t_0$;

(iii) $0 < g(x') \leq c$ for some constant $c$;

(iv) $f_1'(x) \geq K > 0$ and $f_2(x)/f_1(x) \leq k$ for $x \neq 0$, where $k \geq 0$.

If there exists a function $\rho \in C^1(I, (0, \infty))$ such that

\[
\limsup_{t \to \infty} t^{-\alpha} \int_{t_0}^{t} (t - s)^{\alpha-2} \rho(s) \left( (t - s)^2(q(s) - ckh(s)) \right. \\
- \frac{1}{4K} r(s) \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) + \alpha \bigg)^2 ds = \infty
\]

for some $\alpha > 0$, then every infinitely continuable solution of Eq. (1) is oscillatory.

The purpose of this paper is to obtain the extension of Theorem B as well as of other results in [5] concerning the oscillation of Eq. (1) making use of the technique similar to that exploited by Philos [13], Li [12], Grace [4], and Rogovchenko [15] for Eqs. (2)–(4). The results of the paper extend and unify some of results in Grace and Lalli [5, 6], Graef et al. [8], Kamenev [10], Li [12], Philos [13], Rogovchenko [15], Yan [17], and Yeh [18, 19].

1. Oscillation Criteria

In what follows we assume that

(A) the functions $r: I \to R_+$, $p: I \to R$ are continuous, where $I = [t_0, \infty), R = (-\infty, \infty), R_+ = (0, \infty)$;

(B) the functions $Q: I \times R \to R$, $P: I \times R^2 \to R$ are continuous and such that for $x \neq 0$

\[ Q(t, x) \geq q(t)f_1(x) \quad \text{and} \quad P(t, x, x') \leq h(t)f_2(x)g(x'), \]
where \( q: I \to \mathbb{R}, h: I \to \mathbb{R}_+ \), and \( f_1, f_2, g: \mathbb{R} \to \mathbb{R} \) are continuous functions such that
(i) \( xf_1(x) > 0 \) and \( xf_2(x) \geq 0 \) for \( x \neq 0 \);
(ii) \( 0 < g(x') \leq c \) for some constant \( c \);
(iii) \( f_1'(x) \geq K > 0 \) and \( f_2'(x)/f_1(x) \leq k \) for \( x \neq 0 \), where \( k \geq 0 \).

**Theorem 1.** Let assumptions (A), (B), (i)–(iii) be fulfilled and let \( h, H: D = \{(t, s); t \geq s \geq t_0\} \to \mathbb{R} \) be continuous functions such that \( H \) belongs to the class \( P \) and
\[
- \frac{\partial H}{\partial s}(t, s) = h(t, s)\sqrt{H(t, s)} \quad \text{for all} \ (t, s) \in D_0.
\]

If there exists a continuously differentiable function \( \rho: I \to \mathbb{R}_+ \) such that
\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[ H(t, s) \rho(s)(q(s) - ckh(s)) - \frac{\rho(s)r(s)}{4K} \right.
\]
\[
\times \left. \left( h(t, s) + \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right] ds = \infty,
\]
then any regular solution of Eq. (1) is oscillatory.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of Eq. (1) and let \( T_0 \geq t_0 \) be such that \( x(t) \neq 0 \) for all \( t \geq T_0 \), so without loss of generality we may assume that \( x(t) > 0 \) for all \( t \geq T_0 \) since the similar argument holds also for the case when \( x(t) \) is eventually negative. Define
\[
v(t) = \rho(t)r(t)\frac{x'(t)}{f_1(x(t))}.
\]

Then differentiating (7) and making use of (1) we obtain
\[
v'(t) = \frac{\rho(t)}{f_1(x(t))}P(t, x(t), x'(t)) - \frac{p(t)}{r(t)} - \frac{\rho'(t)}{\rho(t)}v(t)
\]
\[
- \frac{\rho(t)}{f_1(x(t))}Q(t, x(t)) - \frac{f_1'(x(t))}{\rho(t)r(t)}v^2(t).
\]
In view of (B) and (iii) we get from (8)

\[ v'(t) \leq \rho(t)(ckh(t) - q(t)) - \left( \frac{p(t)}{r(t)} - \frac{\rho'(t)}{\rho(t)} \right) v(t) - \frac{K}{\rho(t)r(t)} v^2(t). \]

Hence by (9) and (5) we have for all \( t \geq T \geq T_0 \)

\[
\int_T^t H(t, s) \rho(s)(q(s) - ckh(s)) \, ds \\
\leq - \int_T^t H(t, s) v'(s) \, ds \\
- \int_T^t H(t, s) \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) v(s) \, ds \\
- \int_T^t H(t, s) \frac{K}{\rho(s)r(s)} v^2(s) \, ds \\
= -H(t, s)v(s) \bigg|_T^t \\
- \int_T^t \left[ -\frac{\partial H}{\partial s}(t, s)v(s) + H(t, s) \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) v(s) \\
+H(t, s) \frac{K}{\rho(s)r(s)} v^2(s) \right] \, ds \\
= H(t, T)v(T) - \int_T^t \left[ \sqrt{\frac{K H(t, s)}{\rho(s)r(s)}} v(s) \\
+ \frac{1}{2} \sqrt{\frac{\rho(s)r(s)}{K}} \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right) \right]^2 \, ds \\
+ \int_T^t \frac{\rho(s)r(s)}{4K} \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \, ds.
\]

Therefore for all \( t \geq T \geq T_0 \)

\[
\int_T^t H(t, s) \rho(s)(q(s) - ckh(t)) \\
- \frac{\rho(s)r(s)}{4K} \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \, ds
\]
\[
\leq H(t, T) v(T) - \int_T^t \left[ \sqrt{K H(t, s)} v(s) + \frac{1}{2} \sqrt{\frac{\rho(s) r(s)}{K}} \right. \\
\left. \times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \right] ds. \quad (10)
\]

It follows from (10) and from the property \((H_2)\) that for every \(t \geq T_0\)
\[
\int_{T_0}^t \left[ H(t, s) \rho(s) (q(s) - ckh(s)) \right. \\
- \frac{\rho(s) r(s)}{4K} \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \right] ds \\
\leq H(t, T_0) |v(T_0)| \leq H(t, t_0) |v(T_0)|. \quad (11)
\]

Thus by (11) and by the property \((H_2)\) we obtain
\[
\int_{t_0}^t \left[ H(t, s) \rho(s) (q(s) - ckh(s)) \right. \\
- \frac{\rho(s) r(s)}{4K} \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \right] ds \\
\leq H(t, t_0) \int_{t_0}^{T_0} \left| \rho(s) (q(s) - ckh(s)) \right| ds \\
+ H(t, t_0) |v(T_0)| \\
= H(t, t_0) \left[ \int_{t_0}^{T_0} \rho(s) |q(s) - ckh(s)| ds + |v(T_0)| \right]. \quad (12)
\]

It follows from (12) that
\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[ H(t, s) \rho(s) (q(s) - ckh(s)) \right. \\
- \frac{\rho(s) r(s)}{4K} \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \right] ds \\
\leq \int_{t_0}^{T_0} \rho(s) |q(s) - ckh(s)| ds + |v(T_0)|
\]
contradicting the assumption (6). Hence all regular solutions of Eq. (1) are oscillatory.

As an immediate consequence of Theorem 1 we get the following

**Corollary 1.** Let assumption (6) in Theorem 1 be replaced by

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s) \rho(s)(q(s) - c(kh(s))) \, ds = \infty
\]

and

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \rho(s) r(s)
\]

\[
\times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{p'(s)}{p(s)} \sqrt{H(t, s)} \right)^2 ds < \infty.
\]

Then any regular solution of Eq. (1) is oscillatory.

With an appropriate choice of the functions \( H \) and \( h \) one can derive from Theorem 1 a number of oscillation criteria for Eq. (1) as well as for Eqs. (2)–(4). Consider, for example, the function \( H(t, s) = (t - s)^{n-1} \), \((t, s) \in D\) with integer \( n > 2 \). Evidently \( H \) belongs to the class \( P \).

Further, the function \( h(t, s) = (n - 1)(t - s)^{n-3/2} \), \((t, s) \in D\) is continuous on \( I \times I \) and satisfies condition (5). Therefore by Theorem 1 we get the following oscillation criteria.

**Corollary 2 [19, Theorem 1].** Let assumptions (A), (B), (i)–(iii) be fulfilled. If there exists a function \( \rho \in C^1(I, \mathbb{R}_+) \) such that for some integer \( n > 2 \)

\[
\limsup_{t \to \infty} \int_{t_0}^{t} \left( (t - s)^{n-1} \rho(s)(q(s) - c(kh(s)))
\]

\[
- \frac{\rho(s) r(s)}{4K} (t - s)^{n-3} \left( n - 1 + \left( \frac{p(s)}{r(s)} - \frac{p'(s)}{p(s)} \right) (t - s) \right)^2 \right] \, ds = \infty,
\]

then any regular solution of Eq. (1) is oscillatory.
**Corollary 3** [19, Corollary 2]. Let assumptions (A), (B), (i)–(iii) be fulfilled. If there exists a function \( \rho \in C^1(I, R_+) \) such that for some integer \( n > 2 \)

\[
\limsup_{t \to \infty} t^{1-n} \int_{t_0}^{t} (t - s)^{n-1} \rho(s)(q(s) - ckh(s)) \, ds = \infty
\]

\[
\limsup_{t \to \infty} t^{1-n} \int_{t_0}^{t} \rho(s)r(s)(t - s)^{n-3}
\times \left[ n - 1 + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right)(t - s) \right]^2 \, ds < \infty,
\]

then any regular solution of Eq. (1) is oscillatory.

**Theorem 2.** Let assumptions (A), (B), (i)–(iii) be fulfilled. Let the functions \( H \) and \( h \) be defined as in Theorem 1, and suppose also that

\[
0 < \inf_{s \geq t_0} \left[ \liminf_{t_0 \to \infty} \frac{H(t, s)}{H(t, t_0)} \right] \leq \infty. \quad (13)
\]

Suppose that there exist functions \( \rho \in C^1(I, R_+) \) and \( \phi \in C(I, R_+) \) such that

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \rho(s)r(s)
\times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \, ds < \infty \quad (14)
\]

and

\[
\limsup_{t \to \infty} \int_{t_0}^{t} \frac{\phi_2(s)}{\rho(s)r(s)} = \infty. \quad (15)
\]

If for any \( T \geq t_0 \)

\[
\limsup_{t \to \infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[ H(t, s) \rho(s)(q(s) - ckh(s)) - \frac{\rho(s)r(s)}{4K} \right.
\times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \, ds \geq \phi(T), \quad (16)
\]

where \( \phi_+(t) = \max(\phi(t), 0) \), then any regular solution of Eq. (1) is oscillatory.
Proof. Without loss of generality we may assume that there exists a solution \( x(t) \) of Eq. (1) such that \( x(t) > 0 \) on \( [T_0, \infty) \) for some \( T_0 \geq t_0 \).

Define the function \( v(t) \) by (7). Then in the same way as in Theorem 1 we obtain the inequality (10).

It follows from (10) that

\[
\frac{1}{H(t, T)} \int_T^t \left[ H(t, s) \rho(s)(q(s) - ckh(s)) - \frac{\rho(s)r(s)}{4K} \left( h(t, s) + \frac{\rho(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right]^2 ds
\]

\[
\leq v(T) - \frac{1}{H(t, T)} \int_T^t \left[ \sqrt{\frac{KH(t, s)}{\rho(s)r(s)}} v(s) + \frac{1}{2} \sqrt{\frac{\rho(s)r(s)}{K}} \right] ds
\]

for \( t > T \geq T_0 \) and therefore

\[
\limsup_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s) \rho(s)(q(s) - ckh(s)) - \frac{\rho(s)r(s)}{4K} \left( h(t, s) + \frac{\rho(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right]^2 ds
\]

\[
\leq v(T) - \liminf_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ \sqrt{\frac{KH(t, s)}{\rho(s)r(s)}} v(s) + \frac{1}{2} \sqrt{\frac{\rho(s)r(s)}{K}} \left( h(t, s) + \frac{\rho(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right]^2 ds.
\]

It follows from (16) that

\[
v(T) \geq \phi(T) + \liminf_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ \sqrt{\frac{KH(t, s)}{\rho(s)r(s)}} v(s) + \frac{1}{2} \sqrt{\frac{\rho(s)r(s)}{K}} \left( h(t, s) + \frac{\rho(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right]^2 ds,
\]
so
\[ v(T) \geq \phi(T) \quad \text{for all } T \geq T_0 \] (17)
and
\[
\liminf_{t \to \infty} \frac{1}{H(t, T_0)} \int_{T_0}^{t} \left[ \sqrt{\frac{KH(t, s)}{\rho(s)r(s)}} v(s) + \frac{1}{2} \sqrt{\frac{\rho(s)r(s)}{K}} \right] \times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right) ds
\]
\[ \leq v(T_0) - \phi(T_0) = M < \infty. \]

Therefore for all \( t \geq T_0 \)
\[
\liminf_{t \to \infty} \frac{1}{H(t, T_0)} \int_{T_0}^{t} \left[ \sqrt{\frac{KH(t, s)}{\rho(s)r(s)}} v(s) + \frac{1}{2} \sqrt{\frac{\rho(s)r(s)}{K}} \right] \times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right) ds
\]
\[ \geq \liminf_{t \to \infty} \frac{1}{H(t, T_0)} \int_{T_0}^{t} \left[ \frac{KH(t, s)}{\rho(s)r(s)} v^2(s) + \sqrt{H(t, s)} \right] \times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right) v(s) ds. \] (18)

Define for \( t > T_0 \) the functions \( \alpha(t) \) and \( \beta(t) \) as
\[
\alpha(t) = \frac{1}{H(t, T_0)} \int_{T_0}^{t} \frac{KH(t, s)}{\rho(s)r(s)} v^2(s) ds,
\]
\[
\beta(t) = \frac{1}{H(t, T_0)} \int_{T_0}^{t} \sqrt{H(t, s)} \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right) v(s) ds.
\]
Then (18) may be rewritten as
\[
\liminf_{t \to \infty} [\alpha(t) + \beta(t)] < \infty. \] (19)
In order to show that
\[
\int_{T_0}^{\infty} \frac{v^2(s)}{\rho(s)r(s)} \, ds < \infty,
\] (20)
let us suppose by contradiction that
\[
\int_{T_0}^{\infty} \frac{v^2(s)}{\rho(s)r(s)} \, ds = \infty.
\] (21)

By the assumption (13) there exists a positive constant \( \eta \) such that
\[
\inf_{s \geq T_0} \left( \liminf_{i \to \infty} \frac{H(t, s)}{H(t, T_0)} \right) > \eta > 0.
\] (22)

On the other hand, by (21) for any positive number \( \mu \) there exists \( T_1 > T_0 \) such that
\[
\int_{T_0}^{t} \frac{v^2(s)}{\rho(s)r(s)} \, ds \geq \frac{\mu}{\eta} \quad \text{for all } t \geq T_1,
\]
so for all \( t \geq T_1 \)
\[
\alpha(t) = \frac{1}{H(t, T_0)} \int_{T_0}^{t} KH(t, s) d \left[ \int_{T_0}^{s} \frac{v^2(u)}{\rho(u)r(u)} \, du \right] \\
= \frac{1}{H(t, T_0)} \int_{T_0}^{t} K \left[ -\frac{\partial H(t, s)}{\partial s} \right] \left[ \int_{T_0}^{s} \frac{v^2(u)}{\rho(u)r(u)} \, du \right] \, ds \\
\geq \frac{\mu}{\eta} \frac{1}{H(t, T_0)} \int_{T_1}^{t} K \left[ -\frac{\partial H(t, s)}{\partial s} \right] \, ds = K \frac{\mu}{\eta} \frac{H(t, T_1)}{H(t, T_0)}. \] (23)

From (22) we have
\[
\liminf_{t \to \infty} \frac{H(t, s)}{H(t, T_0)} > \eta > 0,
\]
so there exists \( T_2 \geq T_1 \) such that \( H(t, T_1)/H(t, T_0) \geq \eta \) for all \( t \geq T_2 \).

Therefore by (23), \( \alpha(t) \geq K\mu \) for all \( t \geq T_2 \), and since \( \mu \) is arbitrary constant, we conclude that
\[
\lim_{t \to \infty} \alpha(t) = \infty.
\] (24)
Consider a sequence \( \{t_n\}_{n=1}^\infty \) in the interval \((T_0, \infty)\) such that \( \lim_{t \to \infty} t_n = \infty \) and
\[
\lim_{n \to \infty} [\alpha(t_n) + \beta(t_n)] = \lim_{t \to \infty} \inf [\alpha(t) + \beta(t)].
\]

By (19) there exists a natural number \( N \) such that
\[
\alpha(t_n) + \beta(t_n) \leq M \quad \text{for all } n > N \tag{25}
\]
with the same constant \( M \) as above.

Obviously, by (24), \( \lim_{n \to \infty} \alpha(t_n) = \infty \) and thus (25) implies that
\[
\lim_{n \to \infty} \beta(t_n) = -\infty. \tag{26}
\]

Further, it follows from (25) and (26) that for the large values of \( n \)
\[
\frac{\beta(t_n)}{\alpha(t_n)} < \varepsilon - 1 < 0, \tag{27}
\]
where \( \varepsilon \in (0, 1) \).

It follows from (26) and (27) that
\[
\lim_{n \to \infty} \frac{\beta(t_n)}{\alpha(t_n)} \cdot \beta(t_n) = \infty. \tag{28}
\]

Further, by the Schwarz inequality we have for any natural number \( n \)
\[
\beta^2(t_n) = \frac{1}{H^2(t_n, T_0)} \left[ \int_{T_0}^{t_n} \sqrt{H(t_n, s)} \left( h(t_n, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \right) \times \sqrt{H(t_n, s)} \, v(s) \, ds \right]^2
\]
\[
\leq \left[ \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} K \frac{H(t_n, s) v^2(s)}{\rho(s) r(s)} \, ds \right]
\times \left[ \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} \rho(s) r(s) \frac{\sqrt{H(t_n, s)}}{K} \times \left( h(t_n, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t_n, s)} \right)^2 \, ds \right]
\]
\[ \leq \alpha(t_n) \left[ \frac{1}{H(t_n, T_0)} \int_{t_0}^{t_n} \frac{\rho(s)r(s)}{K} \right. \]
\[ \times \left( h(t_n, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t_n, s)} \right)^2 ds \]

and therefore
\[ \frac{\beta^2(t_n)}{\alpha(t_n)} \leq \frac{1}{\eta \ H(t_n, t_0) \ \int_{t_0}^{t_n} \frac{\rho(s)r(s)}{K}} \]
\[ \times \left( h(t_n, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t_n, s)} \right)^2 ds \]

for all \( n \) large enough.

It follows from (28) that
\[ \lim_{n \to \infty} \frac{1}{H(t_n, t_0)} \int_{t_0}^{t_n} \rho(s)r(s) \]
\[ \times \left( h(t_n, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t_n, s)} \right)^2 ds = \infty, \quad (29) \]

so
\[ \limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \rho(s)r(s) \]
\[ \times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 ds = \infty, \]

and the latter equality contradicts the assumption (14). Therefore we have proved that (21) fails, so the inequality (20) holds true. Further, by (17)
\[ \int_{T_0}^{\infty} \frac{\phi^2(s)}{\rho(s)r(s)} ds \leq \int_{T_0}^{\infty} \frac{v^2(s)}{\rho(s)r(s)} ds < \infty \]

which contradicts the condition (15), and this completes the proof. \( \blacksquare \)

The following proposition is the direct consequence of Theorem 2 and uses the same choice of the functions \( H \) and \( h \) as in Corollaries 2 and 3 above.
COROLLARY 4. Let assumptions (A), (B), (i)–(iii) be fulfilled. If there exist two functions \( \rho \in C^3(I, \mathbb{R}_+) \) and \( \phi \in C(I, \mathbb{R}_+) \) such that (15) holds and

\[
\limsup_{t \to \infty} t^{1-n} \int_{t_0}^{t} \rho(s)r(s)(t-s)^{n-3} \left[ n - 1 + \left( \frac{\rho'(s)}{\rho(s)} - \frac{\rho(s)'}{\rho(s)} \right) (t-s) \right] ds < \infty,
\]

\[
\limsup_{t \to \infty} t^{1-n} \int_{t_0}^{t} \left( (t-s)^{n-1} \rho(s)(q(s) - ckh(s)) - \frac{\rho(s)r(s)}{4K} (t-s)^{n-3} \right) \left[ n - 1 + \left( \frac{\rho(s)'}{\rho(s)} - \frac{\rho(s)'}{\rho(s)} \right) (t-s) \right] ds \geq \phi(T)
\]

for all \( T \geq t_0 \) and for some integer \( n > 2 \), then any regular solution of Eq. (1) is oscillatory.

Proof. The only thing to be checked is the condition (13) which with the above choice of the functions \( H \) and \( h \) is fulfilled automatically because

\[
\lim_{t \to \infty} \frac{H(t, s)}{H(t, t_0)} = \lim_{t \to \infty} \frac{(t-s)^{n-1}}{(t-t_0)^{n-1}} = 1
\]

for any \( s \geq t_0 \).

THEOREM 3. Let the functions \( H \) and \( h \) be defined as in Theorem 1, and suppose also that (13) holds true. If there exist functions \( \rho \in C^3(I, \mathbb{R}_+) \) and \( \phi \in C(I, \mathbb{R}_+) \) such that (15) holds and

\[
\liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s) \rho(s)(q(s) - ckh(s)) ds < \infty, \quad (30)
\]

\[
\liminf_{t \to \infty} \frac{1}{H(t, T)} \int_{T}^{t} \left( H(t, s) \rho(s)(q(s) - ckh(s)) - \frac{\rho(s)r(s)}{4K} \right) \left[ h(t, s) + \left( \frac{\rho(s)'}{\rho(s)} - \frac{\rho(s)'}{\rho(s)} \right) \sqrt{H(t, s)} \right] ds \geq \phi(T) \quad (31)
\]

for all \( T \geq t_0 \), where \( \phi_+ \) is the same as in Theorem 2, then any regular solution of Eq. (1) is oscillatory.
Proof. As above we may assume that there exists a solution \( x(t) \) of Eq. (1) such that \( x(t) > 0 \) on \([T_0, \infty)\) for some \( T_0 \geq t_0 \). With the function \( v(t) \) defined by (7), the inequality (10) holds and thus for all \( T \geq T_0 \)

\[
\liminf_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s) \rho(s)(q(s) - ckh(s)) - \frac{\rho(s)r(s)}{4K} \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \right] ds
\]

\[
\leq v(T) - \limsup_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ \frac{\sqrt{KH(t, s)}}{\rho(s)r(s)} v(s) + \frac{1}{2} \frac{\rho(s)r(s)}{K} \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \right] ds.
\]

By (31) we get

\[
v(T) \geq \phi(T) + \limsup_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ \frac{\sqrt{KH(t, s)}}{\rho(s)r(s)} v(s) + \frac{1}{2} \frac{\rho(s)r(s)}{K} \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \right] ds.
\]

Therefore, inequality (17) holds for all \( T \geq T_0 \) and

\[
\limsup_{t \to \infty} \frac{1}{H(t, T_0)} \int_T^t \left[ \frac{\sqrt{KH(t, s)}}{\rho(s)r(s)} v(s) + \frac{1}{2} \frac{\rho(s)r(s)}{K} \times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \right] ds \leq M < \infty
\]

with the same \( M \) as above.
\[ \limsup_{t \to \infty} \left[ \alpha(t) + \beta(t) \right] \]
\[ \leq \limsup_{t \to \infty} \frac{1}{H(t, T_0)} \int_{T_0}^t \left[ \sqrt{\frac{K H(t, s)}{\rho(s) r(s)}} v(s) + \frac{1}{2} \sqrt{\frac{\rho(s) r(s)}{K}} \right] \times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \, ds \leq M < \infty \quad (32) \]

with the same \( \alpha(t) \) and \( \beta(t) \) as in the proof of Theorem 2.

By the assumption \( 31 \) we obtain

\[ \phi(t_0) \leq \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) \rho(s) (q(s) - ckh(s)) \, ds \]
\[ - \frac{1}{4K} \liminf_{t \to \infty} \int_{t_0}^t \rho(s) r(s) \times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \, ds. \quad (33) \]

It follows from \( 30 \) and \( 33 \) that

\[ \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \rho(s) r(s) \times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \, ds < \infty. \quad (34) \]

By \( 34 \) there exists a sequence \( \{ t_n \} \) in the interval \( (T_0, \infty) \) such that \( \lim_{n \to \infty} t_n = \infty \) and

\[ \lim_{n \to \infty} \frac{1}{H(t_n, t_0)} \int_{t_0}^{t_n} \rho(s) r(s) \left( h(t_n, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t_n, s)} \right)^2 \, ds \]
\[ = \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \rho(s) r(s) \times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \, ds < \infty. \quad (35) \]
Suppose now that (21) holds. With the same argument as in Theorem 2 we conclude that (24) holds. By (32) there exists a natural number \( N \) such that (25) is fulfilled. Proceeding as in the proof of Theorem 2 we obtain (29) which contradicts (35) and hence (21) fails. Making use of (17) we get by (20)

\[
\int_{T_0}^{\infty} \frac{\phi_2^2(s)}{\rho(s)r(s)} \, ds \leq \int_{T_0}^{\infty} \frac{v^2(s)}{\rho(s)r(s)} \, ds < \infty
\]

which contradicts the assumption (15), so the proof is now complete. 

For the case when the function \( f_3(x) \) has no continuous derivative, we present the following

**Theorem 4.** Let assumption (iii) in Theorem 1 be replaced by

(iii') \( f_3(x)/x \geq \kappa_1 > 0 \) and \( f_3(x)/x \leq \kappa_2 \) for \( x \neq 0 \),

where \( \kappa_1, \kappa_2 \) are constants, and let the functions \( h, H \) be the same as in Theorem 1. Suppose also that \( q(t) \geq 0 \) for \( t \in I \).

If there exists a continuously differentiable function \( \rho: I \to \mathbb{R}_+ \) such that

\[
\limsup_{t \to \infty} \frac{1}{H(t, T_0)} \int_{T_0}^{t} \left[ H(t, s) \rho(s) (\kappa_1 q(s) - c \kappa_2 h(s)) - \frac{\rho(s)r(s)}{4} \right. \\
\left. \times \left( h(t, s) + \sqrt{H(t, s)} \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \right)^2 \right] \, ds = \infty, \quad (36)
\]

then any regular solution of Eq. (1) is oscillatory.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of Eq. (1). Without loss of generality we assume that \( x(t) > 0 \) for all \( t \geq T_0 \). Define

\[
v(t) = \rho(t)r(t) \frac{x'(t)}{x(t)}, \quad (37)
\]

Then differentiating (37) and making use of (1) we obtain

\[
v'(t) = \frac{\rho(t)}{x(t)} P(t, x(t), x'(t)) - \left( \frac{p(t)}{r(t)} - \frac{\rho'(t)}{\rho(t)} \right) v(t) \\
- \frac{\rho(t)}{x(t)} Q(t, x(t)) - \frac{1}{\rho(t)r(t)} u^2(t). \quad (38)
\]
In view of (B) and (iii') we get from (38)

\[ v'(t) \leq \rho(t)(c\kappa_2 h(t) - \kappa_1 q(t)) - \left( \frac{p(t)}{r(t)} - \frac{\rho'(t)}{\rho(t)} \right)v(t) \]

\[ - \frac{1}{\rho(t)r(t)}v^2(t), \quad (39) \]

and proceeding as in the proof of Theorem 1 we can complete the proof.

Evidently, we can deduce from Theorem 4 corollaries analogous to Corollaries 2 and 3 given above, and we can also prove the following theorems analogous to Theorems 2 and 3 with assumption (iii) replaced by (iii'). The proofs are similar to those of Theorems 2 and 3, and hence they are omitted.

**Theorem 5.** Let the functions \( H \) and \( h \) be defined as in Theorem 1, and suppose also that assumption (13) is fulfilled. Suppose that there exist two functions \( \rho \in C^1(I, \mathbb{R}_+) \) and \( \phi \in C(I, \mathbb{R}_+) \) such that conditions (14) and (15) hold. If for any \( T \geq t_0 \)

\[
\limsup_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s) \rho(s)(\kappa_1 q(s) - c\kappa_2 h(s)) \right.
\]

\[
- \rho(s)r(s) \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \right] ds \geq \phi(T), \quad (40)
\]

where \( \phi_+(t) \) is the same as in Theorem 2, then any regular solution of Eq. (1) is oscillatory.

**Theorem 6.** Let the functions \( H \) and \( h \) be defined as in Theorem 1, and suppose also that (13) holds. If there exist two functions \( \rho \in C^1(I, \mathbb{R}_+) \) and \( \phi \in C(I, \mathbb{R}_+) \) such that (15) holds and

\[
\liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) \rho(s)(\kappa_1 q(s) - c\kappa_2 h(s)) ds < \infty, \quad (41)
\]

\[
\liminf_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s) \rho(s)(\kappa_1 q(s) - c\kappa_2 h(s)) \right.
\]

\[
- \rho(s)r(s) \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right)^2 \right] ds \geq \phi(T) \quad (42)
\]

for all \( T \geq t_0 \), where \( \phi_+(s) \) is the same as in Theorem 2, then any regular solution of Eq. (1) is oscillatory.
For the case when the function $f_1(x)$ has a continuous bounded derivative, we can obtain more sharp oscillation criteria for Eq. (1) similar to those derived for Eq. (2) with $f(x) = x$ by Li [12] and for Eq. (4) by Rogovchenko [15].

**Theorem 7.** Let assumption (iii) in Theorem 1 be replaced by

(iii') $0 < K \leq f'_1(x) \leq K_1$ and $f_2(x)/f'_1(x) \leq k$ for $x \neq 0$,

where $K, K_1, k > 0$ are constants, and let the functions $h, H$ be the same as in Theorem 1.

If there exists a continuously differentiable function $\rho: I \rightarrow \mathbb{R}_+$ such that

$$\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[ H(t, s) \psi(s) - \frac{a(s)r(s)}{4K} \right] \left( h(t, s) + \frac{p(s)}{r(s)} \sqrt{H(t, s)} \right)^2 ds = \infty, \quad (43)$$

where $\psi(t) = a(t)[kch(t) - q(t) + (1/K_1)p(t)\rho(t) - (K/K_1)r(t)\rho^2(t) + (1/K_1)[r(t)\rho(t)]^2]$ and $a(t) = \exp(-2/\rho(u)du)$, then any regular solution of Eq. (1) is oscillatory.

**Proof.** Let $x(t)$ be a nonoscillatory solution of Eq. (1). Without loss of generality we assume that $x(t) > 0$ for all $t \geq T_0$.

We use the generalized Riccati transformation putting

$$v(t) = a(t)r(t) \left[ \frac{x'(t)}{f_1(x(t))} + \frac{\rho(t)}{K_1} \right]. \quad (44)$$

Then differentiating (44) and making use of (1) we obtain

$$v'(t) = -2\rho(t)v(t) + a(t) \frac{P(t, x(t), x'(t))}{f_1(x(t))} - \frac{p(t)}{r(t)}v(t)$$

$$+ \frac{1}{K_1}p(t)\rho(t) - a(t) \frac{Q(t, x(t))}{f_1(x(t))} - \frac{f'_1(x(t))}{a(t)r(t)}v^2(t)$$

$$+ \frac{2}{K_1^2}f'_1(x(t))\rho(t)v(t) - \frac{1}{K_1^2}f'_1(x(t))\rho^2(t) + \frac{1}{K_1}[r(t)\rho(t)]'. \quad (45)$$
In view of (B) and (iii’’) we get from (45)

\[ v'(t) \leq -\psi(t) - \frac{p(t)}{r(t)}v(t) - \frac{K}{a(t)r(t)}v^2(t), \]

where \( \psi(t) \) is defined as above, and proceeding as in the proof of Theorem 1 we can complete the proof of Theorem 7. \( \Box \)

The following theorems are analogous to Theorems 2 and 3. We omit the proofs because they are similar to those of the theorems cited above.

**Theorem 8.** Let the functions \( H \) and \( h \) be defined as in Theorem 1, and suppose also that assumption (13) is fulfilled. Suppose that there exist two functions \( \rho \in C^1(I, \mathbb{R}_+) \) and \( \phi \in C(I, \mathbb{R}_+) \) such that conditions (14) and (15) hold. If for any \( T \geq t_0 \)

\[
\limsup_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s) \psi(s) - \frac{a(s)r(s)}{4K} \right. \\
\times \left. \left( h(t, s) + \frac{p(s)}{r(s)}\sqrt{H(t, s)} \right)^2 \right] ds \geq \phi(T), \tag{46}
\]

where \( a(t), \psi(t), \phi_+(t) \) are the same as in Theorem 7, then any regular solution of Eq. (1) is oscillatory.

**Theorem 9.** Let the functions \( H \) and \( h \) be defined as in Theorem 1, and suppose also that (13) holds. If there exist two functions \( \rho \in C^1(I, \mathbb{R}_+) \) and \( \phi \in C(I, \mathbb{R}_+) \) such that (15) holds and

\[
\liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) \psi(s) ds < \infty, \tag{47}
\]

\[
\liminf_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s) \psi(s) - \frac{a(s)r(s)}{4K} \right. \\
\times \left. \left( h(t, s) + \frac{p(s)}{r(s)}\sqrt{H(t, s)} \right)^2 \right] ds \geq \phi(T) \tag{48}
\]

for all \( T \geq t_0 \), where \( a(t), \psi(t), \phi_+(t) \) are the same as above, then any regular solution of Eq. (1) is oscillatory.
2. ASYMPTOTICS FOR FORCED EQUATION

In this section we study the asymptotic behaviour of solutions of the forced perturbed nonlinear equation

$$[r(t)x'(t)]' + p(t)x'(t) + Q(t, x(t)) = P(t, x(t), x'(t)) + e(t). \tag{49}$$

The main result of this section is the following

**Theorem 10.** Let assumptions (A), (B), (i), (ii) be fulfilled. Then each of the following conditions guarantees that every solution $x(t)$ of Eq. (49) satisfies $\liminf_{t \to \infty} |x(t)| = 0$:

(C$_1$) assumption (iii) holds and there exists a function $\rho \in C^1(I, \mathbb{R}_+)$ such that (6) holds and

$$\int_0^\infty \rho(s)|e(s)|ds = N < \infty; \tag{50}$$

(C$_2$) assumption (iii') holds and there exists a function $\rho \in C^1(I, \mathbb{R}_+)$ such that (36) and (50) hold;

(C$_3$) assumption (iii") holds and there exists a function $\rho \in C^1(I, \mathbb{R}_+)$ such that (43) holds and

$$\int_0^\infty \rho(s)|e(s)|ds = N < \infty,$$  

where the function $e \in C(I, \mathbb{R}_+)$ and $a(t)$ is defined as above.

**Proof.** We shall prove the theorem for the case of condition (C$_1$), the other two cases are considered in the same way. Let $x(t)$ be a solution of Eq. (49) and suppose by contradiction that

$$\liminf_{t \to \infty} |x(t)| = C > 0,$$

so $x(t)$ is nonoscillatory. Without loss of generality we may assume that $x(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. Differentiating the function $v(t)$ defined by (7) we obtain

$$v'(t) = \frac{\rho(t)}{f_3(x(t))}P(t, x(t), x'(t)) - \left(\frac{p(t)}{r(t)} - \frac{\rho'(t)}{\rho(t)}\right)v(t)$$

$$- \frac{\rho(t)}{f_3(x(t))}Q(t, x(t)) - \frac{f_3'(x(t))}{\rho(t)r(t)}v^2(t) + \frac{\rho(t)e(t)}{f_3(x(t))}$$
and thus

\[
v'(t) \leq \rho(t)(ckh(t) - q(t)) - \left( \frac{p(t)}{r(t)} - \frac{\rho'(t)}{\rho(t)} \right) v(t) - \frac{K}{\rho(t)r(t)} v^2(t) + \frac{1}{f_1(C)} \rho(t)|e(t)|.
\]

Hence for all \( t \geq T \geq T_0 \) we get

\[
\int_T^t H(t, s) \rho(s)(q(s) - ckh(s)) \, ds \
\leq -\int_T^t H(t, s) v'(s) \, ds - \int_T^t H(t, s) \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) v(s) \, ds \
- \int_T^t H(t, s) \frac{K}{\rho(s)r(s)} v^2(s) \, ds + \int_T^t H(t, s) \frac{1}{f_1(C)} \rho(s)|e(s)| \, ds
\]

and consequently

\[
\int_T^t H(t, s) \rho(s)(q(s) - ckh(s)) \
- \frac{\rho(s)r(s)}{4K} \left[ h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right]^2 \, ds \
\leq H(t, T)v(T) - \int_T^t \left[ \sqrt{\frac{KH(t, s)}{\rho(s)r(s)}} v(s) + \frac{1}{2} \sqrt{\frac{\rho(s)r(s)}{K}} \right. \
\times \left( h(t, s) + \left( \frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t, s)} \right) \left. \right]^2 \, ds \\
+ \frac{1}{f_1(C)} \int_T^t H(t, s) \rho(s)|e(s)| \, ds.
\]

(52)
By (50), (52), and by the property \((H_2)\) for every \(t \geq T_0\)

\[
\int_{T_0}^t \left[H(t, s) \rho(s) (q(s) - ckh(s))
- \frac{\rho(s)r(s)}{4K} \left[h(t, s) + \left(\frac{r(s)}{\rho(s)} - \frac{\rho'(s)}{\rho(s)}\right)\sqrt{H(t, s)}\right]^2 ds
\leq H(t, T_0)|v(T_0)| + \frac{1}{\int_1(C) H(t, T_0)} \int_T^t \rho(s)|e(s)| ds.
\]

Now the proof proceeds in the same way as in Theorem 1.

**Remark 1.** Some of the results of this paper can be extended for the more general equations involving the term \([r(t)\psi(x(t))x'(t)]'\) in place of \([r(t)x'(t))']\), where \(\psi\) is a positive continuous function on \(I\), as well as for some classes of differential equations with deviating argument.

**Remark 2.** The results of the paper are presented in the form of a high degree of generality and thus they give wide possibilities of deriving the different oscillation criteria with an appropriate choice of the functions \(H\) and \(h\), as, for example, with

\[
H(t, s) = \left[\int_s^t \frac{dz}{\Theta(z)}\right]^{\frac{n-1}{n}}, \quad (t, s) \in D,
\]

where \(n > 2\) is a constant and \(\Theta\) is a positive continuous function on \([t_0, \infty)\) such that

\[
\int_{t_0}^\infty \frac{dz}{\Theta(z)} = \infty
\]

(an important particular case is \(\Theta(z) = z^\alpha\) with \(\alpha\) real).

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