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Riesz bases and their dual modular frames in Hilbert C^*-modules

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Abstract

We investigate dual frames of modular frames and Riesz bases in Hilbert C^*-modules. In Hilbert C^*-modules, a Riesz basis may have many dual modular frames, and it may even admit two different dual modular frames both of which are Riesz bases. We characterize those Riesz bases that have unique duals. In addition, we obtain a necessary and sufficient condition for a dual of a Riesz basis to be again a Riesz basis, and prove some new related results.

Keywords: Frames; Riesz bases; Dual sequences; Hilbert C^*-modules

1. Introduction

Hilbert space frames were originally introduced by Duffin and Schaeffer [2] to deal with some problems in non-harmonic Fourier analysis. Frames can be viewed as redundant bases which are generalizations of Riesz bases. This redundancy property sometimes is extremely important in applications such as signal and image processing, data compression and sampling theory. Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Frames for Hilbert spaces have natural analogues for Hilbert C^*-modules [5]. These frames are called *Hilbert C^*-modular frames* or just simply *modular frames*. Modular frames are not trivial generalizations of Hilbert space frames due to the complex structure of C^*-algebras. It is well known that the theory of Hilbert C^*-modules is quite different from that of Hilbert spaces. For example, we know that, any closed linear subspace in a Hilbert space has an orthogonal complement. But this is no longer true in Hilbert C^*-module setting since not every closed submodule of a Hilbert C^*-module is complemented. Moreover, the Riesz representation theorem for continuous functionals on Hilbert spaces does not hold in Hilbert C^*-modules, and so there exist nonadjointable bounded linear operators on Hilbert C^*-modules. Therefore it is expected that problems about frames in Hilbert C^*-modules are more complicated than those in Hilbert spaces. While some of the results

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about frames in Hilbert spaces can be easily extended to Hilbert C^*-modular frames, many others cannot be obtained by simply modifying the approaches used in Hilbert spaces case (cf. [1–8,13,15,16]).

Riesz bases play important roles in the study of Hilbert space frame theory (cf. [6]). However, we will encounter several obstacles when we deal with Riesz bases and frames for Hilbert C^*-modules. Firstly, unlike the Hilbert space case, not every Hilbert C^*-module admits a Riesz basis (see Example 3.4 in [5]). However, the famous Kasparov's Stabilization Theorem [9] implies that every finitely or countably generated Hilbert C^*-module has a frame. This makes the concept of frames more relevant to Hilbert C^*-modules when dealing with issues involving *bases and expansions*. Secondly, a Riesz basis for a Hilbert C^*-module can have many different dual frames which are not necessarily Riesz bases. This is completely different from the Hilbert space setting in which a Riesz basis has a unique dual and this dual is also a Riesz basis.

The main purpose of this paper is to investigate some basic properties about modular Riesz bases and their duals. These are the properties that are needed for further investigation of the modular frame theory. A Riesz basis always has a canonical dual which is necessarily a Riesz basis. However, it can even have two different duals both of which are Riesz bases. In this paper we obtain a necessary and sufficient condition for a dual of a Riesz basis to be again a Riesz basis. In particular, we characterize those modular Riesz bases that have unique duals. The characterization is given in terms of the properties of the range spaces of the *analysis operators*. As a consequence, we show that when the underlying C^*-algebra is commutative, every modular Riesz basis has a unique Riesz dual (although it may still have other duals that are not Riesz bases, see Example 3.6). Several examples are also given to show the differences between the Riesz bases and their duals in Hilbert C^*-modules with respect to those in Hilbert spaces.

2. Preliminaries

We review some basics about Hilbert C^*-modular frames that will be needed in the proofs of the main results in Section 3. For basic notations and theory for Hilbert C^*-modules see [9–12,14,16,18].

Definition 2.1. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{H}$ be a (left) $\mathcal{A}$-module. Suppose that the linear structures given on A and $\mathcal{H}$ are compatible, i.e. $\lambda(ax) = a(\lambda x)$ for every $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$ and $x \in \mathcal{H}$. If there exists a mapping $\langle \cdot,\cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ with the properties

(1) $\langle x, x\rangle \geqslant 0$ for every $x \in \mathcal{H}$,
(2) $\langle x, x\rangle = 0$ if and only if $x = 0$,
(3) $\langle x, y\rangle = \langle y, x\rangle^*$ for every $x, y \in \mathcal{H}$,
(4) $\langle ax, y\rangle = a\langle x, y\rangle$ for every $a \in \mathcal{A}$, and every $x, y \in \mathcal{H}$,
(5) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for every $x, y, z \in \mathcal{H}$.

Then the pair $\{\mathcal{H}, \langle \cdot,\cdot \rangle\}$ is called a (left-) pre-Hilbert $\mathcal{A}$-module. The map $\langle \cdot,\cdot \rangle$ is said to be an $\mathcal{A}$-valued inner product. If the pre-Hilbert $\mathcal{A}$-module $\{\mathcal{H}, \langle \cdot,\cdot \rangle\}$ is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$, then it is called a *Hilbert $\mathcal{A}$-module*.

We want to mention here that the Cauchy–Schwartz inequality holds in Hilbert C^*-modules.

Proposition 2.2. *Let $\mathcal{H}$ be a pre-Hilbert $\mathcal{A}$-module and $x, y \in \mathcal{H}$. Then*

$$\langle y, x\rangle \langle x, y\rangle \leqslant \big\| \langle x, x\rangle \big\| \langle y, y\rangle.$$

Moreover,

$$\big\| \langle x, y\rangle \big\| \leqslant \|x\| \|y\|.$$

In this paper we focus on finitely and countably generated Hilbert C^*-modules over unital C^*-algebra $\mathcal{A}$. A Hilbert $\mathcal{A}$-module $\mathcal{H}$ is (*algebraically*) *finitely generated* if there exists a finite set $\{x_1, \ldots, x_n\} \subseteq \mathcal{H}$ such that every element $x \in \mathcal{H}$ can be expressed as an $\mathcal{A}$-linear combination $x = \sum_{i=1}^{n} a_i x_i$, $a_i \in \mathcal{A}$. A Hilbert $\mathcal{A}$-module $\mathcal{H}$ is *countably generated* if there exists a countable set of generators (see [5]).

We now recall the definitions of frames and Riesz bases in Hilbert C^*-modules as follows.

Definition 2.3. (See [5].) Let $\mathcal{A}$ be a unital C^*-algebra and $\mathbb{J}$ be a finite or countable index set. A (countable or finite) sequence $\{x_j\}_{j\in\mathbb{J}}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a (standard) *frame* for $\mathcal{H}$ if there exist two constants $C, D > 0$ such that

$$C\langle x, x\rangle \leqslant \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \leqslant D\langle x, x\rangle$$

for every $x \in \mathcal{H}$, where the sum in the middle of the inequality is convergent in norm. The numbers C and D are called *frame bounds*.

Likewise, $\{x_j\}_{j\in\mathbb{J}}$ is called a (standard) *Bessel sequence* with bound D if there exists $D > 0$ such that

$$\sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \leqslant D\langle x, x\rangle$$

for every $x \in \mathcal{H}$, where the sum in the inequality is convergent in norm.

Definition 2.4. (See [5].) A frame $\{x_j\}_{j\in\mathbb{J}}$ for a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a (standard) *Riesz basis* for $\mathcal{H}$ if it satisfies:

(i) $x_j \neq 0$ for all j;
(ii) if an $\mathcal{A}$-linear combination $\sum_{j\in S} a_j x_j$ with coefficients $\{a_j\colon\ j \in S\} \subseteq \mathcal{A}$ and $S \subseteq \mathbb{J}$ is equal to zero, then every summand $a_j x_j$ is equal to zero.

Let us introduce the definition of dual sequences of modular frames.

Definition 2.5. Suppose that $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. Let $\{x_j\}_{j\in\mathbb{J}}$ be a (standard) frame and $\{y_j\}_{j\in\mathbb{J}}$ a sequence of $\mathcal{H}$. Then $\{y_j\}_{j\in\mathbb{J}}$ is said to be a (standard) *dual sequence* of $\{x_j\}_{j\in\mathbb{J}}$ if

$$x = \sum_{j\in\mathbb{J}} \langle x, y_j\rangle x_j \tag{1}$$

holds for all $x \in \mathcal{H}$, where the sum in (1) converges in norm. The pair $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ are called a *dual frame pair* when $\{y_j\}_{j\in\mathbb{J}}$ is also a frame.

Like the Hilbert space frame case, the following result guarantees the existence of a dual for any Hilbert C^*-module frame.

Theorem 2.6. *(See [5].) Let $\{x_j\}_{j\in\mathbb{J}}$ be a standard frame in a finitely or countably generated Hilbert A-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then there exists a unique positive and invertible operator S on $\mathcal{H}$ such that*

$$x = \sum_{j\in\mathbb{J}} \langle x, Sx_j\rangle x_j$$

for every $x \in \mathcal{H}$.

The dual in the above theorem is called the *canonical or standard dual* and it is obviously a frame.

We now introduce a few more notations. For a unital C^*-algebra $\mathcal{A}$, let $l^2(\mathcal{A})$ be the Hilbert $\mathcal{A}$-module defined by

$$l^2(\mathcal{A}) = \left\{\{a_j\}_{j\in\mathbb{J}} \subseteq \mathcal{A}\colon \sum_{j\in\mathbb{J}} a_j a_j^* \text{ converges in } \|\cdot\|\right\}.$$

Let $\{e_j\}_{j=1}^{\infty}$ denote the standard orthonormal basis of $l^2(\mathcal{A})$, where e_j takes value $1_{\mathcal{A}}$ at j and $0_{\mathcal{A}}$ everywhere else. For any Bessel sequence $\{x_j\}_{j\in\mathbb{J}}$ of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$, the associated *analysis operator* $T_X : \mathcal{H} \to l^2(\mathcal{A})$ is defined by

$$T_X x = \sum_{j\in\mathbb{J}} \langle x, x_j\rangle e_j, \quad x \in \mathcal{H}.$$

Note that analysis operator T_X is adjointable and its adjoint T_X^* fulfills $T_X^* e_j = x_j$ for all j.

We end up this section with the following inequality which will be used in the proof of Theorem 3.10.

Lemma 2.7. *Let $\mathcal{A}$ be a C^*-algebra. Suppose that $\{a_j\}_{j\in\mathbb{J}}$ and $\{b_j\}_{j\in\mathbb{J}}$ are two sequences of $\mathcal{A}$ such that both $\sum_{j\in\mathbb{J}} a_j a_j^*$ and $\sum_{j\in\mathbb{J}} b_j b_j^*$ converge in $\mathcal{A}$, then*

$$\sum_{j\in\mathbb{J}} (a_j+b_j)(a_j+b_j)^* \leqslant 2\sum_{j\in\mathbb{J}} \bigl(a_j a_j^* + b_j b_j^*\bigr).$$

3. Duals of Riesz bases in Hilbert C^*-modules

The aim of this section is to have a detailed investigation on the dual sequences of Riesz bases in Hilbert C^*-modules.

In the Hilbert space case, we know that a frame is a Riesz basis if and only if its analysis operator is surjective [6]. This is no longer true for Hilbert C^*-module frames. We begin this section with a characterization of Riesz bases. Let P_n be the projection on $l^2(\mathcal{A})$ that maps each element to its nth component, i.e. $P_n x = \{u_j\}_{j\in\mathbb{J}}$, where

$$u_j = \begin{cases} x_n & \text{if } j = n, \\ 0 & \text{if } j \neq n, \end{cases}$$

for each $x = \{x_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$.

Theorem 3.1. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis if and only if $x_n \neq 0$ and $P_n(Rang(T_X)) \subseteq Rang(T_X)$ for all $n \in \mathbb{J}$, where T_X is the analysis operator of $\{x_j\}_{j\in\mathbb{J}}$.*

Proof. Suppose first that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis.

Note that for any $a = \{a_j\}_{j\in\mathbb{J}}$ in $l^2(\mathcal{A})$, if $\sum_{j\in\mathbb{J}} a_j x_j = 0$, then $a_j x_j = 0$ for all $j \in \mathbb{J}$.

Now for any $a = \{a_j\}_{j\in\mathbb{J}} \in Rang(T_X)^\perp$ and $x \in \mathcal{H}$, we have

$$0 = \sum_{j\in\mathbb{J}} \langle x, x_j\rangle a_j^* = \Bigl\langle x, \sum_{j\in\mathbb{J}} a_j x_j \Bigr\rangle.$$

And so $a_j x_j = 0$ holds for all j. This implies that

$$\sum_{j=1}^{n} \langle x, x_j\rangle a_j^* = \sum_{j=1}^{n} \langle x, a_j x_j\rangle = 0.$$

It follows that $a \in P_n(Rang(T_X))^\perp$, and so $Rang(T_X)^\perp \subseteq P_n(Rang(T_X))^\perp$. Consequently $P_n(Rang(T_X)) \subseteq Rang(T_X)$.

Suppose now that $P_n(Rang(T_X)) \subseteq Rang(T_X)$ for each n. We want to show that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis.

Suppose that $\sum_{j\in\mathbb{J}} a_j x_j = 0$, where $a_j \in \mathcal{A}$.

Fix an $n \in \mathbb{J}$, then $P_n T_X x \in Rang(T_X)$, so there exists $y_n \in \mathcal{H}$ such that $T_X y_n = P_n T_X x$.

Therefore we get

$$\langle y_n, x_j\rangle = \begin{cases} \langle x, x_n\rangle & \text{if } j = n, \\ 0 & \text{if } j \neq n. \end{cases}$$

Now for any $x \in \mathcal{H}$ we have

$$\langle x, a_n x_n\rangle = \langle x, x_n\rangle a_n^* = \sum_{j\in\mathbb{J}} \langle y_n, x_j\rangle a_j^* = \sum_{j\in\mathbb{J}} \langle y_n, a_j x_j\rangle = \Bigl\langle y_n, \sum_{j\in\mathbb{J}} a_j x_j \Bigr\rangle = 0,$$

which implies that $a_n x_n = 0$. □

Note that in Hilbert spaces, if $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis and $\sum_{j\in\mathbb{J}} c_j x_j$ converges for a sequence $\{c_j\} \subseteq \mathbb{C}$, then $\{c_j\} \in l^2$. But this is not the case in the setting of Hilbert C^*-modules. We have the following example.

Example 3.2. Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbb{N}}$ and $v = \{v_j\}_{j\in\mathbb{N}}$ in l^∞, we define

$$uv = \{u_j v_j\}_{j\in\mathbb{N}}, \qquad u^* = \{\bar{u}_j\}_{j\in\mathbb{N}} \quad \text{and} \quad \|u\| = \max_{j\in\mathbb{N}} |u_j|.$$

Then $\mathcal{A} = \{l^\infty, \|\cdot\|\}$ is a C^*-algebra.

Let $\mathcal{H} = c_0$ be the set of all sequences converging to zero. For any $u, v \in \mathcal{H}$ we define

$$\langle u, v\rangle = uv^* = \{u_j \bar{v}_j\}_{j\in\mathbb{N}}.$$

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.

Obviously, $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$.

For each j we let $c_j = \sqrt{j+1}e_{j+1}$.

Then $c_j e_j = 0$ and so $\sum_{j=1}^\infty c_j e_j = 0$.

But $\sum_{j=1}^\infty c_j c_j^* = \sum_{j=2}^\infty j e_j$ does not converge in $\mathcal{A}$. Thus $\{c_j\} \notin l^2(\mathcal{A})$.

Following the definition of Riesz bases in Hilbert C^*-modules, to test a sequence $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis, one need to show that if $\sum_{j\in\mathbb{J}} c_j x_j = 0$ for some sequence $\{c_j\}_{j\in\mathbb{J}} \subseteq \mathcal{A}$, then $c_j x_j = 0$ for each j. We claim that we can restrict the sequence $\{c_j\}_{j\in\mathbb{J}}$ in $l^2(\mathcal{A})$.

Corollary 3.3. *Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a frame of $\mathcal{H}$, then $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis if and only if*

(1) $x_j \neq 0$ *for each* $j \in \mathbb{J}$;
(2) *if* $\sum_{j\in\mathbb{J}} c_j x_j = 0$ *for some sequence* $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$, *then* $c_j x_j = 0$ *for each* $j \in \mathbb{J}$.

Proof. See the proof of Theorem 3.1. □

We have mentioned in the introduction that, contrast to the Hilbert space situation, Riesz bases of Hilbert C^*-modules may possess infinitely many dual frames due to the existence of zero-divisors in the C^*-algebra of coefficients. The following three simple examples show that the dual of Riesz bases of Hilbert C^*-modules are quite different from and more complicated than the Hilbert space cases.

The following example shows that in Hilbert C^*-modules the dual Riesz basis of a Riesz basis is not unique.

Example 3.4. Let $\mathcal{A} = M_{2\times 2}(\mathbb{C})$ denote the C^*-algebra of all 2×2 complex matrices. Let $\mathcal{H} = \mathcal{A}$ and for any $A, B \in \mathcal{H}$ define

$$\langle A, B\rangle = AB^*.$$

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.

Let $E_{i,j}$ be the 2×2 matrix with 1 in the (i, j)th entry and 0 elsewhere, where $1 \leqslant i, j \leqslant 2$.

Then $\{E_{1,1}, E_{2,2}\}$ is a Riesz basis of $\mathcal{H}$ and it is a dual of itself.

One can check that $\{E_{1,1} + E_{2,1}, E_{2,2}\}$ is also a dual Riesz basis of $\{E_{1,1}, E_{2,2}\}$.

It is well known that if $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis and $\{y_j\}_{j\in\mathbb{J}}$ is a dual sequence of $\{x_j\}_{j\in\mathbb{J}}$ in a Hilbert space H, then $\{y_j\}_{j\in\mathbb{J}}$ is a Riesz basis which is the unique dual of $\{x_j\}_{j\in\mathbb{J}}$. The following example shows that this is not the case in Hilbert C^*-modules.

Example 3.5. Suppose $\mathcal{H}$ and $\mathcal{A}$ are the same as in Example 3.2.

Now let $x_j = e_j$ and

$$y_j = \begin{cases} e_1 & \text{if } j = 1, \\ e_j + j e_{j-1} & \text{if } j \neq 1. \end{cases}$$

One can verify that

$$x = \sum_{j\in\mathbb{N}} \langle x, y_j\rangle x_j$$

holds for all $x \in H$. But $\{y_j\}_{j\in\mathbb{N}}$ is not a Riesz basis, even not a Bessel sequence.

Note that even the dual sequence of a Riesz basis in Hilbert C^*-modules is a Bessel sequence, it still has the chance not to be a Riesz basis. We have the following example.

Example 3.6. Suppose $\mathcal{H}$ and $\mathcal{A}$ are the same as in Example 3.2.

Now let $x_j = e_j$ and

$$y_j = \begin{cases} e_1 + e_2 & \text{if } j = 1, 2, \\ e_j & \text{if } j \neq 1, 2. \end{cases}$$

Then $\{y_j\}_{j\in\mathbb{N}}$ is a Bessel sequence, and satisfies

$$x = \sum_{j\in\mathbb{N}} \langle x, y_j\rangle x_j$$

for all $x \in \mathcal{H}$.

Therefore, $\{y_j\}_{j\in\mathbb{N}}$ is a frame of $\mathcal{H}$. But obviously $\{y_j\}_{j\in\mathbb{N}}$ is not a Riesz basis.

The following result was obtained independently by Arambašić [1] and Jing [7]. One of the advantage of this characterization is that it is much easier to compare the norms of two positive elements than to compare two positive elements in C^*-algebras.

Proposition 3.7. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$ and $\{x_j\}_{j\in\mathbb{J}} \subseteq \mathcal{H}$ a sequence. Then $\{x_j\}_{j\in\mathbb{J}}$ is a frame of $\mathcal{H}$ with bounds C and D if and only if*

$$C\|x\|^2 \leqslant \left\| \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \right\| \leqslant D\|x\|^2$$

for all $x \in \mathcal{H}$.

Using the above characterization of modular frames we can easily prove the following result.

Proposition 3.8. *Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Let $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ be two Bessel sequences in $\mathcal{H}$. If $x = \sum_{j\in\mathbb{J}} \langle x, y_j\rangle x_j$ holds for any $x \in \mathcal{H}$, then both $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ are frames of $\mathcal{H}$ and $x = \sum_{j\in\mathbb{J}} \langle x, x_j\rangle y_j$ holds for all $x \in \mathcal{H}$.*

Proof. Let us denote the Bessel bound of $\{y_j\}_{j\in\mathbb{J}}$ by D_Y. For all $x \in \mathcal{H}$ we have

$$\begin{aligned} \|x\|^4 &= \left\| \left\langle \sum_{j\in\mathbb{J}} \langle x, y_j\rangle x_j, x \right\rangle \right\|^2 \\ &= \left\| \sum_{j\in\mathbb{J}} \langle x, y_j\rangle\langle x_j, x\rangle \right\|^2 \\ &\leqslant \left\| \sum_{j\in\mathbb{J}} \langle x, y_j\rangle\langle y_j, x\rangle \right\| \cdot \left\| \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \right\| \\ &\leqslant D_Y\|x\|^2 \cdot \left\| \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \right\|. \end{aligned}$$

Note that in the first inequality we apply the Cauchy–Schwartz inequality in Hilbert $\mathcal{A}$-module $l^2(\mathcal{A})$. It follows that

$$D_Y^{-1}\|x\|^2 \leqslant \left\| \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \right\|.$$

This implies that $\{x_j\}_{j\in\mathbb{J}}$ is a modular frame. Similarly, we can show that $\{y_j\}_{j\in\mathbb{J}}$ is also a frame of $\mathcal{H}$.

It follows directly from Proposition 6.3 in [5] that

$$x = \sum_{j\in\mathbb{J}} \langle x, x_j\rangle y_j$$

holds true for all $x \in \mathcal{H}$. □

To prove our main result, we also need the following lemma.

Lemma 3.9. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Suppose that $\{y_j\}_{j\in\mathbb{J}}$ and $\{z_j\}_{j\in\mathbb{J}}$ are dual frames of $\{x_j\}_{j\in\mathbb{J}}$ with the property that either $Rang(T_Y) \subseteq Rang(T_Z)$ or $Rang(T_Z) \subseteq Rang(T_Y)$, where T_Y and T_Z are the analysis operators of $\{y_j\}_{j\in\mathbb{J}}$ and $\{z_j\}_{j\in\mathbb{J}}$,respectively. Then $y_j = z_j$ for all $j \in \mathbb{J}$.*

Proof. Suppose that $Rang(T_Z) \subseteq Rang(T_Y)$. Then for each $x \in \mathcal{H}$ there exists $y_x \in \mathcal{H}$ such that

$$T_Y y_x = T_Z x.$$

Applying T_X^* on both sides, we arrive at

$$y_x = T_X^* T_Y y_x = T_X^* T_Z x = x,$$

and so $T_Y x = T_Z x$ for all $x \in \mathcal{H}$.

Equivalently,

$$\sum_{j\in\mathbb{J}} \langle x, y_j\rangle e_j - \sum_{j\in\mathbb{J}} \langle x, z_j\rangle e_j = 0,$$

i.e. $\sum_{j\in\mathbb{J}} \langle x, y_j - z_j\rangle e_j$. Hence $y_j = z_j$ for all j. □

We now give a necessary and sufficient condition about the uniqueness of dual frames in Hilbert C^*-modules. We also prove that if a frame has a unique dual frame, then it must be a Riesz basis.

Theorem 3.10. *Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame of $\mathcal{H}$ with analysis operator T_X, then the following statements are equivalent*:

(1) *$\{x_j\}_{j\in\mathbb{J}}$ has a unique dual frame*;
(2) *$Rang(T_X) = l^2(\mathcal{A})$*;
(3) *if $\sum_{j\in\mathbb{J}} c_j x_j = 0$ for some sequence $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$, then $c_j = 0$ for each j.*

In case the equivalent conditions are satisfied, $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis.

Proof. (2) ⇒ (1). Let $\{x_j^*\}_{j\in\mathbb{J}}$ be the canonical dual of $\{x_j\}_{j\in\mathbb{J}}$ with analysis operator T_{X^*}. Then $x_j^* = S_X^{-1} x_j$, where S_X is the frame operator of $\{x_j\}_{j\in\mathbb{J}}$.

Let $\{y_j\}_{j\in\mathbb{J}}$ be any dual frame of $\{x_j\}_{j\in\mathbb{J}}$ with analysis operator T_Y, then

$$Rang(T_Y) \subseteq l^2(\mathcal{A}) = Rang(T_X) = Rang(T_{X^*}).$$

By Lemma 3.9, $y_j = x_j^*$ for all j.

(1) ⇒ (2). Assume on the contrary that $Rang(T_X) \neq l^2(\mathcal{A})$.

By Theorem 15.3.8 in [17], we have

$$l^2(\mathcal{A}) = Rang(T_X) \oplus \operatorname{Ker} T_X^*.$$

Let P_X be the orthogonal projection from $l^2(\mathcal{A})$ onto $Rang(T_X)$, then

$$l^2(\mathcal{A}) = P_X l^2(\mathcal{A}) \oplus P_X^\perp l^2(\mathcal{A}).$$

Therefore $P_X^\perp l^2(\mathcal{A}) = \operatorname{Ker} T_X^* \neq \{0\}$.

Choose e_{j_0} such that $P_X^{\perp} e_{j_0} \neq 0$ and define an operator $U : P_X^{\perp} l^2(\mathcal{A}) \to \mathcal{H}$ by

$$Uw = \langle w, P_X^{\perp} e_{j_0} \rangle x_{j_0}.$$

Then U is an adjointable linear operator.

Now let $\{x_j^*\}_{j\in\mathbb{J}}$ be the canonical dual of $\{x_j\}_{j\in\mathbb{J}}$ with upper bound D_{X^*} and set $y_j = x_j^* + UP_X^{\perp} e_j$.

We have

$$\sum_{j\in\mathbb{J}} \langle x, y_j\rangle \langle y_j, x\rangle = \sum_{j\in\mathbb{J}} \langle x, x_j^* + UP_X^{\perp} e_j\rangle \langle x_j^* + UP_X^{\perp} e_j, x\rangle \tag{2}$$

$$\leqslant 2\biggl(\sum_{j\in\mathbb{J}} \langle x, x_j^*\rangle \langle x_j^*, x\rangle + \sum_{j\in\mathbb{J}} \langle P_X^{\perp} U^* x, e_j\rangle \langle e_j, P_X^{\perp} U^* x\rangle\biggr) \tag{3}$$

$$\leqslant 2\biggl(D_{X^*}\langle x, x\rangle + \sum_{j\in\mathbb{J}} \langle P_X^{\perp} U^* x, P_X^{\perp} U^* x\rangle\biggr), \tag{4}$$

which implies that $\{y_j\}_{j\in\mathbb{J}}$ is a Bessel sequence. Note that in inequality (3) we apply Lemma 2.7.

Now for any $x \in \mathcal{H}$,

$$\sum_{j\in\mathbb{J}} \langle x, UP_X^{\perp} e_j\rangle x_j = T_X^* \sum_{j\in\mathbb{J}} \langle x, UP_X^{\perp} e_j\rangle e_j = T_X^* \sum_{j\in\mathbb{J}} \langle P_X^{\perp} U^* x, e_j\rangle e_j = T_X^* P_X^{\perp} U^* x = 0.$$

This yields that $x = \sum_{j\in\mathbb{J}} \langle x, y_j\rangle x_j$ for all $x \in \mathcal{H}$. By Proposition 3.8, $\{y_j\}_{j\in\mathbb{J}}$ is a dual frame of $\{x_j\}_{j\in\mathbb{J}}$ and is different from $\{x_j^*\}_{j\in\mathbb{J}}$, which contradicts with the uniqueness of the dual frame of $\{x_j\}_{j\in\mathbb{J}}$.

(2) ⇔ (3). Obvious. □

We now characterize the dual sequences of Riesz bases in Hilbert C^*-modules. The following theorem is straightforward.

Theorem 3.11. *Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Let $\{y_j\}_{j\in\mathbb{J}}$ be a sequence of $\mathcal{H}$. Then the following statements are equivalent.*

(1) *$\{y_j\}_{j\in\mathbb{J}}$ is a dual frame of $\{x_j\}_{j\in\mathbb{J}}$;*
(2) *$\{y_j\}_{j\in\mathbb{J}}$ is a dual Bessel sequence of $\{x_j\}_{j\in\mathbb{J}}$;*
(3) *for each $j \in \mathbb{J}$, $y_j = S^{-1}x_j + z_j$, where S is the frame operator of $\{x_j\}_{j\in\mathbb{J}}$, and $\{z_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathcal{H}$ satisfying $\langle x, z_j\rangle x_j = 0$ for all $x \in \mathcal{H}$ and $j \in \mathbb{J}$.*

For the case of a dual sequence of a Riesz basis to be a Riesz basis, we have the following characterization.

Theorem 3.12. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a Riesz basis and $\{y_j\}_{j\in\mathbb{J}}$ a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then $\{y_j\}_{j\in\mathbb{J}}$ is a dual Riesz basis of $\{x_j\}_{j\in\mathbb{J}}$ if and only if for each $j \in \mathbb{J}$, $y_j = S^{-1}x_j + z_j$, where S is the frame operator of $\{x_j\}_{j\in\mathbb{J}}$, and $\{z_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathcal{H}$ with the property that for each $j \in \mathbb{J}$ there exists $b_j \in \mathcal{A}$ such that $z_j = b_j S^{-1} x_j$ and $\langle x, x_j\rangle b_j x_j = 0$ holds for all $x \in \mathcal{H}$.*

Proof. "⇒." Suppose that $\{y_j\}_{j\in\mathbb{J}}$ is a dual Riesz basis of $\{x_j\}_{j\in\mathbb{J}}$ and let $z_j = y_j - S^{-1}x_j$.

Then it is easy to see that $\{z_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathcal{H}$.

Now fix an $n \in \mathbb{J}$.

From $y_n = \sum_{j\in\mathbb{J}} \langle y_n, x_j\rangle y_j$ we can infer that $y_n = \langle y_n, x_n\rangle y_n$, i.e.

$$S^{-1}x_n + z_n = \langle S^{-1}x_n + z_n, x_n\rangle \bigl(S^{-1}x_n + z_n\bigr).$$

Consequently, we have

$$z_n = \langle z_n, x_n\rangle S^{-1}x_n + \langle S^{-1}x_n, x_n\rangle z_n + \langle z_n, x_n\rangle z_n.$$

To show that $\langle S^{-1}x_n, x_n\rangle z_n + \langle z_n, x_n\rangle z_n = 0$, it suffices to show that

$$\langle S^{-1}x_n, x_n\rangle\langle z_n, x\rangle + \langle z_n, x_n\rangle\langle z_n, x\rangle = 0$$

holds for all $x \in \mathcal{H}$.

Note that

$$x = \sum_{j\in\mathbb{J}}\langle x, y_j\rangle x_j = \sum_{j\in\mathbb{J}}\langle x, S^{-1}x_j\rangle x_j + \sum_{j\in\mathbb{J}}\langle x, z_j\rangle x_j = x + \sum_{j\in\mathbb{J}}\langle x, z_j\rangle x_j,$$

which implies that $\sum_{j\in\mathbb{J}}\langle x, z_j\rangle x_j = 0$ and so $\langle x, z_j\rangle x_j = 0$ for all $x \in \mathcal{H}$ and $j \in \mathbb{J}$.

Particularly, we have $\langle x, z_n\rangle x_n = 0$ for all $x \in \mathcal{H}$. This yields that

$$\langle x, z_n\rangle\langle x_n, z_n\rangle = 0 \quad \text{and} \quad \langle x, z_n\rangle\langle x_n, S^{-1}x_n\rangle = 0.$$

Equivalently, $\langle z_n, x_n\rangle\langle z_n, x\rangle = 0$ and $\langle S^{-1}x_n, x_n\rangle\langle z_n, x\rangle = 0$.

Therefore $z_n = b_n S^{-1}x_n$, where $b_n = \langle z_n, x_n\rangle$.

From $\langle x_n, z_n\rangle x_n = 0$, we have

$$\langle y, x_n\rangle\langle x_n, z_n\rangle\langle x_n, x\rangle = 0$$

for all $x, y \in \mathcal{H}$, which is equivalent to $\langle x, x_n\rangle\langle z_n, x_n\rangle\langle x_n, y\rangle = 0$, this implies that

$$\langle x, x_n\rangle b_n x_n = \langle x, x_n\rangle\langle z_n, x_n\rangle x_n = 0.$$

"$\Leftarrow$." Suppose now that for each $j \in \mathbb{J}$ there exists $b_j \in \mathcal{A}$ such that $z_j = b_j S^{-1}x_j$ and $\langle x, x_j\rangle b_j x_j = 0$ holds for all $x \in \mathcal{H}$. Then for all $x, y \in \mathcal{H}$ we have

$$\langle x, x_j\rangle b_j\langle x_j, y\rangle = 0.$$

Equivalently,

$$\langle y, x_j\rangle b_j^*\langle x_j, x\rangle = 0.$$

This implies that $\langle y, x_j\rangle b_j^* x_j = 0$ for all $y \in \mathcal{H}$.

Now for arbitrary $x \in \mathcal{H}$,

$$\begin{aligned}\sum_{j\in\mathbb{J}}\langle x, y_j\rangle x_j &= \sum_{j\in\mathbb{J}}\langle x, S^{-1}x_j\rangle x_j + \sum_{j\in\mathbb{J}}\langle x, z_j\rangle x_j = x + \sum_{j\in\mathbb{J}}\langle x, b_j S^{-1}x_j\rangle x_j = x + \sum_{j\in\mathbb{J}}\langle x, S^{-1}x_j\rangle b_j^* x_j \\ &= x + \sum_{j\in\mathbb{J}}\langle S^{-1}x, x_j\rangle b_j^* x_j = x,\end{aligned}$$

which implies that $\{y_j\}_{j\in\mathbb{J}}$ is a dual sequence of $\{x_j\}_{j\in\mathbb{J}}$.

With the similar argument in (2)–(4), one can easily see that $\{y_j\}_{j\in\mathbb{J}}$ is a Bessel sequence. It follows from Proposition 3.8 that $\{y_j\}_{j\in\mathbb{J}}$ is a dual frame of $\{x_j\}_{j\in\mathbb{J}}$.

To complete the proof, we need to show that $\{y_j\}_{j\in\mathbb{J}}$ is a Riesz basis of $\mathcal{H}$.

Suppose that $\sum_{j\in\mathbb{J}} a_j y_j = 0$, then we have

$$0 = \sum_{j\in\mathbb{J}} a_j\left(S^{-1}x_j + b_j S^{-1}x_j\right) = \sum_{j\in\mathbb{J}} a_j(1 + b_i)S^{-1}x_j.$$

Therefore $a_j(1 + b_j)S^{-1}x_j = 0$, i.e. $a_j y_j = 0$ for all j.

We now show that $y_j \neq 0$ for each $j \in \mathbb{J}$.

Assume on the contrary that $y_n = 0$ for some $n \in \mathbb{J}$. Then $z_n = -S^{-1}x_n$. It follows that

$$0 = \langle x, x_n\rangle b_n x_n = \langle x, x_n\rangle S z_n = -\langle x, x_n\rangle x_n$$

holds for all $x \in \mathcal{H}$.

In particular, letting $x = S^{-1}x_n$, we have $0 = -\langle S^{-1}x_n, x_n\rangle x_n = -x_n$, and so $x_n = 0$, a contradiction. This completes the proof. □

Corollary 3.13. *Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. If $\mathcal{A}$ is commutative, then every Riesz basis of $\mathcal{H}$ has a unique dual Riesz basis.*

Proof. Choose an arbitrary Riesz basis $\{x_j\}_{j\in\mathbb{J}}$ of $\mathcal{H}$. Suppose that $\{S^{-1}x_j + z_j\}_{j\in\mathbb{J}}$ is a dual Riesz basis of $\{x_j\}_{j\in\mathbb{J}}$, where S is the frame operator of $\{x_j\}_{j\in\mathbb{J}}$.

Then by Theorem 3.12, for each $j \in \mathbb{J}$ there exists $b_j \in \mathcal{A}$ such that $z_j = b_j S^{-1}x_j$ and $\langle x, x_j\rangle b_j x_j = 0$ holds for all $x \in \mathcal{H}$.

Since $\mathcal{A}$ is commutative, we have $b_j\langle x, x_j\rangle x_j = 0$ for all $x \in \mathcal{H}$ and $j \in \mathbb{J}$.

Let $x = S^{-1}x_j$. We have

$$0 = b_j\big\langle S^{-1}x_j, x_j\big\rangle x_j = b_j\big\langle x_j, S^{-1}x_j\big\rangle x_j = b_j x_j,$$

which yields that $z_j = b_j S^{-1}x_j = 0$. □

Note that under the conditions of Corollary 3.13, though a Riesz basis has a unique dual Riesz basis, it may have many dual frames. We have the following example.

Example 3.14. Let $\mathcal{A} = D_{2\times 2}(\mathbb{C})$ denote the C^*-algebra of all 2×2 complex diagonal matrices. Let $\mathcal{H} = \mathcal{A}$ and for any $A, B \in \mathcal{H}$ define

$$\langle A, B\rangle = AB^*.$$

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.

It is obvious that $\mathcal{A}$ is commutative.

Let $E_{i,j}$ be the 2×2 matrix with 1 in the (i, j)th entry and 0 elsewhere, where $1 \leqslant i, j \leqslant 2$.

Then $\{E_{1,1}, E_{2,2}\}$ is a Riesz basis of $\mathcal{H}$, and so it has a unique dual Riesz basis which is itself.

But the dual frame of $\{E_{1,1}, E_{2,2}\}$ is not unique. For example, one can verify that $\{E_{1,1} + \alpha E_{2,2}, \beta E_{1,1} + E_{2,2}\}$ is also a dual frame of $\{E_{1,1}, E_{2,2}\}$ for any $\alpha, \beta \in \mathbb{C}$.

The following example shows that the converse of Corollary 3.13 is not true, namely, if every Riesz basis of a Hilbert $\mathcal{A}$-module $\mathcal{H}$ has a unique dual Riesz basis, $\mathcal{A}$ is not necessarily commutative.

Example 3.15. Let

$$\mathcal{H} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \forall a \in \mathbb{C} \right\} \quad \text{and} \quad \mathcal{A} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix} : \forall a, b, c, d, e \in \mathbb{C} \right\}.$$

For any $A, B \in \mathcal{H}$ we define

$$\langle A, B\rangle = AB^*.$$

Then $\mathcal{H}$ is a $\mathcal{A}$-module.

Note that $\mathcal{A}$ is not commutative.

Let

$$E_\alpha = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\{E_\alpha\}$ is a Riesz basis of $\mathcal{H}$.

It is easy to see that any Riesz basis of $\mathcal{H}$ has the form of $\{E_\alpha\}$ for some nonzero $\alpha \in \mathbb{C}$. And one can also check that every dual Riesz basis of $\{E_\alpha\}$ for each nonzero α is unique.

It is also natural to ask

Question 3.16. If one Riesz basis of a Hilbert C^*-module $\mathcal{H}$ has a unique dual Riesz basis, does every Riesz basis of $\mathcal{H}$ have a unique dual Riesz basis?

Unfortunately, the answer to this question is negative. We have the following example.

Example 3.17. Suppose that $\mathcal{A}$ and $\mathcal{H}$ are the same as in Example 3.4.

We already know that $\{E_{1,1}, E_{2,2}\}$ is a Riesz basis of $\mathcal{H}$ whose dual Riesz bases are not unique.

Let I be the 2×2 identity matrix. Then $\{I\}$ is also a Riesz basis of $\mathcal{H}$, but the dual Riesz basis of $\{I\}$ is itself. Of course, it is unique.

Remark 3.18. From the above example we see that the numbers of elements of two Riesz bases for a Hilbert C^*-module may be different which can never happen in Hilbert spaces.

We end this paper with the following conjecture.

Conjecture 3.19. *Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. Then every Riesz basis of $\mathcal{H}$ has a unique dual Riesz basis if and only if the set $\{\langle a, b\rangle\colon \forall a, b \in \mathcal{H}\}$ is commutative, i.e., $\langle a, b\rangle \cdot \langle c, d\rangle = \langle c, d\rangle \cdot \langle a, b\rangle$ for all a, b, c, and d in $\mathcal{H}$.*

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References

[1] L. Arambašić, On frames for countably generated Hilbert C^*-modules, Proc. Amer. Math. Soc. 135 (2007) 469–478.
[2] R. Duffin, A. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952) 341–366.
[3] M. Frank, D. Larson, A module frame concept for Hilbert C^*-modules, Contemp. Math. 247 (1999) 207–233.
[4] M. Frank, D.R. Larson, Modular frames for Hilbert C^*-modules and symmetric approximation of frames, Proc. SPIE 4119 (2000) 325–336.
[5] M. Frank, D. Larson, Frames in Hilbert C^*-modules and C^*-algebras, J. Operator Theory 48 (2002) 273–314.
[6] D. Han, D. Larson, Frames, bases and group representations, Mem. Amer. Math. Soc. 147 (697) (2000).
[7] W. Jing, Frames in Hilbert C^*-modules, PhD thesis, University of Central Florida, 2006.
[8] W. Jing, D. Han, R. Mohapatra, Structured Parseval frames in Hilbert C^*-modules, Contemp. Math. 414 (2006) 275–287.
[9] I. Kaplansky, Algebras of type I, Ann. of Math. 56 (1952) 460–472.
[10] G. Kasparov, Hilbert C^*-modules: The theorem of Stinespring and Voiculescu, J. Operator Theory 4 (1980) 133–150.
[11] E. Lance, Hilbert C^*-Modules—A Toolkit for Operator Algebraists, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge University Press, Cambridge, 1995.
[12] V. Manuilov, E. Troisky, Hilbert C^*-Modules, Amer. Math. Soc., Providence, RI, 2005.
[13] J. Packer, M. Rieffel, Wavelet filter functions, the matrix completion problem, and projective modules over $C(\mathbb{T}^n)$, J. Fourier Anal. Appl. 9 (2003) 101–116.
[14] W. Paschke, Inner product modules over B^*-algebras, Trans. Amer. Math. Soc. 182 (1973) 443–468.
[15] I. Raeburn, S. Thompson, Countably generated Hilbert modules, the Kasparov stabilization theorem, and frames in Hilbert modules, Proc. Amer. Math. Soc. 131 (2003) 1557–1564.
[16] M. Rieffel, Morita equivalence for C^*-algebra and W^*-algebra, J. Pure Appl. Algebra 5 (1974) 51–96.
[17] N. Wegge-Olsen, K-Theory and C^*-Algebras—A Friendly Approach, Oxford University Press, Oxford, 1993.
[18] P. Wood, Wavelets and Hilbert modules, J. Fourier Anal. Appl. 10 (2004) 573–598.